# *Portfolio Decomposition: Modeling Aggregate Loss (Ratio) Distributions*

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## Abstract

There are two things that may be responsible for differences between the expected loss amount (for a contract or an entire portfolio) and the actual loss amount that is experienced: errors in estimating the long term average (parameter error) and random good or bad luck (process risk). This paper presents a method for using historical data to establish a model for process risk. Because the method does not require individual claim data, it is especially suitable for reinsurance companies for whom individual claim data may not be available. It can also be used when data is obtained from the aggregate policy year and accident year calls that are filed with rating bureaus.

Essentially, the method treats the experience of multiple contract years as if each year were a random sample drawn from a single population consisting of all the outcomes that could have occurred. The techniques of Time Series Decomposition are used to restate the historical data on an "as if current levels" basis. Decomposition is then used to isolate the random fluctuations (process variance). A generalization of the Central Limit Theorem allows a model of these fluctuations to be constructed. While derived from aggregate *portfolio* experience, the model's divisibility property allows it to be scaled down to accurately reflect the aggregate loss distribution of an individual contract or policy.

# Portfolio Decomposition: Modeling Aggregate Loss (Ratio) Distributions


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# Portfolio Decomposition: Modeling Aggregate Loss (Ratio) Distributions

Process variance enters the analysis of individual insurance (or reinsurance) contracts at two different levels:

1. In the form of the aggregate loss distribution associated with a cohort, or portfolio, of similar contracts (where *similar* may be inclusive enough to encompass the entire company's book of business), and
2. In the form of the aggregate loss distribution associated with individual contracts.

The first application frequently arises in conjunction with the assignment of surplus that is necessary when modeling return on equity (ROE). The second application arises during the analysis of loss sensitive contract provisions. The determination of the necessary supporting surplus that acts as a cushion against ruin requires a knowledge of the aggregate loss distribution for an entire portfolio of contracts or policies, whereas an analysis of loss sensitive contract provisions (e.g., retrospective rating of individual primary company policies, swing rating which is its reinsurance equivalent, sliding scale contingent commissions, profit sharing agreements, contributory dividends, etc.) requires a knowledge of the aggregate loss distribution at the individual policy or contract level.

There are two sources of uncertainty that may be responsible for differences between the estimated loss amount (for a contract or an entire portfolio) and the actual loss amount that is experienced. These are errors in estimating the long term mean (parameter error) and variation from the mean (process risk). This paper presents a method for using historical data to establish a model for process risk. Because the method does not require individual claim data, it is especially suitable for reinsurance companies for which individual claim data may not be available (e.g., proportional reinsurance is often ceded in the form of a bordereau that displays aggregate loss and premium cessions rather than individual claim detail). It can also be used when data is obtained from the aggregate policy year and accident year calls that are filed with rating bureaus.

The first section of the paper presents a very simplified overview of the method. In this section, several very significant assumptions are made without justification or support. Among these assumptions are that homogenous portfolios consisting of identical *exposure units* exist and that the concept of an exposure unit not only has meaning, but that these units can be counted according to some logical rule. The purpose of the first section is to provide a rationale for the more rigorous treatment that follows. Subsequent sections of the paper deal with how to relax the assumptions that were made in the simplified overview.

Throughout the paper, it is assumed that all companies conduct loss reserve adequacy testing and that a byproduct of this type of analysis is the segregation of individual policies or reinsurance contracts into more or less homogenous portfolios. It is further assumed that all of the data that is available to the reserving actuary is also available for modeling aggregate loss distributions. Note that, under many circumstances, the aggregate loss distribution, aggregate pure premium distribution, and aggregate loss ratio distribution differ only by a scale transformation. Where the distinction is not significant, the three terms have been used almost interchangeably.

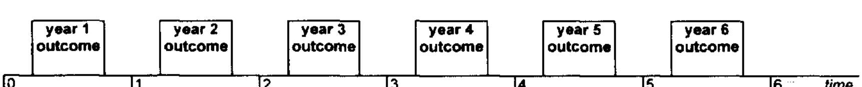
*A Very Simplified Overview of the Method*

An ideal insurance (or reinsurance) portfolio consists of a large group of identical exposure units. Such a portfolio might consist of identical insurance policies (or reinsurance treaties), all covering the same period of time. If the group of policies all renew coverage upon expiration, the renewal portfolio can be considered as if it were a second year in the life of the original portfolio[1]. In an even more relaxed sense of the definition of a portfolio, the indistinguishable nature of the exposure units (i.e., identical) allows a portfolio to live from year to year, even if the particular constituents (i.e., the particular policyholders or ceding companies) differ from year to year. Later, the definition will be further relaxed to allow the *number* of constituents to vary from year to year[2].

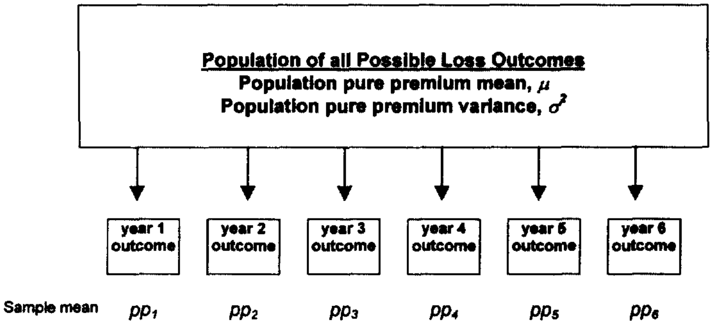
During each year of the portfolio's existence, the $N$ exposure units that make up the portfolio will each experience a loss outcome (where "no loss" may be the most common outcome). The total of the loss outcomes divided by the number of exposure units is the pure premium outcome for the portfolio. In particular, if the loss outcome for the j[th] exposure unit during year $t$ is given by $L_{jt}$, then the pure premium outcome for the year, $pp_t$, is given by

$$pp_t = N^{-1}\sum_{j=1}^{N} L_{jt}.$$

The historical experience of a portfolio is displayed below in timeline form.



If the population does not change over time, the portfolio's historical experience over several years can be considered to be different samples, all drawn from the same population of all possible outcomes. This alternative interpretation is displayed below.




If there are $N$ exposure units in the portfolio, then each sample is of size $N$ (one selected outcome for each exposure unit). The mean of each sample is the incurred pure premium for the year.

If $N$ is large, the Central Limit theorem tells us that the sample means will be distributed Normally with a mean equal to the population mean, $\mu$, and with a variance equal to the variance of the population, $\sigma^2$, divided by $N$. The mean of the historical portfolio pure premiums (mean of the sample means), $m$, can be used as an estimate of the mean of the population. The variance of the portfolio pure premiums from year to year (variance of the sample means), $s^2$, times the number of exposure units, $N$, can be used to estimate the population variance.

Now consider next year's experience for a similar portfolio that consists of $M$ identical exposure units (where $M$ is not necessarily equal to $N$). The portfolio pure premium (i.e., the mean of a sample of size $M$, drawn from the population) is a random variable. As long as $M$ is large, the Central Limit Theorem can be used to estimate its distribution. The distribution of next year's $M$ exposure unit portfolio pure premium will be Normal, with

$$\text{mean} = \mu = m,$$

and

$$\begin{aligned}\text{variance} &= \sigma^2/M \\ &= [N\,s^2]/M \\ &= [N/M]s^2 \\ &= s^2/\alpha_M,\end{aligned}$$

where

$$\alpha_M = [M/N].$$

Notice that, while the historical experience was used to estimate the population mean and variance, the population parameters are simply abstractions. If all that is desired is to use historical experience to determine the distribution of next year's pure premium, then the population parameters need not ever be explicitly determined. Next year's distribution will be Normally distributed with a mean equal to the mean of the historical means, $m$, and a variance equal to the variance of the historical pure premiums divided by the ratio of the prospective portfolio size to the size of the historical portfolios.

The remainder of this paper deals with how to relax many of the assumptions, both implicit and explicit, that were made above so that more realistic situations can be addressed. More specifically, the following assumptions must be addressed:

1. Exposure units can defined in such a way that portfolio size can be measured,
2. If the population of possible outcomes is not stationary (i.e., if it changes over time), then there exists a transformation (i.e., a restatement of the historical experience) that makes it possible to treat the population as if it were stationary,
3. If the measure of exposure units is not stationary, then there exists a transformation (i.e., a restatement of the historical portfolio size) that makes it possible to treat the units as if they were stationary[3],
4. The Central Limit Theorem can be generalized so that the parameters of the population can be estimated *even if the samples are not all of the same size*,
5. There is a way in which to extend the method so that individual policy or contract aggregate loss distributions can be modeled even if $M$ is too small to satisfy the requirements of the Central Limit Theorem, and that
6. The method can be applied to situations in which the portfolio is not perfectly homogeneous (i.e., when it reflects a mixture of different outcome spaces).

## *Assumptions and Issues that Must be Addressed*

### ***Portfolio Size: Exposure Units***

For the proposed method to work, it is not only necessary to have a well defined, homogeneous portfolio but one must also be able to measure its size. In the overview, the sizes of the portfolios were measured in terms of independent exposure units. Clearly the concept of an **exposure unit** is an abstraction. As such, it will not be readily quantifiable for many (if not all) portfolios. To illustrate why this is true, consider two different homogeneous portfolios.

Portfolio 1 consists of basic limits liability policies covering 200,000 vehicles, each with exactly the same manual classification. Vehicle years is an obvious measure of the exposure. All else being equal, associating twice the number of exposure units to a second portfolio consisting of 400,000 vehicles is consistent with our intuitive notion of what the abstraction, "exposure," means.

As the homogeneity condition is relaxed, vehicles with different manual classifications will enter the portfolio as will drivers with different driving records. In this case, vehicle years becomes a less obvious measure of exposure. To the extent that the manual rate relativities reflect the expected loss amounts, premium might actually be a better proxy for the more abstract concept of exposure for the purpose of determining the size of the "relatively" homogeneous portfolio.

Portfolio 2 consists of 60 excess of loss medical malpractice reinsurance contracts. Each contract covers losses in the layer $750,000 excess of $250,000 per claim arising from any one of the individual policies that the primary company issues to small hospitals. Selecting an appropriate measure of exposure for this portfolio is not as straightforward as it was for Portfolio 1. To the extent that the reinsurance contracts are identical, *number of contracts* might be an acceptable measure of exposure. If the contracts differ in the number of policies issued, *policies issued* or the *number of covered physicians* might be a more appropriate measure of the portfolio exposure. If the individual policyholders are not identical, the number of surgical procedures performed might be used as a measure of exposure.

As with the automobile portfolio, as the homogeneity condition is relaxed, all of the proposed measures of exposure begin to lose their luster. None of the measures seems appropriate if different surgical specialties are covered within a portfolio. Again, when the portfolio is allowed to reflect a mixture of exposures, relying upon actuarially sound rates and using premium as a proxy for the more abstract "number of exposure units" is reasonable.

Three things need to be kept in mind when measuring the size of a portfolio.

1. When the portfolio consists of similar but not identical exposure units, premium may be a better measure of the size of the portfolio than the exposure base that is associated with pricing the underlying coverage.

2. Because there is usually more than one candidate that can be used as a proxy for the size of the portfolio and because they will not always produce the same number (e.g., the number of contracts in Portfolio 2 is not necessarily equal to the number of policies, physicians, or surgical procedures), the size of a portfolio is only a relative number. When determining whether the sample size, $N$, is large enough for the Central Limit theorem to be used, no absolute standard exists. This issue is addressed more thoroughly in another section of the paper.

3. If the purpose of measuring the size of a portfolio is to compare it to other similarly distributed portfolios, restated premium (at current rate and exposure base levels) can be an appropriate measure of size, even if it is not a good measure of exposure in any absolute sense.

### *Portfolio Size: Reinsurance Treaty Shares*

Reinsurance treaty portfolios introduce an additional complication when size is measured because it is common (especially in the broker market) for several reinsurers to each take a proportional **share of a treaty**. For example, the reinsurer may accept 30% of the total reinsurance premium for the layer \$750,000 excess of \$250,000 in return for which it pays only 30% of each loss in the layer. Intuitively, it is clear that a portfolio consisting of 25% shares of four identical and independent contracts will not have the same pure premium distribution as a portfolio consisting of 50% shares of two of these contracts. The reason is that the exposure making up the first portfolio composition reflects more *spread*. As shown in Appendix A, the variance of any share of a single contract is the same as the variance of the entire contract. Appendix A also shows that the *effective size* of a reinsurance portfolio that is made up of $m$ individual contracts is given by

$$N_{portfolio} = \left( \sum_{j=1}^{m} \left( n_j^2 / N_j \right) \right)^{-1}$$

where

$N_j$ is the size of 100% of the $j^{th}$ contract (in units of exposure, however measured),

$n_j$ is the percentage of the reinsurer's portfolio volume contributed by the $j^{th}$ contract,

$$n_j \equiv \frac{S_j N_j}{\sum_{k=1}^{m} S_k N_k},$$

where $S_j$ is the percentage share taken by the reinsurer.

In our example, all of the contracts were the same size so $N_j = N$ for every $j$. If four 25% shares are taken, $N_{portfolio} = 4N$. In other words, taking equal *amounts*, *SN*, from four independent contracts of equal size is the same thing as taking the same amount from a single contract that is four times the size. In the case where two 50% shares are taken, $N_{portfolio} = 2N$. Even though both of these situations produce portfolios that have the same premium, the one with twice as many independent contracts acts as if it were twice the size of the other portfolio. The reader is encouraged to consult Appendix A concerning the effective size of a reinsurance portfolio.

### *Restatement to Produce a Stationary Population*

The sample mean is determined by dividing the sum of the loss outcomes by the number of exposure units in the portfolio. When premium is used as a proxy for exposure, the sample means are loss ratios.

As a result of inflationary trends, the loss outcome (dollars of loss) corresponding to a particular event may depend upon when the event occurs and when the loss is settled. To the extent that there is such a time dependence for the portfolio under consideration, the experience of successive years cannot be considered to be the same as multiple loss outcome samples taken from the same population. Clearly, the population of possible outcomes is not stationary over time.

The exposure base used in pricing the primary policy may also be inflation sensitive. For example, *wages* (which are inflation sensitive) are used in Workers Compensation, and *revenues* have been used to rate some liability policies. To the extent that the exposure base inflates at the same rate as the corresponding losses, *pure premium* will be invariant. When such is the case, the portfolio pure premium outcomes from several years can be treated as sample means of multiple samples drawn from a single population ***even if the distribution of the loss outcomes is time dependent.***

If the loss and exposure base inflation trends are not equal, then the pure premium outcomes from different years cannot be taken as the means of samples drawn from a single population. If premium has been selected as the measure of the size of the portfolio (remember, the exposure base that is used to price individual policies need not be adopted as the measure of portfolio exposure units), changes in *rate level adequacy* introduces another factor that can invalidate the assumption that the portfolio incurred loss ratios from successive years can be treated as the means of multiple samples drawn from a single population.

The usual manner in which changes in rate level adequacy is addressed is by restating all historical data on an "as if current levels" basis. If there is sufficient information available, such a restatement is the preferred course of action. If such a procedure could be carried out, individual losses would be trended to a common point in time, the pricing exposure base would be inflated to a common point in time (while not necessarily the same point in time as the losses, using the same common time for both is logical, especially if the common time is the midpoint of the prospective period for which rates are being determined), and historical rates would be replaced by the current rate.

Once the restatement process had been carried out, pure premiums and incurred loss ratios would be equally good measures of the mean of the sample, although loss ratios offer more immunity to changes in the class mix than pure premiums do. The restatement would allow samples drawn from the portfolios of different years to be treated as if they were drawn from a single population. This, of course, is exactly what is assumed when the experience of many years is restated and averaged for the purpose of determining an experience based rate.

