Statistical and Financial Aspects of Self-Insurance Funding

Leigh J. Halliwell, A.C.A.S.

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ABSTRACT

Actuaries are well aware that good ratemaking and reserving involves the estimation not only of expected values, but also of variances. Moreover, they frequently are concerned with the present value of their estimates. These two matters, the statistical matter of estimation and the financial matter of present value, are especially important when actuaries evaluate self-insurance funds for workers' compensation. This paper will demonstrate the usefulness of constrained least-squares estimation for these actuarial evaluations, and will pose some questions of general relevance to actuarial science.

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1) Introduction

Two matters make ratemaking and reserving for a self-insuring entity more problematic than they are for an insurance company, especially in regard to workers' compensation, where self-insurance is common. First, self-insurers do not have the volume of experience that insurance companies have. Of course, the actuary for a self-insurer can supplement the self-insurer's data with data from insurance companies and bureaus. However, most self-insuring entities believe that their experience is better than that of similar entities who buy insurance; and hence they want to be rated and reserved on their own merits. Second, it is the experience of the author that ratemaking and reserving estimates for self-insurers are on a discounted basis. Whereas the discounting of losses is slowly penetrating statutory accounting, it is a commonplace of self-insurance. Therefore, it is much easier for an actuary's opinion to be far wide of the mark when he is dealing with self-insurers.

And the consequences of being far wide of the mark are more serious when the actuary is dealing with self-insurers. An insurance company must maintain a surplus deemed sufficient by regulators to support its written premiums. If an actuary underestimates the losses which have been incurred or will be incurred, the company's surplus will diminish. The company is not happy; but the underestimation is not likely to consume all the surplus and render the company insolvent. And as for the hapless actuary, who is normally an employee of the insurance company -- at the worst he may be asked to find other employment. But with a self-insurer the situation is be different. True, the self-insurer's net worth is like the insurance company's surplus in that it will be diminished in the event that the actuary underestimates. However, self-insurers are unregulated and do not file NAIC annual statements. If an actuary has underestimated, a thinly capitalized company could become bankrupt. And the actuary, who is normally a consultant of the self-insurer, could be sued for malpractice, a fate much worse than unemployment.

So when an actuary is dealing with a self-insurer, he is more apt to opine far wide of the mark and to suffer harsher consequences. The traditional actuarial methods of ratemaking and reserving have sufficed for the insurance companies. These methods can be called deterministic in that the estimates which they produce are point estimates. But when dealing with self-insurers, actuaries need to furnish estimates of the variability about the point, and perhaps to provide rates and reserves which will be adequate to a suitably chosen confidence level.

In the first of the following three sections, traditional actuarial methods are used to determine a pure premium for a entity self-insuring its workers' compensation.[1] Next, a least-squares model is constructed which is constrained by the pure premium. Predicted values and their variances are derived. Finally, the predictions are present valued. The result is a random variable representing the present value of liability, whose mean and variance have been estimated. By positing a loss distribution, one can reserve to any desired level of confidence.

2) Estimation of the Self-Insurer's Pure Premium

Exhibit 1 presents some of the self-insurer's data. Accident year 1994 (actually fiscal accident year beginning 01Apr94) is known at twelve months. This means that the information is as of some time after 31Mar95. So the payroll for 1995 is an estimate. It is common in actuarial evaluations of self-insurance funds to project not only the liability incurred prior to the evaluation, but also the liability expected to occur within the twelve months following. Therefore, the functions of reserving and ratemaking are united, i.e., reserving is to ratemaking merely as loss incurred is to loss to be incurred.

Paid losses are used throughout this exercise. In the actual evaluation, the author looked at case-incurred losses and decided that they were not suitable for a pure premium estimate. The benefit levels were derived from the benefit level changes found in the 1994 NCCI Statistical Bulletin. In mid-1993 the state in question introduced a medical fee schedule, which was thought to reduce more than $1.17 of benefits to around one dollar.

In most actuarial work *losses* are put on the latest level, here 1.03480. However, there is good reason to adjust *exposures* to the latest level. For one thing, aside from measurement error, the exposures (i.e., payroll divided by 100) are nonstochastic. But also, leveling the exposures allows us to keep working with the *actual* losses, with the result that predictions of future loss payments (Exhibits 10 and 13) will be at historical levels. An exposure unit is not what it used to be; hence, for example, the 1988 adjusted payroll is higher than the unadjusted by a factor of 1.13406/1.03480, because 1995 exposures are tamer and so it requires more of them to equal the 1988 exposures.

It may seem backwards to put exposures, rather than losses, on level. If so, then consider this example: Suppose that at the beginning of year B, a medical fee schedule is introduced, which is supposed to reduce costs twenty percent. Year A, the previous year, could have a level of 1.25, with year B having level 1.00. Suppose that the exposures ($100's of payroll) and losses for year A are 10,000 and $200,000 respectively. The pure premium of year A on its own level is $200,000/10,000 = $20.00.

When benefit levels are calculated, the effects of a law change by injury type are weighted according to expected losses by injury type. Therefore, when actual losses are put on another level, it is assumed that these losses are distributed across injury types much as are the expected losses which were used in the benefit level calculation. This may be reasonable for a large sample of losses, such as the accident year of an insurance company. It is probably not reasonable for a self-insurer. If $180,000 of the $200,000 owed to one fatality, in which a worker died instantly and the loss is for the benefit of the dependents, then the medical fee schedule would not affect this claim. It would be erroneous to say that year A's losses would have been $160,000, if they had happened under year B's conditions. The losses, being stochastic, are wedded to what actually happened.

But the exposures are wedded to what might have happened, and embody expected values. Therefore, it is proper to say that year A's 10,000 exposures are equal to 12,500 of year B's exposures, irrespective of what losses actually happened in year A. The pure premium on year B's level is the same: $160,000/10,000 = $200,000/12,500 = $16.00. However, later we are going to set up a design matrix for a linear regression. It is correct to adjust the exposures, rather than the losses, in that the resulting estimate for σ^2 should reflect historical variability.

Adjusting the exposures for benefit changes is an adjustment for a factor external to the self-insurer. Exhibits 2 through 5 will adjust the exposures for apparent factors internal to the self-insurer, viz., changes in the frequency and severity of its losses. Exhibit 2 shows two methods for developing reported claims.[2] The development factors of the chain ladder method are weighted-averaged across all available years, e.g., 1.073 = (203+233+213+207+206+209)/(198+213+164+207+206+197). The additive method uses a rate of newly reported claims per exposure. When it is accumulated, it produces claim counts @84 very close to those produced by the chain ladder method. Only in 1994, developed from a 12th report, is there an integral difference, and even this difference hardly matters in regard to claim frequency.

The frequencies of the additive method are carried over to Exhibit 3, where a seven-point linear regression is found closely to fit the frequencies. It appears that exposure units are decreasing in frequency. If the trend continues, a 1995 exposure unit should have a frequency of 0.0012 claims per $100 of payroll. Therefore, if 1988 exposures had had a frequency of 0.0012 instead of 0.0024, there would have had to have been twice as many of them reasonably to produce the 203 claims that were reported. So the second adjustment of exposures results in exposure units constant in both benefit level and claim frequency.

Exhibit 4 is similar to Exhibit 2; however, it develops paid losses and uses the twice-adjusted exposures. The two methods diverge significantly, and the author chose to carry the chain-ladder-method pure premiums to the next exhibit because they are better fitted by a semi-log regression. Exhibit 5 is similar to Exhibit 3, and yields the thrice and finally adjusted exposures. It is determined that pure premium, adjusted for frequency, is growing at a rate of 5.39 percent per year. Given the constant frequency, this is equivalent to a growth in claim severity. The regression diagnostics are not of high significance; but a 5.39% severity trend is reasonable. If the trend continues, the projected pure premium for 1995 will be $7.12 per $100 of payroll. The thrice adjusted exposures are, therefore, the actual exposures made to look as if they had existed under 1995 benefit, frequency, and severity levels.

With the adjusted exposures we apply three standard development methods in Exhibit 6 in order to arrive at a 1995 pure premium. The chain ladder method is oblivious to exposure, so it is just copied from Exhibit 4. It produces a pure premium of 7.144 =

6,863,447/960,664.75. That it is close, but not equal, to the projected 1995 pure premium of 7.12 is due to the fact that the severity adjustment in Exhibit 5 uses the fitted, rather than the actual, pure premiums. The additive method is affected by the new exposures, and yields a pure premium of 7.023 = 6,746,577/960,664.75. Now a payout pattern (84^{th} considered for now as ultimate) can be constructed, from which the Cape Cod, or Stanard-Bühlmann, method[3] can produce a third estimate of the pure premium, viz., 7.101.

The author wanted to select a conservative estimate, so he averaged the three numbers and added two units of their standard deviation to arrive at 7.213. There is nothing normative in thus making the selection. In particular, one can make no assertion as to the probability that 7.213 will be a rate adequate to cover losses paid within the first eighty-four months. This should be obvious at least with respect to what actuaries call process risk. However, one might think that the so-called parameter risk is captured in the standard deviation of 0.062. One might be tempted to argue that according to Chebyshev's inequality, at least 75% of the probability must lie within two standard deviations of the mean. Therefore, we should be at least 75% confident that 7.213 is greater than the true pure premium. But this is faulty statistical inference on several counts: 1) Chebyshev's inequality holds for the true parameters μ and σ^2, not for estimates of them, 2) It is always possible that there are unused methods which would yield results which significantly change the estimates, and, most important, 3) the distinction between parameter and process risk is purely theoretical. One cannot calculate confidence limits for each one separately; rather, the variance of a predicted value in a statistical model includes both.

Therefore, up to this point (Exhibit 6) the actuary has developed only a pure premium, i.e., an estimate of certain expected losses per unit of exposure. One could multiply this estimate (7.213) times the thrice adjusted exposures to arrive at losses @84 for all accident years. Then one could subtract the losses paid, to arrive at losses to be paid. However, there is no measure of how drastically the actual losses to be paid might vary from those expected to be paid. Deriving such a measure by the use of a constrained least-squares model is the subject of the next section.

