Portfolio Decomposition: A Building Block Approach to Loss Development

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Abstract

Loss development triangles are a fundamental component of the loss reserve adequacy testing process. These triangles may not be ideal for the immediate application and, almost certainly, will be less efficacious when used for other applications. Whenever possible, the patterns and associated age-to-ultimate loss development factors disclosed by the triangles are adjusted to correct for the less than ideal correspondence between the intended application and the historical data.

These adjustments are typically performed at a macroscopic level. The methods frequently begin by determining the *average maturity* of the data that is to be developed. This average maturity is then compared with the average maturity of the historical data as of each valuation date. If a match can be found (i.e., if the data that is to be developed has the same average maturity as the historical data at one of its valuations), then it is assumed that the corresponding age-to-ultimate loss development factor can be applied to the losses under investigation. If no match is found, the required development factor is determined by interpolating between historical valuations whose average maturities lie on either side of the required maturity.

This paper presents an alternative adjustment method that is based upon a microscopic model of loss development. More specifically, losses making up the actual triangle are expressed in terms of an aggregation of the losses arising from single, infinitesimal, units of exposure. The development pattern for losses arising from these infinitesimal exposure units is modeled in such a way that it reproduces the observed macroscopic patterns. Once the model is established, the infinitesimal building blocks can be separated and recombined to reflect the aggregate losses corresponding to other groupings, at any desired valuation date.

Portfolio Decomposition: A Building Block Approach to Loss Development

Loss development triangles are a fundamental component of the loss reserve adequacy testing process. Individual triangles represent a compromise between the desire to work with homogeneous data and the need to have a credible volume of data from which to discern historical development patterns. Decisions made in forming the triangles will be influenced by their intended use as well as the availability of historical data. As a result, the triangles may not be ideal for the immediate application and, almost certainly, will be less efficacious when used for other applications. Whenever appropriate, the patterns and age-to-ultimate (ATU) loss development factors disclosed by the triangles are adjusted to correct for the less than ideal correspondence between the intended application and the available historical data.

These adjustments are typically performed at a macroscopic level. For example, the methods frequently begin by determining the *average maturity* of the loss data that is to be developed. This average maturity is then compared with the average maturity of the historical data at each valuation date. If a match can be found (*i.e.*, if the data that is to be developed has the same average maturity as the historical data at one of its valuations), then it is assumed that the corresponding age-to-ultimate loss development factor can be applied to the losses under investigation. If no match is found, the required development factor can be found by interpolating between historical valuations whose average maturities lie on either side of the required maturity. Since methods of this type do not require any subdivision of the data below the portfolio level, they will be referred to collectively as Top-Down methods.

This paper presents an alternative method of adjustment that is based upon a microscopic model of loss development. More specifically, losses making up the actual triangle are expressed in terms of an aggregation of the losses arising from single, infinitesimal, units of exposure. The development pattern for losses arising from these infinitesimal exposure units is modeled in such a way that it still reproduces the observed macroscopic patterns. Once the model is established, the infinitesimal exposure units can be separated and recombined to reflect the aggregate losses corresponding to other groupings, at any desired valuation date. The microscopic approach will be referred to as the Bottom-Up method.

Underlying the Top-Down methodology is the implicit assumption that it is meaningful to associate an *average maturity* with an exposure period and that there is a functional relationship between this average maturity and the expected future loss development. Once a reserving triangle has been expressed in terms of losses that arise from fundamental exposure units, it becomes obvious that the average maturity assumption of the Top-Down method is not universal. The paper specifies a condition that must be met when the Top-Down method is used and presents a situation in which the condition is met. Since the Bottom-Up method does not depend upon the assignment of an average maturity, this condition need not be met when the Bottom-Up method is used. Because there is no condition that must be met for the Bottom-Up method that is not also required by the Top-Down method, the Bottom-Up method is clearly the more versatile approach.

While primary companies are not immune to situations in which historical development patterns must be modified to fit specific applications, reinsurance companies encounter these situations more often. There are two major reasons for this. First of all, reinsurers have less control over the quality of their data. Where a primary company may be able to sort data according to report year, policy year, or accident year, a reinsurer rarely captures all three primary company fields. Second, the nature of excess of loss reinsurance usually results in a small volume of claim data. To obtain a sufficient volume of data to produce stable development patterns, a large degree of heterogeneity must be allowed. A few reinsurance examples illustrate when adjustments are necessary.

- Consider a portfolio consisting of Risk Attaching[1] and Loss Occurring[2] excess of loss medical malpractice liability insurance treaties. While the bulk of the treaties are effective on January 1st, many treaties are also effective on July 1st Suppose that losses arising from the treaties are grouped according the year in which the treaty was written (i.e., by contract year) and valued at successive year ends (12/31/XX). Loss development patterns are observed and ATU factors are derived. The intended use of these patterns is to develop the current year-end diagonal to the estimated ultimate loss amount for each contract year.

  If there has been a shift in the mix of business over time (*e.g.*, a change in the distribution of premiums by effective date or a shift in the proportion of treaties that are Risk Attaching *vs.* Loss Occurring), then historical patterns will not be appropriate. For example, an increase in the proportion of Risk Attaching treaties written toward the end of the year means that their will be a higher percentage of less mature claims at a December 31st valuation date than would be reflected in the historical data. Some adjustment would be required to reflect the less mature nature of the more recent contract years.

- Even if all of the treaties develop identically when time is measured from the individual treaty effective date, it would be very unlikely that any treaty would develop in the same manner as the entire contract year. In other words, it would be unlikely that any single treaty written in 1995 and valued 24 months after its effective date would have the same percentage of its ultimate loss reported as the 1995 contract year would have as of 12/31/96 (i.e., at 24 months). Adjustments to the development pattern would be required before the triangle could be used to predict the loss development of a single treaty. Such single treaty layer loss development factors are needed whenever experience rating is performed.

- A client submits historical data for which the valuation dates are not uniformly spaced (e.g., 12/31/94, 12/31/95, 3/31/96, 6/30/97, 3/31/98). This data is to be used to experience rate a treaty. Before the link ratios from successive valuations can be compared, there must be some adjustment to the data.

- The primary company changes the rate at which it settles claims (i.e., the disposal rate) in a known manner. An increase in disposal rates means that data will be more mature (i.e., closer to being fully emerged and developed) at recent valuation dates than it was historically. Before the historical development patterns can be used, adjustments must be made.

While all of these examples have been described in terms of changes in average maturity, it must be emphasized that, as intuitive as the concept of average maturity may be, it will be demonstrated that the concept is neither necessary nor easily quantifiable.

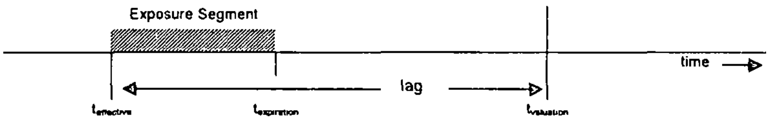
The first section of the paper establishes the foundation that underlies both the analysis of the Top-Down methodology and the development of the Bottom-Up methodology. In the second part of the paper, the Bottom-Up methodology is presented in a manner that is especially suited to a spreadsheet application. Finally, the validity of the Top-Down methodology is shown to depend upon a very restrictive assumption concerning the nature of the underlying development pattern.

Fundamental Exposure Unit, ΔE

Consider a segment of time during which the insurance or reinsurance company is exposed to loss. The obligation may be triggered by a loss occurring during the time segment (an occurrence form) or by a loss being reported during the time segment (a claims-made form). The nature of the trigger is not material to our discussion (although it is material to the insurance or reinsurance contract). The exposed segment $[t_{effective}, t_{expiration}]$ has an effective, or starting time, and an ending, or expiration time. The interval may either exclude on include the end points. Only events that occur between the two times trigger an obligation (loss) associated with the exposed segment.

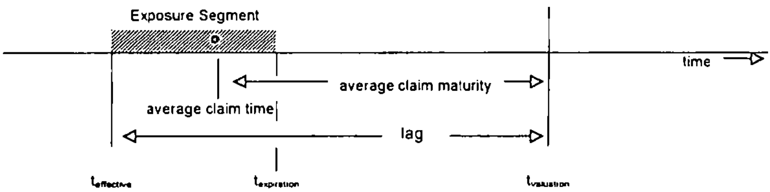
As time elapses between $t_{effective}$ and $t_{expiration}$, loss obligations (i.e., claims) may come about. After $t_{expiration}$, no new loss obligations can be incurred by the exposure segment. Note that the insurance company need not be aware of all the claims immediately after $t_{expiration}$, as any unknown claims will be incurred but not reported (IBNR).

