Modelling Asset Variability in Assessing Insurer Solvency

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Abstract:

A considerable amount of research in the development of methods to evaluate the probability that an insurer will become insolvent has been performed by the British Solvency Working Party and the Finnish Working Party on solvency. The result of the work of each of these two groups is a comprehensive model which simulates the future cash flows of an insurance company. Procedures to model the variability of asset values and asset cash flows are an integral part of these models.

In this paper the models used by the British Working party and the Finnish Working Party to model asset values and asset cash flows will be introduced. The paper will describe how these models can be applied in the United States. Application of the models will be illustrated using historical stock and bond return data.

INTRODUCTION

The British Solvency Working Party and the Finnish Working Party on solvency have pioneered the development of models for predicting the probability that an insurance company will become insolvent and for determining solvency margin requirements. The work of each of these groups has relied heavily on risk theory.

In its simplest form, risk theory is used to evaluate the probability that insured losses will exceed beginning surplus plus premium income received during a specified time period. This is expressed as:

$$\varepsilon = \Pr\{X_t \geq U_o + (1+a)\, P_t\}^1$$

where: X_t is the aggregate losses for the period
U_o is the beginning surplus
P_t is the pure premium or expected losses for the period

The factor a is a safety loading which contributes a cushion against adverse deviation to surplus. The quantity ε is known as the probability of ruin.

For this basic formulation of the risk theory model, a probability distribution for X can be derived and used to compute ruin probabilities. Risk theory models such as the one described above have been used to compute surplus requirements and risk margins.

The simple one period model can be extended and to include multi year and multi variable risk processes. The variables which are added incorporate into the model as stochastic components funds which flow into and out of the company. Among these variables are premium income, expenses, and yield on investments, as well as aggregate losses. Multi year dependencies are incorporated into the model. The more general model is denoted by Pentikainen et. al. as follows[2]:

$$U_t = U_{t-1} + P_t + I_t - X_t - E_t - D_t + U_t^{new}$$

where:

P_t = Premium Income at time t
I_t = Investment income at time t
X_t = Aggregate Losses at time t
E_t = Administrative Expenses at time t
D_t = Dividends paid to shareholders at time t
U_t^{New} = New capital from issue of new shares.

A stochastic function is developed for each component of the general model. Because of the complexity of the general model, simulation is used to derive a probability distribution of insolvency and determine appropriate solvency margins.

The casualty actuarial literature contains a great deal more material about modelling the random loss process than about modelling any of the other factors contributing to the variability of insurers' surplus. This paper will focus on only one aspect of the overall model which has not been addressed frequently in the actuarial literature in the United States: assets. A goal of this paper is to introduce procedures developed by European actuaries for modelling asset change and investment income. These procedures are viable tools for use in stochastic simulation. These techniques will then be applied to actual investment data to illustrate how such models are parameterized. The procedures for incorporating the models into simulation will then be presented and procedures for assessing the reasonableness of the models will be discussed. Finally, the application of the model to the study of the impact asset variability on solvency will be presented.

In the discussions which follow, two definitions of returns are used. They are:

$$X_t = \frac{A_t}{A_{t-1}} - 1$$

$$X_t^1 = \ln \left(\frac{A_t}{A_{t-1}}\right)$$

where A_t is the value of an asset at time t.

When the absolute value of returns is small, both definitions give similar return values. The second definition is convenient to work with when analyzing the product of returns over a number of time periods. This is the definition which is used by the British Working Party. The first definition is used by the Finish Working Party.

INVESTMENT MODELS

The Wilkie Model

One of the most widely used procedures for modelling investment returns in actuarial science is a series of models developed by Wilkie[3]. The Wilkie investment model was developed for use in simulation, and was incorporated by the British Working Party[4] into a comprehensive solvency monitoring model. The model was developed with a view towards capturing the long term behavior of investments. Therefore the parameters selected do not always provide the most accurate short term predictions of investment returns.

The Wilkie model is composed of four interrelated components: the Retail Price Index, an index of gross equity dividends, the gross dividend yield and the gross yield on consoles. The retail price index is used as an input variable in predicting all the investment series.

Wilkie's four models are:

(1) Q(t): Retail Price Index

$$\Delta \ln(Q_t) - \mu_Q - \phi\,(\Delta \ln(Q_{t-1}) - \mu_Q) + \sigma_Q z_t$$

For this model and all subsequent models z is an independent normal (0,1) variable. The retail price index model can be reexpressed as a model of the inflation rate i_t' where $1+i_t' = Q_t/Q_{t-1}$.

$$\ln(1+i'_t) - \mu_i = \phi\,(\ln(1+i'_{t-1}) - \mu_i) + \sigma_i\, z_t$$

or

$$i_t - \mu_i = \phi\,(i_{t-1} - \mu_i) + \sigma_i\, z_t$$

This model is an autoregressive model. Procedures for deriving the parameters of this model are described in Appendix I.