3) Constrained Least-Squares Estimation

We will estimate a (7×1) vector β, representing the pure premiums of losses paid within the first through seventh years. But our estimate will be constrained by our belief that the total pure premium, or the sum of the elements of β, should equal 7.213. So the model whose parameters β and σ^2 are to be estimated is: $\mathbf{Y} = X\beta + \mathbf{e}$, where $\mathbf{e} \sim [0, \sigma^2 I]$, subject to H_0: $R\beta = r$. Appendix A explains these terms and derives the constrained estimator for β. In the present problem, the constraint that the elements of β sum to 7.213 is specified by having R as a (1×7) matrix of ones and r as the (1×1) matrix [7.213].

Exhibit 7 applies Appendix A to the problem. The regressor matrix $X_{(28\times7)}$ indicates that the paid loss during a given age should be proportional to the adjusted exposure. The constrained estimator β^* is (transposed) [1.780 1.942 1.263 0.863 0.542 0.467 0.355]. The table below the model shows these values, along with their standard errors, t statistics, and significances. The standard errors are simply the square roots of the diagonal elements of the matrix Var[β^*]. It can be seen that the standard errors increase with age.

If we had not used the constraint, our estimate for β would have been $(X'X)^{-1}$ $X'Y$, which too is shown in Exhibit 7 as [1.173 1.934 1.253 0.850 0.525 0.440 0.298], the sum of which is 7.073. So the constraint has caused a total increase of 2.0% in the estimate, which distributes by age as [0.4% 0.4% 0.8% 1.5% 3.3% 6.0% 19.3%].

This is a very important feature of constrained least-squares estimation. A constraint will impose a swing away from the unconstrained estimate in the path of least resistance, i.e., the estimate will change most where it has the highest variance. In this case, the total has to go from 7.073 to 7.213, or up by two percent. But most of this is achieved by increasing the pure premiums of the later ages, which have the highest standard errors and t statistics, and have the lowest significances. This is as it ought to be. A payout pattern has to be calculated whenever losses are discounted. However, frequently pure premiums are derived from case-incurred development. It is not correct, for example, to derive a payout pattern where the paid-development pure premium is 1.00, to select an incurred-development pure premium of 1.20, and then to assume that payouts by age will be as according to the paid-development pure premium, but scaled up by 20%. One should consider the relative uncertainty of the payouts by age, and scale accordingly. Usually this means that the payments at later ages will receive more scaling.

Up to this point paid losses have been observed to the 84th month. In Exhibit 8 an attempt is made to project to ultimate. According to the rating bureau of the state in question, ninety percent of the losses should be paid by the 84th month. This implies that the pure premium for payments beyond the 84th month should be 0.801. This can be appended to β^* to make it an (8×1) vector, but we need also to make $\mathrm{Var}[\beta^*]$ an (8×8) matrix. The author had no data from which to estimate the variance of β^*_{84+}. Since it was estimated from external data, it is reasonable to assume that its covariance with the other betas should be zero. One might argue that its variance should be great due to its great uncertainty. One the other hand, β^*_{84+} represents the sum of many years of payments, and, though the variance of each year's payment may be great, in the sum there is the possibility of cancellation. The author simply noted that β^*_{84+} is approximately equal to the sum of β^*_{72} and β^*_{84} , and so took its variance to be the $\mathrm{Var}[\beta^*_{72} + \beta^*_{84}]$, which is 0.136191-2*0.06894+.214445 = 0.213.

The first page of Exhibit 9 shows the predicted values of the losses yet to be paid, i.e., those values which change the loss triangle into a loss rectangle (cf. Exhibit 10). Because all adjustments were made to the exposures, the prediction vector $E[Y_0|Y] = X_0\beta^*$ is at historical levels. Because there are thirty-six predictions, the variance of the predictions is a (36×36) matrix, and is shown on the second page of Exhibit 9. As shown in Appendix A, the formula for $Var[\mathbf{Y}_0|\mathbf{Y}]$ is $X_0Var[\beta^*]X_0' + \sigma^2I_{36}$. By referring back to Exhibit 7 one can see that the formula for $Var[\beta^*]$ has a σ^2 factor, and that σ^2 is 6.3×10^9. The number of degrees of freedom in the denominator of the formula for σ^2 is 28-7+1. As shown in Appendix A, each independent constraint adds a degree to this denominator, which serves to reduce σ^2.

The elements of $E[\mathbf{Y}_0|\mathbf{Y}]$ are "wound around" the lower right triangle of the incremental table of Exhibit 10, and are accumulated in the next table. As to the bottom table of Exhibit 10, the standard deviation for Total 1988-1994 is the square root of the variance, which is the sum of all the elements in the upper left (28×28) submatrix of $Var[\mathbf{Y}_0|\mathbf{Y}]$. The standard deviation for Total 1995 is the square root of the sum of all the elements in the lower right (8×8) submatrix of $Var[\mathbf{Y}_0|\mathbf{Y}]$. And the standard deviation of Grand Total is the square root of the sum of all the elements of $Var[\mathbf{Y}_0|\mathbf{Y}]$ itself.

4) The Present Value of Future Losses

Most yield curves are deceptive. When one hears, for example, that the 30-year treasury rate is 7.00%, this means that a 30-year treasury bond is selling at a price equal to the sum of its coupons and principal discounted at 7.00%. But only the final coupon and the principal is paid at the end of thirty years. The other coupons are received semi-annually in the meantime. So the yield of 7.00% applies to a long stream of cash flows, and it is not necessarily true that a 30-year zero coupon bond should be discounted at 7.00%.

However, the coupons and principals of bonds are atomized, or stripped, and packaged according to date. These are traded in a U.S. Treasury Strip market; and in investment periodicals, such as The Wall Street Journal and Barron's, one can look up their yields, which are in effect yields on zero coupon bonds. Shown in the bottom table of Exhibit 11 are strip yields as of 31Mar95, the evaluation date, and the derived discount factors. For example, $0.971 = (1.0603)^{-0.5}$. The middle table of Exhibit 11 assumes that payments within an age will on average occur midway through the age, except for payments after the 84th month, which are assumed to be paid at the 102th month. These assumptions are merely commonsensical, do not unduly affect the outcome, and seem to be on the conservative side.

Exhibit 12 shows the discounted value of the expected future losses. The "ΔPaid" column is $E[\mathbf{Y}_0|\mathbf{Y}]$ from Exhibit 9. The "Time" and "Discount" columns come from Exhibit 11. For a time, t, greater than 6.5 years the discount factor is $(1.0721)^{-t}$. Therefore, the present, or discounted, value of the predicted paid losses is the product of the prediction and the discount factor. The discounted losses are "wound around" the lower right triangle of Exhibit 13.

Consider the "Discount" column of Exhibit 12 diagonalized to form the (36×36) matrix Λ. Then the "PV[ΔPaid]" column can be expressed as $\Lambda E[\mathbf{Y}_0|\mathbf{Y}]$. In fact, consider $\mathbf{Y}_0|\mathbf{Y}$ as a random vector, which it is. Then $\Lambda\mathbf{Y}_0|\mathbf{Y}$ is the random vector representing the present value of $\mathbf{Y}_0|\mathbf{Y}$, which we will call $PV[\mathbf{Y}_0|\mathbf{Y}]$. But by basic theorems about random vectors, $E[PV[\mathbf{Y}_0|\mathbf{Y}]] = E[\Lambda\mathbf{Y}_0|\mathbf{Y}] = \Lambda E[\mathbf{Y}_0|\mathbf{Y}]$. Moreover, $Var[PV[\mathbf{Y}_0|\mathbf{Y}]] = Var[\Lambda\mathbf{Y}_0|\mathbf{Y}] = \Lambda Var[\mathbf{Y}_0|\mathbf{Y}]\ \Lambda'$. Although not shown in the exhibits, this (36×36) matrix was calculated, so that the numbers in the "Std Dev" column of Exhibit 13 could be calculated. The manner of this calculation is the same as that described in connection with Exhibit 10, except that the matrix used here is $\Lambda Var[\mathbf{Y}_0|\mathbf{Y}]\ \Lambda'$, instead of $Var[\mathbf{Y}_0|\mathbf{Y}]$. This is the correct way of discounting a random vector. The usual way is to apply a discount factor to the remaining expected payments of an accident year, with consideration of the age of the accident year. But this renders impossible the calculation of the variance of the discounted losses.

Therefore, the present value of all unpaid losses incurred from 1988 to 1994, or incurrable in 1995, is \$2,974,348, with a standard deviation of \$565,639.[4] The unbiasedness and the efficiency of least-squares estimation do not depend on the assumption that the error term is a normal variate. The assumption of normality is required only in connection with the *t* and F tests, on which no reliance was placed here. We can model the present value of all unpaid losses as a lognormal random variable with mean and standard deviation as given above. By the method of moments we find that the present value is *e* raised to the power of a normal random variable of mean 14.888 and standard deviation of 0.188. Exhibit 14 shows confidence limits which follow therefrom. For example, if one wanted to be 90% confident of having enough assets on hand as of 31Mar95 to make all future loss payments for claims which will have occurred on or before 31Mar96, then one would need $3,720,342. The reasoning is that 90% of the probability of a standard normal curve is below 1.282 (Z_p). But our normal random variable $Y = 14.888+0.188Z$. Therefore, $Y_p = 14.888+0.188\times1.282 = 15.129$. And $e^{15.129}$ = 3,720,342.

The choice of a confidence level might be in the hands of the self-insured entity. An entity which had only $3,000,000 with which to fund its liabilities might press the actuary to opine that three million is adequate. If the actuary used traditional techniques he might figure thus: "I need 115,000×8.014 = $931,960 for 1995 alone. According to the chain ladder method (Exhibit 6), 1988-1994 losses @84th will be $6,863,447. Since this is 90% of ultimate (Exhibit 8), the ultimate losses for 1988-1994 will be $7,626,052, of which $5,026,994 has already been paid. Thus I need $2,599,058 in reserve for 1988-1994; and the total with 1995 is $3,531,018. A reasonable 15% discount would bring this to the desired three million." So a plausible argument exists for a three million dollar reserve. But if the actuary were to have Exhibit 14, he would realize that there is almost a 50% chance that the liability will be greater. Probably most actuaries would not want to opine "adequate" at this point. If the client persisted, at least the actuary could protect himself by opining that three million is adequate to cover the expected discounted loss, but that due to random variation, the probability of actual adequacy is less than 55%.