Losses incurred during the exposure segment can be valued at any valuation time, $t_{valuation}$. The valuation lag is the elapsed time between the effective time and the valuation time. Effective times, Expiration times, and Valuation times are usually far enough apart that they are specified by dates rather than times in the usual sense of the word. Lags are measured in months.



Claims can be incurred throughout the exposure segment. A straight arithmetic mean of the claim times is the average claim time. When times are identified by dates, the average claim time is called an average claim date. If claims are incurred uniformly over the course of the exposure segment, the average claim date will be located at the midpoint of the exposure segment. While the average claim date will always fall between the effective date and expiration date (inclusive), it need not fall halfway between the two points. Consider, for example, a six month crop-hail insurance policy that was effective on June 1, 1995. It is highly unlikely that claims would fall symmetrically around the September 1, 1995 exposure period midpoint.

The time interval from the average claim time to the valuation time is called the average claim maturity as of the valuation time.



It is possible for the valuation time to be to the left of the expiration time. When that occurs, the exposure segment ends at the valuation time. That is to say, the valuation time is the latest possible time for which information can be known. Claims to the right of the valuation time cannot have been incurred as of the valuation.

As the valuation time is moved to the right from an initial location between the effective date and the expiration date, the lag increases, the average claim time either remains fixed or moves to the right (depending upon the distribution of claims within the exposure segment) while the average claim maturity can increase or decrease (depending upon relative rates with which the average claim date moves to the right as the valuation time increases). Once the valuation time becomes greater than the expiration time, the average claim time stops moving (all claims having been incurred). Additional increases in the lag time result in a corresponding increase in both the average claim maturity and the lag.

Following the ideas of differential calculus, allow the length of the exposure segment to approach zero by moving the expiration time toward the effective time. As the length of the segment approaches zero, two things happen:

- the average claim maturity approaches the lag minus half of the length of the exposure period, and
- the average claim time, exposure segment effective time, and expiration time become indistinguishable, so that the timing of individual claims is not significant within the infinitesimal exposure segment. Because the three times are almost equal, there will be no loss of generality if the actual distribution of claims within the infinitesimal segment is replaced by a uniform distribution of claim events.

For the purpose of this paper, we shall assume that these things happen (to a sufficient degree of accuracy) when the length of the exposure unit has been reduced to one month. These one month long exposure units are the fundamental building blocks, ΔE, of the Bottom-Up method. If one month is not sufficiently refined for a particular application, the limit can be taken down to exposure days or less, but with each succeeding decrease in size, the model formulas become more complicated to write.

The i^{th} exposure month, ΔE_i, experiences ultimate loss L_i. As of a particular valuation lag, t (greater than one month), L_iC_{it} will be reported. Here C_{it} is the completion ratio (percentage of the ultimate amount that is reported) for the i^{th} exposure month as of lag t and L_i is the ultimate loss amount arising from ΔE_i. If t is less than one month, the reported loss is given by tL_iC_{it} where the factor t reflects the fact that only part of the exposure unit's losses will have been incurred as of the valuation date. This follows from the assumption that losses can be treated as if they are distributed uniformly throughout ΔE_i.

In order to facilitate the decomposition of aggregate loss amounts into components corresponding to the basic building blocks, two assumptions are made:

- *Losses arising from all exposure months exhibit identical development patterns. In other words, C_t is independent of the exposure month, i.*
- The function C_t is known (or can be determined).

The first assumption does not imply that all losses develop identically. It only implies that the aggregation of losses that occur within an exposure month develops independently of the particular month of occurrence. Furthermore, the assumption does not mean that all exposure months must experience the same aggregate loss. The aggregate loss is denoted by the variable L_i. Not only can L_i reflect seasonality and long term trends, but it can also be zero (note that a claim-free month can develop from zero at lag t to an ultimate value of zero).

Building a Policy

Reported losses arising from an insurance policy with a one year term can be expressed in terms of losses arising from twelve fundamental exposure units Assume that the policy losses are subject to seasonality. The ultimate loss arising from each month can be expressed as the product of a seasonality factor, s_{month} and a constant factor, L. Furthermore, let the seasonality factors add to 12.

Therefore, each policy will experience an ultimate loss equal to $12L$ but these losses will not be uniformly distributed over the year (*i.e.*, while they are uniformly distributed during each month, the distribution is not uniform at the macroscopic, full year, level).

In terms of the building blocks, a policy with a January 1st effective date will experience the development displayed below:

Lag Month	Reported Loss
1	Ls_{Jar}
2	$L(s_{Jan}C_2+ s_{Feb}($
3	$L(s_{Jan}C_3+ s_{Feb}C_2+ s_{Mar}($
4	$L(s_{Jan}C_4+ s_{Feb}C_3+ s_{Mar}C_2+ s_{Api}$
5	$L(s_{Jan}C_5+ s_{Feb}C_4+ s_{Mar}C_3+ s_{Apr}C_2+ s_{May}($
5.5	$s_{Jan}LC_{5.5}+ s_{Feb}LC_{4.5}+ s_{Mar}LC_{3.5}+ s_{Apr}LC_{2.5}+ s_{May}LC_{1.5}+0.5s_{June}LC$
6	$s_{Jan}LC_6+ s_{Feb}LC_5+ s_{Mar}LC_4+ s_{Apr}LC_3+ s_{May}LC_2+ s_{June}L$
.	.
10	$L(s_{Jan}C_{10}+s_{Feb}C_9+s_{Mar}C_8+s_{Apr}C_7+s_{May}C_6+s_{June}C_5+s_{July}C_4+s_{Aug}C_3+s_{Sept}C_2+ s_{Oct}($
11	$L(s_{Jan}C_{11}+s_{Feb}C_{10}+s_{Mar}C_9+s_{Apr}C_8+s_{May}C_7+s_{June}C_6+s_{July}C_5+s_{Aug}C_4+s_{Sept}C_3+s_{Oct}C_2+s_{Nov}($
12	$L(s_{Jan}C_{12}+s_{Feb}C_{11}+s_{Mar}C_{10}+s_{Apr}C_9+s_{May}C_8+s_{June}C_7+s_{July}C_6+s_{Aug}C_5+s_{Sept}C_4+s_{Oct}C_3+s_{Nov}C_2+s_{Dec}($
13	$L(s_{Jan}C_{13}+s_{Feb}C_{12}+s_{Mar}C_{11}+s_{Apr}C_{10}+s_{May}C_9+s_{June}C_8+s_{July}C_7+s_{Aug}C_6+s_{Sept}C_5+s_{Oct}C_4+s_{Nov}C_3+s_{Dec}($
14	$L(s_{Jan}C_{14}+s_{Feb}C_{13}+s_{Mar}C_{12}+s_{Apr}C_{11}+s_{May}C_{10}+s_{June}C_9+s_{July}C_8+s_{Aug}C_7+s_{Sept}C_6+s_{Oct}C_5+s_{Nov}C_4+s_{Dec}($
.	.
$\omega+11$	$L(s_{Jan}1.00+s_{Feb}1.00+s_{Mar}1.00+s_{Apr}1.00+s_{May}+s_{June}1.00+s_{July}1.00+s_{Aug}1.00+s_{Sept}1.00+s_{Oct}1.00$ $+s_{Nov}1.00+s_{Dec}1.00) = 12L$

By the end of January, one lag month has elapsed for this January effective policy. One exposure month will have been earned and a loss $s_{Jan}L$ will have been incurred. By the end of the month, $s_{Jan}LC_1$ will have been reported. By the end of February, the lag for the January exposure month will be 2 and the reported loss will have matured to $s_{Jan}LC_2$. At the same time (lag month 2), the February exposure month will be earned. Losses arising from February, $s_{Feb}L$, will be reported as $s_{Feb}LC_1$. As a result, the policy reported loss as of lag 2 (measured from the policy effective date) is $L(s_{Jan}C_2 + s_{Feb}C_1)$.

The pattern of losses from each exposure month aging by one month and the addition a new exposure month at each policy lag continues until all of the exposure months have been earned. For a policy of one year term that is effective on January 1st, this occurs on December 31st, or lag 12. After that point, each of the twelve months continues to age but no additional months are added (see lags 12 and beyond).

At some point in time, ω, there will be no further loss development on losses arising from an exposure month. In other words,

$$C_j = 1.000 \text{ when } j \geq \omega$$

By lag $\omega + 11$, all of the exposure months that make up the policy will be fully developed. Since the seasonality indices were arbitrarily forced to add to 12, the ultimate policy loss will be $12L$.