When there is insufficient information for restating the historical details (e.g., proportional reinsurance) or when such a process would involve a prohibitive amount of work, an alternative approach, based upon the techniques of time series decomposition, can be used (see, for example, Makridakis [1]). Time series decomposition is based upon the assumption that, at any time $t$, the portfolio loss ratio (or pure premium) can be expressed as the product of three functions of time, $T_t$, $C_t$, and $R_t$. In symbolic form,

$$ILR_t = T_t x C_t x R_t.$$

$T$ reflects the long-term expected loss ratio trend. In insurance terms, it reflects the degree to which the pricing exposure base trend is exceeded by (for positive trend) or exceeds (for negative trend) the loss trends (both frequency and severity). $C$ reflects cyclic changes in the expected loss ratio. It is in $C$ that the insurance (reinsurance) pricing cycle would be reflected. $R$ reflects the random fluctuations (process risk) that cause the actual incurred loss ratio to differ from its expected value. $R$ has a mean equal to 1.000 (i.e., no long term deviation from the expected loss ratio). For a sufficiently large portfolio, $R$ will be symmetrically distributed (good and bad luck of a given magnitude to be equally likely) with most of its weight near the expected value (large deviations from the expected are much less likely than small deviations in either direction). We note that while none of the insurance interpretations (or the *a priori* notions regarding the shape of $R$ as a function of $t$) are critical to the method of times series decomposition, they do form the basis of reasonableness tests to which the results must be subjected.

While the usual objective of time series decomposition is to isolate the non-random components, $TxC$, the objective in the portfolio decomposition application is to isolate the random component and then determine its distribution. An example illustrates how the technique of series decomposition can be used to isolate $R_t$.

*Exhibit 1* demonstrates the method using some rather well behaved (and completely fictitious) data. The column of data titled "ILR" displays ultimate (reinsurance portfolio) incurred loss ratios for each of the 38 contract years, 1960-1997. While the data is fictitious, it could have been the byproduct of a loss reserve adequacy test for the portfolio. In practice, some of these ultimate loss ratios would be estimates as of the most recent valuation date but, for now, we will assume that they are actual ultimate incurred loss ratios, thereby sidestepping the issue of potential bias introduced by the estimation process (both with respect to the loss ratios and their distribution). These issues will be addressed later.

The 38 loss ratios make up a time series. Decomposition of this series begins by observing that the random errors tend to offset each other over time. If a moving average is taken over a suitably large number of years, the random fluctuations should cancel, leaving only the effects of trend and cycle. If a five year period is suitably long, taking a five-year moving average of the *ILR* should result in a series of 34 $T_t x C_t$'s where the time associated with each of the five-year moving averages corresponds to the third year included in the moving average (e.g., the average of the first five years, 1960-1964, was assigned to 1962). Of course, a five year period may not be sufficiently long to remove all of the process variance, but its effect should be greatly diminished by taking a five year moving average. Note that, while a seven year moving average might result in less surviving process variance, it would also decrease the number of TxC points from 34 to 32. There is always an issue of balance regarding the number of points to be included in the moving average and the number of points that remain for analysis (each additional point included in the moving average reduces the number of remaining points). Including more points in the moving average reduces the amount of residual randomness, which is desirable; at the same time, a reduction in the number of moving averages reduces the ability of the analysis to detect rapid, non-random, changes (i.e., there is a reduction in resolution when each moving average reflects a longer period of time).

The trend component can be isolated by fitting a trend curve to the points, $T_t x C_t$. In this example, the trend model,

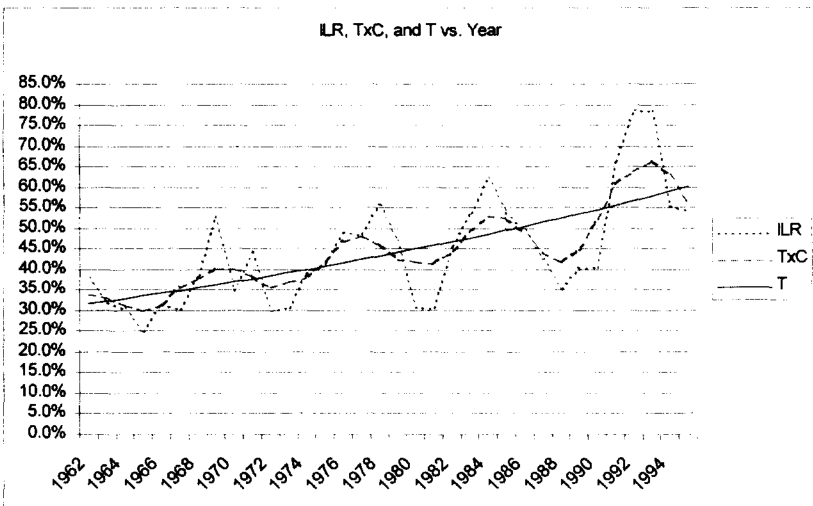
$$T_t = T_{1960}(1+trend)^{(t-1960)},$$

was fit to the TxC series using the method of least square errors. The result was

$$T_{1960} = 30.5\%,$$

$$trend = 1.96\%.$$

The graph, below, displays the original data (ILR), moving average (TxC) and trend component (T) corresponding to the data that is displayed in Exhibit 1.




The degree to which TxC differs from the graph of T can be attributed to the presence of a cycle.

Because the data for this example was generated from known functions, we can compare the model to "reality," something that cannot be done in practice. The fictitious data was generated from a trend curve,

$$T_t = 0.30(1.020)^{(t-1960)}.$$

The model overestimated the initial loss ratio and underestimated the annual trend. There are two reasons for these errors. First of all, the underlying cycle was eight years long. As a result, the 38 years did not reflect five full cycles. In this example, the inclusion of a partial cycle introduced a small bias toward understating the long-term trend for the 38 years. Second, while the random component (generated by means of a random number generator) was rescaled so that its 38-year mean was unity, it did not have a mean of unity over every five year period. As a result, there was some residual process variance left in the moving averages. In practice, cycles are not regular (varying in both length and severity) and the random variance factor cannot be expected to average to unity over short periods of time. Because the model of $T$ and $C$ (which is nothing more than the moving average at time $t$ divided by trend component at time $t$) can be distorted by the mixing of cycle with trend and by the presence of residual process variance, both components should be subjected to a reasonableness test.

Once a trend rate and cycle function have been selected, each of the historical loss ratios can be restated to reflect a common point in time (i.e., placed at a common point in the cycle and trended to a common point in time). These restated loss ratios should differ from each other only as a result of process variance. The *restated* loss ratios make up 34 (or 38 if C can be extrapolated) samples taken from the same population. For consistency, Exhibit 1 displays both the cyclic component and the trend component in the form of indices. The restated ILR is determined by multiplying each data point, $ILR_t$, by a restatement factor,

$$(TrendIndex_{1997}/TrendIndex_t)(100/CycleIndex_t).$$

This effectively restates each loss ratio to an "as if 1997 inflation levels" and "as if at the midpoint of the pricing cycle" basis. The fact that 1997 does not appear to be the midpoint of a pricing cycle is not important. The fact that all the loss ratios are restated to a common point in the cycle and common level of inflation is all that is important [4].

### *Restatement to Produce a Consistent Measure of Exposure (Portfolio Size)*

Were it not for the possibility that the portfolio may have changed in actual size (e.g., as a result of true growth or contraction), a restatement of the historical loss ratios to a common level of rate *adequacy* would be all that was necessary. The population loss ratio mean and loss ratio variance could be estimated directly from the restated incurred loss ratios. When there is a possibility that the portfolio size has changed over time (in an absolute sense), it is necessary to remove the impact of changing *rate levels* (not just rate level adequacy) and *pricing exposure base inflation* from the measurement of the historical portfolio size. Once these changes in the measurement "yardstick" have been removed, true size changes can be determined and reflected in the parameter estimates.

Where the sample means (loss ratios) were concerned, only the extent to which premiums and losses changed by different factors was relevant. In other words, only changes in rate level adequacy were relevant. Premiums for a portfolio can change even if relative rate adequacy does not. In some cases, these changes may be indicative of a change in the size of the underlying exposure, and in some cases they may not be. An example of the former would be a change in premium resulting from an increase in the number of identical policies written. This clearly reflects a change in the size of the portfolio. An example of the latter would be a rate change. Whether or not the rate change resulted in a change in rate level adequacy, it clearly does not reflect a change in the size of the portfolio. If premium is selected as the proxy for exposure, then restatements reflecting more than rate level adequacy are necessary.

Clearly, premiums need to be restated to reflect a common *rate level* before they can be used to measure the size of a portfolio. Premiums can also change as a result of exposure base inflation, even in the absence of a rate change. As long as rate level adequacy does not slip, an increase in insured value will have no effect on the aggregate loss ratio distribution. Because of this, historical premiums must be restated to reflect a common level of exposure base inflation as well as a common rate level before they can be used as a measure of true portfolio size.

Ideally, historical premiums should be directly restated Unfortunately, a rate level history for an entire portfolio is rarely available. It is more likely, when full information is lacking, that more will be known concerning losses than rate changes or the pricing exposure base. Frequently, information regarding severity trends may be derived during loss reserve adequacy testing. Econometric data may provide additional information regarding changes in loss severity (e.g., Consumer Price Data or combinations of selected indices such as the Masterson Indices) and frequency. Industry data may be a source of information regarding frequency trends (e.g., the National Council On Compensation Insurance Annual Statistical Bulletin [2]).

Once the impact of changes in rate level adequacy has been eliminated by means of restatement to a common level of rate adequacy, any remaining premium changes (the product of rate and exposure changes) will reflect loss trends (including law and/or benefit changes) exactly. For example, if the ratio of two successive TxC components is 0.90, that means that rate level adequacy has increased by 11.1% (i.e., 0.90 = 1/1.111). If, during the same time, inflated 20%, one can conclude that premiums must have increased by a total of 33.3%. Here, 33.3% not only offsets the inflation but also results in more adequate rates.

In more mathematical detail, let $L_X$ and $Rate_X$, be the Loss and Rate, for year $X$. Further, let $Exposure_X$ be the exposure base used to determine the premium during year X. Note that it would be very unusual for $Exposure_X$ to be anything more than a proxy for the true exposure.

In terms of these variables, an 11.1% increase in rate level adequacy implies

$$L_{X+1}/(Rate_{X+1}*Exposure_{X+1}) = 0.90*[L_X/(Rate_X*Exposure_X)].$$

Note that, to the extent that the actual number of exposure units change, the loss amount will change proportionately, leaving the rate level adequacy unchanged. In other words, the ratio *L/Exposure* is invariant to changes in the actual number of exposure units. Changes in rate level adequacy reflect factors that can distort the measurement of portfolio size (*i.e.*, it reflects only those factors that do *not* depend upon the portfolio size). The implied change in premium (as far as rate level adequacy is concerned) can be attributed to two factors: changes in the rates themselves and those changes in the proxy used to measure exposure (*i.e.*, the exposure base) that are not related to true changes in size of the population.

In the example, if losses per exposure unit increase by 20%, and $\Delta$ represents the unknown fractional change in the combined rate and exposure base product (that portion that is unrelated to true changes in the portfolio size), then

$$0.90^*[L_X/(Rate_X{}^*Exposure_X)] = 1.20^*L_X/([1+\Delta]Rate_X{}^*Exposure_X).$$

Solving for $\Delta$,

$$[1+\Delta]=1.20/0.90 = 1.333 \text{ or a } 33.3\% \text{ increase.}$$

The conclusion is that the first 33.3% of premium increase must be attributed to changes other than changes in the number of exposure units (i.e., the portfolio size). Any remaining premium change (which might be a decrease) can be attributed to a change in the size of the portfolio.

Another simple example, based upon a different set of well behaved fictitious data, illustrates the required restatement process. The first four columns of Exhibit II display hypothetical data as it might appear together with the results of a series decomposition of the type performed in Exhibit I. As was the case for Exhibit I, the new data is well behaved but no more realistic than the data underlying Exhibit I. It was selected to simplify the illustration of the premium restatement procedure.

For each contract year, there would be a record of the historical premium (restated to reflect reinsurance shares, if necessary), the *TxC* loss ratio component, and a set of loss indices, $\{LossIndex_t\}$, that reflect both severity and frequency. It is assumed that such indices can be obtained. Their derivation is outside the scope of this paper.

Exhibit IIa displays information that could only be known to a privileged, or "all-knowing," observer. The additional information is disclosed only to demonstrate the validity of the restatement procedure. In particular, the privileged observer knows:

- what the true historical exposure (the abstraction) was,
- that the exposure base used to price the primary policies, while related to the true exposure, was inflating relative to the true exposure at a rate of 3.0% per year (e.g., the true exposure might have been products sold but dollars of sales may have been used to determine the premium for individual policies),
- what the historical rates were (where the historical rate times the pricing exposure base equals the historical premium), and
- that losses began as 2.75 times the true exposure and losses per exposure unit increased at a rate of 10% per year.

Of course, none of this detail is directly disclosed by the portfolio data, only those columns that are displayed on Exhibit II would be known.

As was previously discussed, changes in TxC reflect changes in rate level adequacy. Column (8) quantifies these annual changes. Column (10) is the factor necessary to restate historical premiums on an "as if current rate level adequacy" basis. Similarly, Column (11) consists of factors that allow losses to be restated on an "as if current loss levels" basis. The composite factor displayed in Column (12) is simply the loss factor divided by the rate level adequacy factor. Finally, Column (13) displays the restated historical premiums. To see that the year-to-year changes in restated premium mirror the percentage change in the true exposure, Column (13) was rescaled, making the 1960 size equal to 100.00. As can be seen, all of the other entries are then equal to the true exposure. As a result, the relative restated premium is the same as the relative true exposure (i.e., for any pair of years, the ratios are the same). Changes in the true exposure are indicative of changes in portfolio size, N, from year to year.

***Central Limit Theorem Estimators when the Samples are of Different Size***

It is unlikely that a given portfolio would remain constant in size over time. The number of policies or reinsurance contracts in a portfolio frequently change over time, giving rise to **samples of different sizes**. The Central Limit Theorem makes statements about the distribution of sample means when the samples are all of size $N$. These statements must be generalized when the samples are of different sizes. In particular, while the distribution of the mean for any single sample of size $N$ continues to be Normal with mean $\mu$ and variance $\sigma^2/N$, estimators of the population mean and variance are no longer equal to the mean, $m$, and $N$ times the variance of the sample means, $Ns^2$, of previously drawn samples when the samples are of different sizes.

Appendix B provides support for the following *generalized estimators* of the population mean and variance.

$$Estimator(pp_{population}) = m' = \sum_{j=1}^{T} \alpha_j pp_j \Big/ \sum_{j=1}^{T} \alpha_j$$

$$Estimator(Var[pp_{population}]) = N_1 (T-1)^{-1} \sum_{j=1}^{T} \alpha_j (pp_j - m')^2.$$

Where

$pp_j$ is the pure premium experienced by the $j^{th}$ sample (i.e., during the $j^{th}$ year),

$\alpha_j$ is the relative size of the $j^{th}$ sample (i.e., $\alpha_j = N_j/N_1$),

T is the number of samples (i.e., the number of years), and

$N_1$ is the size of the first sample.

For large sample size, the distribution of the sample mean is Normal with variance equal to $Var[pp_{population}]/N$. It, therefore, follows that the estimate of the variance of the pure premium (i.e., the sample mean) for a sample of size $N_1$ is given by

$$Var[pp_{population}]/N_1 = (T-1)^{-1}\sum_{j=1}^{T}\alpha_j(pp_j - m')^2.$$

The variance of the $k^{th}$ sample is $N_1/N_k$ as large, or

$$Var[pp_{population}]/N_k = (T-1)^{-1}\sum_{j=1}^{T}\alpha_j(pp_j - m')^2/\alpha_k,$$

Since $N_1/N_k = 1/\alpha_k$.