(2) Y_t: Stock Dividend Yield

$$\ln(Y_t) = v_Y\, i_t + NY_t$$

$$NY_t - \mu_y = \phi_y(NY_{t-1} - \mu_y) + ey_t$$

$$ey_t = \sigma_y\, z_t$$

The dividend yield can be viewed as the product of 2 factors. The first factor is an inflation dependant factor and the second factor is a random component which is independent of inflation. This model is a bivariate autoregressive model. Procedures for estimating the parameters of this model are discussed in Appendix II.

3) D_t: Index of gross equity dividends

The index of gross equity dividends corresponds approximately to the return on the stock market. The model for the index is

$$\Delta \ \ln(D_t) = \nu_d\left(\frac{\alpha}{(1-(1-\alpha))B}\right) \Delta \ \ln(Q_t) + \omega \ \Delta d$$
$$+\mu_d + \delta_y \ ey_{t-1} + ed_t + \theta \ ed_{t-1}$$

$$ed_t = \sigma_d \, z_t$$

In this model B is the backwards shift operator, and the term

$$\frac{\alpha}{(1-(1-\alpha)B)} \ \Delta \ \ln(Q_t)$$

is an infinite sequence of lagged inflation effects. The parameter α is the smoothing parameter used to compute an exponentially smoothed average of past inflation rates. Exponential smoothing is used by Wilkie to model expected inflation. This term can be denoted $E(i_t)$ or the expected inflation rate and the term $\Delta\ln(D_t)$ can be denoted d_t. Then the Wilkie model for equity dividends can be redefined as follows:

$$d_t = \nu_d \ E(i_t)+\omega_d \ i_t + \mu_d+ \delta \ ey_{t-1} + ed_t + \theta \ ed_{t-1}$$

Thus the dividend index depends on expected inflation, the current actual inflation rate, the lagged residual from the dividend yield model, and the lagged residual for the dividend index model.

(4) C_t: Console yield:

$$C_t - v_1 \; E(i) + NC_t$$

$$\ln(NC_t) - \mu_c + \phi_1(\ln(NC_{t-1}) - \mu_c) + \phi_2(\ln(NC_{t-2}) - \mu_c)$$
$$+ \phi_3(\ln(NCt-3 - \mu_c) + \delta_c \; ey_t + \sigma_c \; z_t$$

Console yields consist of a factor related to inflation and a factor which is independent of inflation. The factor which is independent of inflation is related to past values of itself and the residual of the stock dividend yield series.

Application of Wilkie's Model

The data on which the parameter estimates for the Wilkie model are based are from United Kingdom sources. In addition some of the investments are different from those which are found in the United States. For instance, consoles which are non redeemable fixed income bonds, have no equivalent in the United States. For these reasons one would expect models built using data from United States sources to incorporate different financial series and to have different parameter estimates even when similar series are used. Despite these considerations, a model can be built to simulate the behavior of United States securities with a structure similar to the structure used by Wilkie. The following model is proposed:

(1) i_t: The inflation rate

For a model similar to Wilkie's, the returns for a security are related to inflation and possibly to the returns on other securities. The inflation series is modeled as a first order autoregressive process like the one used by Wilkie to forecast the Retail Price Index. Thus, the model for inflation is:

$$i_t - \mu_i = \phi_i (i_{t-1} - \mu_i) + \sigma_i z_t$$

where i_t, the inflation rate equals $\ln(CPI_t/CPI_{t-1})$ and CPI_t is the consumer price index at time t.

The procedure used to derive ϕ_i is illustrated in Appendix I. This procedure was applied to annual inflation rates from 1926 through 1987 and resulted in an estimate of .66. The estimate of μ_i, the mean of the series is .03 It should be noted that if the inflation rate is modelled using only values from the past 20 years, μ_i is approximately .05.

A residual or error term is computed for each inflation rate using:

$$e_t = i_t - \mu_i - .66 (i_{t-1} - \mu_i)$$

The standard deviation of the residual is the estimate of σ_i. For the inflation rate series an estimate of .037 was obtained.

(2) SI: Stock Market Dividend Income

The proposed model relates stock market dividend income to expected inflation and a random residual term which is independent of inflation. The model was derived using regression analysis. The regression analysis indicated that dividend income was related to expected inflation rather than actual inflation. An exponentially weighted average of past inflation rates is used to model expected inflation. This model is