As of any (policy) lag month, an ATU loss development factor can be determined. For example, the 13 month-to-ultimate factor for the January effective policy is given by:

$$12/(s_{Jan}C_{13}+s_{Feb}C_{12}+s_{Mar}C_{11}+s_{Apr}C_{10}+s_{May}C_9+s_{June}C_8+s_{July}C_7+s_{Aug}C_6+s_{Sept}C_5+s_{Oct}C_4+s_{Nov}C_3+s_{Dec}C_2).$$

For lags less than 12, there are two possible development factors. If the partial policy period is to be developed to reflect the ultimate loss for the full year, then the reported loss would be divided into $12L$, as above. If the ultimate loss for the stub period is desired, then the numerator would be the sum of the seasonality indices for the exposed months times L. For example,

$$3{:}\textit{Full Policy Ult} = 12L/(s_{Jan}C_3 + s_{Feb}C_2 + s_{Mar}C_1)$$

develops the three month stub (January 1 – March 30, as of March 30) to a full year's ultimate loss whereas

$$3{:}\textit{Stub period Ult} = (s_{Jan} + s_{Feb} + s_{Mar})/\ (s_{Jan}C_3 + s_{Feb}C_2 + s_{Mar}C_1)$$

takes the same losses to the ultimate loss incurred during the period January 1 to March 30.

Because the conversion,

$$3{:}\textit{Stub period Ult} = ((s_{Jan} + s_{Feb} + s_{Mar})/12) * 3{:}\textit{Full Policy Ult},$$

is trivial, the remainder of this paper will concern itself with deriving expressions for age-to-full ultimate development factors.

Finally, note that the reported loss as of lag 5.5 has been displayed. As of lag 5.5, half of the June exposure month will be earned. If an exposure month is sufficiently short to satisfy the conditions required of a fundamental exposure unit, then the losses can be treated as if they were uniformly distributed throughout the month, hence, the factor of 0.5 for June.

A policy effective on February 1st would be constructed in a similar manner. In fact, the formulas for the reported loss would be almost identical with those displayed above except that the seasonality indices would all be offset by one month. For example:

Lag Month	Reported Loss
1	$Ls_{Feb}C_1$
2	$L(s_{Feb}C_2 + s_{Mar}C_1)$
3	$L(s_{Feb}C_3 + s_{Mar}C_2 + s_{Apr}C_1)$
.	.
.	.
.	.

<u>*Bottom-Up Decomposition of a Loss Development Triangle*</u>

A loss development triangle displays the aggregate incurred loss arising from portfolios of similar exposures as they were reported at different valuation dates. Aside from the subject business, portfolios are usually identified according to the period during which there was exposure to loss (e.g., report years, policy years, accident years, reinsurance contract years, etc.). To be more specific, an individual portfolio might consist of claims-made medical malpractice insurance policies for which there was exposure to reported loss during 1994 (i.e., Report Year 1994). The reason for casting the historical experience into triangle form is the expectation that, by doing so, recurring development patterns will be observed. If these patterns continue into the future, then an application of these historical patterns to less mature exposure periods will provide reasonable estimates of the ultimate loss amount arising from the "greener" periods. The patterns may include second order changes such as a constant rate of change in link ratios from exposure period to exposure period (see, for example, Berquist and Sherman [1]).

These patterns will only be meaningful to the extent that the portfolios of losses arising from different years can be expected to develop in a similar manner. The patterns will be detectable only to the extent that there is sufficient data so that random fluctuations from the true patterns will offset themselves once the data is aggregated. Satisfying these two conditions involves making compromises between having restrictive entry criteria to insure the homogeneity of the individual exposures making up each portfolio and relaxing the portfolio definition enough to obtain a sufficient volume of data. For the purpose of this paper, it will be assumed (at least initially) that these issues have been addressed and that the appropriate ATU development factors have been derived for the aggregate data as of each valuation date.

The Bottom-Up method consists of expressing the reported losses for each cell of the triangle (i.e., for each portfolio and valuation date) in terms of the completion ratios corresponding to the basic building blocks of exposure. A model for the completion ratios, $\{C_t\}$, is devised such that the expressions for each triangle cell results in the same aggregate completion as was selected from an analysis of the empirical (triangle) data. Once the $\{C_t\}$ are known, the exposure units can be recombined into different groupings and, by means of the model, the aggregate completion and corresponding ATU development factor can be determined.

A quick example illustrates the method. Consider a captive insurance company that writes twelve month Workers Compensation policies, all effective on the first of January. It has been writing policies for the same set of 100 policyholders for over 40 years. At each year-end, the company values all of its claims. As a result of a loss reserve adequacy test, the following policy year incurred loss development factors were derived.

Empirical Development Factors

Lag	ATU	% Reported
12	16.667	6%
24	2.500	40%
36	1.667	60%
48	1.333	75%
60	1.111	90%
72	1.053	95%
84	1.000	100%
96	1.000	100%
108	1.000	100%

The last column represents the empirical completion ratios for the cohort of policies. These completion ratios can also be expressed in terms of the completion of the underlying exposure units. From the expression for the reported loss on a January effective policy, the reported loss as of lag 24 is

$$L(s_{Jan}C_{24}+s_{Feb}C_{23}+s_{Mar}C_{22}+s_{Apr}C_{21}+s_{May}C_{20}+s_{June}C_{19}+s_{July}C_{18}+s_{Aug}C_{17}+s_{Sept}C_{16}+s_{Oct}C_{15}+s_{Nov}C_{14}+s_{Dec}C_{13})$$

which represents a

$$(s_{Jan}C_{24}+s_{Feb}C_{23}+s_{Mar}C_{22}+s_{Apr}C_{21}+s_{May}C_{20}+s_{June}C_{19}+s_{July}C_{18}+s_{Aug}C_{17}+s_{Sept}C_{16}+s_{Oct}C_{15}+s_{Nov}C_{14}+s_{Dec}C_{13})/12$$

completion ratio. In the absence of seasonality, this reduces to

$$(C_{24}+C_{23}+C_{22}+C_{21}+C_{20}+C_{19}+C_{18}+C_{17}+C_{16}+C_{15}+C_{14}+C_{13})/12.$$

From the empirical analysis, we know that

$$(C_{24}+C_{23}+C_{22}+C_{21}+C_{20}+C_{19}+C_{18}+C_{17}+C_{16}+C_{15}+C_{14}+C_{13})/12 = 0.40,$$

or, more compactly written,

$$\sum_{j=13}^{24} C_j = 12 * \text{Empirical Ratio (at lag = 24)}.$$

The Bottom-Up method consists of finding a set of $\{C_j\}$ that satisfies

$$\sum_{j=g-11}^{g} C_j = 12 * \text{Empirical Ratio (at lag} = g)$$

for all of the empirical lags, g.

Let us assume, for the sake of the illustration, that a set of $\{C_j\}$ that reproduce the empirical observations can be found. Further, assume that the company writes an additional 100 policies that are identical to the first 100 with the exception of the policy effective date. The new policies are effective on July 1st Clearly, the percentage of the policy year's reported loss for the combined set of 200 policies will not be 40% at the end of the second year. In terms of the $\{C_j\}$, the reported loss as of 12/31/X+1 will consist of 100 policies at lag 24 plus 100 policies at lag 18, or

$$[100\, L(C_{24}+C_{23}+C_{22}+C_{21}+C_{20}+C_{19}+C_{18}+C_{17}+C_{16}+C_{15}+C_{14}+C_{13})]$$
$$+[100L(C_{18}+C_{17}+C_{16}+C_{15}+C_{14}+C_{13}+C_{12}+C_{11}+C_{10}+C_{9}+C_{8}+C_{7})]$$

Since the total ultimate incurred loss is

$$100*12*L+100*12*L = 2400\, L,$$

the completion ratio as of 12/31/X+1 can quickly be calculated even though the mix of effective dates within the portfolio has changed from 100% January to 50% January and 50% July. The desired completion ratio is

$$[(C_{24}+C_{23}+C_{22}+C_{21}+C_{20}+C_{19})+2(C_{18}+C_{17}+C_{16}+C_{15}+C_{14}+C_{13})+(C_{12}+C_{11}+C_{10}+C_{9}+C_{8}+C_{7})]/24.$$

Conversely, if the portfolios had consistently reflected this mixture of January and July effective policies, then the expression above could have been equated to the empirical completion ratio. Once a $\{C_j\}$ that reproduces the empirical completion ratios for this mixture of effective dates has been determined, they can be reconstituted to produce the completion ratios corresponding to a single policy with either effective date (lags now being measured from the policy effective date rather than from the portfolio effective date).