*A More Realistic Example*

Exhibit III illustrates an application of the methodology to a portfolio consisting of similar general casualty excess of loss reinsurance treaties. Over the course of twenty-seven years (contract years 1969-1995) portfolio premiums have increased from approximately $2,000,000 to almost $100,000,000. During the same period, incurred loss ratios ranged from a low of less than 40% to a high in excess of 350%. Some of this loss ratio volatility was due to process risk and some (perhaps most) was due to the presence of at least two reinsurance pricing cycles.

Supporting Exhibit IIIa displays the actual loss ratios for the twenty-seven year period together with the series decomposition that allows the loss ratios to be restated on an "as if common rate level adequacy" basis. The first graph displays the actual data, *TxC* component, and isolated trend component, *T*.

The second graph allows for a reasonableness test of the decomposition. Rather than graphing *C vs. Year*, the graph displays the reciprocal of C. When a soft market forces rate level adequacy to slip, incurred loss ratios increase. As a result, C moves in the opposite direction from rate levels. Graphing the reciprocal of C makes the graph more intuitive. The soft reinsurance market of the early to mid 1980's is clearly evident in the graph.

Because this is a reinsurance portfolio and because the reinsurer took less than 100% shares of most of the contracts that it wrote, its premium is not a true indicator of the size of the reinsured entities. Exhibit IIIb displays individual account premium detail for the 1971 contract year. During that year, 35 clients were reinsured. While the reinsurer earned premium equal to $2,120,969, that amount represented over $30,000,000 of premium on a 100% basis. As a result of the manner in which the reinsurer authorized shares, the reinsurer experienced the same variability as if it had reinsured a single $14,193,426 client –regardless of the share taken. While no supporting exhibits were prepared for the other years, the premium for each of the other contract years was similarly adjusted.

Exhibit IIIc begins with the historical premiums (after adjustment to reflect reinsurance shares) and concludes with historical premiums restated on an 'as if current rate levels" premium. Because only relative portfolio size is important, the restated premiums were rescaled to make the 1971 premium equal to 1.000. Column (14) displays the $\alpha$'s of the estimator formulas.

All of the components are brought together in Exhibit III, where the two estimators, $m'$ and $Var$, were determined. As previously shown, these estimators are the estimated mean and variance of the loss ratio distribution for a portfolio the size of the first one drawn; more precisely, they correspond to the size of the portfolio that is assigned a relative weight equal to 1.000 (i.e., $Var = Var[ILR|\alpha{=}1.00]$). The expected mean loss ratio is independent of portfolio size, and the expected variance of any portfolio of size $\alpha_j$ (measure relative to the initial sample size) is given by $Var/\alpha_j$.

The objective of this exercise was to find the distribution of the process variance, $R$, not the distribution of the incurred loss ratios. If each of the restated $ILR$'s in Column (4) is divided by the mean, $m'$ (which is nothing more than an estimate of the restated $TxC$ component), the result is a column of $R$'s. The mean is automatically equal to 1.000 and the variance is equal to the variance of the loss ratios divided by the square of the mean loss ratio. Finding the $R$'s from the $ILR$'s is nothing more than a scale transformation.

Exhibit III concludes with a display of the expected variance, both for the $ILR$ distribution and for the corresponding $R$ distribution, for portfolios of various sizes. All of the $ILR$ distributions are Normal with mean, $m'$, and the indicated variance, while all of the $R$ distributions are Normal with a mean of unity and the indicated variance.

While the Normal distribution is particularly easy to use and it has an intuitive appeal, wanting it to work isn't evidence that it is an appropriate model. The test displayed in Exhibit IIId begins with the assumption that the random component, *R*, is Normally distributed with the estimated parameters. While a failure to reject the null hypothesis is not proof that the hypothesis is true, the result of the $\chi^2$ test is evidence that there is no compelling reason to reject the hypothesis.

*An Alternative Model: The Gamma Distribution*

The parameters of the Normal model, $\sigma$ in particular, can be adjusted to reflect changes in the size of a portfolio. This adjustment follows directly from the Central limit theorem. If $\sigma_j$ is known, then $\sigma_k$ is given by

$$\sigma_j\sqrt{N_j/N_k}$$

where $N_j/N_k$ is the ratio of the size of the $j^{th}$ portfolio to the size of the $k^{th}$ portfolio. One assumption that underlies the Central Limit Theorem is that $N$ is large. That the adjustment is not appropriate when $N_k$ is very small quickly becomes obvious. As $N_k$ decreases, the standard deviation of the model distribution becomes very large. As a result, the model allows for a significant probability of negative loss ratios for small portfolios (e.g., the size of a single account or policy).

The more realistic example (Exhibit III) indicates that $\sigma_I$=0.097 when $N_I$ = \$224,626,501 (the restatement of \$14,193,426). A typical client (e.g., #2 on Exhibit IIIb) has approximately \$550,602 of premium on a 100% basis. The

Normal model predicts a random component, R, with a standard deviation equal to 0.492 (=0.097 *(14,193,426/550,602)$^{1/2}$ ). The probability that $R$ will be negative (i.e., fall more than approximately 2.00 standard deviations below the mean) is 0.0212. A significant probability that an account's loss ratio will be negative is not a realistic expectation for most lines of business.

There is another model that is almost Normal for large $N$, but whose reproductivity/divisibility properties allow it to scale down to the size of an individual account. This is particularly useful if the model is to be derived at the portfolio level and applied at the individual account level. The model is the Gamma distribution, whose pdf is given by

$$\Gamma_{a,r}(x) \equiv \frac{a^{r}}{\Gamma(x)} x^{r-1} e^{-ax}; \begin{bmatrix} x>0 \\ a>0 \\ r>0 \end{bmatrix}.$$

The Gamma distribution has the following properties (Hewett [3]):

- It is divisible. That is to say that if $\Gamma_{a,r}(x)$ is the appropriate model for the aggregate distribution of a portfolio consisting of $N$ independent units of exposure, then $\Gamma_{a,rM/N}(x)$ is the appropriate model for the aggregate distribution of a portfolio consisting of $M$ independent units of exposure.
- The mean of $x$ when $x$ is distributed $\Gamma_{a,r}(x)$ is $r/a$ from which it follows that,
- When the mean of the distribution is known to be unity, $a = r$, and
- When $x$ is distributed $\Gamma_{a,r}(x)$, the variance of $x$, $\sigma^2$, is $r/a^2$, or $1/r$ when the mean is known to be unity.
- The mode of x, when $x$ is distributed $\Gamma_{a,r}(x)$ is given by $(r-1)/a$ which becomes $(r-1)/r$ when the mean is known to be unity.

Exhibit IV displays the results of a $\chi^2$ test for the Gamma distribution. It is the Gamma equivalent of Exhibit IIId. Since $R_t$ has a mean equal to unity,

$$a = r = 1/\sigma^2.$$

When the estimated portfolio variance is approximately 0.097, corresponding to an $\alpha = 1.000$ size portfolio (i.e., for the 1971 portfolio),

$$a = r = 10.345.$$

As was the case for the Normal distribution, there is no compelling reason to reject the Gamma distribution. Exhibit Va discloses that for a portfolio this large, the Gamma distribution is almost Normal.

As the size of the portfolio is decreased to 15% of the original size, both models spreadout. The Normal distribution allows a significant probability for negative loss ratios whereas the mode of the Gamma shifts to the left while remaining in the first quadrant (see Exhibit Vb). Exhibit Vc displays the gradual shift of the Gamma distribution as the portfolio size becomes progressively smaller.

The decreasing mode (the mean is always equal to unity) of the Gamma is consistent with reality. That this is so can be seen by allowing the portfolio size to decrease to that of a few exposure units. The most frequent loss outcome for a single exposure unit is often "no claim," yet the expected loss for a large aggregation of such exposure units is rarely zero. A mean equal to unity retains the long term expected average outcome while acknowledging that the most likely outcome is something less than an average loss. It's the possibility of extremely large losses that pushes the mean above the mode.

## *Applications of the Methodology*

### Allocation of Surplus

Surplus provides a cushion against unanticipated events. As such, the entire company surplus is available to meet the company's unanticipated obligations, regardless of their source. Strictly speaking, surplus is indivisible and cannot be allocated to lines of business. At the same time, writing one additional unit of exposure increases the amount of surplus that is required (either as a result of Risk Based Capital requirements or to maintain the probability of ruin below some desired amount). In a sense, this additional surplus can be associated with, or allocated to, the particular line of business. To the extent that all lines of business do not have the same marginal surplus requirement, business decisions regarding the mix of business and acceptable profit margins can be influenced by such an allocation of surplus. This paper accepts the premise that, at least for the purpose of assisting in business decisions, surplus can be allocated to lines of business. It also accepts the idea that the role of surplus is to keep the company's probability of ruin below some arbitrarily selected amount. As a result, the establishment of a reserve-to-surplus leverage ratio for a given portfolio requires knowledge of the aggregate loss distribution for the entire portfolio of contracts.

More specifically, the actual loss ratio experience for a given portfolio can differ from the anticipated experience for two reasons:

1. The anticipated result was erroneous. In other words, parameter error was present.
2. The anticipated result was correct, but random bad or good luck resulted in the actual result being different from the anticipated result. In other words, process risk resulted in the unanticipated difference.

It is the process variance, as measured by $R_t$, that should be reflected in the allocation of surplus when determining the return on equity associated with a particular contract. For a given expected loss ratio, the distribution of $R_t$ would determine how much surplus would be necessary to protect the company against ruin up to some preselected confidence level. The marginal supporting surplus required as a result of introducing an additional contract would depend upon the additional process variance resulting from the introduction of the new contract. The additional process variance would depend upon the portfolio to which the contract was assigned as well as the existing mix of business and any correlations between portfolios.

The resulting return on equity (see Bingham [4] and Bender [5] for a discussion concerning how to measure ROE once surplus has been allocated) will not reflect the presence of potential parameter error. The appropriate reflection of parameter error is in selecting an ROE target. For example, an 8% ROE might be sufficient reward for placing the company surplus at risk if the parameters (e.g., expected loss ratio and payout timing) can be estimated to a high degree of certainty, whereas the target might increase to 15% if less credible estimates are available.

Further discussion regarding how to measure ROE and how to select an appropriate target are beyond the scope of this paper.

## Evaluating Loss Sensitive Contract Provisions

Frequently, a portfolio is made up of policies or reinsurance contracts that are subject to loss sensitive elements at the individual contract level. Examples of this would be a portfolio consisting of swing rated reinsurance treaties or retrospectively rated Workers Compensation policies. Because the loss sensitive premium is calculated for each contract separately, substituting the estimated ultimate loss ratio for the entire portfolio into the loss sensitive rating formula will not necessarily produce the best estimate of the ultimate aggregate premium (see Bender [6]).

Charles H. Berry ([7]) proposed a method of estimating the ultimate premium return for such a portfolio. Essentially, his method consists of credibility weighting the reported premium return ratio (to standard or subject premium) with an *a priori* premium return ratio. Over time, more credibility is given to the reported ratio and less is given to the *a priori* ratio. Berry's *a priori* ratio is based on the relationship between historical aggregate portfolio loss ratios and return premium ratios. In his discussion of Berry's method, Roy K. Morell [8] noted a significant limitation to the methodology. Its success depends upon the historical portfolios being subject to similar rating parameters (e.g., swing maximums and minimums) and consisting of similar risks (i.e., exposure units). If there have been material changes in either, the *a priori* estimates will not be appropriate.

As an alternative to using the historical relationship between aggregate loss and aggregate premium, the portfolio aggregate loss ratio distribution could be scaled down to the size of an individual contract using the divisibility property of the Gamma distribution. The individual contract distribution could be used to simulate the possible loss outcomes and to determine the corresponding return premium ratio together with the associated probability of occurrence for each contract in the current portfolio. The aggregate loss ratio and corresponding aggregate expected return premium ratio could serve as the *a priori* estimates for the portfolio. The Gamma Distribution could also be used to determine the sensitivity of the aggregate premium return to changes in the aggregate loss ratio, just as Table M was used by Bender [6].

The results of the sensitivity analysis would be used to develop the reported aggregate return premium ratio to reflect the impact of IBNR loss. At the same time, the sensitivity analysis could be used to revise the *a priori* aggregate return premium ratio to reflect the additional loss ratio information. As of any valuation, the estimated ultimate aggregate return premium ratio would be the credibility weighted average of the developed return premium ratio and the revised *a priori* aggregate return premium ratio.

For example, assume that a portfolio was rated to produce a 60% loss ratio to standard premium and to return 10% of standard premium in the form of retrospective (swing) rated premium adjustments. Further, assume that the expected sensitivity of the formula is 25% (i.e., for every 100 additional points of loss ratio, the return premium ratio decreases by 25 percentage points) for this portfolio. The portfolio's reserve history might look something like Exhibit VI.

While reserving retrospectively rated policies is beyond the scope of this paper, the reader is encouraged to compare the suggestion to the methodology proposed by Berry.

*Qualifications and Caveats*

Two of the major assumptions regarding the composition of each portfolio are unlikely to be strictly met in reality. These assumptions are that the units of exposure are *identically distributed* and *independent*. While the criteria defining each portfolio can be adjusted to minimize the degree to which either of these assumptions is not met, the cost of doing so will always be a reduction in the size of the portfolio. This is an example of the ubiquitous conflict between obtaining homogeneity while maintaining a credible volume of data. Fortunately, neither of the assumptions has to be met in order for the methodology to be applied. To the extent that the assumptions are not met, parameter error may be introduced into the models.

To see how parameter error arises when the exposure units are not identical requires only that the assumption of identically distributed random variables be removed from the requirements of the Central Limit Theorem. In 1901, Liapounov proved a more general form of the Central Limit Theorem ([9]) that applies when the outcomes for different exposure units are not necessarily identically distributed. In particular, he assumed that a set of random variables, $\{X_j\}$, is distributed with means $\{\mu_j\}$ (not necessarily equal) and variances $\{\sigma^2_j\}$ (again, not the same for all $j$).

Defining $Y_N$, the sum of the $X_j$ for a sample of size $N$, as follows,

$$Y_N \equiv \sum_{j=1}^{N} X_j ,$$

Liapounov proved that the sample mean of a finite number of random variables, $Y_N/N$, will be distributed Normally with mean,

$$\begin{aligned} \mu_N &= N^{-1} \sum_{j=1}^{N} \mu_j \\ &= \langle \mu_j \rangle , \end{aligned}$$

and variance

$$\sigma_N^2 = \left(\sum_{j=1}^{N} \sigma_j^2\right) / N^2$$
$$= \left(N^{-1} \sum_{j=1}^{N} \sigma_j^2\right) / N$$
$$= \langle \sigma_j^2 \rangle / N$$

where the brackets, <··> indicate taking the average of the indicated sub-population parameters.

It is as if the samples were drawn from a population of exposure units with *identically* distributed losses where the mean pure premium equals the average of the means of the actual distributions and where the variance of the pure premium equals the average of the variances of the actual distributions. For example, consider a portfolio consisting of 50 exposure units whose pure premiums are distributed with a mean of 30.00 and variance 4.00 and 75 exposure units whose pure premiums are distributed with a mean equal to 50.00 and variance equal to 9.00. The 125 exposure unit portfolio will experience Normally distributed pure premiums with a mean equal to 42.00 ( [50*30+75*50]/125 ) and variance equal to 0.056 (the average variance of the population,[50*4+75*9]/125 = 7, divided by the size of the sample, 7/125 =0.056). This is the same distribution of sample means that 125 identically distributed exposure units with mean 42.000 and variance 7.00 would have produced.