$$\ln(SI_t) = -3.6 + 15\ E(i_{t-1}) + es_t$$

$$es_t = .9\ es_{t-1} + .01\ z_t$$

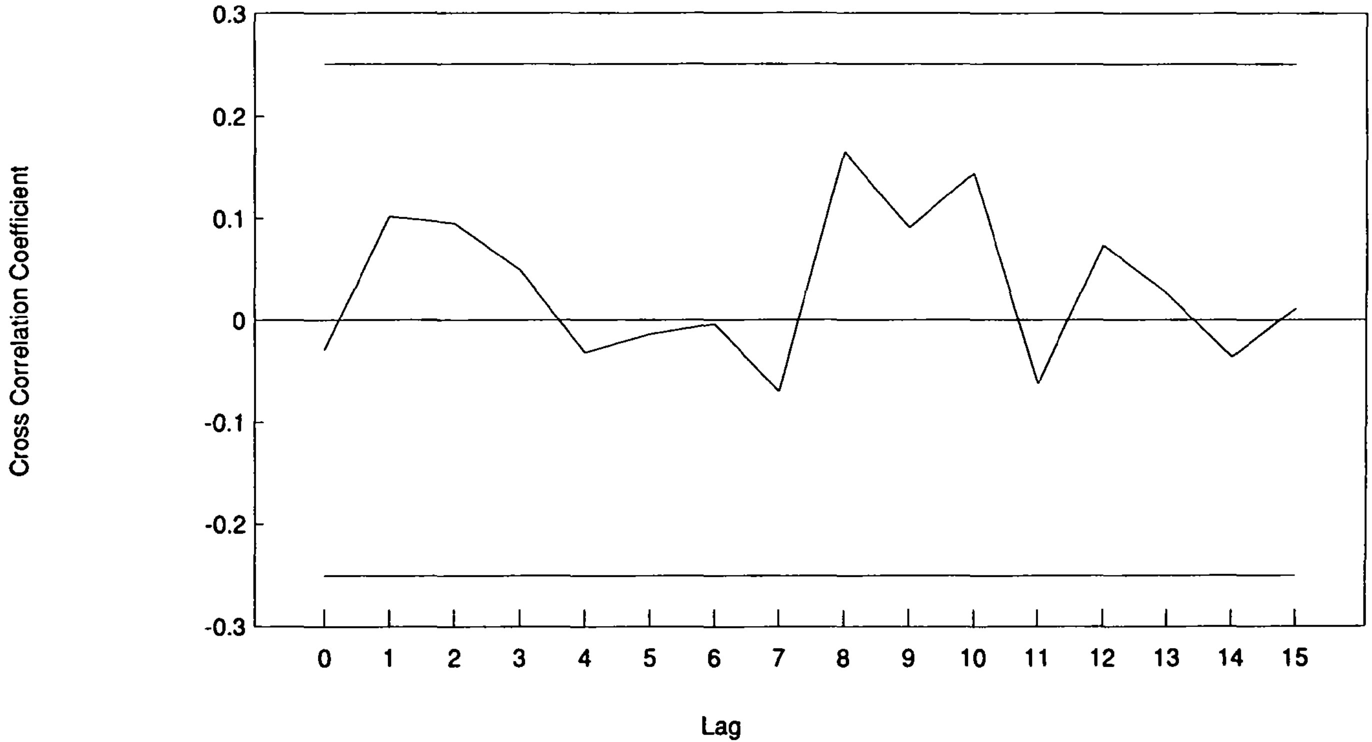
(3)SCAP: Stock Market Capital Appreciation.

The change in an index of stock equity shares such as the Dow Jones Average or the S&P 500 index can be used as a measure of stock capital appreciation. The correlation coefficients (or correlation function) between inflation rates and the stock market return can be used to study the relationship between these two series. Figure 1 presents the cross correlations between the stock series (the percentage change in the S&P 500 index) and the inflation series. Note that none of the cross correlation coefficients are significant, indicating that inflation does not significantly affect stock market returns. This result is contrary to the findings other researchers.

The stock market series exhibited no significant autocorrelations as well as no significant cross correlation with inflation. The stock index series can therefore be modeled as a white noise series with mean of .045 and standard deviation of .2. That is

Figure 1

Stock Capital Appreciation



Note: Based on stock and inflation rates from 1926 - 1987. Rates are from Ibbotson and Sinquefield, Stocks, Bonds, Bills and Inflation, 1989.

$$S\&P_t = .045 + .2\ Z_t$$

(4) BCA_t: Long Term Government Bond Capital Appreciation

A procedure for modelling the relationship between the change in asset values for government bond and the inflation rate is described in Appendix II. A model derived for long term government bond capital appreciation is

$$BCA_t = -.8(i_t - \mu_i) + NY_t$$

$$NY_t = .07\ z_t$$

This model indicates that the market value of bonds varies inversely with the inflation rate.

(5)INC: Long Term Government Bond Income:

The total return on government bonds is composed of capital appreciation, investment income and reinvestment returns. For the purposes of this analysis, reinvestment returns which are typically small will be ignored. If Y_t is the return on bonds at time t then:

$$(1 + Y_t') = (1 + BCA_t')\ (1 + INC_t')$$

where $\exp(BCA_t) = (1 + BCA_t')$, BCA_t' is the percentage change in market value of the bond in year t and INC_t is the income from coupons received during year t. These two items must be derived separately since they are treated differently for accounting and taxation purposes and involve different cash flows. The value for $(1 + Y_t')$ can be derived from the values for $(1 + BCA_t')$ and $(1 + INC_t')$.