The simple example begs two important issues:

1. For any but the simplest applications, the expressions containing the C_j become very long and complicated, and
2. An appropriate model for the C_j must be found.

Fortunately, both of these issues are relatively easy to address. The Bottom-Up methodology adapts well to spreadsheet software and a very simple model consisting of mixed Weibull curves fits a wide range of applications. The iterative solving features of many spreadsheet packages makes the selection of appropriate Weibull parameters a simple matter.

To illustrate just how complicated the formulas can become, we consider two more complicated examples, and then outline a spreadsheet approach that makes the task far less intimidating.

More Complex Applications: Reinsurance Treaties.

During the first month of a loss occurring (LO) reinsurance contract, twelve exposure months will be covered. Assuming that the contract is effective on January 1st (and that all of the policies are effective on the first of the month), the twelve exposure months would consist of the January earned exposure on the policies written during January together with the eleven exposures earned in January and arising from policies that were written during each of the previous eleven months. During the second month of the LO contract, twelve more "first" months will be earned while losses arising from the previously earned twelve exposure months will gain one month of maturity. This pattern will continue until twelve LO contract months, each consisting of twelve exposure months have been earned (a total of 144 earned exposure months).

Even in the case of a stationary portfolio (no growth or inflation) where there is no seasonality (either with respect to premium volume or incurred loss), expressions for the reported loss as of any lag (as measured from the treaty effective date) are fairly complicated, as shown below.

Simple Loss Occurring Reinsurance Treaty

Lag	Reported Loss
1	$12LC_1$
2	$12LC_1 + 12LC_2$
3	$12LC_1 + 12LC_2 + 12LC_3$
.	.
12	$12L(C_1 + C_2 + C_3 + \ldots C_{10} + C_{11} + C_{12})$
13	$12L(C_2 + C_3 + C_4 + \ldots C_{11} + C_{12} + C_{13})$
.	.

During the first month of a risk attaching (RA) reinsurance contract, only one exposure month is earned, the first month of policies written during the first month of the reinsurance contract. During the second month of the contract, losses from two new exposure months are added (the second month of the original policy and the first month of policies written during the second month of the RA contract). In addition to these new losses, the losses incurred during the first exposure month age one month during the second treaty month. By the time the RA contract expires, 144 exposure months will be earned (twelve earned exposure months from each of twelve policy effective months, assuming a one year RA contract covers policies of a one year term).

Again, assuming that the portfolio is stationary (no growth), exhibits no seasonality (either in losses or in the distribution of written premium), and that all policies are written for a full year term incepting on the first of the month, losses will develop as shown below.

Simple Risk Attaching Reinsurance Treaty

Lag Month	Reported Loss
1	$1LC_1$
2	$L(2C_1 + 1C_2)$
3	$L(3C_1 + 2C_2 + 1C_3)$
.	.
12	$L(12C_1 + 11C_2 + 10C_3 + 9C_4 + 8C_5 + 7C_6 + 6C_7 + 5C_8 + 4C_9 + 3C_{10} + 2C_{11} + 1C_{12})$
13	$L(11C_1 + 12C_2 + 11C_3 + 10C_4 + 9C_5 + 8C_6 + 7C_7 + 6C_8 + 5C_9 + 4C_{10} + 3C_{11} + 2C_{12} + 1C_{13})$
14	$L(10C_1 + 11C_2 + 12C_3 + 11C_4 + 10C_5 + 9C_6 + 8C_7 + 7C_8 + 6C_9 + 5C_{10} + 4C_{11} + 3C_{12} + 2C_{13} + 1C_{14})$
.	.
23	$L(1C_1 + 2C_2 + 3C_3 + 4C_4 + 5C_5 + 6C_6 + 7C_7 + 8C_8 + 9C_9 + 10C_{10} + 11C_{11} + 12C_{12} + 11C_{13} + 10C_{14} + 9C_{15} + 8C_{16} + 7C_{17} + 6C_{18} + 5C_{19} + 4C_{20} + 3C_{21} + 2C_{22} + 1C_{23})$
24	$L(1C_2 + 2C_3 + 3C_4 + 4C_5 + 5C_6 + 6C_7 + 7C_8 + 8C_9 + 9C_{10} + 10C_{11} + 11C_{12} + 12C_{13} + 11C_{14} + 10C_{15} + 9C_{16} + 8C_{17} + 7C_{18} + 6C_{19} + 5C_{20} + 4C_{21} + 3C_{22} + 2C_{23} + 1C_{24})$
.	.
n	$L(1C_{n-22} + 2C_{n-21} + 3C_{n-20} + .. 12C_{n-11} + . .3C_{n-2} + 2C_{n-1} + 1C_n)$

Overview of a Bottom-Up Spreadsheet

Writing an expression for the aggregate losses arising from a portfolio consisting of a mixture of loss occurring and risk attaching treaties with effective dates distributed throughout year can be an intimidating task, especially if one attempts to do it in a single step. Add seasonality to the distribution of primary policy written premium and attempt to reflect seasonality in loss dates and the situation becomes overwhelming. The solution is to break the task into smaller pieces. Because the methodology involves building larger units from basic building blocks, it is a natural spreadsheet application.

The spreadsheet begins with two columns, one for the lag month and one for the corresponding completion ratio for a single exposure month. In the beginning, the completion ratio column contains arbitrary values that hold a place for the model that has yet to be determined. The valuation lags should run well beyond the ω of loss development. Once a model has been selected, these two columns will display the j's and corresponding C_j's.

A third column consisting of seasonality indices would be added next. While there is no requirement that the indices add up to twelve, the author has consistently required that they do so. The reason for doing so is that, in the absence of seasonality, each exposure month in a twelve month policy would contribute the same amount of loss to the policy which would experience twelve times the loss of a single exposure month. If seasonality is present, the loss contributions will not be equal but it is convenient to let the total remain equal to twelve.

Following the example previously described, a single January effective policy could be modeled by combining the appropriate seasonality indices and fundamental completion ratios. In this section of the spreadsheet, a column indicating the lag as measured *from the policy effective date* would be used. For example, the first few formulas for reported losses arising from policies effective in January and Febraury would be

Lag Month	Single Policies: January Effectives	Single Policies: February Effectives
1	$Ls_{Jan}C_1$	$Ls_{Feb}C_1$
2	$L(s_{Jan}C_2+ s_{Feb}C_1)$	$L(s_{Feb}C_2+ s_{Mar}C_1)$
3	$L(s_{Jan}C_3+ s_{Feb}C_2+ s_{Mar}C_1)$	$L(s_{Feb}C_3+ s_{Mar}C_2+ s_{Apr}C_1)$
4	$L(s_{Jan}C_4+ s_{Feb}C_3+ s_{Mar}C_2+ s_{Apr}C_1$	$L(s_{Feb}C_4+ s_{Mar}C_3+ s_{Apr}C_2+ s_{May}C_1$
5	$L(s_{Jan}C_5+ s_{Feb}C_4+ s_{Mar}C_3+ s_{Apr}C_2+ s_{May}C_1)$	$L(s_{Feb}C_5+ s_{Mar}C_4+ s_{Apr}C_3+ s_{May}C_2+ s_{June}C_1$
⋮	⋮	⋮

Many lines of business are not written uniformly throughout the year. There are relatively few commercial insurance policies written with February effectives dates versus January effective dates. To the extent that the line of business experiences seasonality in its effective dates, it might be desirable to replace all of the L's in the second column with policy effective month specific, L_{Jan} and all of the L's in the next column with L_{Feb}. The intent of introducing ultimate loss amounts that depend upon the policy effective month is to reflect the proportion of writings by effective month. When policies are combined to form portfolios, only the *relative* size of the L's is significant. The author prefers to force the sum to 144L (i.e., twelve effective months of 12L each gives 144L for a policy year).

It is in the primary policy section of the spreadsheet that inflation or a changing mix of effective dates *could* be modeled. For example, L *could* be given a policy effective month *and* year designation. Doing so would increase the required number of columns from 12 (one for each effective month) to 12N, where N is the number of effective years represented in the loss development triangle. If the methodology used to determine the empirical completion ratios did not take these factors into account, it is unlikely that the additional Bottom-Up complexity is justified. After all, the objective is to be able to decompose and reconstitute a reserving portfolio, not necessarily to refine the analysis of the aggregate loss triangle.