When the historical portfolios differ in size and the exposure units are not identically distributed, the generalized estimators will provide the population average mean and variance as long as the proportion of each sub-distribution remains the same as the sample size changes. As long as it is the distribution of a similarly distributed portfolio that is to be modeled, the mixture introduces no parameter error. However, if one type of exposure is to be modeled or if the composition of the portfolio has changed over time, then possibility of parameter error must be considered. Parameter errors could be material if the Gamma distribution for a heterogeneous portfolio is scaled down to the size of an individual contract in the portfolio. *In the case of an extremely heterogeneous portfolio, scaling down the distribution to model an the loss ratio distribution for individual constituent (contract, policy, or treaty) should be performed only as a last resort.*

To the extent that the random loss ratio fluctuations of different exposure units within a single portfolio are correlated, those correlations will be reflected in the volatility of the historical experience. Depending upon the application, correlations within a portfolio may or may not introduce parameter error. There will be no parameter error if the application involves modeling the aggregate loss ratio distribution of a portfolio that is similar to the portfolios that generated the historical data. If the application involves using the historical portfolio experience to model the aggregate loss ratio distribution of a single exposure unit, correlations within the portfolio will result in an inappropriate model; in particular, the model variance may be misstated.

Refining the criteria that define a portfolio can often significantly reduce correlations within a portfolio. For example, one could exclude the reflection of any exposure to catastrophe loss by eliminating catastrophe losses from the data and modeling potential catastrophe losses separately. It is common to treat catastrophe losses (and the corresponding exposure) separately when testing loss reserve adequacy or when pricing a cohort of policies.

Likewise, increasing the homogeneity of the portfolio might reduce correlations that result from the manner in which exposure was quantified. For example, a portfolio consisting of a mixture of $500,000 excess of $500,000 loss reinsurance treaties together with $2,500,000 excess of $500,000 reinsurance treaties may have internal correlations simply by virtue of the proxy in terms of which exposure units are measured. Treaties with a $2,500,000 limit clearly represent more exposure than those with a $500,000 limit. If multiple exposure units are assigned to treaties with the larger limit, correlations will be introduced (between the exposure units assigned to a single treaty). Forming two separate portfolios will allow each type of treaty to be treated as a single exposure unit. Such a separation is not unique to the modeling process. It might also be prudent when attempting to model loss development patterns.

Even if portfolio criteria can be suitably refined without sacrificing predictive credibility, some sources of parameter error will remain. One significant source lies in the method, and another lies in the data itself.

The methodology involved determining the random component, $R_t$, by restating the actual loss ratios, $ILR_t$, to an "as if common point in time basis" and then dividing the restated loss ratios by the mean of the historical loss ratios. It was argued that the only reason why the restated historical loss ratios differ is the presence of random fluctuations. Therefore, it followed that the division isolated the random component. Strictly speaking, this is true only if the restated loss ratios were divided by the *population* mean. The method involves dividing by an *estimate* of the population mean, not the *true* population mean. The estimate is, itself, a random variable.

From Appendix B, we see that the estimator, $m'$, is Normally distributed with a mean equal to the population mean and a variance equal to the population variance divided by the size of the super sample ($N$, the sum of all of the exposure units when all of the samples are combined into a single sample). The result of the division process is not $R$ but rather, it is a stochastic variable, $Z$. $Z$ is equal to the quotient of two Normally distributed stochastic variables, the *ILR* (which is distributed $N[\mu, \sigma^2]$) and the estimator of $\mu$ (which is distributed $N[\mu,\sigma^2/N]$). The distribution of the quotient of two Normally distributed variables is a Cauchy distribution, not a Normal distribution. For a sufficiently large number of years and for sufficiently large portfolios, $N$ will be very large and $\sigma^2/N$ will be vanishingly small. As a result, the estimator for $\mu$ can be treated as if it were not a random variable. Restricting the application of the method to cases where the super sample (all individual samples combined to form a single gigantic sample) size is very large avoids the problem of dealing with a distribution such as the Cauchy distribution, which does not have any moments.

Using loss ratio estimates that arise from loss reserve adequacy tests may introduce an additional source of parameter error. When the method was described, it was assumed that the ultimate loss ratios for each of the historical years were known with certainty. In practice, some of the more recent year's loss ratios will be estimates as of a particular valuation date. To the extent that a Loss

Ratio or Bornhuetter/Ferguson methodology was used to estimate the ultimate loss ratios, random deviations of the more recent loss ratios from the expected will be tempered by the estimates. The reserving methodology automatically reduces the variance of the estimates. As a result, the variance of $R_t$ will be understated. On the other hand, a projection methodology could result in an overstatement of the true variance. Such data induced bias can be detected by applying the methodology to successively shorter historical periods (e.g., 1969-1997, 1969-1996, 1969-1995, etc.) to see if the resulting $\sigma^2$ exhibits a constant trend. If $\sigma^2$ consistently increases as more recent years are eliminated, then there is evidence that the more recent estimates may be masking the true variance. On the other hand, a decreasing $\sigma^2$ would be evidence that the reserving methodology is introducing a misleading indication of actual loss ratio volatility.

*Summary*

The parameters of the aggregate loss ratio distribution corresponding to a large portfolio of identical and independent exposure units can be determined by examining the historical loss experience of portfolios made up of the same type of exposure units. The methodology of time series decomposition allows the historical experience to be restated to a common point in time, making it possible to consider the experience from different years to be equivalent to taking multiple samples from a single year. Even if the *number* of exposure units (i.e., the *size* of the portfolios) varies from historical portfolio to portfolio, a generalized form of the estimators for the population parameters, $\mu$ and $\sigma^2$, make it possible to model the aggregate loss distribution in terms of a Normal distribution with size dependent parameters.

While a Normal model with size dependent parameters performs satisfactorily for large portfolios, it does not produce a realistic distribution for smaller aggregations of exposure (e.g., small portfolios or individual reinsurance contracts). The Gamma distribution was proposed as an alternative to the Normal distribution for several reasons. It is approximately Normal in the limit as the number of independent exposure units increases without bound. For large portfolios, the two distributions are virtually indistinguishable, making it possible to smoothly make the transition from the Normal distribution produced by application of the Central Limit Theorem to the almost identical Gamma distribution. Unlike the Normal distribution, the Gamma distribution has a divisibility property that allows it to be size transformed in a precise manner. Aside from this attractive, but purely mathematical feature of the Gamma distribution, Hewitt previously reported that the Gamma distribution produces results that are consistent with reality, especially when compared to the Table M tabulation of Workers Compensation loss ratios.

Foot Notes

1. Strictly speaking, the life of a portfolio of exposure units consists of the period in which the exposure units are in force together with the time over which losses that arise from the portfolio run off. Under this definition of the life of a portfolio, the renewal of a group of contracts (primary policies, reinsurance treaties, etc.) results in the formation of a new portfolio. In another sense, if all of the contracts renew, then the new portfolio can be considered to be the second year of the original portfolio. In this paper, the context will make clear which sense of the word "portfolio" applies.

2. Portfolios, as defined, cannot actually *change size*. In more precise terms, the historical data consists of the experience of many different portfolios, one for each of the historical contract years. If the same number of identical and independent exposure units make up all of the portfolios, they are all of the same size. If the number varies from year to year, the historical data reflects the historical exposure of portfolios of different size. If the exposure units are indistinguishable, these different portfolios can be treated as if they were a single portfolio with a changing size.

3. The second and third assumptions, while closely related, are not the same. It is possible for the loss outcomes to be inflation sensitive, making the population of possible outcomes time dependent without any corresponding change in the measured size of the portfolio. Conversely, an inflation sensitive exposure base, such as Workers Compensation Payroll, could result in a changing "yardstick" being used to measure portfolio size without any

corresponding change in the possible pure premium outcomes (e.g., Workers Compenstion indemnity loss pure premium for which the losses and exposure would both change at the same rate, leaving the ratio invariant). Because the two inflation sensitivities (outcomes and size) can be different, the restatement of the population of outcomes and measurement of the portfolio size must be addressed separately.

4. To be more precise, once the loss ratios ratios have been restated to reflect a common level of rate adequacy, they may be considered to be multiple samples drawn from the same population. Depending on how the methodology is to be applied, an additional restatement to some particular time might be required. This restatement will involve multiplication of all of the loss ratios by the same restatement factor (taking them all from cycle point 100, and the midpoint of 1997 to the desired time and cycle point). Multiplying all of the loss ratios by the same factor introduces nothing new; it is simply a change in scale.

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## Appendix A: Distribution of Portfolio Pure Premium, Mean and Variance

Consider a portfolio consisting of the loss experience of many independent and identical exposure units. These exposure units may arise from reinsuring several different client companies. Let $l_{ij}$ be the loss outcome for the $i^{th}$ exposure unit arising from the $j^{th}$ client. Further, let $N_j$ be the number of independent exposure units arising from the $j_{th}$ client. Since the exposure units are identical and independent, we may assume that

$$E[l_{ij}] = \mu$$

and

$$E[(l_{ij}-\mu)^2] = Var[l_{ij}] = \sigma^2,$$

both of which will be independent of $i$ and $j$.

The loss outcome for the $j^{th}$ client is given by

$$L_j = \sum_{i=1}^{N_j} l_{ij}$$

and the corresponding account pure premium, $pp_j$, is given by $L_j/N_j$.

- It can be demonstrated that the expected account pure premium is equal to the expected pure premium of the individual exposure units, $\mu$.

$$E[pp_j] = E[N_j^{-1}\sum_{i=1}^{N_j} l_{ij}]$$

$$= N_j^{-1}\sum_{i=1}^{N_j} E[l_{ij}]$$

$$= \mu$$

The variance of the account pure premium is equal to the variance of the individual exposure unit pure premium divided by the number of exposure units making up the account.

$$Var(pp_j) = Var[L_j/N_j] =$$

$$= N_j^{-2} Var[L_j]$$

$$= N_j^{-2} \sum_{i=1}^{N_j} Var[l_{ij}]$$

$$= \sigma^2 / N_j$$

These results confirm the notion that all clients have the same pure premium, regardless of size but that the larger clients have less volatility (i.e., the variance of the pure premiums is smaller).

Now, suppose that only a portion, $S_j$, $(0 < S_j \leq 1)$ of a client's loss is reflected in the portfolio. In other words, for every loss $l_{ij}$ that is incurred, only $S_j l_{ij}$ is reflected in the portfolio. The corresponding exposure contribution is given by $S_j N_j$. An important conclusion that can be drawn is that both the expected pure premium of a share and the variance of the share's pure premium are independent

$$E[pp_j] = (S_j N_j)^{-1} \sum_{i=1}^{N_j} E[S_j l_{ij}]$$

$$= N_j^{-1} \sum_{i=1}^{N_j} E[l_{ij}]$$

$$= \mu,$$

The proof is trivial, since the pure premium is independent of the share,

and, similarly,

$$Var\ [pp_j] = Var\ [(S_j N_j)^{-1} \sum_{i=1}^{N_j} S_j l_{ij}]$$

$$= Var\ [N_j^{-1} \sum_{i=1}^{N_j} l_{ij}]$$

$$= \sigma^2 / N_j .$$

Now, consider a portfolio that is made up of shares of *m* different contracts. The portfolio loss is

$$L = \sum_{j=1}^{m} \sum_{i=1}^{N_j} S_j l_{ij}$$

The corresponding portfolio exposure is given by

$$N = \sum_{j=1}^{m} S_j N_j$$

The portfolio pure premium is the ratio of the two, L/N,

$$pp_{portfolio} = \frac{\sum_{j=1}^{m} S_j \sum_{i=1}^{N_j} l_{ij}}{\sum_{k=1}^{m} S_k N_k}$$

The expected portfolio pure premium is $\mu$,

$$E[pp_{portfolio}] = \frac{\sum_{j=1}^{m} S_j \sum_{i=1}^{N_j} E[l_{ij}]}{\sum_{k=1}^{m} S_k N_k}$$

$$= \frac{\sum_{j=1}^{m} S_j \sum_{i=1}^{N_j} \mu}{\sum_{k=1}^{m} S_k N_k} =$$

$$= \frac{\sum_{j=1}^{m} S_j N_j \mu}{\sum_{k=1}^{m} S_k N_k}$$

$$= \mu.$$

The percentage of the portfolio exposure, $n$, contributed by the $j^{th}$ contract is given by

$$n_j \equiv \frac{S_j N_j}{\sum_{k=1}^{m} S_k N_k}$$

In terms of $n$, the variance of the portfolio pure premium can be written as

$$Var[pp_{portfolio}] = Var \sum_{j=1}^{m} \sum_{i=1}^{N_j} n_j \left[ l_{ij} / N_j \right]$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{N_j} \left[ n_j / N_j \right]^2 \sigma^2$$

$$= \sum_{j=1}^{m} \left( n_j^2 / N_j \right) \sigma^2$$

$$= \sigma^2 / N_{portfolio}$$

where

Consider two examples:

Two identical clients each have an exposure equal to N. A reinsurance portfolio of size ½N is formed in two ways.

1. A 25% share of each contract is written. Since each contract makes up half of the portfolio, $n_1 = n_2 = ½$. $N_{portfolio} = 2N$. In other words, taking equal shares of two identical contracts produces the same variance as taking a share of a single contract for a client with twice the exposure.

$$N_{portfolio} \equiv \left( \sum_{j=1}^{m} \left( n_j^2 / N_j \right) \right)^{-1}.$$

2. A 50% share of a single contract is written. This produces the same amount of premium and expected loss but, $n_1 = 1$ and $n_2 = 0$. Therefore, $N_{portfolio} = N$ which is consistent with the fact that the contract variance is independent of the share of the contract that is taken.

As would be expected, the portfolio variance depends not only upon how large the portfolio is but also the manner in which the portfolio is formed. All else being equal, the portfolio that is made up of small shares of many large contracts will have a smaller pure premium variance than one of equal size that consists of large shares of small contracts. This is nothing more than an application of the concept of *spread*.

## Appendix B: Generalized Central Limit Theorem

The Central Limit Theorem is appropriate for situations in which T independent samples, each of size N, are drawn (either with replacement or from a population that is so large that that act of drawing the sample has virtually no effect upon the probabilities affecting subsequently drawn samples). If N is large, then the Central Limit Theorem states that the sample means are distributed Normally with a mean equal to the population mean and a variance equal to the population variance divided by N.

Symbolically,

let $l_{ij}$ be the $i^{th}$ element in the $j^{th}$ sample,

$\mu$ be the population mean, and

$\sigma^2$ be the population variance.

In terms of the individual, independent sample elements, the mean of the $j^{th}$ sample, $\langle l \rangle_j$, is given by

$$\langle l \rangle_j = N^{-1} \sum_{i=1}^{N} l_{ij}$$

and the mean of the sample means is given by

$$\langle l \rangle = T^{-1} \sum_{j=1}^{T} \langle l \rangle_j$$

The variance of the sample means is given by

$$Var(\langle l \rangle_j) = s^2 = (T-1)^{-1} \sum_{j=1}^{T} \left( \langle l \rangle_j - \langle l \rangle \right)^2$$

If N is very large, the Central Limit Theorem states that $\langle l \rangle_j$ is Normally distributed

$$N(\mu, \sigma^2/N),$$

and that $\langle l \rangle$ is an unbiased estimator of $\mu$ and $Ns^2$ is an unbiased estimator of the population variance. Note that there are no restrictions regarding how the $l_{ij}$ are distributed or even that they be independent (see DeGroot [9]).

If all that is desired is to use the T samples to estimate the distribution of the mean of an additional sample of size M (large, but not necessarily equal to N), it is necessary to know neither the population variance, N nor M. Knowledge of the variance of the sample means, $s^2$ and the *ratio* of N to M is all that is necessary. The distribution of the mean of the $T+1^{st}$ sample will be Normal with mean $\mu$, estimated by $\langle l \rangle$, and variance $\sigma^2/M$, estimated by $s^2(N/M)$.

In the Generalized version of the Central Limit Theorem, the condition that the sample sizes all be equal to N is relaxed. T samples of sizes $\{N_j\}$, where $Nj$ is the size of the $j^{th}$ sample, are drawn from a population (with replacement). Alternatively, one can denote the size of the $jth$ sample by $N_1 \alpha_j$, where $\alpha_j$ is the *ratio* of the size of the $j^{th}$ sample to that of the first sample drawn.