5) Conclusion

The major point of this paper is that in self-insurance funding, where surplus may not exist and where discounting is common, the variance of the losses is as important as the expected value. The traditional techniques of loss development are unable to treat the variance. Statistical modeling is required, which just about means least-squares estimation. The traditional techniques need not be jettisoned; they can supply constraints to the least-squares estimation. Then the results, or the predictions, can be discounted without nullifying the variance estimates. Finally, the actuary can qualify his opinion along the [0,1] continuum of confidence level, rather than be limited to the binary judgment of "adequate" or "not adequate."

Perhaps this is how all actuarial work ought to be, even in instances where surplus exists. If the liabilities of an insurance company were explicitly reserved in the aggregate to a sufficiently high confidence level, then the company would need little or no surplus. Either that, or else surplus could be redefined so as to include the "contingency reserve," i.e., that amount by which the reserve exceeds the expected value of liability. This certainly would have far reaching implications to insurance regulation, and particularly to risk-based capital.

Notes

[1] The example in this paper is of individual self-insurance. Individual self-insurance of workers' compensation is permitted in all states except North Dakota and Wyoming [8, pp.1-5]. Thirty-two states permit group self-insurance of workers' compensation, whereby a number of employers (usually of similar businesses) jointly and severally pool their exposures. Group self-insurance is similar to individual self-insurance in that typically the pool is backed by little or no surplus, and reserves are discounted. In the event of a shortfall the members are assessed.

[2] These two methods are the first and fourth of Stanard's methods [6, pp.130f.].

[3] See Stanard [6, p. 131] and Patrik [5, pp. 352-354].

[4] The self-insurers for whom the author has worked have all had per-occurrence excess and aggregate stop-loss reinsurance. This seems to be a legal requirement. Therefore, that the author evaluated the direct loss, rather than the net loss, is solely a conservative measure. Furthermore, with this self-insurer an attachment to a reinsurance layer was unlikely.

Note also that the losses treated in this paper are exclusive of loss adjustment expense. Unallocated loss adjustment expense is usually a budgetary item. The author has treated allocated loss adjustment expense (ALAE) as proportional to losses, assuming that the payout pattern is the same and that the correlation between losses and ALAE is 100 percent. Both of these assumptions are conservative, since ALAE tends to be paid more slowly than losses and the variance of losses and ALAE together is maximized at a 100% correlation.

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EXHIBIT 1

Fund Data

Year	Age	Claims	Paid	ΔPaid
1988	12	198	266,354	266,354
1988	24	203	432,926	166,572
1988	36	203	465,255	32,329
1988	48	203	518,865	53,610
1988	60	203	526,989	8,124
1988	72	203	543,913	16,924
1988	84	203	583,022	39,109
1989	12	213	246,981	246,981
1989	24	233	606,361	359,380
1989	36	234	835,377	229,016
1989	48	234	904,916	69,539
1989	60	234	1,023,551	118,635
1989	72	234	1,123,843	100,292
1990	12	164	203,178	203,178
1990	24	213	578,946	375,768
1990	36	213	855,563	276,617
1990	48	213	930,475	74,912
1990	60	213	1,016,903	86,428
1991	12	207	395,630	395,630
1991	24	207	656,273	260,643
1991	36	207	823,982	167,709
1991	48	207	1,094,674	270,692
1992	12	206	207,698	207,698
1992	24	206	382,313	174,615
1992	36	206	544,953	162,640
1993	12	197	167,681	167,681
1993	24	209	447,859	280,178
1994	12	120	215,740	215,740

Year	Payroll	Benefit Level	Adj[1] Payroll
1988	8,099,386	1.13406	88,762.95
1989	8,935,838	1.14336	98,732.89
1990	9,183,206	1.15571	102,562.07
1991	9,606,490	1.16438	108,094.36
1992	10,136,257	1.17229	114,830.24
1993	10,871,000	1.12704	118,400.19
1994	11,353,832	1.02945	112,951.32
1995	11,500,000	1.03480	115,000.00

[1] Adjusted to 1995 Benefit Level (00)

EXHIBIT 2

Development of Reported Claims

Chain Ladder Method

Year	Adj[1] Payroll	@12	@24	@36	@48	@60	@72	@84	Frequency
1988	88,762.95	198	203	203	203	203	203	203	0.0023
1989	98,732.89	213	233	234	234	234	234	234	0.0024
1990	102,562.07	164	213	213	213	213	213	213	0.0021
1991	108,094.36	207	207	207	207	207	207	207	0.0019
1992	114,830.24	206	206	206	206	206	206	206	0.0018
1993	118,400.19	197	209	209	209	209	209	209	0.0018
1994	112,951.32	120	129	129	129	129	129	129	0.0011

Development Factor:			1.073	1.001	1.000	1.000	1.000	1.000

Additive Method

Year	Adj[1] Payroll	@12	@24	@36	@48	@60	@72	@84
1988	88,762.95	198	5	0	0	0	0	0
1989	98,732.89	213	20	1	0	0	0	0
1990	102,562.07	164	49	0	0	0	0	0
1991	108,094.36	207	0	0	0	0	0	0
1992	114,830.24	206	0	0	0	0	0	0
1993	118,400.19	197	12	0	0	0	0	0
1994	112,951.32	120	15	0	0	0	0	0

Rate:		0.00175	0.00014	0.00000	0.00000	0.00000	0.00000	0.00000

Additive Method Accumulated

Year	Adj[1] Payroll	@12	@24	@36	@48	@60	@72	@84	Frequency
1988	88,762.95	198	203	203	203	203	203	203	0.0023
1989	98,732.89	213	233	234	234	234	234	234	0.0024
1990	102,562.07	164	213	213	213	213	213	213	0.0021
1991	108,094.36	207	207	207	207	207	207	207	0.0019
1992	114,830.24	206	206	206	206	206	206	206	0.0018
1993	118,400.19	197	209	209	209	209	209	209	0.0018
1994	112,951.32	120	135	136	136	136	136	136	0.0012

EXHIBIT 3

Frequency Trend

Year	Frequency	Fitted
1988	0.0023	0.0024
1989	0.0024	0.0023
1990	0.0021	0.0021
1991	0.0019	0.0019
1992	0.0018	0.0017
1993	0.0018	0.0016
1994	0.0012	0.0014

Linear Regression	
-0.0001696	0.00259405
2.8467E-05	0.00012731
87.6%	0.00015063
35.4823357	5
8.0509E-07	1.1345E-07

Year	Payroll	Adj[1] Payroll	Claim Freq	Adj[2] Payroll
1988	8,099,386	88,762.95	0.0024	173,900.71
1989	8,935,838	98,732.89	0.0023	179,904.77
1990	9,183,206	102,562.07	0.0021	172,828.72
1991	9,606,490	108,094.36	0.0019	167,339.89
1992	10,136,257	114,830 24	0.0017	162,033.28
1993	10,871,000	118,400.19	0.0016	150,847.22
1994	11,353,832	112,951.32	0.0014	128,428.21
1995	11,500,000	115,000.00	0.0012	115,000.00

[1] Adjusted to 1995 Benefit Level (00)
[2] Adjusted to 1995 Benefit and Frequency Levels (00)

EXHIBIT 4

Paid Loss Development

Chain Ladder Method

Year	Adj² Payroll	@12	@24	@36	@48	@60	@72	@84	Pure Prem
1988	173,900.71	266,354	432,926	465,255	518,865	526,989	543,913	583,022	3.35
1989	179,904.77	246,981	606,361	835,377	904,916	1,023,551	1,123,843	1,204,651	6.70
1990	172,828.72	203,178	578,946	855,563	930,475	1,016,903	1,093,778	1,172,424	6.78
1991	167,339.89	395,630	656,273	823,982	1,094,674	1,193,801	1,284,049	1,376,376	8.23
1992	162,033.28	207,698	382,313	544,953	630,669	687,778	739,772	792,964	4.89
1993	150,847.22	167,681	447,859	594,230	687,696	749,970	806,665	864,667	5.73
1994	128,428.21	215,740	450,281	597,444	691,416	754,026	811,028	869,344	6.77

Development Factor:		2.087	1.327	1.157	1.091	1.076	1.072

Additive Method

Year	Adj² Payroll	@12	@24	@36	@48	@60	@72	@84
1988	173,900.71	266,354	166,572	32,329	53,610	8,124	16,924	39,109
1989	179,904.77	246,981	359,380	229,016	69,539	118,635	100,292	40,459
1990	172,828.72	203,178	375,768	276,617	74,912	86,428	57,258	38,868
1991	167,339.89	395,630	260,643	167,709	270,692	67,741	55,440	37,634
1992	162,033.28	207,698	174,615	162,640	109,447	65,593	53,682	36,440
1993	150,847.22	167,681	280,178	153,015	101,892	61,065	49,976	33,924
1994	128,428.21	215,740	206,275	130,274	86,748	51,989	42,548	28,883

Rate:	1.500	1.606	1.014	0.675	0.405	0.331	0.225

Additive Method Accumulated

Year	Adj² Payroll	@12	@24	@36	@48	@60	@72	@84	Pure Prem
1988	173,900.71	266,354	432,926	465,255	518,865	526,989	543,913	583,022	3.35
1989	179,904.77	246,981	606,361	835,377	904,916	1,023,551	1,123,843	1,164,302	6.47
1990	172,828.72	203,178	578,946	855,563	930,475	1,016,903	1,074,161	1,113,029	6.44
1991	167,339.89	395,630	656,273	823,982	1,094,674	1,162,415	1,217,855	1,255,488	7.50
1992	162,033.28	207,698	382,313	544,953	654,400	719,993	773,675	810,115	5.00
1993	150,847.22	167,681	447,859	600,874	702,766	763,830	813,806	847,731	5.62
1994	128,428.21	215,740	422,015	552,289	639,037	691,026	733,574	762,457	5.94