Once individual policies have been built, policy years or risk attaching reinsurance treaties can be constructed. For example, a risk attaching reinsurance treaty that is effective on January 1st begins its life by "seeing" the first exposure month of all policies that are effective in January. The spreadsheet reference to the first cell of the reported loss for policies effective in January is trivial. The reported loss is simply $L_{Jan}s_{Jan}C_1$. During February, losses arising from the January exposure month become more mature by one month, $L_{Jan}s_{Jan}C_2$ while the February exposure month of policies that became effective in January introduces a new set of losses, $L_{Jan}s_{Feb}C_1$. During February, the January effective treaty also sees losses arising from the first exposure month of policies that were effective in February, $L_{Feb}s_{Feb}C_1$. The aggregate loss arising during the first two months (i.e., by lag 2, as measured from the treaty effective date) of a risk attaching treaty that incepts in January is

$$L_{Jan}S_{Jan}C_2+L_{Jan}S_{Feb}C_1+L_{Feb}S_{Feb}C_1,$$

but this is nothing more than the sum of the contents of the cell reflecting the second lag of a January policy and the cell reflecting the first lag of a February policy. The intermediate step of forming primary policies that reflect the relative volume of business written during each effective month reduces the number of terms from 144 completion ratios to 12 references to the individual policy section.

If seasonality of loss and/or differences in premium volume are present at the primary policy level, there will be twelve risk attaching treaty development patterns, one corresponding to each treaty effective month. As was the case when primary policy writing was not uniform, treaty effective dates may not be equally populated. In fact, the preference for January effective dates is more pronounced at the reinsurance level than at the primary company level. To reflect the actual distribution, the losses in each column (for each treaty effective date) could be multiplied by a weighting factor that reflects the relative amount of portfolio ultimate loss contributed by each treaty effective date). Absent any evidence to the contrary, premium can be used as a proxy for the expected loss weight.

Similarly, a loss occurring reinsurance treaty can be built up from fundamental exposure units. Assuming a stationary mix of business (i.e., no year subscript on the L's), the first month of a January effective loss occurring reinsurance treaty sees losses arising from the January exposure of primary policies written during any of the preceding eleven months plus the current month. The reported loss as of treaty lag 1 is given by

$$L_{Jan}s_{Jan}C_1 + L_{Dec}s_{Jan}C_1 + L_{Nov}s_{Jan}C_1 + L_{Oct}s_{Jan}C_1 + L_{Sept}s_{Jan}C_1 + \ldots L_{Mar}s_{Jan}C_1 + L_{Feb}s_{Jan}C_1 = 12Ls_{Jan}C_1$$

since the relative loss weights add up to reflect twelve policy effective months. At lag 2, the reported losses for a January effective loss occurring treaty are

$$12L(s_{Jan}C_2 + s_{Feb}C_1)$$

which equals $12L(C_2+C_1)$ in the absence of seasonality

As with the primary policies and risk attaching treaties, the entire column of loss occurring reported losses for each treaty effective month can be adjusted to reflect the relative proportion of loss contributed to the portfolio by treaties of that type (loss occurring vs. risk attaching) and effective date.

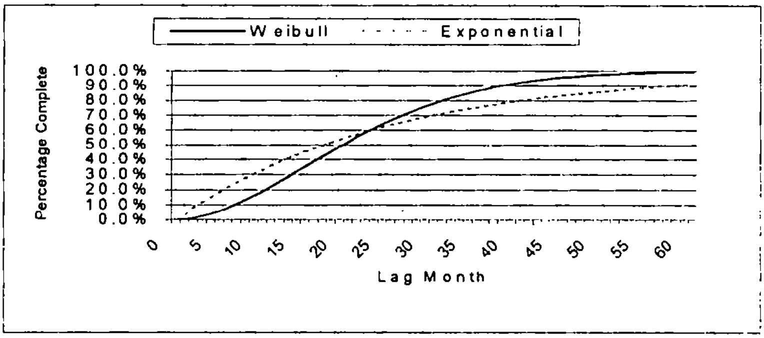
In a similar fashion, the individual risk attaching and loss occurring treaties can be combined to form the reported portfolio loss as of different lags (as measured from the beginning of the contract year).

The Weibull Model

A very general model is the Weibull curve. Because of its versatility and the fact that it frequently provides a good fit when interpolating ATU loss development factors (LDF's), it was a likely candidate. The Weibull is especially attractive because it involves only two parameters.

The equation for the two-parameter Weibull curve is given below:

The graphs of a Weibull model with c = 1.8, d=300 and c=1.0, d=25 are displayed below.



When c=1.00, the Weibull becomes the familiar exponential curve, another favorite for those who must model loss development tail factors [2]. Both curves are reasonable selections for the development of losses arising out of a fundamental exposure unit.

To allow for a mixture of different types of fundamental loss, the final model consisted of a weighted sum of Weibull curves,

$$C_t = 1 - \sum_{j=1}^{4} w_j e^{-t^{c_j}/d_j}$$

where the w_j represents weights which are required to add to 1.000.

An iterative spreadsheet solving package can be used to select the model parameters and weights that minimize the sum of the squared differences between the model completion ratios and empirical completion ratios at typical valuation lags such as "at 12 months," "at 24 months," "at 36 months," etc. Even with so few parameters, the author has been able to obtain remarkably good fits for several different reserving portfolios.

Some Applications of the Bottom-Up Method

Once the model has been specified and the $\{C_l\}$ are known, the $\{C_l\}$ can be reconstituted in a number of ways. Examples include:

- Reserving portfolios are typically valued annually, semi-annually, or quarterly. The $\{C_l\}$ are defined for each lag month. Even when the entire portfolio is valued less often than monthly, the expressions for the completion at these valuations will probably involve the C_l's for several months. As a result, the model will have few, if any, C_l's that were not reflected in the fitting process. Since the aggregate completion of the portfolio at every month end can be expressed in terms of the $\{C_l\}$, the model can be used to interpolate completion factors (and hence ATU loss development factors).
- A given portfolio, consisting of one type of data (accident year, for example) can be used to specify the model parameters. The model can then be used to express the corresponding completion on a policy year basis.
- Aggregate policy year calls of Workers Compensation data could be used to specify the model. The model could then be used to determine the corresponding Unit Statistical Plan report loss development. In the aggregate calls, all losses are valued as of a common *date*. Since policies incept throughout the year, a common valuation *date* results in a mixture of individual policy *lags*. Unit Statistical Plan reports are filed for each policy at lags that are measured from the policy inception. As a result, at first report, all USP losses are valued at exactly 18 months from the policy inception. Since only five years of USP data is filed, there is always a problem

modeling USP loss development beyond 5th report. Since both the aggregate calls and USP losses can be expressed in terms of the $\{C_{t}\}$, the model can be used to shuffle the data into the required form without actually recasting the aggregate triangles.

- Reinsurance Association of America (RAA) data reflects excess of loss reinsurance layer loss development on an accident year basis. If some reasonable assumptions regarding the mix of treaties by effective date and treaty type can be made, then a model can be fit to their triangles. The model can be used to determine the development pattern for a single risk attaching or loss occurring treaty.

- Consider a coverage that has been written for many years. The policy year development patterns are well known. How will the development pattern corresponding to a pro rata risk attaching reinsurance treaty change if *each policy* is now to be subject to a five year commutation (all claims were previously covered until they were reported and closed)? Commutations will begin at the beginning of the fifth year of the reinsurance treaty and continue until the last covered policy is commuted at the end of the fifth year.

  The historical *paid* loss data (without commutation) could be used to specify a model. Then, the primary policy section of the spreadsheet could be used to determine the impact of the commutation at lag 60 (as measured from the beginning of each policy). For example, the present value of the future payments might be determined and assumed to be the value of the settlement, NPV(payments beyond policy lag 60).

  The historical *incurred* loss data (without commutation) would be used to specify an incurred loss model. Without changing the parameters, the primary policy section of the incurred loss worksheet would be modified by replacing all of the completion ratios beyond lag 59 with the lag 59 completion plus the NPV(payments beyond policy lag 60). The spreadsheet would then appropriately reflect the commutation in all of the treaty aggregates. Note that because the ultimate loss will be less than 144*L*, the completion ratios will have to be renormalized to reflect the smaller number.

A Top-Down Methodology

As mentioned at the beginning of this paper, there is another method for decomposing and reconstituting a portfolio called the Top Down method. The Top Down method does not require any knowledge of the portfolio below the macroscopic, or aggregate, level. Were it not for an extremely restricting condition that frequently goes unnoticed, its intuitive appeal would easily defeat the Bottom-Up Methodology.

More specifically, the Top Down methodology requires a knowledge of only the average claim maturity at each valuation. The implicit assumption is that the completion ratios (or loss development factors) depend only upon the average claim maturity as of a given valuation date. Once the average claim maturity of a portfolio is known, future loss development can be determined. We will show that the implied assumption is true only if the mean of any set of consecutive completion ratios $\{C_x, C_{x+1}, C_{x+2} \ldots C_{x+n}\}$ is equal to the completion ratio corresponding to the mean time, $C_{\text{mean of } x, x+1, x+2, \ldots x+n}$. While this condition is met if the completion ratios are a linear function of time, this is a very severe restriction, indeed.