In developing an investment model, we have compared long term bond income to the console yield model developed by Wlkie. In Wilkie's model, console yields are related to the current inflation rate and the expected inflation rate, where the expected inflation rate is modelled as an exponentially smoothed average of past inflation rates.

The graph of the correlations between long-term bond income and inflation is used to examine the relationship between bond income and inflation. Looking at the cross correlations on Figure 2 between LBI_t and i_t, the high correlations beginning at lag 1 indicates that there is a significant relationship between long term bond income and a average of past inflation rates. This suggests that long term bond income is a function of expected inflation. In the Wilkie model console yields were related to the residual of the stock dividend series as well. A model was investigated in which long term bond income was related to inflation, expected inflation, and the residual of the stock dividend series. The following model was fit to the bond income data:

$$\ln(INC_t) = -3.7 + 10\ E(i_t) + .05*ey_{t-1} + 1.13*i_{t-1} + ein_t$$

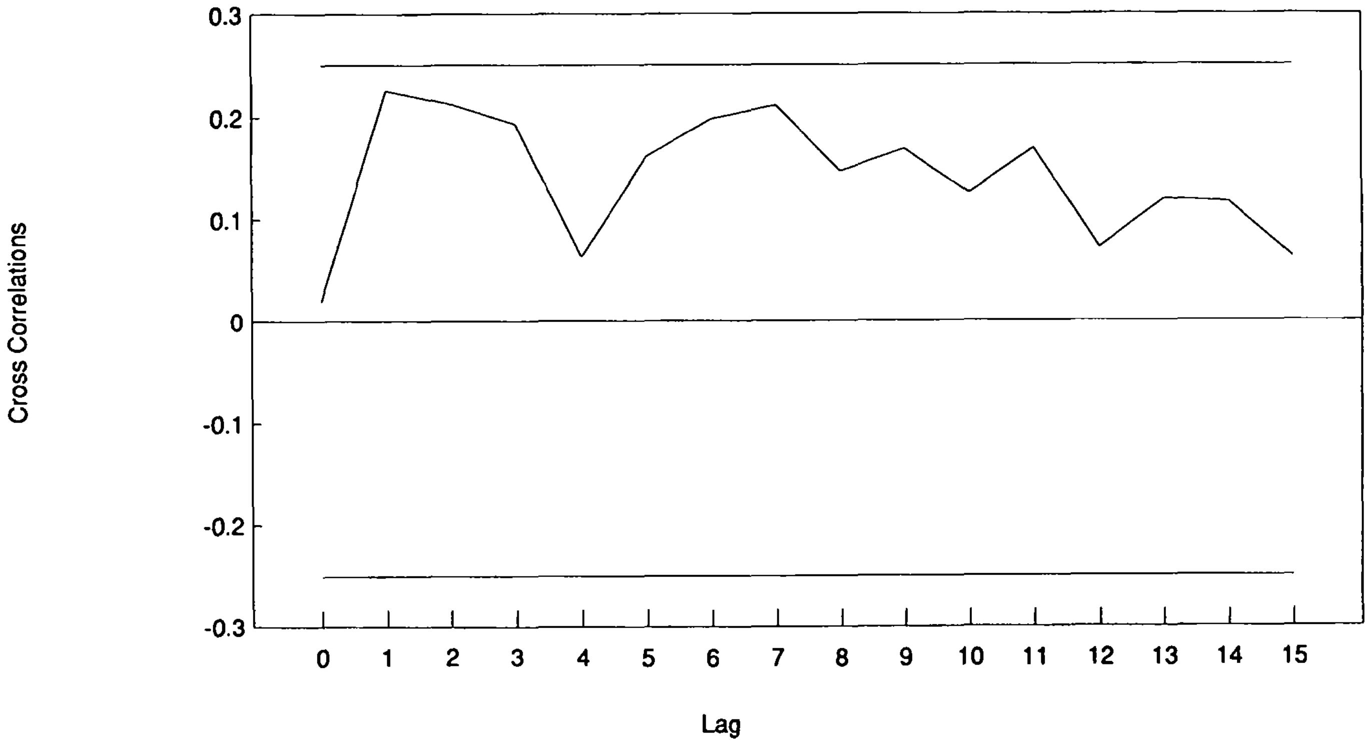
$$ein_t = .94\ ein_{t-1} + .095\ z_t$$

Other Bond Series

While the derivation of parameters for other bond series will not be illustrated in this paper, insurance companies hold significant proportions of their assets in bonds other than

Figure 2

ln(Long Term Bond Income)



Note: Based on dividend and inflation rates from 1926 - 1987. Rates are from Ibbotson and Sinquefield, <u>Stocks, Bonds, Bills and Inflation,</u> 1989.

long term government bonds. Among these are short term and intermediate term government bonds, municipal bonds and corporate bonds. Procedures similar to those presented above can be used to derive parameters for other bond series.