The conclusion of the Central Limit theorem is unchanged. The distribution of the sample mean of the $T+1^{st}$ sample will be distributed Normally with a mean equal to the population mean and a variance equal to the population variance divided by the size of the $T+1^{st}$ sample, $M$. This conclusion is still a statement relating the distribution of multiple samples of size $M$ to statistics associated with the entire population, but the estimators of $\mu$ and $\sigma^2$ need to be modified when the historical samples are not all of the same size.

When the samples differ in size, the **estimator of the population mean** becomes the weighted mean of the sample means,

$$\langle l\rangle = \sum_{j=1}^{T} N_j \langle l\rangle_j \Big/ \sum_{j=1}^{T} N_j$$

$$= \sum_{j=1}^{T} \alpha_j \langle l\rangle_j \Big/ \sum_{j=1}^{T} \alpha_j$$

$$= \sum_{j=1}^{T} \sum_{i=1}^{N_j} l_{ij} \Big/ \sum_{j=1}^{T} N_j$$

When all of the samples are the same size, all of the $\alpha$'s are equal to one and the estimator becomes the same as it was for the Central Limit Theorem.

The estimator, $\langle l\rangle$, is itself a random variable. If T samples are drawn and the estimator is determined, it will not be exactly the same estimator that would be determined if another set of T samples were drawn. As a result, $\langle l\rangle$ has a distribution.

If the sample sizes, $N_j$, are large,

- the estimator, $\langle l\rangle$, will be Normally distributed with mean $\mu$ and variance

$$Var(\langle l\rangle) = \sigma^2 / \sum_{j=1}^{T} N_j$$

- the estimator is unbiased, i.e., the expectation of the estimator, E[<l>] = $\mu$, and

- the estimator has minimum variance for all estimators that are linear functions of the sample means.

To prove the first assertion (that the estimator is Normally distributed with the indicated mean and variance), observe that the last form of the definition is simply a sum over all of the observations, regardless of which sample gives rise to the observation. It is identical to the mean of a single, super, sample made up of all T of the individual samples. Thought of in that way, the conventional Central Limit Theorem states that the mean of this sample of size

$$N = \sum_{j=1}^{T} N_j$$

is distributed Normally with mean $\mu$, and variance $\sigma^2/N$. This proves the first assertion regarding <l> and the other two follow immediately from the conventional Central Limit Theorem.

The generalized **estimator for the population variance**, $\sigma^2$, is given by $Ns^2$ where

$$s^2 = (T-1)^{-1}\left(\sum_{j=1}^{T} N_j\left(\langle l\rangle_j - \langle l\rangle\right)^2 \Big/ \sum_{j=1}^{T} N_j\right)$$

$$= (T-1)^{-1}\left(N_1\sum_{j=1}^{T} \alpha_j\left(\langle l\rangle_j - \langle l\rangle\right)^2 \Big/ N_1\sum_{j=1}^{T} \alpha_j\right)$$

and $N$ is the size of the super sample (i.e., all T samples combined and considered as a single sample).

$Ns^2$ is an unbiased estimator of $\sigma^2$. To show this, begin by considering the expectation of a single term in the numerator,

$$E\left(\langle l\rangle_1-\langle l\rangle\right)^2 = E\left(\frac{\sum_{j=1}^{T} N_j\langle l\rangle_j}{\sum_{j=1}^{T} N_j}-\frac{\sum_{j=1}^{T} N_j}{\sum_{j=1}^{T} N_j}\langle l\rangle_1\right)^2$$

$$=\left(\sum_{j=1}^{T} N_j\right)^{-2} E\left(\sum_{j=2}^{T} N_j\langle l\rangle_j-\left(\sum_{j=2}^{T} N_j\right)\langle l\rangle_1\right)^2$$

$$=\left(\sum_{j=1}^{T} N_j\right)^{-2} E\left(\sum_{j=2}^{T} N_j\left(\langle l\rangle_j-\langle l\rangle_1\right)\right)^2$$

$$=\left(\sum_{j=1}^{T} N_j\right)^{-2} Var\left(\sum_{j=2}^{T} N_j\left(\langle l\rangle_j-\langle l\rangle_1\right)\right)$$

where the last line follows from the fact that the expectation for all of the sample means are equal to $\mu$, hence the mean of the quantity $\langle l\rangle_j-\langle l\rangle_1$ is zero.

Continuing,

$$E\left(\langle l\rangle_1-\langle l\rangle\right)^2=\left(\sum_{j=1}^{T}N_j\right)^{-2}Var\left(\sum_{j=2}^{T}N_j\left(\langle l\rangle_j-\langle l\rangle_1\right)\right)$$

$$=\left(\sum_{j=1}^{T}N_j\right)^{-2}Var\left(\sum_{j=2}^{T}N_j\langle l\rangle_j-\sum_{j=2}^{T}N_j\langle l\rangle_1\right)$$

$$=\left(\sum_{j=1}^{T}N_j\right)^{-2}\left(\sum_{j=2}^{T}N_j^2Var\left(\langle l\rangle_j\right)+\left(\sum_{j=2}^{T}N_j\right)^{2}Var\left(\langle l\rangle_1\right)\right)$$

From the Generalized Central Limit Theorem, $Var(\langle l\rangle_j) = \sigma^2/N_j$ so,

$$\left(\sum_{j=1}^{T}N_j\right)^{-2}\left(\sum_{j=2}^{T}N_j^2Var\left(\langle l\rangle_j\right)+\sum_{j=2}^{T}N_j^2Var\left(\langle l\rangle_1\right)\right)=\sigma^2\left(\sum_{j=1}^{T}N_j\right)^{-2}\left(\sum_{j=2}^{T}N_j+\left(\sum_{j=2}^{T}N_j\right)^{2}/N_1\right)$$

$$=\sigma^2\left(\sum_{j=1}^{T}N_j\right)^{-1}\left(\frac{\sum_{j=2}^{T}N_j}{N_1}\right)$$

$$=\sigma^2\left(\sum_{j=1}^{T}N_j\right)^{-1}\left(\frac{\sum_{j=1}^{T}N_j-N_1}{N_1}\right)$$

$$=\sigma^2\left(N_1^{-1}-\left(\sum_{j=1}^{T}N_j\right)^{-1}\right)$$

While the derivation awarded special status to the first sample, any one of the samples could have been given special treatment. In general, then,

$$E\left(\langle l\rangle_k-\langle l\rangle\right)^2=\sigma^2\left(N_k^{-1}-\left(\sum_{j=1}^{T}N_j\right)^{-1}\right).$$

The expected value of the estimator of the variance, $NE(s^2)$, is given by

$$NE(s^2)=(T-1)^{-1}\sum_{k=1}^{T}N_j E\left(\langle l\rangle_k-\langle l\rangle\right)^2$$

$$=\sigma^2(T-1)^{-1}\sum_{k=1}^{T}N_k\left(N_k^{-1}-\left(\sum_{j=1}^{T}N_j\right)^{-1}\right)$$

$$=\sigma^2$$

Because the expected value of the population variance estimator is the population variance, the estimator is unbiased.

The conclusion of all this is that the Central Limit Theorem can be used when samples of different sizes are drawn from a single population. The estimators of the population mean and variance are similar to those used when all samples are of the same size except that weighted averages must be taken. The weights are the sample sizes. Note that the estimators when all samples are the same size are special cases of the more general expressions.

In the case when $l_{ij}$ is a loss outcome for the $i^{th}$ exposure unit in the $j^{th}$ sample, <I>$_j$ is the sample pure premium. In terms of portfolio pure premiums, the estimators become:

$$Estimator(pp_{population}) = m' = \sum_{j=1}^{T} \alpha_j pp_j \Big/ \sum_{j=1}^{T} \alpha_j$$

$$Estimator(Var[pp_{population}]) = Ns^2 = N_1 (T-1)^{-1} \sum_{j=1}^{T} \alpha_j (pp_j - m')^2$$

## Loss Ratio Series Decomposition

Exhibit I

| Year | ILR (1) | Isolated TxC (2) | Isolated T (3) | Isolated C (4) | Trend Index (5) | Cycle Index (6) | Restated ILR (7) |
|---|---|---|---|---|---|---|---|
| 1960 | 29.9% | | | | | | |
| 1961 | 39.9% | | | | | | |
| 1962 | 38.0% | 34.1% | 31.7% | 1.07 | 103.96 | 107.50 | 69.85% |
| 1963 | 32.0% | 33.1% | 32.3% | 1.02 | 106.01 | 102.39 | 60.53% |
| 1964 | 30.5% | 31.4% | 32.9% | 0.95 | 108.09 | 95.44 | 60.75% |
| 1965 | 25.0% | 29.9% | 33.6% | 0.89 | 110.21 | 89.03 | 52.21% |
| 1966 | 31.7% | 31.2% | 34.2% | 0.91 | 112.37 | 91.19 | 63.45% |
| 1967 | 30.3% | 35.7% | 34.9% | 1.02 | 114.58 | 102.24 | 53.18% |
| 1968 | 38.6% | 37.7% | 35.6% | 1.06 | 116.83 | 105.86 | 64.15% |
| 1969 | 52.9% | 40.2% | 36.3% | 1.11 | 119.12 | 110.73 | 82.30% |
| 1970 | 34.9% | 40.1% | 37.0% | 1.08 | 121.46 | 108.44 | 54.39% |
| 1971 | 44.2% | 38.5% | 37.7% | 1.02 | 123.84 | 102.03 | 71.83% |
| 1972 | 30.1% | 35.7% | 38.5% | 0.93 | 126.27 | 92.87 | 52.62% |
| 1973 | 30.5% | 37.0% | 39.2% | 0.94 | 128.75 | 94.38 | 51.50% |
| 1974 | 39.0% | 38.0% | 40.0% | 0.95 | 131.28 | 95.02 | 64.24% |
| 1975 | 41.4% | 41.7% | 40.8% | 1.02 | 133.86 | 102.23 | 62.06% |
| 1976 | 49.1% | 46.7% | 41.6% | 1.12 | 136.48 | 112.32 | 65.78% |
| 1977 | 48.5% | 48.3% | 42.4% | 1.14 | 139.16 | 113.96 | 62.77% |
| 1978 | 55.6% | 46.2% | 43.2% | 1.07 | 141.89 | 106.84 | 75.24% |
| 1979 | 47.1% | 42.5% | 44.1% | 0.96 | 144.68 | 96.30 | 69.41% |
| 1980 | 30.7% | 41.9% | 45.0% | 0.93 | 147.52 | 93.14 | 45.88% |
| 1981 | 30.4% | 41.5% | 45.8% | 0.90 | 150.42 | 90.49 | 45.88% |
| 1982 | 45.6% | 44.5% | 46.7% | 0.95 | 153.37 | 95.17 | 64.08% |
| 1983 | 53.6% | 49.1% | 47.7% | 1.03 | 156.38 | 103.02 | 68.30% |
| 1984 | 62.1% | 52.9% | 48.6% | 1.09 | 159.45 | 108.97 | 73.40% |
| 1985 | 53.8% | 52.4% | 49.5% | 1.06 | 162.58 | 105.69 | 64.24% |
| 1986 | 49.7% | 48.6% | 50.5% | 0.96 | 165.77 | 96.30 | 63.90% |
| 1987 | 42.6% | 44.3% | 51.5% | 0.86 | 169.02 | 85.98 | 60.22% |
| 1988 | 35.0% | 41.6% | 52.5% | 0.79 | 172.34 | 79.24 | 52.65% |
| 1989 | 40.3% | 44.9% | 53.5% | 0.84 | 175.73 | 83.85 | 56.17% |
| 1990 | 40.4% | 52.1% | 54.6% | 0.95 | 179.18 | 95.41 | 48.52% |
| 1991 | 66.1% | 60.8% | 55.7% | 1.09 | 182.69 | 109.17 | 68.07% |
| 1992 | 78.6% | 63.8% | 56.8% | 1.12 | 186.28 | 112.47 | 77.01% |
| 1993 | 78.4% | 66.6% | 57.9% | 1.15 | 189.94 | 115.06 | 73.69% |
| 1994 | 55.6% | 63.1% | 59.0% | 1.07 | 193.66 | 107.00 | 55.11% |
| 1995 | 54.2% | 55.7% | 60.2% | 0.93 | 197.47 | 92.61 | 60.83% |
| 1996 | 48.9% | | 61.4% | | 201.34 | | |
| 1997 | 41.5% | | 62.6% | | 205.29 | | |

(1) Historical Data

(2) Five year moving average of historical data

(3) Model of trend, fit to the isolated TxC of column (2). (3) $=30.5*(1.0196)^{(year-1960)}$

(4) Implied Cycle, TxC/T = (2)/(3)

(5) $=100*(1.0196)^{(year-1960)}$

(6) =100*(4)

(7) =(1)year*[(5)1997/(5)year]*[100/(6)year]