The Top Down methodology is most often encountered in two applications. The first involves adjusting loss development patterns to reflect a change in the distribution of written premium. The second involves extracting the development pattern for a single entity (such as a single, Risk Attaching reinsurance treaty) from the pattern determined for a mixture (such as from a portfolio consisting of Loss Occurring and Risk Attaching treaties that were written throughout a contract year).

Regardless of the particular application, the Top-Down method begins by determining average claim date and claim maturity corresponding to each lag. This can be a rather complicated task if losses exhibit seasonality. To simplify the illustration, assume that all policies have a one year term and that claims occur uniformly throughout the year. At the individual policy level, the average claim dates and corresponding maturity as of several lags (as measured from the policy effective date) are displayed below.

Average Maturity for an Individual Policy

Lag Month	Average Claim Date	Average Claim Maturity
1	0.5	0.5
2	1.0	1.0
3	1.5	1.5
4	2.0	2.0
5	2.5	2.5
6	3.0	3.0
7	3.5	3.5
8	4.0	4.0
9	4.5	4.5
10	5.0	5.0
11	5.5	5.5
12	6.0	6.0
13	6.0	7.0
14	6.0	8.0
15	6.0	9.0
16	6.0	10.0
17	6.0	11.0
18	6.0	12.0
19	6.0	13.0
20	6.0	14.0
21	6.0	15.0
22	6.0	16.0
23	6.0	17.0
24	6.0	18.0

Notice that the average claim date increases by one half month for every month that the lag increases until the lag reaches twelve, after which time the average claim date remains fixed. The shift results because each additional month of lag introduces another month of exposure during the first twelve months. After lag 12, the policy has expired, and no new incurred losses are possible. On the other hand, the average claim maturity (the time between the average claim date and the valuation date) continues to increase as the lag increases, even when the lag exceeds 12.

According to the Top Down method, this chart is all that is necessary to adjust the development pattern in the previously introduced simple example in which

100 July effective policies are to be added to a portfolio that has historically consisted of 100 policies with January effective dates. As the logic goes, on 12/31/X+1, the January policies are at lag 24 and the July policies are at lag 18. Since lag 24 represents an average claim maturity equal to 18 months and lag 18 represents an average claim maturity equal to 12 months, the average claim maturity for the combined portfolio as of 12/31/X+1 is 15 months. Of course, this is precisely correct. If claims are incurred uniformly, then the average claim date for the mixture is 15 months prior to 12/31/X+1. The error arises when the table is used in the reverse order to infer that the new portfolio with an average claim maturity equal to 15 months will have the same completion ratio as the original portfolio valued as of lag 21.

In terms of the fundamental building blocks, the completion ratio of the original portfolio at lag 21 is given by

$$(C_{21}+C_{20}+C_{19}+C_{18}+C_{17}+C_{16}+C_{15}+C_{14}+C_{13}+C_{12}+C_{11}+C_{10})/12.$$

There is no *a priori* reason to believe that this expression is equal to the correct completion ratio for the new portfolio (see the simple Bottom-Up illustration),

$$[(C_{24}+C_{23}+C_{22}+C_{21}+C_{20}+C_{19})+2(C_{18}+C_{17}+C_{16}+C_{15}+C_{14}+C_{13})+(C_{12}+C_{11}+C_{10}+C_{9}+C_{8}+C_{7})]/24.$$

If, however, the completion ratio for each exposure unit is **a linear function** of time,

$$C_t=mt+b,$$

then,

$$(C_{21}+C_{20}+C_{19}+C_{18}+C_{17}+C_{16}+C_{15}+C_{14}+C_{13}+C_{12}+C_{11}+C_{10})/12 = 15.5m+b$$

and

$$[(C_{24}+C_{23}+C_{22}+C_{21}+C_{20}+C_{19})+2(C_{18}+C_{17}+C_{16}+C_{15}+C_{14}+C_{13})+ (C_{12}+C_{11}+C_{10}+C_{9}+C_{8}+C_{7})]/24 = 15.5m+b$$

so the two expressions are equal.

There is another application for which the Top-Down method appears to provide an intuitively attractive solution but fails to perform. It involves the extraction of the loss development pattern for a single loss occurring or risk attaching treaty from a reserving portfolio that is composed of contract years that reflect a mixture of loss occurring and risk attaching treaties with different effective dates.

The method begins by setting up two claim maturity tables, one for loss occurring treaties, and one for risk attaching treaties. Again, for simplicity, we assume that there is no growth and that losses are not seasonal. Because a loss occurring treaty covers twelve exposure months (one arising from each of the twelve policy effective months) during each month that it is in force, the average claim dates and maturities are identical to those of a single policy. A risk attaching treaty slowly builds up from seeing a single exposure month (the first month of policies that share the effective date of the treaty) to seeing 144 exposure months by the time the policies that were effective during each month that the treaty was in force (twelve months) have fully earned their twelve exposure months.

For simplicity, assume that all policies incept at the beginning of their effective month. At the end of the first month of a risk attaching treaty, the average claim date is 0.5. A month later, the average claim date increases to 1.167, which is the average of one exposure month with 0.50 and two with an average claim date equal to 1.50 months. For example, during January, a January effective treaty sees the first exposure month of all policies that became effective in January. The average claim is incurred on January 15th. During February, the January treaty sees two additional exposure months, the second month of all policies that became effective in January and the first exposure month of policies that became effective in February. The average February claim occurs in mid-February (time equals 1.5). The average of the three dates is 1.167. Risk attaching average claim dates and claim maturities as of each of the first 25 lags are displayed below.

Average Claim Dates for Reinsurance Treaties

	Risk Attaching		Loss Occurring	
Lag Month	Average Claim Date	Average Claim Maturity	Average Claim Date	Average Claim Maturity
1	0.50	0.50	0.50	0.50
2	1.17	0.83	1.00	1.00
3	1.83	1.17	1.50	1.50
4	2.50	1.50	2.00	2.00
5	3.17	1.83	2.50	2.50
6	3.83	2.17	3.00	3.00
7	4.50	2.50	3.50	3.50
8	5.17	2.83	4.00	4.00
9	5.83	3.17	4.50	4.50
10	6.50	3.50	5.00	5.00
11	7.17	3.83	5.50	5.50
12	7.83	4.17	6.00	6.00
13	8.41	4.59	6.00	7.00
14	8.92	5.08	6.00	8.00
15	9.39	5.61	6.00	9.00
16	9.81	6.19	6.00	10.00
17	10.19	6.81	6.00	11.00
18	10.53	7.47	6.00	12.00
19	10.83	8.17	6.00	13.00
20	11.08	8.92	6.00	14.00
21	11.30	9.70	6.00	15.00
22	11.42	10.58	6.00	16.00
23	11.50	11.50	6.00	17.00
24	11.50	12.50	6.00	18.00
25	11.50	13.50	6.00	19.00
26	11.50	14.50	6.00	20.00
⋮	⋮	⋮	⋮	⋮

As the argument goes, each portfolio valuation represents some average claim maturity. *The average claim maturity is the weighted average of the claim* maturities of the individual contracts that make up the portfolio. The weights are the ultimate loss amounts for each type of contract. Assuming that there is no correlation between the contract effective date or treaty type and the expected ultimate loss ratio, treaty premiums are a readily accessible proxy for the ultimate loss amounts.

For example, assume that a particular contract year consists of the following distribution of loss (premiums) by type of treaty and effective date.

Effective	Type	Loss Weight
January 1, 1995	RA	25.0%
January 1, 1995	LO	25.0%
July 1, 1995	RA	25.0%
July 1, 1995	LO	25.0%

then the average claim maturity of the portfolio as of 12/31/96 would be 12.49 months

Effective	Weight	Type	Lag	Maturity
January 1, 1995	0.25	RA	24	12.50
January 1, 1995	0.25	LO	24	18.00
July 1, 1995	0.25	RA	18	7.47
July 1, 1995	0.25	LO	18	12.00
Portfolio	1.00	N/A	24	***12.49***

In other words, the portfolio, valued at 24 months reflects an average claim maturity equal to 12.49 months. The portfolio at 24 months lag has a completion ratio associated with it. The Top-Down method assumes that the completion ratio can also be associated with a 12.49 month claim maturity. Similarly for every other valuation, an average portfolio claim maturity is determined and associated with the portfolio completion ratio at the point in time.