EXHIBIT 5

Loss Trend

Year	Pure Prem	Fitted
1988	3.35	5.38
1989	6.70	5.61
1990	6.78	5.83
1991	8.23	6.06
1992	4.89	6.29
1993	5.73	6.52
1994	6.77	6.75

Semi-Log Regression	
1.0539196	4.75130558
0.05622039	0.25142523
14.9%	0.29749034
0.87256611	5
0.07722254	0.44250252

Year	Payroll	Adj[1] Payroll	Adj[2] Payroll	Pure Prem	Adj[3] Payroll
1988	8,099,386	88,762.95	173,900.71	5.38	131,332.20
1989	8,935,838	98,732.89	179,904.77	5.61	141,672.24
1990	9,183,206	102,562.07	172,828.72	5.83	141,677.29
1991	9,606,490	108,094.36	167,339.89	6.06	142,577.99
1992	10,136,257	114,830.24	162,033.28	6.29	143,285.58
1993	10,871,000	118,400.19	150,847.22	6.52	138,261.75
1994	11,353,832	112,951.32	128,428.21	6.75	121,857.69
1995	11,500,000	115,000.00	115,000.00	7.12	115,000.00

[1] Adjusted to 1995 Benefit Level (00)

[2] Adjusted to 1995 Benefit and Frequency Levels (00)

[3] Adjusted to 1995 Benefit, Frequency, and Trend Levels (00)

EXHIBIT 6

Pure Premium Estimates

Chain Ladder Method

Year	Adj[3] Payroll	@12	@24	@36	@48	@60	@72	@84
1988	131,332.20	266,354	432,926	465,255	518,865	526,989	543,913	583,022
1989	141,672.24	246,981	606,361	835,377	904,916	1,023,551	1,123,843	1,204,651
1990	141,677.29	203,178	578,946	855,563	930,475	1,016,903	1,093,778	1,172,424
1991	142,577 99	395,630	656,273	823,982	1,094,674	1,193,801	1,284,049	1,376,376
1992	143,285.58	207,698	382,313	544,953	630,669	687,778	739,772	792,964
1993	138,261.75	167,681	447,859	594,230	687,696	749,970	806,665	864,667
1994	121,857.69	215,740	450,281	597,444	691,416	754,026	811,028	869,344
Total	960,664.75	1,703,262	3,554,959	4,716,804	5,458,711	5,953,019	6,403,048	6,863,447
						Pure Premium:		7.144

Additive Method

Year	Adj[3] Payroll	@12	@24	@36	@48	@60	@72	@84
1988	131,332.20	266,354	166,572	32,329	53,610	8,124	16,924	39,109
1989	141,672.24	246,981	359,380	229,016	69,539	118,635	100,292	42,188
1990	141,677 29	203,178	375,768	276,617	74,912	86,428	60,830	42,190
1991	142,577 99	395,630	260,643	167,709	270,692	73,299	61,217	42,458
1992	143,285.58	207,698	174,615	162,640	120,528	73,663	61,520	42,669
1993	138,261.75	167,681	280,178	171,372	116,302	71,080	59,363	41,173
1994	121,857.69	215,740	234,932	151,040	102,504	62,647	52,320	36,288
Rate:		1.773	1.928	1.239	0.841	0.514	0.429	0.298

Additive Method Accumulated

Year	Adj[3] Payroll	@12	@24	@36	@48	@60	@72	@84
1988	131,332.20	266,354	432,926	465,255	518,865	526,989	543,913	583,022
1989	141,672.24	246,981	606,361	835,377	904,916	1,023,551	1,123,843	1,166,031
1990	141,677.29	203,178	578,946	855,563	930,475	1,016,903	1,077,733	1,119,923
1991	142,577.99	395,630	656,273	823,982	1,094,674	1,167,973	1,229,190	1,271,648
1992	143,285.58	207,698	382,313	544,953	665,481	739,144	800,665	843,333
1993	138,261.75	167,681	447,859	619,231	735,534	806,614	865,977	907,150
1994	121,857.69	215,740	450,672	601,712	704,216	766,863	819,183	855,471
Total	960,664.75	1,703,262	3,555,350	4,746,074	5,554,161	6,048,037	6,460,504	6,746,577
						Pure Premium:		7.023

Payout Ratios to 84th

	@12	@24	@36	@48	@60	@72	@84
Chain Ladder	1,703,262	3,554,959	4,716,804	5,458,711	5,953,019	6,403,048	6,863,447
Additive	1,703,262	3,555,350	4,746,074	5,554,161	6,048,037	6,460,504	6,746,577
Sum	3,406,524	7,110,310	9,462,877	11,012,872	12,001,055	12,863,552	13,610,024
Payout	25.0%	52.2%	69.5%	80.9%	88.2%	94.5%	100.0%

Cape Cod (Stanard-Buhlmann) Method

Year	Adj[3] Payroll	Latest Age	Latest Paid	Payout	Payroll Alloc
1988	131,332.20	84	583,022	100.0%	131,332.20
1989	141,672.24	72	1,123,843	94 5%	133,901.91
1990	141,677.29	60	1,016,903	88 2%	124,928.29
1991	142,577.99	48	1,094,674	80.9%	115,370.35
1992	143,285.58	36	544,953	69.5%	99,624.65
1993	138,261.75	24	447,859	52.2%	72,232.34
1994	121,857.69	12	215,740	25.0%	30,500.40
Total	960,664.75		5,026,994		707,890.15
			Pure Premium:		7 101

Pure Premium @84th Results	
Chain Ladder	7.144
Additive	7.023
Cape Cod	7.101
Mean	7.090
Std Dev	0.062
Selected	7 213

EXHIBIT 7

Constrained Regression

Y (28x1)

Year	Adj' Payroll	Age	ΔPaid
1988	131,332.20	12	266,354
1988	131,332.20	24	166,572
1988	131,332.20	36	32,329
1988	131,332.20	48	53,610
1988	131,332.20	60	8,124
1988	131,332.20	72	16,924
1988	131,332.20	84	39,109
1989	141,672.24	12	246,981
1989	141,672.24	24	359,380
1989	141,672.24	36	229,016
1989	141,672.24	48	69,539
1989	141,672.24	60	118,635
1989	141,672.24	72	100,292
1990	141,677.29	12	203,176
1990	141,677.29	24	375,766
1990	141,677.29	36	276,617
1990	141,677.29	48	74,912
1990	141,677.29	60	86,426
1991	142,577.99	12	395,630
1991	142,577.99	24	260,643
1991	142,577.99	36	167,709
1991	142,577.99	48	270,692
1992	143,285.58	12	207,698
1992	143,285.58	24	174,615
1992	143,285.58	36	162,640
1993	138,261.75	12	167,681
1993	138,261.75	24	280,176
1994	121,857.89	12	215,740

X (28x7)

β_{12}	β_{24}	β_{36}	β_{48}	β_{60}	β_{72}	β_{84}
131332						
	131332					
		131332				
			131332			
				131332		
					131332	
						131332
141672						
	141672					
		141672				
			141672			
				141672		
					141672	
141677						
	141677					
		141677				
			141677			
				141677		
142578						
	142578					
		142578				
			142578			
143286						
	143286					
		143286				
138262						
	138262					
121858						

Model. $Y = X\beta + e$, where $e \sim (0, \sigma^2 I)$, given H_0: $R\beta = r$,
where R = [1 1 1 1 1 1 1] and r = [7.213]

	β^*	Std[β^*]	t statistic	Signif
β_{12}	1.780	0.212	8.40	100.0%
β_{24}	1.942	0.224	8.67	100.0%
β_{36}	1.283	0.243	5.19	100.0%
β_{48}	0.863	0.271	3.19	99.6%
β_{60}	0.542	0.310	1.75	90.6%
β_{72}	0.467	0.369	1.27	78.1%
β_{84}	0.355	0.463	0.77	54.9%
$\Sigma(\beta)$	7.213			

R (1x7)

1	1	1	1	1	1	1

X'X (7x7)

1.3E+11						
	1.2E+11					
		9.8E+10				
			7.8E+10			
				5.7E+10		
					3.7E+10	
						1.7E+10

$(X'X)^{-1}$ (7x7)

7.6E-12						
	8.5E-12					
		1E-11				
			1.3E-11			
				1.7E-11		
					2.7E-11	
						5.8E-11

$(X'X)^{-1}R'$ (7x1)

7.6E-12
8.5E-12
1E-11
1.3E-11
1.7E-11
2.7E-11
5.8E-11

$R(X'X)^{-1}R'$ (1x1)

1.4E-10

$H = [R(X'X)^{-1}R']^{-1}$ (1x1)

7.1E+09

$M = I_7 - (X'X)^{-1}R'HR$ (7x7)

0.94648	-0.05352	-0.05352	-0.05352	-0.05352	-0.05352	-0.05352
-0.06029	0.93971	-0.06029	-0.06029	-0.06029	-0.06029	-0.06029
-0.07202	-0.07202	0.92798	-0.07202	-0.07202	-0.07202	-0.07202
-0.09104	-0.09104	-0.09104	0.90896	-0.09104	-0.09104	-0.09104
-0.12329	-0.12329	-0.12329	-0.12329	0.87671	-0.12329	-0.12329
-0.1896	-0.1896	-0.1896	-0.1896	-0.1896	0.8104	-0.1896
-0.41024	-0.41024	-0.41024	-0.41024	-0.41024	-0.41024	0.58976

$(X'X)^{-1}X'Y$ (7x1)

1.773
1.934
1.253
0.850
0.525
0.440
0.298
7.073

$\beta^* = M(X'X)^{-1}X'Y + (X'X)^{-1}R'Hr$ (7x

1.780
1.942
1.263
0.863
0.542
0.467
0.355
7.213

$\sigma^2 = e^{*\prime}e^*/(28-7+1)$

8.3E+09

$M(X'X)^{-1}$ (7x7)