## Portfolio Size

Exhibit II

| Contract Year (1) | Historical Premium (5) | Isolated TxC (7) | LossIndex (9) | ΔTxC (8) | On Level Factor: adequacy (10) | On Level Factor: Loss (11) | On Level Factor: Composite (12) | Restated Premium (13) | "N" (14) |
|---|---|---|---|---|---|---|---|---|---|
| 1960 | 400.00 | 68.8% | 100.00 | 1.000 | 1.088 | 34.004 | 31.261 | 12,504.46 | 100.00 |
| 1961 | 412.00 | 73.4% | 110.00 | 1.068 | 1.019 | 30.913 | 30.351 | 12,504.46 | 100.00 |
| 1962 | 424.36 | 78.4% | 121.00 | 1.068 | 1.019 | 28.102 | 27.591 | 11,708.72 | 93.64 |
| 1963 | 437.09 | 83.7% | 133.10 | 1.068 | 1.019 | 25.548 | 25.083 | 10,963.62 | 87.68 |
| 1964 | 450.20 | 89.4% | 146.41 | 1.068 | 1.019 | 23.225 | 22.803 | 10,265.94 | 82.10 |
| 1965 | 463.71 | 95.5% | 161.05 | 1.068 | 1.019 | 21.114 | 20.730 | 9,612.65 | 76.87 |
| 1966 | 477.62 | 102.0% | 177.16 | 1.068 | 1.019 | 19.194 | 18.845 | 9,000.94 | 71.98 |
| 1967 | 501.79 | 108.9% | 194.87 | 1.068 | 1.019 | 17.449 | 17.132 | 8,596.71 | 68.75 |
| 1968 | 526.98 | 116.3% | 214.36 | 1.068 | 1.019 | 15.863 | 15.575 | 8,207.48 | 65.64 |
| 1969 | 553.22 | 124.2% | 235.79 | 1.068 | 1.019 | 14.421 | 14.159 | 7,832.98 | 62.64 |
| 1970 | 580.57 | 132.7% | 259.37 | 1.068 | 1.019 | 13.110 | 12.872 | 7,472.91 | 59.76 |
| 1971 | 609.06 | 141.7% | 285.31 | 1.068 | 1.019 | 11.918 | 11.701 | 7,126.94 | 57.00 |
| 1972 | 638.74 | 151.3% | 313.84 | 1.068 | 1.019 | 10.835 | 10.638 | 6,794.74 | 54.34 |
| 1973 | 669.65 | 161.6% | 345.23 | 1.068 | 1.019 | 9.850 | 9.671 | 6,475.96 | 51.79 |
| 1974 | 701.84 | 172.6% | 379.75 | 1.068 | 1.019 | 8.954 | 8.791 | 6,170.24 | 49.34 |
| 1975 | 689.40 | 196.6% | 417.72 | 1.139 | 0.955 | 8.140 | 8.525 | 5,877.20 | 47.00 |
| 1976 | 712.49 | 212.8% | 459.50 | 1.082 | 1.005 | 7.400 | 7.364 | 5,246.69 | 41.96 |
| 1977 | 736.01 | 230.4% | 505.45 | 1.083 | 1.005 | 6.727 | 6.696 | 4,928.10 | 39.41 |
| 1978 | 759.97 | 249.5% | 555.99 | 1.083 | 1.005 | 6.116 | 6.088 | 4,626.76 | 37.00 |
| 1979 | 784.34 | 270.2% | 611.59 | 1.083 | 1.004 | 5.560 | 5.536 | 4,341.90 | 34.72 |
| 1980 | 809.14 | 292.7% | 672.75 | 1.083 | 1.004 | 5.054 | 5.033 | 4,072.77 | 32.57 |
| 1981 | 795.83 | 317.1% | 740.02 | 1.083 | 1.004 | 4.595 | 4.577 | 3,642.38 | 29.13 |
| 1982 | 781.77 | 343.6% | 814.03 | 1.084 | 1.004 | 4.177 | 4.162 | 3,253.42 | 26.02 |
| 1983 | 766.94 | 372.4% | 895.43 | 1.084 | 1.004 | 3.797 | 3.784 | 2,902.16 | 23.21 |
| 1984 | 751.32 | 403.8% | 984.97 | 1.084 | 1.003 | 3.452 | 3.441 | 2,585.19 | 20.67 |
| 1985 | 904.51 | 355.8% | 1,083.47 | 0.881 | 1.235 | 3.138 | 2.542 | 2,299.38 | 18.39 |
| 1986 | 1240.04 | 304.0% | 1,191.82 | 0.854 | 1.273 | 2.853 | 2.241 | 2,778.92 | 22.22 |
| 1987 | 1679.29 | 270.5% | 1,311.00 | 0.890 | 1.222 | 2.594 | 2.122 | 3,563.72 | 28.50 |
| 1988 | 2242.17 | 247.6% | 1,442.10 | 0.915 | 1.188 | 2.358 | 1.984 | 4,449.25 | 35.58 |
| 1989 | 2276.44 | 264.4% | 1,586.31 | 1.068 | 1.019 | 2.144 | 2.105 | 4,791.03 | 38.31 |
| 1990 | 2311.24 | 286.5% | 1,744.94 | 1.083 | 1.004 | 1.949 | 1.941 | 4,486.15 | 35.88 |
| 1991 | 2070.07 | 351.9% | 1,919.43 | 1.228 | 0.886 | 1.772 | 2.000 | 4,140.66 | 33.11 |
| 1992 | 2096.63 | 382.2% | 2,111.38 | 1.086 | 1.002 | 1.611 | 1.608 | 3,371.44 | 26.96 |
| 1993 | 2122.93 | 415.2% | 2,322.52 | 1.086 | 1.001 | 1.464 | 1.462 | 3,104.28 | 24.83 |
| 1994 | 2148.92 | 451.2% | 2,554.77 | 1.087 | 1.001 | 1.331 | 1.330 | 2,857.47 | 22.85 |
| 1995 | 2206.07 | 490.4% | 2,810.24 | 1.087 | 1.001 | 1.210 | 1.209 | 2,667.61 | 21.33 |
| 1996 | 1912.86 | 533.3% | 3,091.27 | 1.087 | 1.000 | 1.100 | 1.100 | 2,103.45 | 16.82 |
| 1997 | 1934.43 | 580.1% | 3,400.39 | 1.088 | 1.000 | 1.000 | 1.000 | 1,934.43 | 15.47 |

Exhibit IIa displays the assumptions that underlie this exhibit. Column numbers on this exhibit refer to their position in Exhibit IIa.

(8) = 1.000 for 1960 and $(7)_{current}/(7)_{prior}$ for all other years. Note that an increasing TxC denotes rate adequacy slippage.

(10)current= (8)1997/(8)current

(11)current= (9)1997/(9)current

(12)=(11)/(10)

(13)=(5)*(12)

(14)=(13) times a factor that makes the first entry 100. [i.e., (14) is a rescaled version of (13)

| Contract Year | Exposure Units | Pricing Exposure Base | Historical Rate | Historical Premium | Historical Loss | Isolated (TxC) | ΔTxC | LossIndex | On Level Factor adequacy | | | Restated Premium | "N" |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | adequacy | Loss | Composite | | |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) |
| 1960 | 100 | 100.00 | 4.00 | 400.00 | 275.00 | 68.8% | 1.000 | 100.00 | 1.088 | 34.004 | 31.261 | 12,504.46 | 100.00 |
| 1961 | 100 | 103.00 | 4.00 | 412.00 | 302.50 | 73.4% | 1.068 | 110.00 | 1.019 | 30.913 | 30.351 | 12,504.46 | 100.00 |
| 1962 | 100 | 106.09 | 4.00 | 424.36 | 332.75 | 78.4% | 1.068 | 121.00 | 1.019 | 28.102 | 27.591 | 11,708.72 | 93.64 |
| 1963 | 100 | 109.27 | 4.00 | 437.09 | 366.03 | 83.7% | 1.068 | 133.10 | 1.019 | 25.548 | 25.083 | 10,963.62 | 87.68 |
| 1964 | 100 | 112.55 | 4.00 | 450.20 | 402.63 | 89.4% | 1.068 | 146.41 | 1.019 | 23.225 | 22.803 | 10,265.94 | 82.10 |
| 1965 | 100 | 115.93 | 4.00 | 463.71 | 442.89 | 95.5% | 1.068 | 161.05 | 1.019 | 21.114 | 20.730 | 9,612.65 | 76.87 |
| 1966 | 100 | 119.41 | 4.00 | 477.62 | 487.18 | 102.0% | 1.068 | 177.16 | 1.019 | 19.194 | 18.845 | 9,000.94 | 71.98 |
| 1967 | 102 | 125.45 | 4.00 | 501.79 | 546.62 | 108.9% | 1.068 | 194.87 | 1.019 | 17.449 | 17.132 | 8,596.71 | 68.75 |
| 1968 | 104 | 131.74 | 4.00 | 526.98 | 613.07 | 116.3% | 1.068 | 214.36 | 1.019 | 15.863 | 15.575 | 8,207.48 | 65.64 |
| 1969 | 106 | 138.31 | 4.00 | 553.22 | 687.34 | 124.2% | 1.068 | 235.79 | 1.019 | 14.421 | 14.159 | 7,832.98 | 62.64 |
| 1970 | 108 | 145.14 | 4.00 | 580.57 | 770.34 | 132.7% | 1.068 | 259.37 | 1.019 | 13.110 | 12.872 | 7,472.91 | 59.76 |
| 1971 | 110 | 152.27 | 4.00 | 609.06 | 863.07 | 141.7% | 1.068 | 285.31 | 1.019 | 11.918 | 11.701 | 7,126.94 | 57.00 |
| 1972 | 112 | 159.69 | 4.00 | 638.74 | 966.64 | 151.3% | 1.068 | 313.84 | 1.019 | 10.835 | 10.638 | 6,794.74 | 54.34 |
| 1973 | 114 | 167.41 | 4.00 | 669.65 | 1082.29 | 161.6% | 1.068 | 345.23 | 1.019 | 9.850 | 9.671 | 6,475.96 | 51.79 |
| 1974 | 116 | 175.46 | 4.00 | 701.84 | 1211.40 | 172.6% | 1.068 | 379.75 | 1.019 | 8.954 | 8.791 | 6,170.24 | 49.34 |
| 1975 | 118 | 183.84 | 3.75 | 689.40 | 1355.52 | 196.6% | 1.139 | 417.72 | 0.955 | 8.140 | 8.525 | 5,877.20 | 47.00 |
| 1976 | 120 | 192.56 | 3.70 | 712.49 | 1516.34 | 212.8% | 1.082 | 459.50 | 1.005 | 7.400 | 7.364 | 5,246.69 | 41.96 |
| 1977 | 122 | 201.65 | 3.65 | 736.01 | 1695.77 | 230.4% | 1.083 | 505.45 | 1.005 | 6.727 | 6.696 | 4,928.10 | 39.41 |
| 1978 | 124 | 211.10 | 3.60 | 759.97 | 1895.93 | 249.5% | 1.083 | 555.99 | 1.005 | 6.116 | 6.088 | 4,626.76 | 37.00 |
| 1979 | 126 | 220.94 | 3.55 | 784.34 | 2119.16 | 270.2% | 1.083 | 611.59 | 1.004 | 5.560 | 5.536 | 4,341.90 | 34.72 |
| 1980 | 128 | 231.18 | 3.50 | 809.14 | 2368.08 | 292.7% | 1.083 | 672.75 | 1.004 | 5.054 | 5.033 | 4,072.77 | 32.57 |
| 1981 | 124 | 230.68 | 3.45 | 795.83 | 2523.49 | 317.1% | 1.083 | 740.02 | 1.004 | 4.595 | 4.577 | 3,642.38 | 29.13 |
| 1982 | 120 | 229.93 | 3.40 | 781.77 | 2686.29 | 343.6% | 1.084 | 814.03 | 1.004 | 4.177 | 4.162 | 3,253.42 | 26.02 |
| 1983 | 116 | 228.94 | 3.35 | 766.94 | 2856.42 | 372.4% | 1.084 | 895.43 | 1.004 | 3.797 | 3.784 | 2,902.16 | 23.21 |
| 1984 | 112 | 227.67 | 3.30 | 751.32 | 3033.72 | 403.8% | 1.084 | 984.97 | 1.003 | 3.452 | 3.441 | 2,585.19 | 20.67 |
| 1985 | 108 | 226.13 | 4.00 | 904.51 | 3217.91 | 355.8% | 0.881 | 1,083.47 | 1.235 | 3.138 | 2.542 | 2,299.38 | 18.39 |
| 1986 | 115 | 248.01 | 5.00 | 1240.04 | 3769.12 | 304.0% | 0.854 | 1,191.82 | 1.273 | 2.853 | 2.241 | 2,778.92 | 22.22 |
| 1987 | 126 | 279.88 | 6.00 | 1679.29 | 4542.61 | 270.5% | 0.890 | 1,311.00 | 1.222 | 2.594 | 2.122 | 3,563.72 | 28.50 |
| 1988 | 140 | 320.31 | 7.00 | 2242.17 | 5552.08 | 247.6% | 0.915 | 1,442.10 | 1.188 | 2.358 | 1.984 | 4,449.25 | 35.58 |
| 1989 | 138 | 325.21 | 7.00 | 2276.44 | 6020.04 | 264.4% | 1.068 | 1,586.31 | 1.019 | 2.144 | 2.105 | 4,791.03 | 38.31 |
| 1990 | 138 | 334.96 | 6.90 | 2311.24 | 6622.05 | 286.5% | 1.083 | 1,744.94 | 1.004 | 1.949 | 1.941 | 4,486.15 | 35.88 |
| 1991 | 138 | 345.01 | 6.00 | 2070.07 | 7284.25 | 351.9% | 1.228 | 1,919.43 | 0.886 | 1.772 | 2.000 | 4,140.66 | 33.11 |
| 1992 | 138 | 355.36 | 5.90 | 2096.63 | 8012.68 | 382.2% | 1.086 | 2,111.38 | 1.002 | 1.611 | 1.608 | 3,371.44 | 26.96 |
| 1993 | 138 | 366.02 | 5.80 | 2122.93 | 8813.95 | 415.2% | 1.086 | 2,322.52 | 1.001 | 1.464 | 1.462 | 3,104.28 | 24.83 |
| 1994 | 138 | 377.00 | 5.70 | 2148.92 | 9695.34 | 451.2% | 1.087 | 2,554.77 | 1.001 | 1.331 | 1.330 | 2,857.47 | 22.85 |
| 1995 | 140 | 393.94 | 5.60 | 2206.07 | 10819.44 | 490.4% | 1.087 | 2,810.24 | 1.001 | 1.210 | 1.209 | 2,667.61 | 21.33 |
| 1996 | 120 | 347.79 | 5.50 | 1912.86 | 10201.18 | 533.3% | 1.087 | 3,091.27 | 1.000 | 1.100 | 1.100 | 2,103.45 | 16.82 |
| 1997 | 120 | 358.23 | 5.40 | 1934.43 | 11221.30 | 580.1% | 1.088 | 3,400.39 | 1.000 | 1.000 | 1.000 | 1,934.43 | 15.47 |

## Application of the Generalized Central Limit Theorem

| Contract Year | Portfolio Premium | Relative Size, $\alpha_j$ | Restated *ILR* | Wt'd ILR | $(ILR\text{-}m)^2$ | Wt'd square |
|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5)=(3)*(4) | (6) | (7)=(3)*(6) |
| 1971 | 2,120,969 | 1.000 | 56.83% | 56.83% | 0.00346 | 0.00346 |
| 1972 | 2,911,068 | 0.956 | 59.44% | 56.81% | 0.00108 | 0.00103 |
| 1973 | 3,743,812 | 0.917 | 77.81% | 71.35% | 0.02278 | 0.02089 |
| 1974 | 5,013,270 | 0.910 | 87.11% | 79.25% | 0.05947 | 0.05410 |
| 1975 | 8,152,331 | 1.130 | 60.74% | 68.64% | 0.00039 | 0.00044 |
| 1976 | 14,309,591 | 1.438 | 48.75% | 70.10% | 0.01952 | 0.02807 |
| 1977 | 18,575,401 | 1.483 | 70.18% | 104.08% | 0.00557 | 0.00826 |
| 1978 | 21,062,760 | 1.758 | 56.90% | 100.03% | 0.00338 | 0.00595 |
| 1979 | 28,171,973 | 2.260 | 54.25% | 122.59% | 0.00717 | 0.01621 |
| 1980 | 25,932,066 | 2.205 | 60.83% | 134.11% | 0.00036 | 0.00079 |
| 1981 | 28,123,798 | 1.845 | 53.15% | 98.07% | 0.00916 | 0.01690 |
| 1982 | 23,631,441 | 1.295 | 94.12% | 121.92% | 0.09859 | 0.12771 |
| 1983 | 27,239,321 | 1.104 | 95.36% | 105.25% | 0.10654 | 0.11758 |
| 1984 | 42,815,673 | 1.506 | 77.93% | 117.39% | 0.02312 | 0.03483 |
| 1985 | 74,217,430 | 1.892 | 34.57% | 65.40% | 0.07922 | 0.14987 |
| 1986 | 79,620,062 | 1.580 | 30.00% | 47.40% | 0.10705 | 0.16914 |
| 1987 | 57,935,225 | 1.010 | 51.98% | 52.51% | 0.01154 | 0.01166 |
| 1988 | 69,423,482 | 1.925 | 59.50% | 114.56% | 0.00104 | 0.00199 |
| 1989 | 72,130,055 | 2.118 | 64.83% | 137.31% | 0.00044 | 0.00094 |
| 1990 | 69,647,663 | 1.910 | 72.05% | 137.64% | 0.00870 | 0.01662 |
| 1991 | 99,383,529 | 2.014 | 71.02% | 143.03% | 0.00690 | 0.01389 |
| 1992 | 93,804,805 | 1.745 | 76.72% | 133.90% | 0.01961 | 0.03422 |
| 1993 | 94,513,634 | 1.570 | 59.15% | 92.86% | 0.00127 | 0.00200 |
| Total | 960,479,379 | 35.57 | | 2231.04% | 0.59636 | 0.83655 |

Column (2) reflects historical data. The first entry is the total of Column (2) on Exhibit IIIb.
Column (3) is taken from Column (14) of Exhibit IIIc.
Column (4) is the last column on Exhibit IIIa.

| Estimators when $\alpha = 1.000$ | | | |
|---|---|---|---|
| | $ILR_t$ | ===> | $R_t$ |
| $m'$ = | 62.7% | ===> | 1.000 |
| *Var* = | 0.038 | ===> | 0.097 |

Once the distribution for the incurred loss ratio has been determined, the distribution for the random component follow as a change of scale.