The Finnish Working Party Model and its Application

A procedure for modelling asset variability was developed by the Finnish Working Party[5] in conjunction with the development of a comprehensive model for evaluating insurer's financial strength.

Like Wilkie, the Finnish Working party first models the inflation rate. The return for each security is then a function of the inflation rate. The Finnish working party separately models variability in asset values and variability in income for each category of assets that a company owns. Wilkie separately models asset values and income only for stocks. The Finnish Working Party describes a general model which can be applied to any kind of security. The model, therefore can potentially be used to model any kind of asset a company may own. Thus it is more general than the Wilkie procedure which models only stocks and government bonds (consoles). There are three components of the general model which will be described: 1) the inflation rate, 2) the model for asset change and 3) the model for investment income.

(1) i_t The inflation rate

Like Wilkie, the Finnish Working Party models the inflation rate as a first order autoregressive process. The model for inflation is:

$$i_t' - \mu_i = \phi\ (i_{t-1}' - \mu_i) + \sigma_i\ \varepsilon_t$$

where ε_t is an independent identically distributed random shock term. The term i_t' is the change in the price index at time t (i.e. $i_t' = \exp(i_t)-1$).

The derivation of the parameter ϕ for this model is illustrated in Appendix 1. The estimated parameter value for ϕ using inflation rates from 1926-1987, .65, is very close to the parameter value of .66 for i_t.

Figure 3 which displays the distribution of the residual of the model fit to inflation indicates that the random shock term has significant positive skewness. This positive skewness was cited by Pentikainen et. al. as their reason for using the gamma distribution to model the shock term ε_t. The assumption used by Wilkie, and the usual assumption for time series analysis is that ε_t is normally distributed.

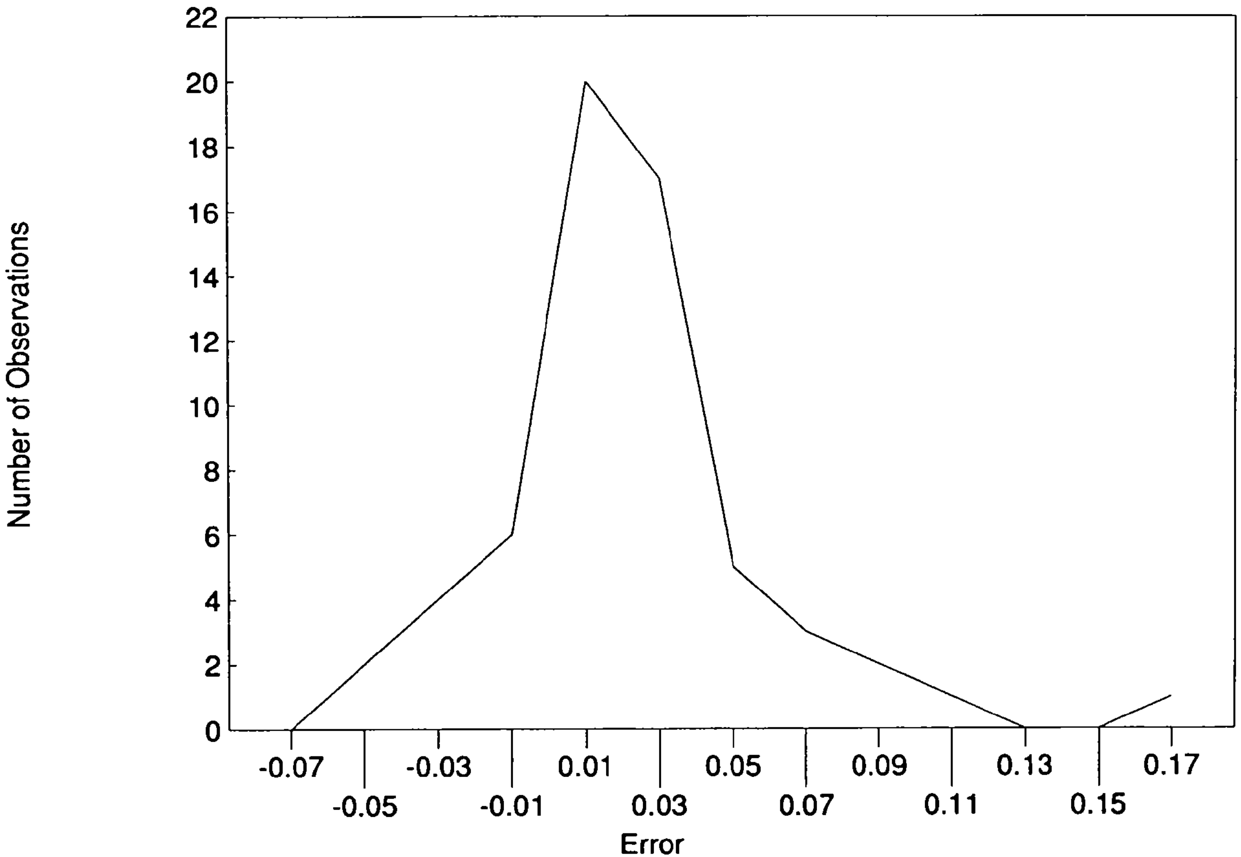
The Finnish Working Party assumes that $\varepsilon + 2$ is distributed gamma (4,2). This is equivalent to assuming ε_t follows a three parameter gamma distribution:

$$f(x) = \frac{1}{B\ \Gamma(a)}\left(\frac{x - s}{B}\right)^{a - 1} \exp\left(-\frac{(x-s)}{B}\right)$$

where a=4, B=2 and s= -2. A procedure which can be used to derive the parameters of gamma distributed random variables is described in Appendix V.

Figure 3

Distribution of Inflation Rate Residuals



Note: Based on inflation rates from 1926 - 1987. Rates are from Ibbotson and Sinquefield, Stocks, Bonds, Bills and Inflation, 1989.

2) Changes in asset value for the kth security.

Define the change in asset value a_{kt}

$$a_{kt} = A_{kt}/A_{k,t-1} - 1$$

where A_{kt} is the asset value of the k^{th} security at time t.

The Finnish working party models the change in asset value as the product of a mean, an inflation effect and a residual effect which is the effect of all other economic variables.

$$(1 + a_{kt}) = (1 + \bar{a}) (1 + \iota_{kt}) (1 + n_{kt})$$

where $\bar{a}$ is the sample mean of changes in asset value for security k, ι_t is inflation effects at time t which is modeled as

$$\iota_t - \sum_{s \geq 1} \delta_s (i_{t-s} - \bar{i})$$

n_{kt} is a residual effect net of inflation. The variable n_{kt} follows an autoregressive process

$$n_{kt} - \sum_{s \geq 1} b_s n_{t-s} + e_t$$

The random shock term , e_t is gamma distributed. For the examples presented by Pentikainen et. al. , the order of the autoregression never exceeded 2.

To illustrate the application of this model to actual asset return data, the parameters for long term bond capital appreciation are estimated. The example is shown on Exhibit 1.

First, $\bar{a}$ is estimated as the average of the percent change in long term bond capital appreciation. Then $(1 + a_{kt})$ is divided by $(1 + \bar{a})$ and 1 is subtracted from the result. Then the correlation function between this and the inflation rate series is analyzed. Figure 4 presents the cross correlation function between the inflation and the long term bond series. This chart indicates that there is a significant negative correlation between long term bond capital appreciation and inflation at lag 0. Time series estimation procedures suggest that $\delta=-.81$. Once δ has been estimated, it is used in the following formula to compute the residual term.

$$(1 + n_{kt}) = (1 + a_{kt})/[(1 + \bar{a})\ (1 + \delta_k(i_t - \bar{i}))]$$

Since this residual is assumed to follow an autoregressive process, ordinary least squares regression can be used to derive the residual parameters.

3) Investment Income

Investment income is frequently expressed as a percentage of the current value or market value of an asset. Therefore, as prices increase on assets, income as a percent of current value may drop, even through the actual value of the coupon or dividend remains the same. Because of the variability added to the calculation, the Finnish working party derives a value called stabilized current value[6].