7.2E-12	-4.6E-13	-5.4E-13	-6.9E-13	-9.3E-13	-1.4E-12	-3.1E-12
-4.6E-13	8E-12	-6.1E-13	-7.8E-13	-1.1E-12	-1.6E-12	-3.5E-12
-5.4E-13	-6.1E-13	9.4E-12	-9.3E-13	-1.3E-12	-1.9E-12	-4.2E-12
-6.9E-13	-7.8E-13	-9.3E-13	1.2E-11	-1.6E-12	-2.4E-12	-5.3E-12
-9.3E-13	-1.1E-12	-1.3E-12	-1.6E-12	1.5E-11	-3.3E-12	-7.1E-12
-1.4E-12	-1.6E-12	-1.9E-12	-2.4E-12	-3.3E-12	2.2E-11	-1.1E-11
-3.1E-12	-3.5E-12	-4.2E-12	-5.3E-12	-7.1E-12	-1.1E-11	3.4E-11

$Var[\beta^*] = \sigma^2 M(X'X)^{-1}$ (7x7)

0.0449	-0.00286	-0.00342	-0.00432	-0.00585	-0.00899	-0.01946
-0.00286	0.05021	-0.00385	-0.00486	-0.00659	-0.01013	-0.02192
-0.00342	-0.00385	0.05924	-0.00581	-0.00787	-0.0121	-0.02619
-0.00432	-0.00486	-0.00581	0.07335	-0.00995	-0.0153	-0.0331
-0.00585	-0.00659	-0.00787	-0.00995	0.09581	-0.02072	-0.04483
-0.00899	-0.01013	-0.0121	-0.0153	-0.02072	0.13619	-0.06894
-0.01946	-0.02192	-0.02619	-0.0331	-0.04483	-0.06894	0.21444

EXHIBIT 8

84th to Ultimate

According to collateral data, payments @84th are about 90% of ultimate. So:

Pure Premium at 84th	7.213
Ratio to Ultimate	90%
Pure Premium at Ultimate	8.014
β84+	0.801
Var[β84+]	0.213
Std[β84+]	0.461

	β*	Var[β*]							
β12	1.780	0.044896	-0.00286	-0.00342	-0.00432	-0.00585	-0.00899	-0.01946	0
β24	1.942	-0.00286	0.050215	-0.00385	-0.00486	-0.00659	-0.01013	-0.02192	0
β36	1.263	-0.00342	-0.00385	0.059236	-0.00581	-0.00787	-0.0121	-0.02619	0
β48	0.863	-0.00432	-0.00486	-0.00581	0.073349	-0.00995	-0.0153	-0.0331	0
β60	0.542	-0.00585	-0.00659	-0.00787	-0.00995	0.095805	-0.02072	-0.04483	0
β72	0.467	-0.00899	-0.01013	-0.0121	-0.0153	-0.02072	0.136191	-0.06894	0
β84	0.355	-0.01946	-0.02192	-0.02619	-0.0331	-0.04483	-0.06894	0.214445	0
β84+	0.801	0	0	0	0	0	0	0	0.212751

EXHIBIT 9

Prediction of Unpaid Losses

			E[Y_0|Y](36x1)	X_0 (36x8)							
Year	Adj[3] Payroll	Age	ΔPaid	β12	β24	β36	β48	β60	β72	β84	β84+
1988	131,332.20	>84	105,254								131332.2
1989	141,672.24	84	50,335							141672.2	
1989	141,672.24	>84	113,541								141672.2
1990	141,677.29	72	66,144						141677.3		
1990	141,677.29	84	50,337							141677.3	
1990	141,677.29	>84	113,545								141677.3
1991	142,577.99	60	77,289					142578			
1991	142,577.99	72	66,565						142578		
1991	142,577.99	84	50,657							142578	
1991	142,577.99	>84	114,267								142578
1992	143,285.58	48	123,692				143285.6				
1992	143,285.58	60	77,672					143285.6			
1992	143,285.58	72	66,895						143285.6		
1992	143,285.58	84	50,908							143285.6	
1992	143,285.58	>84	114,834								143285.6
1993	138,261.75	36	174,622			138261.7					
1993	138,261.75	48	119,355				138261.7				
1993	138,261.75	60	74,949					138261.7			
1993	138,261.75	72	64,550						138261.7		
1993	138,261.75	84	49,123							138261.7	
1993	138,261.75	>84	110,808								138261.7
1994	121,857.69	24	236,661		121857.7						
1994	121,857.69	36	153,904			121857.7					
1994	121,857.69	48	105,194				121857.7				
1994	121,857.69	60	66,057					121857.7			
1994	121,857.69	72	56,891						121857.7		
1994	121,857.69	84	43,295							121857.7	
1994	121,857.69	>84	97,661								121857.7
1995	115,000.00	12	204,739	115000							
1995	115,000.00	24	223,342		115000						
1995	115,000.00	36	145,243			115000					
1995	115,000.00	48	99,274				115000				
1995	115,000.00	60	62,339					115000			
1995	115,000.00	72	53,690						115000		
1995	115,000.00	84	40,858							115000	
1995	115,000.00	>84	92,165								115000

EXHIBIT 9 (Cont'd)

Variance of Unpaid Losses

$$Var[Y_0|Y] = X_0 Var[\beta^*]X_0' + \sigma^2 I_{36}$$

EXHIBIT 10

Ultimate Losses

Incremental

Year	@12	@24	@36	@48	@60	@72	@84	84+
1988	266,354	166,572	32,329	53,610	8,124	16,924	39,109	105,254
1989	246,981	359,380	229,016	69,539	118,635	100,292	50,335	113,541
1990	203,178	375,768	276,617	74,912	86,428	66,144	50,337	113,545
1991	395,630	260,643	167,709	270,692	77,289	66,565	50,657	114,267
1992	207,698	174,615	162,640	123,692	77,672	66,895	50,908	114,834
1993	167,681	280,178	174,622	119,355	74,949	64,550	49,123	110,808
1994	215,740	236,661	153,904	105,194	66,057	56,891	43,295	97,661
1995	204,739	223,342	145,243	99,274	62,339	53,690	40,858	92,165

Cumulative

Year	@12	@24	@36	@48	@60	@72	@84	@Ult
1988	266,354	432,926	465,255	518,865	526,989	543,913	583,022	688,276
1989	246,981	606,361	835,377	904,916	1,023,551	1,123,843	1,174,178	1,287,719
1990	203,178	578,946	855,563	930,475	1,016,903	1,083,047	1,133,384	1,246,929
1991	395,630	656,273	823,982	1,094,674	1,171,963	1,238,528	1,289,184	1,403,452
1992	207,698	382,313	544,953	668,645	746,318	813,213	864,121	978,955
1993	167,681	447,859	622,481	741,836	816,785	881,335	930,458	1,041,266
1994	215,740	452,401	606,305	711,499	777,556	834,447	877,742	975,403
1995	204,739	428,082	573,324	672,599	734,938	788,628	829,486	921,651

	Paid	E[Unpaid]	Std Dev
Total 1988-1994:	5,026,994	2,595,006	657,623
Total 1995:	0	921,651	230,189
Grand Total:	5,026,994	3,516,658	729,701

EXHIBIT 11

Yield Information

	Predicted Payments							
Year	@12	@24	@36	@48	@60	@72	@84	84+
1988								105,254
1989							50,335	113,541
1990						66,144	50,337	113,545
1991					77,289	66,565	50,657	114,267
1992				123,692	77,672	66,895	50,908	114,834
1993			174,622	119,355	74,949	64,550	49,123	110,808
1994		236,661	153,904	105,194	66,057	56,891	43,295	97,661
1995	204,739	223,342	145,243	99,274	62,339	53,690	40,858	92,165

	Payment Time from 31Dec94							
Year	@12	@24	@36	@48	@60	@72	@84	84+
1988								1.5
1989							0.5	2.5
1990						0.5	1.5	3.5
1991					0.5	1.5	2.5	4.5
1992				0.5	1.5	2.5	3.5	5.5
1993			0.5	1.5	2.5	3.5	4.5	6.5
1994		0.5	1.5	2.5	3.5	4.5	5.5	7.5
1995	0.5	1.5	2.5	3.5	4.5	5.5	6.5	8.5

	U.S. Treasury Strip Yields as of 31Mar95							
Time	0.5	1.5	2.5	3.5	4.5	5.5	6.5	>6.5
Maturity	Aug 1995	Aug 1996	Aug 1997	Aug 1998	Aug 1999	Aug 2000	Aug 2001	Aug 2002
Yield	6.03%	6.36%	6.84%	6.99%	7.04%	7.14%	7.15%	7.21%
Discount	0.971	0.912	0.848	0.789	0.736	0.684	0.638	Variable

EXHIBIT 12

Present Value of Unpaid Losses

Year	Age	Time	ΔPaid	Discount	PV[ΔPaid]
1988	>84	1.5	105,254	0.912	95,956
1989	84	0.5	50,335	0.971	48,883
1989	>84	2.5	113,541	0.848	96,232
1990	72	0.5	66,144	0.971	64,236
1990	84	1.5	50,337	0.912	45,890
1990	>84	3.5	113,545	0.789	89,633
1991	60	0.5	77,289	0.971	75,059
1991	72	1.5	66,565	0.912	60,685
1991	84	2.5	50,657	0.848	42,934
1991	>84	4.5	114,267	0.736	84,133
1992	48	0.5	123,692	0.971	120,123
1992	60	1.5	77,672	0.912	70,811
1992	72	2.5	66,895	0.848	56,697
1992	84	3.5	50,908	0.789	40,187
1992	>84	5.5	114,834	0.684	78,585
1993	36	0.5	174,622	0.971	169,584
1993	48	1.5	119,355	0.912	108,811
1993	60	2.5	74,949	0.848	63,523
1993	72	3.5	64,550	0.789	50,956
1993	84	4.5	49,123	0.736	36,168
1993	>84	6.5	110,808	0.638	70,733
1994	24	0.5	236,661	0.971	229,833
1994	36	1.5	153,904	0.912	140,308
1994	48	2.5	105,194	0.848	89,157
1994	60	3.5	66,057	0.789	52,145
1994	72	4.5	56,891	0.736	41,888
1994	84	5.5	43,295	0.684	29,628
1994	>84	7.5	97,661	0.593	57,937
1995	12	0.5	204,739	0.971	198,832
1995	24	1.5	223,342	0.912	203,612
1995	36	2.5	145,243	0.848	123,100
1995	48	3.5	99,274	0.789	78,368
1995	60	4.5	62,339	0.736	45,899
1995	72	5.5	53,690	0.684	36,742
1995	84	6.5	40,858	0.638	26,081
1995	>84	8.5	92,165	0.553	51,000