If, for example, one wanted to know the completion ratio for a LO treaty valued 14 months after its effective date, all that would be necessary is to determine the average claim maturity of the treaty at lag 14 and then look up the portfolio completion ratio corresponding to the same average claim maturity. If the required portfolio maturity does not correspond to one of the portfolio valuations, a simple linear interpolation would provide the desired completion ratio. Unfortunately, as was shown in the previous example, averaging claim maturities only works if the fundamental exposure units have linear development patterns.

A corollary to this erroneous logic is that the completion ratios for a single risk attaching treaty can be easily determined by means of a simple mapping from the completion ratios for a single loss occurring treaty. Because of the mapping, only one set of development factors would be necessary. An appropriate mapping (based on average maturity) would indicate where to enter a LO table to obtain the appropriate loss development factor for a RA treaty. As will be demonstrated, such a mapping exists but there is no way in which to derive it from average claim maturity.

Intuitively, it makes sense to believe that a risk attaching treaty valued as of a certain lag, n, will experience as much future development as a loss occurring treaty at some point in its lifetime. That is to say, for every $RA\text{-}LDF_n$, there exists a $LO\text{-}LDF_m$ such that

$$RA\text{-}LDF_n = LO\text{-}LDF_m$$

A mapping consists of finding the set of all ordered pairs, (m,n) for which the equality is valid. The Top-Down method attempts to accomplish the mapping by comparing the *average claim maturity* of the two types of contracts. That is to say, the equality is assumed to hold whenever the corresponding average claim maturities are equal.

Once all of the subject exposure is earned (i.e., after lag 23 for an RA contract), the difference in average claim maturity for a RA and LO contract is 5.5 months (see the table of average claim maturities) if a uniform month-to-month distribution of policies is assumed. For $n \geq 23$, the average claim maturity mapping suggests

$$f(n) = m = n\text{-}5.5.$$

In other words, for lags greater than or equal to 23 months, the loss development factor for a risk attaching treaty can be found by selecting a loss occurring development factor corresponding to a lag that is 5.5 months less.

Using the expressions for the *RA-LDF's* and *LO-LDF*'s in terms of their component single exposure completion ratios, we can test the validity of this mapping. As an example, consider $RA\text{-}LDF_{29.5}$, the risk attaching ATU loss development factor associated with a 29.5 month lag (as measured from the contract effective date). The maturity mapping asserts that

$$RA\text{-}LDF_{29.5} = LO\text{-}LDF_{29.5\text{-}5.5} = LO\text{-}LDF_{24}$$

From the building block expressions for the completion ratio at each lag, the mapping implies

$$(1C_{7.5} + 2C_{8.5} + 3C_{9.5} + \ldots 12C_{18.5} + \ldots 3C_{27.5} + 2C_{28.5} + 1C_{29.5})/144$$

$$= (C_{13} + C_{14} + C_{15} + \ldots C_{22} + C_{23} + C_{24})/12.$$

Since different C's appear on the right and left sides of the equation, it is clear that the relationship cannot be true in general. The degree to which it is approximated depends upon the relationship between C's at different lags. As in the previous example, the equality is satisfied when C_t is a linear function of t. For, in that case, both sides of the equation reduce to $18.5m+b$.

For n<23, the mapping is more complicated, but no more accurate. The appropriate way to proceed is to determine the $\{C_t\}$ and then build both sets (LO and RA) completion ratios and LDF's from the basic building blocks. That is to say, use a Bottom-Up approach.

Summary

A loss reserving portfolio can be thought of as reflecting the aggregate loss experience of many individual fundamental exposure units. Once the aggregate data has been used to establish a model of the fundamental exposure unit completion ratios, the units can be reconstituted to form a wide variety of other portfolios. A fairly simple model, mixed Weibull curves, produces good fits for many reserving portfolios.

Once the completion of aggregate portfolios has been expressed in terms of completion ratios for fundamental exposure units, a competing method (the Top Down method) can be scrutinized. This scrutiny reveals that the Top Down method imposes a relationship among the fundamental completion ratios, a relationship that is satisfied when they are linear functions of time. Because such a relationship is unlikely to occur, the Top Down method is seen to be far less versatile than the Top Down method.

Appendix: An Application of the Bottom-Up Methodology

As a practical example of the Bottom Up Methodology, consider the derivation of policy year ATU loss development factors for Alabama Workers Compensation medical losses from National Council on Compensation Insurance (NCCI) accident year data.

Column 2 on Exhibit A-1 displays cumulative (ATU) accident year development factors for Alabama case incurred medical losses as they appear in the 1998 Statistical Bulletin [3]. The factors were based upon a five year average of historical link ratios. The NCCI data was valued as of 12/31/96. Column 3 simply restates the NCCI factors in the form of completion ratios.

If, at the level of the entire industry, there is no exposure growth during the year, there is no loss seasonality, and all policies are effective on the first of the month, then

$$\text{Reported loss at AY lag g} = 12L \sum_{j=g-11}^{g} C_j.$$

The corresponding completion ratios are given by

$$\text{Completion ratio at AY lag g} = \frac{1}{12} \sum_{j=g-11}^{g} C_j.$$

A particularly simple model for $\{C_i\}$ produced an excellent fit to the actual accident year completion ratios. The model consisted of the sum of two Weibull curves,

$$C_t = 1 - 0.678\, e^{-\left(t^{0.496} / 1.549\right)} - 0.322\, e^{-\left(t^{01.633} / 3596.268\right)}.$$

Column 4 displays the corresponding accident year completion ratios at each lag. The model C_t's, evaluated at monthly lags (as measured from the beginning of the fundamental unit of exposure --- the exposure month), are displayed on Exhibit A-2.

Once the C_t's are known, they can be reconstituted into policy years. Column 6 of Exhibit A-2 displays year-end policy year completion ratios. The specific combination of C_t's,

$$(1C_{g-22}+2C_{g-21}+3C_{g-20}+4C_{g-19}+5C_{g-18}+6C_{g-17}+7C_{g-16}+8C_{g-15}+9C_{g-14}+10C_{g-13}+11C_{g-12}+12C_{g-11}$$
$$+11C_{g-10}+10C_{g-9}+9C_{g-8}+8C_{g-7}+7C_{g-6}+6C_{g-5}+5C_{g-4}+4C_{g-3}+3C_{g-2}+2C_{g-1}+1C_g)/144,$$

is consistent with the assumption that policies are written uniformly throughout the year (with inception dates on the first of each month). Column 8 displays the corresponding ATU development factors.

The ATU development factors imply a set of corresponding policy year age-to-age link ratios. The implied link ratios, which were derived by fitting a Bottom-Up model to accident year data, can be compared to historical policy year link ratios for Alabama Workers Compensation medical case incurred loss data. Exhibit A-3 provides just such a comparison. The historical policy year link ratios were obtained from data contained in the NCCI policy year calls as displayed in their *Loss Development Exhibits for Alabama, Evaluated as of 12/31/95* [4].

It is clear that the model link ratios are consistent with the actual link ratios. The model always produced a link ratio that fell between the minimum and maximum historical link ratio. In many cases, the model link ratio was very near the six year average or medial average (average excluding the highest and lowest) of the historical link ratios. As an added bonus, the model provides us with a tail factor, even though no data beyond accident years at lag 96 was used.

Foot Notes

Reinsurance contracts deal with losses on underlying primary company policies. If reinsurance coverage for a particular loss event depends upon:

1. whether the underlying policy became effective during the reinsurance contract regardless of when the loss event occurred, then the contract is said to be Risk Attaching (*i.e.*, depending upon when the underlying risk, rather than the loss event, attaches to the treaty).
2. whether the loss event took place (or the claim was made, in the case of claims-made policies) during the reinsurance contract regardless of when the underlying policy became effective, then the reinsurance contract is said to be Loss Occurring (*i.e.*, depending upon when the loss occurs, rather than the underlying policy's effective date).

References

[1] Berquist, James R and Richard E. Sherman, "Loss Reserve Adequacy Testing: A Comprehensive, Systematic Approach," *PCAS* LXIV, 1977, p.123.

[2] Keatinge, Clive L, "Modeling Losses with the Mixed Exponential Distribution, *PCAS, LXXXVI*, 1999

[3] *Annual Statistical Bulletin*, 1998 Edition, National Council on Compensation Insurance, Inc.

[4] "Loss Development Exhibits, Alabama Evaluated as of 12/95", National Council on Compensation Insurance, Inc.