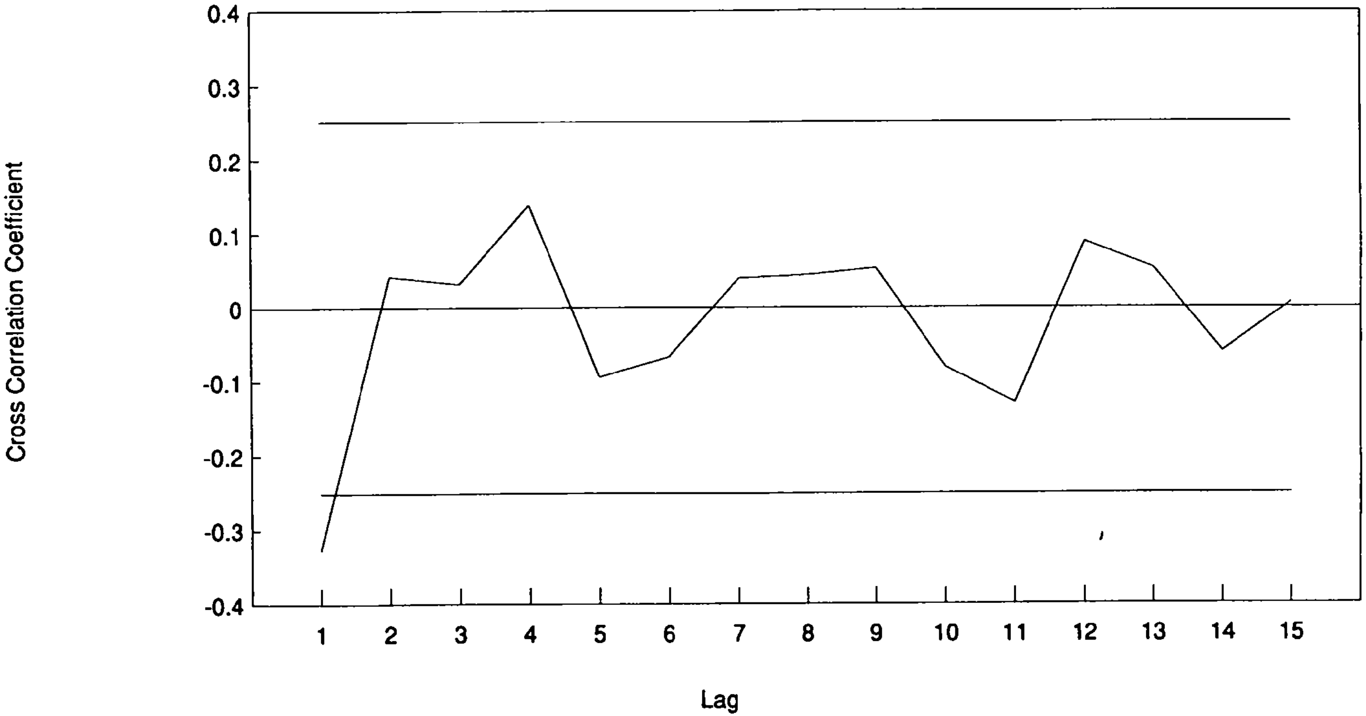
Estimation of Asset Change Parameters Using Finnish Working Party Model for Long Term Govt Bonds Capital Appreciation

(1) Long Term Bond Capital Appreciation	(2) 1 + Capital Appreciation	(3) Deviation From Mean	(4) Bond Residual	(5) (1+iota)	(6) Residual Independent Of Inflation
0.1430	1.1430	0.0872			
0.0492	1.0492	-0.0019	-0.0586	0.9882	0.0100
-0.1051	0.8949	-0.1488	-0.1475	0.9946	-0.1442
0.1275	1.1275	0.0725	0.1692	0.9902	0.0831
0.1250	1.1250	0.0701	0.0230	0.9868	0.0844
0.2188	1.2188	0.1594	0.1138	0.9905	0.1705
0.1597	1.1597	0.1032	-0.0004	0.9846	0.1204
0.1104	1.1104	0.0562	-0.0109	0.9732	0.0853
0.2221	1.2221	0.1625	0.1260	0.9757	0.1915
0.0115	1.0115	-0.0378	-0.1435	0.9622	-0.0001
0.1313	1.1313	0.0761	0.1006	0.9511	0.1314
0.1548	1.1548	0.0984	0.0490	0.9561	0.1489
-0.0969	0.9031	-0.1409	-0.2049	0.9731	-0.1172
0.0876	1.0876	0.0345	0.1261	0.9727	0.0636
0.3261	1.3261	0.2614	0.2390	0.9296	0.3570
0.0166	1.0166	-0.0330	-0.2029	0.9024	0.0716
0.2415	1.2415	0.1810	0.2024	0.9439	0.2512
0.1225	1.1225	0.0677	-0.0499	0.9615	0.1104
-0.0175	0.9825	-0.0654	-0.1094	0.9458	-0.0119
0.0838	1.0838	0.0309	0.0734	0.9278	0.1112
0.1883	1.1883	0.1303	0.1103	0.8935	0.2651
0.1164	1.1164	0.0620	-0.0228	0.9008	0.1789
-0.0666	0.9334	-0.1121	-0.1524	0.9285	-0.0438
-0.0127	0.9873	-0.0609	0.0120	0.9690	-0.0309
0.0039	1.0039	-0.0451	-0.0055	0.9696	-0.0152
0.0159	1.0159	-0.0336	-0.0043	0.9684	-0.0021
0.1577	1.1577	0.1012	0.1231	0.9698	0.1355
0.0239	1.0239	-0.0261	-0.0918	0.9910	-0.0172

Notes:
(1): Based on simulated data
(3): (2)/(1+ mean of (1)) - 1
(4): (4) - .65 * prior (3). Used to fit transfer function as described in Appendix II.
(5): -.81 * (inflation rate - mean inflation rate) + 1
(6): (1 + (3))/(5)

Figure 4

Long Term Bond Capital Appreciation
(Finnish Model)



Note: Based on inflation rates from 1926 - 1987. Rates are from Ibbotson and Sinquefield, Stocks, Bonds, Bills and Inflation, 1989.

The formula is:

$$\bar{A}_{kt} = (1+\bar{a}_k)^{t-t_0} A_{kt_0}$$

where $\bar{a}$ is the average annual change in asset value for assets in category k.