EXHIBIT 13

Discounted Unpaid Losses

Incremental Losses

Year	@12	@24	@36	@48	@60	@72	@84	84+
1988								95,956
1989							48,883	96,232
1990						64,236	45,890	89,633
1991					75,059	60,685	42,934	84,133
1992				120,123	70,811	56,697	40,187	78,585
1993			169,584	108,811	63,523	50,956	36,168	70,733
1994		229,833	140,308	89,157	52,145	41,888	29,628	57,937
1995	198,832	203,612	123,100	78,368	45,899	36,742	26,081	51,000

	Paid	E[Unpaid]	Std Dev
Total 1988-1994:	0	2,210,714	523,034
Total 1995:	0	763,634	177,017
Grand Total:	0	2,974,348	565,639

EXHIBIT 14

Lognormal Confidence Limits

$X \sim e^Y$, where $Y \sim N(\mu, \sigma^2)$

E[X]	2,974,348
Std[X]	565,639
CV[X]	0.190
σ^2	0.036
σ	0.188
μ	14.888

Confidence (p)	Z_p	Y_p	X_p
99.5%	2.576	15.373	4,748,221
99.0%	2.326	15.326	4,530,101
97.5%	1.960	15.257	4,227,820
95.0%	1.645	15.198	3,984,027
90.0%	1.282	15.129	3,720,342
75.0%	0.674	15.015	3,318,103
50.0%	0.000	14.888	2,921,980

Appendix A: Constrained Least-Squares Estimation

For another treatment of this subject see Judge [3, pp. 235-240]. If the reader needs to review matrix algebra, Judge's Appendix A [3, pp. 919-983] is recommended.

The problem is to estimate β in the model $\mathbf{Y} = X\beta + \mathbf{e}$, under the hypothesis, or constraint, H_0: $R\beta = r$. $\mathbf{Y}_{(t\times 1)}$ is observed; $X_{(t\times k)}$ is the design, or regressor, matrix, and the rank of X is k. The error term $\mathbf{e}$ is a $(t\times 1)$ random vector whose mean is $0_{(t\times 1)}$ and whose variance is $\sigma^2 I_t$ (I_t is a $(t\times t)$ identity matrix). Let R be $(j\times k)$ and of rank j. This means that the j rows of R, each of which is a constraint on β, are independent of each other.

As a consequence of their being of full rank, both $X'X_{(k\times k)}$ and $RR'_{(j\times j)}$ are nonsingular. This implies that $\{\beta\colon R\beta = r\}$ is not empty, because one solution is $\beta = R'(RR')^{-1}r$. X'X and RR' are also positive definite (Judge [3, pp. 960f.]). $j \le k$; otherwise, the rank of R could not be j. These conditions guarantee that $R(X'X)^{-1} R'_{(j\times j)}$ is nonsingular, so we will define $H = (R(X'X)^{-1} R')^{-1}$.

The object is to find the β which minimizes $(\mathbf{Y} - X\beta)'(\mathbf{Y} - X\beta)$ subject to the constraint. Optimization under constraint is accomplished by the Lagrange multiplier $\lambda_{(j\times 1)}$:

$$\Lambda(\beta,\lambda) = (\mathbf{Y} - X\beta)'(\mathbf{Y} - X\beta) + 2\lambda'(R\beta - r)$$
$$\frac{\partial\Lambda}{\partial\beta} = 2X'X\beta - 2X'\mathbf{Y} + 2R'\lambda$$
$$\frac{\partial\Lambda}{\partial\lambda} = 2(R\beta - r)$$

For a treatment of the rules of matrix differentiation see Judge [3, pp.967-969]. A similar use of the Lagrange multiplier is found in Halliwell [2, Appendix A]. The optimization is accomplished by setting the derivatives to 0 and solving for β^* and λ^*:

$$X'X\beta^* + R'\lambda^* = X'\mathbf{Y}$$
$$R\beta^* = r$$

It is convenient to view this as one equation in partitioned matrices:

$$\begin{bmatrix} X'X & R' \\ R & 0 \end{bmatrix}\begin{bmatrix} \beta^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} X'\mathbf{Y} \\ r \end{bmatrix}$$

$$\begin{bmatrix} \beta^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} X'X & R' \\ R & 0 \end{bmatrix}^{-1}\begin{bmatrix} X'\mathbf{Y} \\ r \end{bmatrix}$$

However, $\begin{bmatrix} X'X & R' \\ R & 0 \end{bmatrix}^{-1} = \begin{bmatrix} (I_k - (X'X)^{-1}R'HR)(X'X)^{-1} & (X'X)^{-1}R'H \\ HR(X'X)^{-1} & -H \end{bmatrix}$, where H was defined above. One can verify that this is the inverse by multiplying it by the matrix and coming out with an identity matrix. The value of λ^* does not concern us; but:

$$\begin{aligned}\beta^* &= (I_k - (X'X)^{-1}R'HR)(X'X)^{-1}X'\mathbf{Y} + (X'X)^{-1}R'Hr \\ &= M(X'X)^{-1}X'\mathbf{Y} + (X'X)^{-1}R'Hr\end{aligned}$$

The matrix $M = I_k-(X'X)^{-1}R'HR$ is important, and some of its properties need to be appreciated. First, $RM = 0_{(j\times k)}$, because $RM = R[I_k-(X'X)^{-1}R'HR] = R - R(X'X)^{-1}R'HR =$

$R - H^{-1}HR = R - R = 0$. Second $MM = M$, because $MM = [I_k-(X'X)^{-1}R'HR]M = M - (X'X)^{-1}R'HRM = M - (X'X)^{-1}R'H(0) = M$. This is to say that M is idempotent. Finally, $M(X'X)^{-1} = (X'X)^{-1} - (X'X)^{-1}R'HR(X'X)^{-1}$, which is symmetric. Therefore, $M(X'X)^{-1} = [M(X'X)^{-1}]' = (X'X)^{-1} M'$.

As a check, $R\beta^* = R[M(X'X)^{-1}X'\mathbf{Y} + (X'X)^{-1}R'Hr] = 0 + R(X'X)^{-1}R'Hr = H^{-1}Hr = r$.

We will now derive the mean and the variance of β^*:

$$\begin{aligned}
\beta^* &= M(X'X)^{-1}X'\mathbf{Y} + (X'X)^{-1}R'Hr \\
&= M(X'X)^{-1}X'\{X\beta + \mathbf{e}) + (X'X)^{-1}R'Hr \\
&= M\beta + M(X'X)^{-1}X'\mathbf{e} + (X'X)^{-1}R'Hr \\
&= (I_k - (X'X)^{-1}R'HR)\beta + M(X'X)^{-1}X'\mathbf{e} + (X'X)^{-1}R'Hr \\
&= \beta + M(X'X)^{-1}X'\mathbf{e} + (X'X)^{-1}R'Hr - (X'X)^{-1}R'HR\beta \\
&= \beta + M(X'X)^{-1}X'\mathbf{e} - (X'X)^{-1}R'H(R\beta - r) \\
&= \beta + M(X'X)^{-1}X'\mathbf{e} - (X'X)^{-1}R'H(0) \\
&= \beta + M(X'X)^{-1}X'\mathbf{e}
\end{aligned}$$

The simplification of the last two lines rests upon the truth of the hypothesis H_0: $R\beta=r$. In the remainder of the analysis the truth of H_0 is implicit.

Since $E[\mathbf{e}] = 0$, $E[\beta^*] = \beta$, so β^* is unbiased (given H_0). Moreover, $Var[\beta^*] = Var[M(X'X)^{-1}X'\mathbf{e}] = M(X'X)^{-1}X'Var[\mathbf{e}]X(X'X)^{-1}M'$ (cf. Judge [3 p. 42-44] and Halliwell [2, p. 3]) $= M(X'X)^{-1}X'(\sigma^2 I_t)X(X'X)^{-1}M' = \sigma^2 M(X'X)^{-1}X'X(X'X)^{-1}M' = \sigma^2 M(X'X)^{-1}M' = \sigma^2 MM(X'X)^{-1} = \sigma^2 M(X'X)^{-1}$.

The variance of the unconstrained estimator $\beta = (X'X)^{-1}X'\mathbf{Y}$ is $\sigma^2(X'X)^{-1}$. So the difference between the two variances $\mathrm{Var}[\beta]-\mathrm{Var}[\beta^*] = \sigma^2(X'X)^{-1} - \sigma^2 M(X'X)^{-1} = \sigma^2(I_k - M)(X'X)^{-1} = \sigma^2(X'X)^{-1}R'HR(X'X)^{-1} > 0_{(k\times k)}$. The meaning of the matrix inequality is that the matrix before it is positive definite (cf. Judge [3 p.239]). Therefore, $\mathrm{Var}[\beta] > \mathrm{Var}[\beta^*]$. As expected, constraining β leads to a tighter estimator.

Let us now treat the fitted and residual vectors. The fitted vector is $\mathbf{Y}^* = X\beta^*$. The residual vector is $\mathbf{e}^* = \mathbf{Y}-\mathbf{Y}^* = X\mathrm{b}+\mathbf{e}-X\beta^* = \mathbf{e}-X(\beta^*-\beta) = \mathbf{e}-X[M(X'X)^{-1}X'\mathbf{e}] = (I_t - XM(X'X)^{-1}X')\mathbf{e} = (I_t - X(X'X)^{-1}M'X')\mathbf{e} = N\mathbf{e}$. The matrix N is both symmetric and idempotent; therefore, $\mathrm{Var}[\mathbf{e}^*] = \mathrm{Var}[N\mathbf{e}] = N\mathrm{Var}[\mathbf{e}]N' = \sigma^2 NN' = \sigma^2 N$. The sum of the squared residuals is $\mathbf{e}^{*\prime}\mathbf{e}^* = \mathbf{e}'N'N\mathbf{e} = \mathbf{e}'N\mathbf{e}$. The expectation of $\mathbf{e}'N\mathbf{e}$ will be σ^2 times the degrees of freedom. To take the expectation one must know the trace operator and its properties (Judge [3 pp. 926-928]).