Model Fit to Accident Year Data **Exhibit A-1**

Alabama Workers Compensation Case Incurred Medical Loss

Model Based Upon Accident Year Loss Development Triangle

Model: $C_t = 1 - 0.678 * \exp(-t^{0.496}/1.549) - 0.322 * \exp(-t^{1.633}/3596.268)$

						ATU (Comparision)	
Lag	ATU	% Complete	Model	Weight	SE	Actual	Model
(1)	(2)	(3)	(4)	(5)	(6)	(7)=(2)	(8) = $(4)^{-1}$
12	1.922	52.03%	52.00%	1.00	0.00000	1.922	1.923
24	1.561	64.06%	64.29%	1.00	0.00001	1 561	1.555
36	1.462	68 40%	68.04%	1.00	0.00001	1.462	1.470
48	1.415	70.67%	70.55%	1.00	0.00000	1.415	1.417
60	1.378	72.57%	72.76%	1.00	0 00000	1.378	1.374
72	1.343	74.46%	74.88%	1.00	0.00002	1.343	1.335
84	1.294	77.28%	76.97%	1.00	0.00001	1.294	1.299
96	1.265	79.05%	79.02%	1.00	0.00000	1.265	1.265
				Sum of SE	0.00005		

(1) Measured from the beginning of the accident year

(2) Cumulative 5 year development factors taken directly from the NCCI Annual Statistical Bulletin, 1998 Edition [3],

(3) =1/(2)

(4) From Exhibit A-2

(5) Can be varied to allow for better fit in some regions. Here, all points were weighted equally

(6) $=(5)*[(3)-(4)]^2$

Alabama Workers Compensation Medical Loss **Exhibit A-2.1**

Model C_t

Lag (1)	Curve 1 (2)	Curve 2 (3)	Combined (4)=1-(2)-(3)	AY (5)	PY (6)	AY ATU $(7)=(5)^{-1}$	PY ATU $(8)=(6)^{-1}$
1	0.35559	0 32192	32.25%				
2	0.27286	0.32173	40.54%				
3	0.22278	0 32147	45.58%				
4	0.18782	0.32115	49.10%				
5	0.16162	0.32077	51.76%				
6	0.14111	0.32034	53.86%				
7	0.12456	0.31986	55.56%				
8	0 11092	0.31934	56.97%				
9	0.09947	0.31878	58.17%				
10	0.08974	0.31818	59.21%				
11	0.08137	0.31754	60.11%				
12	0.07410	0.31686	60.90%	52.00%	25.91%	1.923	3.859
13	0.06775	0.31615	61.61%				
14	0.06215	0.31541	62.24%				
15	0.05719	0.31463	62.82%				
16	0.05277	0.31382	63.34%				
17	0.04881	0.31298	63 82%				
18	0.04526	0.31211	64.26%				
19	0.04205	0.31121	64.67%				
20	0.03914	0.31029	65.06%				
21	0.03650	0.30933	65.42%				
22	0.03409	0.30835	65 76%				
23	0.03189	0.30735	66.08%				
24	0.02988	0.30632	66.38%	64.29%	60.49%	1.555	1.653
25	0.02803	0.30527	66.67%				
26	0.02633	0.30419	66.95%				
27	0.02476	0.30309	67.21%				
28	0.02331	0 30197	67.47%				
29	0.02197	0.30083	67 72%				
30	0.02073	0.29967	67.96%				
31	0.01958	0 29848	68.19%				
32	0.01851	0.29728	68.42%				
33	0.01751	0 29606	68.64%				
34	0.01658	0.29482	68.86%				
35	0 01572	0.29356	69 07%				
36	0.01490	0 29229	69.28%	68.04%	66.51%	1 470	1.504
37	0.01415	0.29099	69 49%				
38	0 01344	0.28968	69 69%				
39	0.01277	0.28836	69.89%				
40	0.01214	0 28702	70.08%				
41	0.01156	0.28566	70.28%				
42	0.01100	0.28429	70.47%				
43	0 01048	0 28291	70 66%				
44	0 01000	0.28151	70.85%				
45	0.00953	0.28010	71.04%				
46	0.00910	0 27868	71.22%				
47	0 00869	0.27724	71.41%				
48	0.00830	0 27579	71.59%	70.55%	69.45%	1 417	1 440
49	0.00793	0.27433	71.77%				
50	0 00758	0.27286	71 96%				
51	0.00726	0 27138	72 14%				
52	0 00694	0.26989	72.32%				

Exhibit A-2.2

Lag (1)	Curve 1 (2)	Curve 2 (3)	Combined (4)=1-(2)-(3)	AY (5)	PY (6)	AY ATU $(7)=(5)^{-1}$	PY ATU $(8)=(6)^{-1}$
53	0.00665	0.26839	72.50%				
54	0.00637	0.26688	72.68%				
55	0.00610	0.26536	72.85%				
56	0.00585	0.26383	73.03%				
57	0.00561	0.26229	73.21%				
58	0.00538	0.26075	73.39%				
59	0.00516	0.25919	73.56%				
60	0.00496	0.25763	73.74%	72.76%	71.76%	1.374	1.393
61	0.00476	0.25606	73.92%				
62	0.00457	0.25449	74.09%				
63	0.00439	0.25291	74.27%				
64	0.00422	0.25132	74.45%				
65	0.00406	0.24973	74.62%				
66	0.00391	0.24813	74.80%				
67	0.00376	0.24653	74.97%				
68	0.00362	0.24492	75.15%				
69	0.00348	0.24331	75.32%				
70	0.00335	0.24169	75.50%				
71	0.00323	0.24007	75.67%				
72	0.00311	0.23845	75.84%	74.88%	73.91%	1.335	1.353
73	0.00300	0.23682	76.02%				
74	0.00289	0.23519	76.19%				
75	0.00279	0.23356	76.37%				
76	0.00269	0.23193	76.54%				
77	0.00259	0.23029	76.71%				
78	0.00250	0.22865	76.88%				
79	0.00241	0.22701	77.06%				
80	0.00233	0.22537	77.23%				
81	0.00225	0.22373	77.40%				
82	0.00217	0.22209	77.57%				
83	0.00210	0.22044	77.75%				
84	0.00203	0.21880	77.92%	76.97%	76.02%	1.299	1.316
85	0.00196	0.21716	78.09%				
86	0.00189	0.21551	78.26%				
87	0.00183	0.21387	78.43%				
88	0.00177	0.21223	78.60%				
89	0.00171	0.21058	78.77%				
90	0.00166	0.20894	78.94%				
91	0.00160	0.20730	79.11%				
92	0.00155	0.20567	79.28%				
93	0.00150	0.20403	79.45%				
94	0.00145	0.20240	79.62%				
95	0.00141	0.20076	79.78%				
96	0.00136	0.19913	79.95%	79.02%	78.08%	1.265	1.281

(2) $=0.678 * \exp(t^{0.498}/1.549)$

(3) $=0.322 * \exp(t^{1.633}/3596.268)$

(4) $=1-(2)-(3)$

(5) $= (C_n+C_{n-1}+C_{n-2}+C_{n-3} \ldots +C_{n-11})/12$, see text for detailed description

(6) $= (C_n+2C_{n-1}+3C_{n-2}+4C_{n-3} \ldots +1C_{n-11})/144$, see text for detailed description

Alabama Workers Compensation Medical Case Incurred Exhibit A-3
Comparison of Model vs. Actual Policy Year Link Ratios

Medical Paid + Case Development Factors

Policy Year	12:24	24:36	36:48	48:60	60:72	72:84	84:96	96:Ult
1983							1 040	
1984						1.027	1.035	
1985					1.029	1.031	1.081	
1986				1 040	1.040	1.028	1 052	
1987			1.064	1.061	1.035	1.028	1 007	
1988		1.133	1.059	1.058	1.018	1.014	1.045	
1989	2.574	1.126	1.057	1 028	1.019	1.042		
1990	2.422	1.117	1.048	1.009	1.038			
1991	2.242	1.144	1.045	1.040				
1992	2.092	1.012	1.045					
1993	1 893	1.089						
1994	2.411							
Average	2 272	1.104	1.053	1.039	1.030	1.028	1.043	
Avg (excl Hi&Lo)	2.292	1.116	1 052	1.042	1.030	1 029	1.043	
Min	1.893	1.012	1.045	1.009	1.018	1.014	1 007	
Model	***2.334***	***1.100***	***1.044***	***1.033***	***1.030***	***1.028***	***1.027***	***1.281***
Max	2.574	1.144	1.064	1.061	1.040	1.042	1.081	

Link ratios were taken directly from the NCCI Loss Development Exhibits for Alabama, Evaluated as of 12/95 [4].

The policy years reflect approximately the same loss data as the accident year development factors displayed on Exhibit A-1