$m'$ = Total (5)/Total(3)

$Var\ [ILR\ |\alpha=1.00]$ = Total(7)/ (number of years -1)

$Var[R\,|\alpha=1.00] = Var[ILR\,|\alpha=1.00]/m'^2$

| Portfolio Estimators for *Var* at other $\alpha$ | | | |
|---|---|---|---|
| Restated $N_{portfolio}$ | Size, $\alpha_j$ | $Var[ILR_t]$ | $Var[R_t]$ |
| 224,626,501 | 1.000 | 0.038 | 0.097 |
| 336,939,752 | 1.500 | 0.025 | 0.064 |
| 449,253,002 | 2.000 | 0.019 | 0.048 |
| 561,566,253 | 2.500 | 0.015 | 0.039 |
| 673,879,504 | 3.000 | 0.013 | 0.032 |
| 786,192,754 | 3.500 | 0.011 | 0.028 |

$Var[ILR \text{ or } R\,|\alpha_k] = Var[ILR \text{ or } R\,|\alpha=1.00]/\alpha_k$

## Loss Ratio Series Decomposition

Exhibit IIIa

| Contract Year | Est Ult Loss Ratio ILR (1) | 5-year Moving Avg TxC (2) | Isolated T (3) | Isolated C (4) | Trend Index (5) | Cycle Index (6) | Restated ILR (7) |
|---|---|---|---|---|---|---|---|
| 1969 | 367.46% | | | | | | |
| 1970 | 275.03% | | | | | | |
| 1971 | 227.32% | 270.44% | 252.7% | 1.070 | 89.596 | 107.025 | 56.83% |
| 1972 | 219.59% | 249.81% | 239.2% | 1.044 | 84.807 | 104.444 | 59.44% |
| 1973 | 262.81% | 228.36% | 226.4% | 1.009 | 80.274 | 100.868 | 77.81% |
| 1974 | 264.32% | 205.18% | 214.3% | 0.957 | 75.984 | 95.745 | 87.11% |
| 1975 | 167.78% | 186.77% | 202.8% | 0.921 | 71.923 | 92.076 | 60.74% |
| 1976 | 111.40% | 154.51% | 192.0% | 0.805 | 68.078 | 80.472 | 48.75% |
| 1977 | 127.55% | 122.88% | 181.7% | 0.676 | 64.440 | 67.615 | 70.18% |
| 1978 | 101.49% | 120.60% | 172.0% | 0.701 | 60.995 | 70.106 | 56.90% |
| 1979 | 106.19% | 132.36% | 162.8% | 0.813 | 57.735 | 81.283 | 54.25% |
| 1980 | 156.37% | 173.81% | 154.1% | 1.128 | 54.649 | 112.770 | 60.83% |
| 1981 | 170.17% | 216.49% | 145.9% | 1.484 | 51.729 | 148.389 | 53.15% |
| 1982 | 334.83% | 240.54% | 138.1% | 1.742 | 48.964 | 174.188 | 94.12% |
| 1983 | 314.87% | 223.26% | 130.7% | 1.708 | 46.347 | 170.802 | 95.36% |
| 1984 | 226.48% | 196.51% | 123.7% | 1.588 | 43.869 | 158.829 | 77.93% |
| 1985 | 69.96% | 136.82% | 117.1% | 1.168 | 41.525 | 116.823 | 34.57% |
| 1986 | 36.44% | 82.12% | 110.9% | 0.741 | 39.305 | 74.078 | 30.00% |
| 1987 | 36.34% | 47.27% | 104.9% | 0.451 | 37.204 | 45.051 | 51.98% |
| 1988 | 41.38% | 47.03% | 99.3% | 0.473 | 35.216 | 47.348 | 59.50% |
| 1989 | 52.24% | 54.49% | 94.0% | 0.580 | 33.334 | 57.964 | 64.83% |
| 1990 | 68.73% | 64.50% | 89.0% | 0.725 | 31.552 | 72.486 | 72.05% |
| 1991 | 73.77% | 70.23% | 84.2% | 0.834 | 29.866 | 83.377 | 71.02% |
| 1992 | 86.39% | 76.13% | 79.7% | 0.955 | 28.269 | 95.489 | 76.72% |
| 1993 | 70.02% | 80.04% | 75.5% | 1.061 | 26.758 | 106.052 | 59.15% |
| 1994 | 81.76% | | 71.4% | | 25.328 | | |
| 1995 | 88.24% | | 67.6% | | 23.974 | | |

(1) Historical Data

(2) Five year moving average of historical data

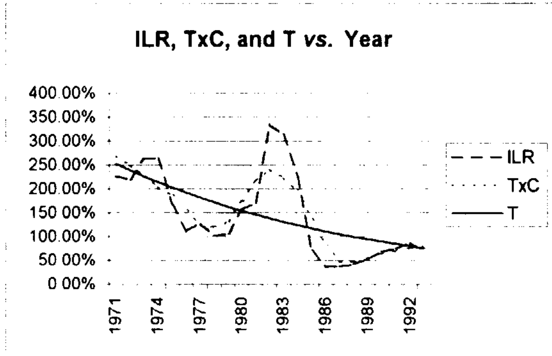
(3) Model of trend, fit to the isolated TxC of column (2). (3) $=30.5*(1.0196)^{(year-1969)}$

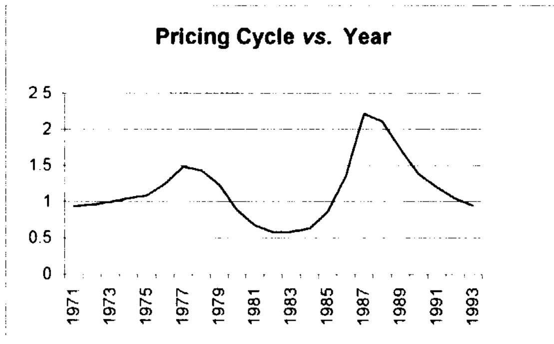
(4) Implied Cycle, TxC/T = (2)/(3)

(5) $=100*(1.0196)^{(year-1969)}$

(6) =100*(4)

(7) =(1)year*[(5)1997/(5)year]*[100/(6)year]




Least square error trend model: $TxC = 266.95*(1-0.0534)^{(yr-1969)}$

**Effective Portfolio Size in 1971**

Exhibit IIIb

Contract Year: 1971

| Client | Portfolio Premium | Percent Cover | Percent Taken | 100% Basis | $N_j$ | $n_j$ | $n_j^2/N_j$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 1 | 57,803 | 90% | 40.0% | 160,565 | 160,565 | 0.0273 | 4.626E-09 |
| 2 | 66,072 | 100% | 12.0% | 550,602 | 550,602 | 0.0312 | 1.763E-09 |
| 3 | 62,945 | 100% | 18.0% | 349,695 | 349,695 | 0.0297 | 2.519E-09 |
| 4 | 55,266 | 100% | 10.0% | 552,655 | 552,655 | 0.0261 | 1.229E-09 |
| 5 | 54,678 | 100% | 20.0% | 273,389 | 273,389 | 0.0258 | 2.431E-09 |
| 6 | 74,950 | 100% | 10.0% | 749,495 | 749,495 | 0.0353 | 1.666E-09 |
| 7 | 73,365 | 100% | 10.0% | 733,647 | 733,647 | 0.0346 | 1.631E-09 |
| 8 | 52,155 | 100% | 20.0% | 260,777 | 260,777 | 0.0246 | 2.319E-09 |
| 9 | 44,847 | 100% | 5.0% | 896,948 | 896,948 | 0.0211 | 4.985E-10 |
| 10 | 72,712 | 100% | 10.0% | 727,119 | 727,119 | 0.0343 | 1.616E-09 |
| 11 | 71,820 | 100% | 10.0% | 718,201 | 718,201 | 0.0339 | 1.597E-09 |
| 12 | 43,423 | 100% | 10.0% | 434,231 | 434,231 | 0.0205 | 9.653E-10 |
| 13 | 53,135 | 100% | 2.0% | 2,656,766 | 2,656,766 | 0.0251 | 2.362E-10 |
| 14 | 43,794 | 100% | 10.0% | 437,935 | 437,935 | 0.0206 | 9.735E-10 |
| 15 | 71,257 | 100% | 10.0% | 712,568 | 712,568 | 0.0336 | 1.584E-09 |
| 16 | 48,097 | 100% | 20.0% | 240,487 | 240,487 | 0.0227 | 2.138E-09 |
| 17 | 50,748 | 100% | 20.0% | 253,740 | 253,740 | 0.0239 | 2.256E-09 |
| 18 | 62,084 | 100% | 5.0% | 1,241,672 | 1,241,672 | 0.0293 | 6.9E-10 |
| 19 | 70,094 | 85% | 15.0% | 549,756 | 549,756 | 0.0330 | 1.987E-09 |
| 20 | 82,714 | 100% | 15.0% | 551,428 | 551,428 | 0.0390 | 2.758E-09 |
| 21 | 57,830 | 100% | 5.0% | 1,156,599 | 1,156,599 | 0.0273 | 6.428E-10 |
| 22 | 71,779 | 100% | 1.0% | 7,177,880 | 7,177,880 | 0.0338 | 1.596E-10 |
| 23 | 63,955 | 100% | 5.0% | 1,279,091 | 1,279,091 | 0.0302 | 7.108E-10 |
| 24 | 47,139 | 100% | 10.0% | 471,388 | 471,388 | 0.0222 | 1.048E-09 |
| 25 | 48,772 | 100% | 10.0% | 487,719 | 487,719 | 0.0230 | 1.084E-09 |
| 26 | 74,841 | 100% | 5.0% | 1,496,824 | 1,496,824 | 0.0353 | 8.318E-10 |
| 27 | 74,084 | 100% | 25.0% | 296,335 | 296,335 | 0.0349 | 4.117E-09 |
| 28 | 44,138 | 100% | 15.0% | 294,251 | 294,251 | 0.0208 | 1.472E-09 |
| 29 | 43,995 | 100% | 15.0% | 293,300 | 293,300 | 0.0207 | 1.467E-09 |
| 30 | 48,169 | 100% | 30.0% | 160,565 | 160,565 | 0.0227 | 3.212E-09 |
| 31 | 76,883 | 100% | 10.0% | 768,832 | 768,832 | 0.0362 | 1.709E-09 |
| 32 | 70,557 | 100% | 20.0% | 352,785 | 352,785 | 0.0333 | 3.137E-09 |
| 33 | 73,538 | 50% | 5.0% | 2,941,520 | 2,941,520 | 0.0347 | 4.087E-10 |
| 34 | 51,082 | 100% | 10.0% | 510,818 | 510,818 | 0.0241 | 1.136E-09 |
| 35 | 62,250 | 100% | 100.0% | 62,250 | 62,250 | 0.0293 | 1.384E-08 |
| Total | 2,120,969 | | 6.9% | 30,801,830 | | | 7.046E-08 |

**$N_{portfolio}$= 14,193,426**

*Column (1) is a client contract identifier. It could be the contract number or the name of the client.*

*Colums (2)-(4) relate to the premium ceded to this particular reinsurance company, the percentage of the total premium ceded to all reinsurers in total, and the percentage of the placement that this particular reinsurer accepted.*

*Column (5) = (2)/[(3)(4)]. It represents the premium that would have been written had 100% of the business been placed and had a single reinsurer accepted 100% of the placement.*

*Column (6)=(5)*

*Column (7) = (2)client/(2)total*

*Column (8) = $(7)^2/(6)$*

$N_{portfolio}$=1/sum(8)

## Restated Effective Portfolio Size

Exhibit IIIc

| Contract | $N_{portfolio}$ | Isolated | | | On Level Factor | | | Restated $N_{portfolio}$ | |
|---|---|---|---|---|---|---|---|---|---|
| Year | Premium | TxC | LossIndex | ΔTxC | Adequacy | Loss | Composite | Premium | $\alpha$ |
| (1) | (5) | (7) | (9) | (8) | (10) | (11) | (12) | (13) | (14) |
| 1971 | 14,193,426 | 270.44% | 11.412 | 1.000 | 1.051 | 16.637 | 15.826 | 224,626,501 | 1.000 |
| 1972 | 19,091,248 | 249.81% | 14.835 | 0.924 | 1.138 | 12.799 | 11.246 | 214,697,754 | 0.956 |
| 1973 | 24,061,300 | 228.36% | 19.286 | 0.914 | 1.150 | 9.845 | 8.561 | 205,979,386 | 0.917 |
| 1974 | 31,575,645 | 205.18% | 25.072 | 0.898 | 1.170 | 7.573 | 6.472 | 204,364,734 | 0.910 |
| 1975 | 50,319,813 | 186.77% | 32.593 | 0.910 | 1.155 | 5.825 | 5.044 | 253,819,452 | 1.130 |
| 1976 | 86,558,657 | 154.51% | 40.038 | 0.827 | 1.271 | 4.742 | 3.732 | 323,011,228 | 1.438 |
| 1977 | 110,115,274 | 122.88% | 47.483 | 0.795 | 1.322 | 3.999 | 3.025 | 333,106,893 | 1.483 |
| 1978 | 122,363,170 | 120.60% | 54.928 | 0.981 | 1.071 | 3.457 | 3.227 | 394,867,336 | 1.758 |
| 1979 | 160,390,551 | 132.36% | 62.633 | 1.097 | 0.958 | 3.031 | 3.165 | 507,575,310 | 2.260 |
| 1980 | 144,685,401 | 173.81% | 69.296 | 1.313 | 0.801 | 2.740 | 3.423 | 495,208,996 | 2.205 |
| 1981 | 142,840,047 | 216.49% | 77.522 | 1.246 | 0.844 | 2.449 | 2.902 | 414,489,368 | 1.845 |
| 1982 | 126,628,059 | 240.54% | 87.333 | 1.111 | 0.946 | 2.174 | 2.298 | 290,970,632 | 1.295 |
| 1983 | 143,041,514 | 223.26% | 96.721 | 0.928 | 1.133 | 1.963 | 1.733 | 247,910,720 | 1.104 |
| 1984 | 220,340,672 | 196.51% | 103.511 | 0.880 | 1.194 | 1.834 | 1.536 | 338,395,876 | 1.506 |
| 1985 | 374,303,495 | 136.82% | 110.761 | 0.696 | 1.510 | 1.714 | 1.135 | 424,927,861 | 1.892 |
| 1986 | 393,519,772 | 82.12% | 120.196 | 0.600 | 1.751 | 1.580 | 0.902 | 354,908,674 | 1.580 |
| 1987 | 280,616,252 | 47.27% | 128.558 | 0.576 | 1.826 | 1.477 | 0.809 | 226,940,464 | 1.010 |
| 1988 | 329,535,791 | 47.03% | 136.905 | 0.995 | 1.057 | 1.387 | 1.312 | 432,472,550 | 1.925 |
| 1989 | 335,535,545 | 54.49% | 147.594 | 1.159 | 0.907 | 1.286 | 1.418 | 475,778,788 | 2.118 |
| 1990 | 317,508,163 | 64.50% | 158.173 | 1.184 | 0.888 | 1.200 | 1.352 | 429,136,831 | 1.910 |
| 1991 | 386,092,174 | 70.23% | 167.834 | 1.089 | 0.966 | 1.131 | 1.172 | 452,362,632 | 2.014 |
| 1992 | 357,131,160 | 76.13% | 178.354 | 1.084 | 0.970 | 1.065 | 1.098 | 392,039,124 | 1.745 |
| 1993 | 352,633,199 | 80.04% | 189.866 | 1.051 | 1.000 | 1.000 | 1.000 | 352,633,199 | 1.570 |

Column numbers on this exhibit are consistent with those on Exhibit II

(5) = Historical portfolio premiums, Nportfolio, reflect treaty shares. The supporting detail for 1971 appears on Exhibit IIIb.

(7) Column (2) from Exhibit IIIa

(8) = 1.000 for 1971 and $(7)_{current}/(7)_{prior}$ for all other years. Note that an increasing TxC denotes rate adequacy slippage.

(9) Indices are consistent with Masterson Bodily Injury (other than automobile) indices.