The trace of a matrix is a (1×1) matrix whose element is the sum of the diagonal elements of the matrix. Obviously $\mathrm{Tr}[A+B] = \mathrm{Tr}[A]+\mathrm{Tr}[B]$. But also it can be shown that as long as the matrices are conformable to multiplication, $\mathrm{Tr}[AB]=\mathrm{Tr}[BA]$. Hence,

$$
\begin{aligned}
Tr[N] &= Tr[I_t - XM(X'X)^{-1}X'] \\
&= Tr[I_t] - Tr[XM(X'X)^{-1}X'] \\
&= [t] - Tr[XM(X'X)^{-1}X'] \\
&= [t] - Tr[M(X'X)^{-1}X'X] \\
&= [t] - Tr[M] \\
&= [t] - Tr[I_k - (X'X)^{-1}R'HR] \\
&= [t] - [k] + Tr[(X'X)^{-1}R'HR] \\
&= [t] - [k] + Tr[R(X'X)^{-1}R'H] \\
&= [t] - [k] + Tr[H^{-1}H] \\
&= [t] - [k] + Tr[I_j] \\
&= [t] - [k] + [j] \\
&= [t - k + j]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E[\mathbf{e}^{*\prime}\mathbf{e}^{*}] &= E[\mathbf{e}'N\mathbf{e}] \\
&= E[Tr[\mathbf{e}'N\mathbf{e}]] \\
&= E[Tr[N\mathbf{e}\mathbf{e}']] \\
&= Tr[E[N\mathbf{e}\mathbf{e}']] \\
&= Tr[NE[\mathbf{e}\mathbf{e}']] \\
&= Tr[NVar[\mathbf{e}]] \\
&= Tr[N\sigma^2 I_t] \\
&= \sigma^2 Tr[N] \\
&= \sigma^2[t - k + j]
\end{aligned}
$$

The j independent constraints add j degrees of freedom. The estimator $\mathbf{e}^{*\prime}\mathbf{e}^{*}/(t-k+j)$ is, therefore, an unbiased estimator of σ^2.

With estimates of β and σ^2 in hand, one can build predictions from a prediction design matrix X_0. According to the model $\mathbf{Y}_0 = X_0\beta + \mathbf{e}_0 = X_0\beta^* + [\mathbf{e}_0 - X_0(\beta^* - \beta)] = X_0\beta^* + \mathbf{h}$. $\mathbf{h}$ is an error term with mean 0. Because there is no covariance between $\mathbf{e}_0$ and β^*, Var[$\mathbf{h}$] =

$Var[\mathbf{e}_0] + Var[X_0\beta^*] = \sigma^2 I + X_0 Var[\beta^*] X_0'$. Hence, $E[\mathbf{Y}_0|\mathbf{Y}] = X_0\beta^*$ and $Var[\mathbf{Y}_0|\mathbf{Y}] = X_0 Var[\beta^*] X_0' + \sigma^2 I$.

An important property is the lack of covariance between β^* and $\mathbf{e}^*$:

$$
\begin{aligned}
Cov[\beta^*, \mathbf{e}^*] &= Cov[\beta^* - \beta, \mathbf{e}^*] \\
&= Cov[M(X'X)^{-1}X'\mathbf{e},\ \ (I_t - XM(X'X)^{-1}X')\mathbf{e}] \\
&= M(X'X)^{-1}X'Cov[\mathbf{e},\mathbf{e}](I_t - X(X'X)^{-1}M'X') \\
&= \sigma^2 M(X'X)^{-1}X'(I_t - X(X'X)^{-1}M'X') \\
&= \sigma^2 M(X'X)^{-1}X' - \sigma^2 M(X'X)^{-1}M'X' \\
&= \sigma^2 M(X'X)^{-1}X' - \sigma^2 MM(X'X)^{-1}X' \\
&= \sigma^2 M(X'X)^{-1}X' - \sigma^2 M(X'X)^{-1}X' \\
&= 0_{(k\times t)}
\end{aligned}
$$

Both β^* and $\mathbf{e}^*$ are linear functions of $\mathbf{e}$; so if $\mathbf{e}$ is multivariate normal, then so too are β^* and $\mathbf{e}^*$ (Anderson [1, pp. 24-26]). And there is a theorem that if two normal random vectors are of zero covariance, then they are independent (Anderson [1, pp. 26-29] and Judge [3, p. 50]). Moreover, if $\mathbf{e}$ is multivariate normal with variance $\sigma^2 I_t$, then $\mathbf{e}^{*\prime}\mathbf{e}^*/\sigma^2$ is χ^2-distributed with t-k+j degrees of freedom (Judge [3, p. 52]), and still independent of β^*. This means that $\dfrac{\beta^* - \beta}{\sqrt{\dfrac{\mathbf{e}^{*\prime}\mathbf{e}^*}{t-k+j}}} \sim \dfrac{N[0_{(k\times 1)}, M(X'X)^{-1}]}{\sqrt{\dfrac{\chi^2_{t-k+j}}{t-k+j}}}$, where numerator and denominator are independent. This is a multivariate form of a *t*-distributed random vector, and the basis for the *t* test. But this test is appropriate only if the error term is normally distributed.

One statistic which may be inapplicable in a constrained model is the $\mathbf{r}^2$ statistic, which is $(\mathbf{Y^{*\prime}Y^*})/(\mathbf{Y'Y})$. By definition, $\mathbf{Y} = \mathbf{Y^*+e^*}$. Therefore, $\mathbf{Y'Y} = (\mathbf{Y^*+e^*})'(\mathbf{Y^*+e^*}) = \mathbf{Y^{*\prime}Y^*+2Y^{*\prime}e^*+e^{*\prime}e^*}$. In the unconstrained model it is guaranteed that $\mathbf{Y^{*\prime}e^*} = 0_{(1\times 1)}$, so $\mathbf{Y'Y} = \mathbf{Y^{*\prime}Y^*+e^{*\prime}e^*}$, and $0 < \mathbf{r}^2 = \mathbf{Y^{*\prime}Y^*}/\,\mathbf{Y'Y} < 1$. But in a constrained model:

$$
\begin{aligned}
\mathbf{Y}^{*\prime}\,\mathbf{e}^* &= (X\beta^*)'\mathbf{e}^* \\
&= \beta^{*\prime} X'\mathbf{e}^* \\
&= \beta^{*\prime} X'N\mathbf{e} \\
&= \beta^{*\prime} X'(I_t - X(X'X)^{-1}M'X')\mathbf{e} \\
&= \beta^{*\prime}(X' - M'X')\mathbf{e} \\
&= \beta^{*\prime}(I_k - M')X'\mathbf{e} \\
&= \beta^{*\prime}(R'HR(X'X)^{-1})X'\mathbf{e} \\
&= (\beta + M(X'X)^{-1}X'\mathbf{e})'R'HR(X'X)^{-1}X'\mathbf{e} \\
&= (\beta' + \mathbf{e}'X(X'X)^{-1}M')R'HR(X'X)^{-1}X'\mathbf{e} \\
&= \beta'R'HR(X'X)^{-1}X'\mathbf{e} + \mathbf{e}'X(X'X)^{-1}M'R'HR(X'X)^{-1}X'\mathbf{e} \\
&= \beta'R'HR(X'X)^{-1}X'\mathbf{e} + \mathbf{e}'X(X'X)^{-1}(RM)'(X'X)^{-1}X'\mathbf{e} \\
&= \beta'R'HR(X'X)^{-1}X'\mathbf{e} + \mathbf{e}'X(X'X)^{-1}(0)'(X'X)^{-1}X'\mathbf{e} \\
&= \beta'R'HR(X'X)^{-1}X'\mathbf{e} \\
&= (R\beta)'HR(X'X)^{-1}X'\mathbf{e} \\
&= r'HR(X'X)^{-1}X'\mathbf{e}
\end{aligned}
$$

The expectation of $\mathbf{Y^{*\prime}e^*}$ is 0; however, $\mathbf{Y^{*\prime}e^*}$ is guaranteed to be zero if and only if its variance is 0. So,

$$
\begin{aligned}
Var[\mathbf{Y}^{*\prime}\,\mathbf{e}^*] &= r'HR(X'X)^{-1}X'Var[\mathbf{e}]X(X'X)^{-1}R'Hr \\
&= \sigma^2 r'HR(X'X)^{-1}R'Hr \\
&= \sigma^2 r'HH^{-1}Hr \\
&= \sigma^2 r'Hr
\end{aligned}
$$

Since σ^2 is not zero, this variance is zero if and only if $\mathrm{r} = 0_{(j\times 1)}$, which will happen if and only if $0_{(k\times 1)} \in \{\beta\text{: } \mathrm{R}\beta = \mathrm{r}\}$. So if the constraint excludes the origin of the β-space, then the $\mathbf{r}^2$ statistic is inapplicable.

One can consider the unconstrained model as a special case of the constrained, viz., unconstrained as having zero constraints. Then R is $(0\times k)$ and r is (0×1). The author believes that matrices with a zero dimension can be consistently defined, though he has not seen the idea in print. Nonetheless, a matrix with a zero dimension has to be of rank zero. So $R(X'X)^{-1}R'$ is a (0×0) matrix, and one can argue that its inverse, which is H, exists. Then $Var[\mathbf{Y}^{*\prime}\mathbf{e}^{*}]$ is a (1×1) matrix, one of whose factors is of rank 0; and therefore it is of rank zero, and hence $0_{(1\times 1)}$. In passing, note that if the number of constraints is zero, then $M = I_k$, and $\beta^* = (X'X)^{-1}X'\mathbf{Y}$, as it should.

Appendix B will show how to transform a constrained model into an unconstrained one, as a result of which the $\mathbf{r}^2$ of the unconstrained model can be taken as the $\mathbf{r}^2$ of the constrained.

Appendix B: Transforming a Constrained Model

This appendix presupposes Appendix A, and is an alternative to the constrained regression of Exhibit 7.

Consider the constraint on β: $R\beta = r$, where R is $(j\times k)$ and of rank j, β is $(k\times 1)$, and r is $(j\times 1)$. As mentioned in Appendix A, one solution for β is $R'(RR')^{-1}r$. $R'(RR')^{-1}$ is an example of a generalized, or Penrose-Moore, inverse of R, and is denoted as R^{+} (cf. Judge [3, pp. 939f.]). Therefore, $R(\beta - R^{+}r) = R\beta - RR^{+}r = r - I_j r = 0_{(j\times 1)}$.

From theorems of matrix rank it can be shown that all the members of $\{\xi: R\xi = 0\}$ are of the form $\xi = W\gamma$, where W is $(k\times[k-j])$ of rank k-j such that RW=0, and where γ is any $([k-j]\times 1)$ vector. In other words, we know the existence of a one-to-one matching between the points of k-space which satisfy the constraint and the points of [k-j]-space.

The transformation of the model is:

$$\begin{aligned} \mathbf{Y} &= X\beta + \mathbf{e} \quad \textit{subject to } H_0\!: R\beta = r \\ \mathbf{Y} - XR^{+}r &= X(\beta - R^{+}r) + \mathbf{e} \quad \textit{still subject to } H_0 \\ \mathbf{Y} - XR^{+}r &= XW\gamma + \mathbf{e} \quad \textit{without constraint on } \gamma \end{aligned}$$

The design matrix of the transformed model is $XW_{(t\times[k-j])}$, necessarily of rank k-j. One can estimate $\gamma = [(XW)'(XW)]^{-1}(XW)'(Y - XR^{+}r)$, and then transform back to $\beta^{*} = W\gamma + R^{+}r$.

Let us take the example of Exhibit 7, and transform it. R = [1 1 1 1 1 1 1] and r = [7.213]. Therefore, $R^{+}r = R'(RR')^{-1}r$ is a (7×1) matrix each of whose elements is 7.213/7 = 1.0304. [k-j] = [7-1] = 6, so a (7×6) matrix, W, of rank 6 is needed such that RW=0. Of the many suitable matrices perhaps the most simple is:

$$W = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed model is:

$$\begin{aligned} &\mathbf{Y}_t = X_t\gamma + \mathbf{e} \\ &\textit{where } \mathbf{Y}_t = \mathbf{Y} - XR^{+}r \\ &\textit{and } X_{t\ (28\times 6)} = XW \end{aligned}$$

$$\therefore \quad \gamma_{(6\times 1)} = (X_t{}' X_t)^{-1} X_t{}' \mathbf{Y}_t = \begin{bmatrix} 0.9117 \\ 0.2326 \\ -0.1672 \\ -0.4883 \\ -0.5636 \\ -0.6751 \end{bmatrix}$$

$$\sigma^2 = (Y_t - X_t\gamma)'(Y_t - X_t\gamma)/(28-6) = 6.27166 \times 10^9$$

$$Var[\gamma] = \sigma^2 (X_t{}' X_t)^{-1} = \begin{bmatrix} 0.0502 & -0.0038 & -0.0049 & -0.0066 & -0.0101 & -0.0219 \\ -0.0038 & 0.0592 & -0.0058 & -0.0079 & -0.0121 & -0.0262 \\ -0.0049 & -0.0058 & 0.0733 & -0.0099 & -0.0153 & -0.0331 \\ -0.0066 & -0.0079 & -0.0099 & 0.0958 & -0.0207 & -0.0448 \\ -0.0101 & -0.0121 & -0.0153 & -0.0207 & 0.1362 & -0.0689 \\ -0.0219 & -0.0262 & -0.0331 & -0.0448 & -0.0689 & 0.2144 \end{bmatrix}$$

Finally, transforming back, $\beta^* = W\gamma + R^+r = \begin{bmatrix} 1.780 \\ 1.942 \\ 1.263 \\ 0.863 \\ 0.542 \\ 0.467 \\ 0.355 \end{bmatrix}$, as agrees with Exhibit 7. One can also verify that $Var[\beta^*] = Var[W\gamma] = WVar[\gamma]W'$ will agree with the $Var[\beta^*]$ of Exhibit 7.

As mentioned in Appendix A, $\mathbf{r}^2$ is ill defined in most constrained models, as is the case with the model of Exhibit 7. However, the transformed model has an $\mathbf{r}^2 = [(X_t\gamma)'(X_t\gamma)] / [\mathbf{Y}_t'\ \mathbf{Y}_t]$, which is 60.7%. It seems reasonable to attribute this $\mathbf{r}^2$ to the untransformed model as well.

Appendix B Exhibit

Transformed Regression

Model: $Y = X\beta + e$, where $e \sim (0, \sigma^2 I)$, given H_0: $R\beta = r$,
where R = [1 1 1 1 1 1 1] and r = [7 213]

$R^* = R'(RR')^{-1}$

0 142857
0 142857
0 142857
0 142857
0.142857
0.142857
0.142857

R^*r

1 030418
1.030418
1 030418
1.030418
1.030418
1.030418
1.030418

W (7x6)

-1	-1	-1	-1	-1	-1
1					
	1				
		1			
			1		
				1	
					1

Transformed Model $Y_t = X_t\gamma + e$, where
$Y_t = Y - XR^*r$, and
$X_t = XW$

Y_t (28x1)	X_t (28x6)						Y_t hat	e_t hat
131,027	-131332	-131332	-131332	-131332	-131332	-131332	98,489	32,538
31,245	131332 2						119,734	-88,489
-102,988		131332 2					30,543	-133,541
-81,717			131332 2				-21 954	-59,763
-127,203				131332.2			-64,134	-63,069
-118,403					131332.2		-74,012	-44,391
-96 218						131332 2	-88,666	-7,552
100,999	-141672	-141672	-141672	-141672	-141672	-141672	106,243	-5,244
213,398	141672 2						129,161	84,237
83,034		141672 2					32,948	50,087
-76,443			141672 2				-23,682	-52,760
-27,347				141672 2			-69,184	41,837
-45,690					141672 2		-79,839	34,150
57,191	-141677	-141677	-141677	-141677	-141677	-141677	106,247	-49,056
229,781	141677 3						129,166	100,615
130 630		141677.3					32,949	97,682
-71,075			141677.3				-23,683	-47,392
-59,559				141677 3			-69,186	9,627
248 715	-142578	-142578	-142578	-142578	-142578	-142578	106,923	141 793
113,728	142578						129,987	-16,259
20,794		142578					33,158	-12,364
123,777			142578				-23,834	147,611
60,054	-143286	-143286	-143286	-143286	-143286	-143286	107,453	-47,399
26,971	143285 6						130,632	-103,661
14,996		143285 6					33,323	-18,327
25 214	-138262	-138262	-138262	-138262	-138262	-138262	103,686	-78,472
137 711	138261.7						126,052	11,659
90,176	-121858	-121858	-121858	-121858	-121858	-121858	91,384	-1,208

$X_t'Y_t$ (6x1)	$X_t'X_t$ (6x6)					
7 85E+09	2 5E+11	1 32E+11	1.32E+11	1 32E+11	1.32E+11	1 32E+11
-7.6E+10	1 32E+11	2 3E+11	1.32E+11	1.32E+11	1 32E+11	1.32E+11
-1.1E+11	1 32E+11	1 32E+11	2 1E+11	1 32E+11	1,32E+11	1.32E+11
-1 3E+11	1.32E+11	1 32E+11	1 32E+11	1.9E+11	1.32E+11	1 32E+11
-1 2E+11	1.32E+11	1.32E+11	1 32E+11	1.32E+11	1 7E+11	1 32E+11
-1.1E+11	1 32E+11	1.32E+11	1 32E+11	1.32E+11	1 32E+11	1 49E+11

2.13E+11	1 38E+11
60 7%	39 3%
df	22
σ^2	6 27E+09

$(X_t'X_t)^{-1}$ (6x6)

8 01E-12	-6 1E-13	-7 8E-13	-1.1E-12	-1.6E-12	-3 5E-12
-6.1E-13	9 45E-12	-9 3E-13	-1.3E-12	-1.9E-12	-4 2E-12
-7 8E-13	-9 3E-13	1.17E-11	-1 6E-12	-2.4E-12	-5 3E-12
-1 1E-12	-1.3E-12	-1 6E-12	1 53E-11	-3 3E-12	-7.1E-12
-1 6E-12	-1.9E-12	-2 4E-12	-3 3E-12	2.17E-11	-1.1E-11
-3 5E-12	-4 2E-12	-5 3E-12	-7 1E-12	-1.1E-11	3 42E-11

γ (6x1)	Var[γ] (6x6)						Std[γ]	t statistic
0 91169	0 050215	-0 00385	-0 00486	-0 00659	-0 01013	-0 02192	0 224	4 07
0 232562	-0 00385	0 059236	-0 00581	-0 00787	-0 0121	-0 02619	0 243	0 96
-0 16716	-0 00486	-0 00581	0 073349	-0 00995	-0 0153	-0 0331	0 271	-0 62
-0 48834	-0 00659	-0 00787	-0 00995	0 095805	-0 02072	-0 04483	0 310	-1.58
-0 56355	-0 01013	-0 0121	-0 0153	-0 02072	0 136191	-0 06894	0 369	-1.53
-0 67513	-0.02192	-0 02619	-0 0331	-0 04483	-0 06894	0 214445	0 463	-1 46

$\beta = W\gamma + R^*r$	Var[β] = W Var[γ] W'						
1.780	0 044896	-0 00286	-0 00342	-0 00432	-0 00585	-0 00899	-0 01946
1 942	-0 00286	0 050215	-0 00385	-0 00486	-0 00659	-0 01013	-0.02192
1 263	-0 00342	-0 00385	0 059236	-0 00581	-0 00787	-0 0121	-0.02619
0 883	-0 00432	-0 00486	-0 00581	0 073349	-0 00995	-0 0153	-0 0331
0 542	-0 00585	-0 00659	-0 00787	-0 00995	0 095805	-0 02072	-0 04483
0 467	-0 00899	-0 01013	-0 0121	-0 0153	-0 02072	0 136191	-0 06894
0 355	-0 01946	-0 02192	-0 02619	-0 0331	-0 04483	-0 06894	0 214445