(10)current= (8)1993/(8)current

(11)current= (9)1993/(9)current

(12)=(11)/(10)

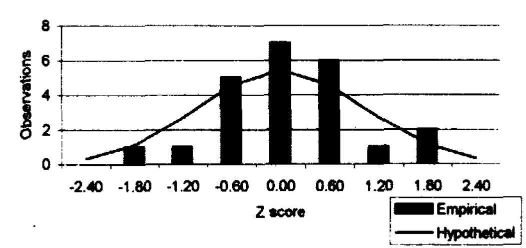
(13)=(5)*(12)

(14)=(13) times a factor that makes the first entry 1.000. [i.e., (14) is a rescaled version of (13)

# $\chi^2$ Test for Normal Distribution

Exhibit IIId

| Contract Year | Portfolio Size | Adjusted ILR | $R_t$ | $Z_R$ |
|---|---|---|---|---|
| 1971 | 1.000 | 56.83% | 0.906 | -0.302 |
| 1972 | 0.956 | 59.44% | 0.948 | -0.168 |
| 1973 | 0.917 | 77.81% | 1.241 | 0.774 |
| 1974 | 0.910 | 87.11% | 1.389 | 1.251 |
| 1975 | 1.130 | 60.74% | 0.968 | -0.101 |
| 1976 | 1.438 | 48.75% | 0.777 | -0.716 |
| 1977 | 1.483 | 70.18% | 1.119 | 0.383 |
| 1978 | 1.758 | 56.90% | 0.907 | -0.298 |
| 1979 | 2.260 | 54.25% | 0.865 | -0.434 |
| 1980 | 2.205 | 60.83% | 0.970 | -0.097 |
| 1981 | 1.845 | 53.15% | 0.847 | -0.491 |
| 1982 | 1.295 | 94.12% | 1.501 | 1.610 |
| 1983 | 1.104 | 95.36% | 1.520 | 1.674 |
| 1984 | 1.506 | 77.93% | 1.242 | 0.780 |
| 1985 | 1.892 | 34.57% | 0.551 | -1.443 |
| 1986 | 1.580 | 30.00% | 0.478 | -1.678 |
| 1987 | 1.010 | 51.98% | 0.829 | -0.551 |
| 1988 | 1.925 | 59.50% | 0.949 | -0.165 |
| 1989 | 2.118 | 64.83% | 1.034 | 0.108 |
| 1990 | 1.910 | 72.05% | 1.149 | 0.478 |
| 1991 | 2.014 | 71.02% | 1.132 | 0.426 |
| 1992 | 1.745 | 76.72% | 1.223 | 0.718 |
| 1993 | 1.570 | 59.15% | 0.943 | -0.183 |
| Mean | | 62.7% | 1.000 | |
| Variance | | | 0.097 | |
| Std Dev | | | 0.311 | |




## $\chi 2$ Test

| z> | z<= | Midpoint z | Expected Count if N(1.000, 0.311) | Empirical Count | Empirical Weight | Uncorrected Chi-square | Corrected* Chi-square |
|---|---|---|---|---|---|---|---|
| -2.70 | -2.10 | -2.40 | 0 | 0 | 0.000 | | |
| -2.10 | -1.50 | -1.80 | 1 | 1 | 1.580 | <= | <= |
| -1.50 | -0.90 | -1.20 | 3 | 1 | 1.892 | 1.117 | 1.368 |
| -0.90 | -0.30 | -0.60 | 5 | 5 | 7.553 | 0.044 | 0.001 |
| -0.30 | 0.30 | 0.00 | 5 | 7 | 11.661 | 0.458 | 0.214 |
| 0.30 | 0.90 | 0.60 | 5 | 6 | 9.576 | 0.459 | 0.197 |
| 0.90 | 1.50 | 1.20 | 3 | 1 | 0.910 | 0.320 | 0.103 |
| 1.50 | 2.10 | 1.80 | 1 | 2 | 2.399 | >= | >= |
| 2.10 | 2.70 | 2.40 | 0 | 0 | 0.000 | | |
| | | Total | 23 | 23 | 35.571 | 0.422 | 0.266 |

Empirical normal

$\mu =$ 1.000

$\sigma =$ 0.311

Degrees of freedom

23 = k = # of observations

1 = *m = # of estimated parameters*

22 = degrees of freedom

Chi-square

Uncorrected $\chi 2 =$ 0.422

Corrected $\chi 2 =$ 0.266

$\chi$ 2 at .95 = 32.67

$\chi$ 2 at .99 = 41.40

$\chi 2$ at .05 = 11.59

$\chi 2$ at .01 = 8.90

Conclusion: at 95% - Fit is good ==> CANNOT REJECT null hypothesis

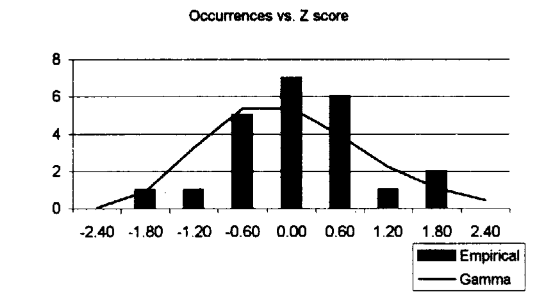
at 99% - Fit is good ==> CANNOT REJECT null hypothesis

* *Corrected to reflect the application of a continuous distribution to discrete data*

## $\chi^2$ Test for a Gamma Distribution

**Exhibit IV**

| Contract Year | Portfolio Size | Adjusted ILR | $R_t$ | $z_R$ |
|---|---|---|---|---|
| 1971 | 1.000 | 56.83% | 0.906 | -0.302 |
| 1972 | 0.956 | 59.44% | 0.948 | -0.168 |
| 1973 | 0.917 | 77.81% | 1.241 | 0.774 |
| 1974 | 0.910 | 87.11% | 1.389 | 1.251 |
| 1975 | 1.130 | 60.74% | 0.968 | -0.101 |
| 1976 | 1.438 | 48.75% | 0.777 | -0.716 |
| 1977 | 1.483 | 70.18% | 1.119 | 0.383 |
| 1978 | 1.758 | 56.90% | 0.907 | -0.298 |
| 1979 | 2.260 | 54.25% | 0.865 | -0.434 |
| 1980 | 2.205 | 60.83% | 0.970 | -0.097 |
| 1981 | 1.845 | 53.15% | 0.847 | -0.491 |
| 1982 | 1.295 | 94.12% | 1.501 | 1.610 |
| 1983 | 1.104 | 95.36% | 1.520 | 1.674 |
| 1984 | 1.506 | 77.93% | 1.242 | 0.780 |
| 1985 | 1.892 | 34.57% | 0.551 | -1.443 |
| 1986 | 1.580 | 30.00% | 0.478 | -1.678 |
| 1987 | 1.010 | 51.98% | 0.829 | -0.551 |
| 1988 | 1.925 | 59.50% | 0.949 | -0.165 |
| 1989 | 2.118 | 64.83% | 1.034 | 0.108 |
| 1990 | 1.910 | 72.05% | 1.149 | 0.478 |
| 1991 | 2.014 | 71.02% | 1.132 | 0.426 |
| 1992 | 1.745 | 76.72% | 1.223 | 0.718 |
| 1993 | 1.570 | 59.15% | 0.943 | -0.183 |
| Mean | | 62.7% | 1.000 | |
| Variance | | | 0.097 | |
| Std Dev | | | 0.311 | |




### $\chi 2$ *Test*

| z> | z<= | Midpoint z | Expected Count $\Gamma_{r,\lambda}(x)$ (r= 10.345) | Empirical Count | Empirical Weight | Uncorrected Chi-square | Corrected* Chi-square |
|---|---|---|---|---|---|---|---|
| -2.70 | -2.10 | -2.40 | 0 | 0 | 0.000 | | |
| -2.10 | -1.50 | -1.80 | 1 | 1 | 1.580 | <= | <= |
| -1.50 | -0.90 | -1.20 | 3 | 1 | 1.892 | 1.204 | 1.561 |
| -0.90 | -0.30 | -0.60 | 5 | 5 | 7.553 | 0.025 | 0.003 |
| -0.30 | 0.30 | 0.00 | 5 | 7 | 11.661 | 0.477 | 0.226 |
| 0.30 | 0.90 | 0.60 | 4 | 6 | 9.576 | 1.107 | 0.640 |
| 0.90 | 1.50 | 1.20 | 2 | 1 | 0.910 | 0.167 | 0.023 |
| 1.50 | 2.10 | 1.80 | 1 | 2 | 2.399 | >= | >= |
| 2.10 | 2.70 | 2.40 | 0 | 0 | 0.000 | | |
| | | Total | 23 | 23 | 35.571 | 0.593 | 0.402 |

**Empirical Gamma**

mean= 1.000
std dev = 0.311
$r$= 10.345

**Degrees of freedom**

23 = k = # of obversations
1 = *m = # of estimated parameters*
22 = degrees of freedom

**Chi-square**

Uncorrected $\chi 2$ = 0.593
Corrected $\chi 2$ = 0.402

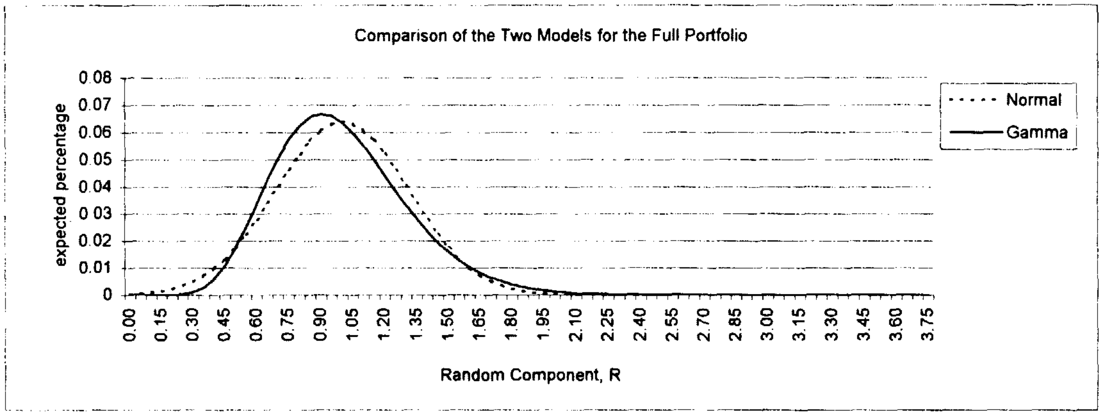
$\chi$ 2 at .95 = 32.67
$\chi$ 2 at .99 = 41.40
$\chi 2$ at .05 = 11.59
$\chi 2$ at .01 = 8.90

Conclusion: at 95% - Fit is good ==> CANNOT REJECT null hypothesis

at 99% - Fit is good ==> CANNOT REJECT null hypothesis

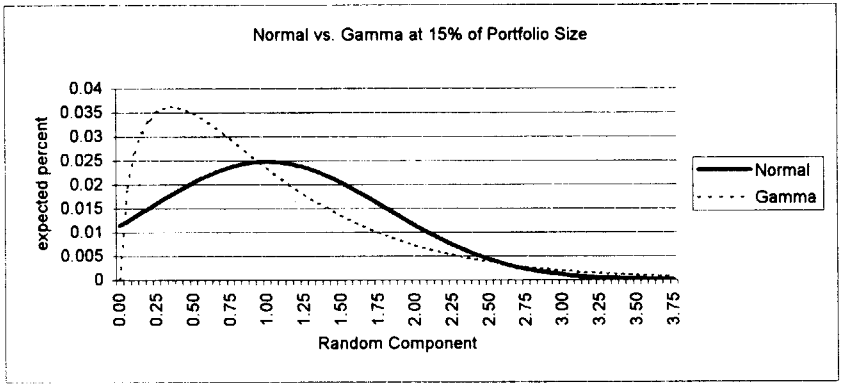
* *Corrected to reflect the application of a continuous distribution to discrete data*

Exhibit Va



Comparison of the Two Models for the Full Portfolio
expected percentage
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.00 0.15 0.30 0.45 0.60 0.75 0.90 1.05 1.20 1.35 1.50 1.65 1.80 1.95 2.10 2.25 2.40 2.55 2.70 2.85 3.00 3.15 3.30 3.45 3.60 3.75
Random Component, R
Normal
Gamma

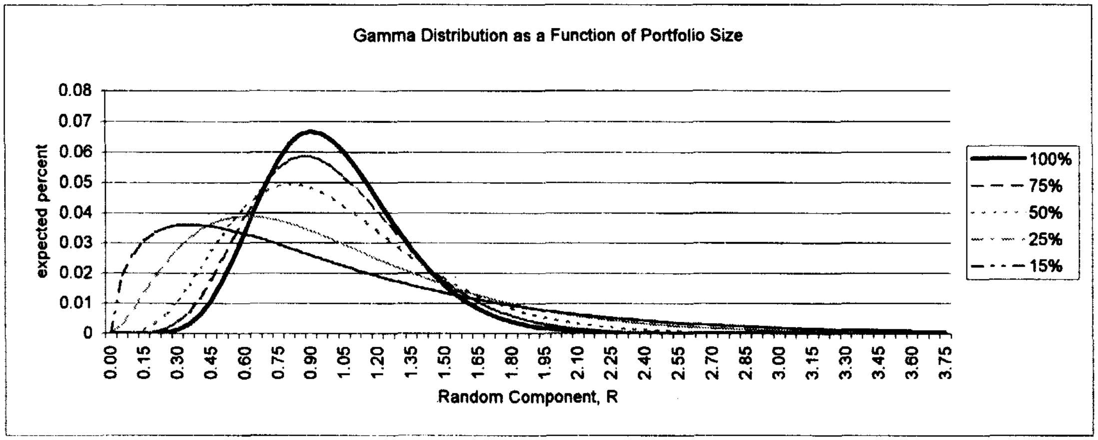
Exhibit Vb




Normal Parameters: $\mu$ =1.000, $\sigma^2$ = 0.097/0.15

Gamma parameter: $r$ = 10.345 * 0.15

Exhibit Vc


Gamma Distribution as a Function of Portfolio Size
expected percent
Random Component, R
100%
75%
50%
25%
15%

**Reserving Loss Sensitive Elements** Exhibit VI

| Valuation | Loss Ratio | | | Return Premium Ratio | | | | |
|---|---|---|---|---|---|---|---|---|
| Months | Reported | IBNR | Est Ult | Reported | Developed | *a priori* | Weight | Est Ult |
| | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 0 | 0.00% | 60.00% | 60.00% | 0.00% | -15.00% | 10.00% | 0.00 | 10.00% |
| 12 | 25.00% | 55.00% | 80.00% | 30.00% | 16.25% | 5.00% | 0.05 | 5.56% |
| 24 | 40.00% | 32.00% | 72.00% | 25.00% | 17.00% | 7.00% | 0.20 | 9.00% |
| 36 | 55.00% | 18.00% | 73.00% | 10.00% | 5.50% | 6.75% | 0.50 | 6.13% |
| 48 | 70.00% | 4.00% | 74.00% | 9.00% | 8.00% | 6.50% | 0.70 | 7.55% |
| 60 | 70.00% | 5.00% | 75.00% | 8.00% | 6.75% | 6.25% | 0.80 | 6.65% |
| 72 | 75.00% | 0.00% | 75.00% | 8.40% | 8.40% | 6.25% | 0.90 | 8.19% |
| 84 | 75.00% | 0.00% | 75.00% | 8.40% | 8.40% | 6.25% | 1.00 | 8.40% |

*(1) From company data*
*(2) Determined by means of standard loss reserve development techniques*
*(3) =(1)+(2)*
*(4) From company data*
*(5) =(4)-.25*(2), reflects expected 25% sensitivity to future loss development.*
*(6) =10%-.25*[(3)-60%], reflects expected 25% sensitivity to changes in expected aggregate loss ratio*
*(7) Illustrative weights increase over time from 0% initially to 100% by 84 months*
*(8) =(7)*(5)+[1-(7)]*(6)*