The income for the series is

$$y_{kt} = \frac{\underline{\text{INCOME}_t}}{\bar{A}_{k,t-1}}$$

where INCOME is the actual dollars of income and

$$(1 + y_{kt}) = (1 + \bar{y})\ (1 + \iota_{kt})\ (1 + n_{kt})$$

where ι_{kt} and n_{kt} are defined as for asset value changes.

The procedure for estimating the parameters of the income model is the same as for the asset model.

Simulation of Asset Value Change and Investment Income

The models derived using an approach similar to Wilkie's all assume that the random shock variable for the model follows the normal distribution. Therefore for each component of the model, the only simulated item is a normal (0,1) random variable. Initial, or starting values are needed for all asset categories modelled. These may be based on the most recent actual observed values, or on some judgementally selected values.

Wilkie suggests running the simulation with several different initial values to test sensitivity of results to initial conditions[7].

Since inflation is a factor determining the values of other investment categories it is simulated first. The procedure to simulate inflation is:

1) Select an initial value for inflation at time 0, i_0 and expected inflation $E(i_0)$.
2) Generate a normal (0,1) random variable, z_t
3) Compute i_t the simulated inflation rate for period t as

   $i_t = \mu_i + \phi(i_{t-1} - \mu_i) + \sigma_i z_t$

   where μ_i, σ_i and ϕ are the estimated parameters for the model
4) Using this simulated inflation rate, expected inflation is updated as

   $E(i_t) = i_{t-1} + (1 - \alpha)(i_{t-1} - E(i_{t-1}))$.

   where α is the smoothing parameter.

Using the simulated expected inflation rate, simulate the stock dividend series.

1) Select an initial value for es, the regression residual
2) Generate a standard normal random variable, z_t
3) Compute $es_t = \phi_s\, es_{t-1} + \sigma z_t$
4) compute $\ln(SI_t) = a + b\,E(i_{t-1}) + es_t$

To simulate stock capital appreciation:

1) Generate a normal random variable, z_t
2) Compute $S_t = \mu_s + \sigma_s z_t$

   where μ_s, and σ_s are parameter estimates

Long term bond capital appreciation is modelled as follows

1) Select an initial value for NY_0, the return series net of inflation

2) Generate a normal random variable z_t

3) Calculate $NY_t = \phi_Y NY_{t-1} + \sigma_y z_t$

4) Compute $Y_t = \mu_y + v_Y E(i_{t-1}) + NY_t$

To simulate long term bond income:

1) Select an initial value for ei, the residual from the long term bond regression

2) Generate a standard normal variable

3) Compute $ei_t = \phi_{in} ei_{t-1} + \sigma_{in} z_t$

4) $\ln(INC_t) = a + b_1 E(i_t) + b_2 ey_{t-1} + b_3 i_{t-1} + ei_t$

Assets modelled using the procedures described by the Finnish Working Party are simulated in a very similar way. Instead of a normal random variable, a random gamma variable is generated as the random component of each equation. A procedure for generating a gamma variable is described by Press et. al[8].

First inflation is simulated and then the other asset changes and returns which are dependent upon inflation are simulated. The following procedure would be used to model the change in asset value for a particular asset:

1) Generate a random inflation shock term, ε_{it} from a gamma distribution.

2) Simulate a random inflation rate:

$$i_t = \mu_i + \phi(i_{t-1} - \mu_i) + \varepsilon_{it}$$

3) Generate a random inflation shock term ε_{kt} from a gamma (a_k, B_k) distribution where a_k and B_k are the gamma parameters of the random shock term for asset category k

4) Compute $n_{kt} = b_1\, n_{k,t-1} + b_2\, n_{k,t-2} + \ldots\, b_n\, n_{k,t-n} + \varepsilon_{kt}$

5) Compute $\iota_{kt} = \delta_1\, (i_{t-1} - \bar{i}) + \ldots\, \delta_n\, (i_{t-n} - \bar{i})$

6) Compute $(1 + a_{kt}) = (1 + \bar{a}_k)\, (1 + \iota_{kt})\, (1 + n_{kt})$

For each asset category compute

$$A_t = A_{t-1}\, (1 + a_{kt})$$

A similar procedure is used to simulate the investment income for each of the asset categories.

Model Evaluation

In this paper, several procedures for modeling investment series have been introduced. A summary of some methods for evaluating the reasonableness of the fitted models will be presented below.

For the time series models presented in this paper, the residual or random shock term is assumed to be independent and identically distributed and uncorrelated with prior shock terms. A preliminary indication of whether the residuals are random can be obtained from a graph of the autocorrelation coefficients. The autocorrelation coefficients are compared to $\pm \frac{1.96}{\sqrt{n}}$. An additional test for autocorrelation is known as the Portmanteau test[9].

The Portmanteau test assesses whether the pooled autocorrelation coefficients are significant. The statistic for this test is:

$$Q = \sum_{k=1}^{h} \hat{\rho}_k^2$$

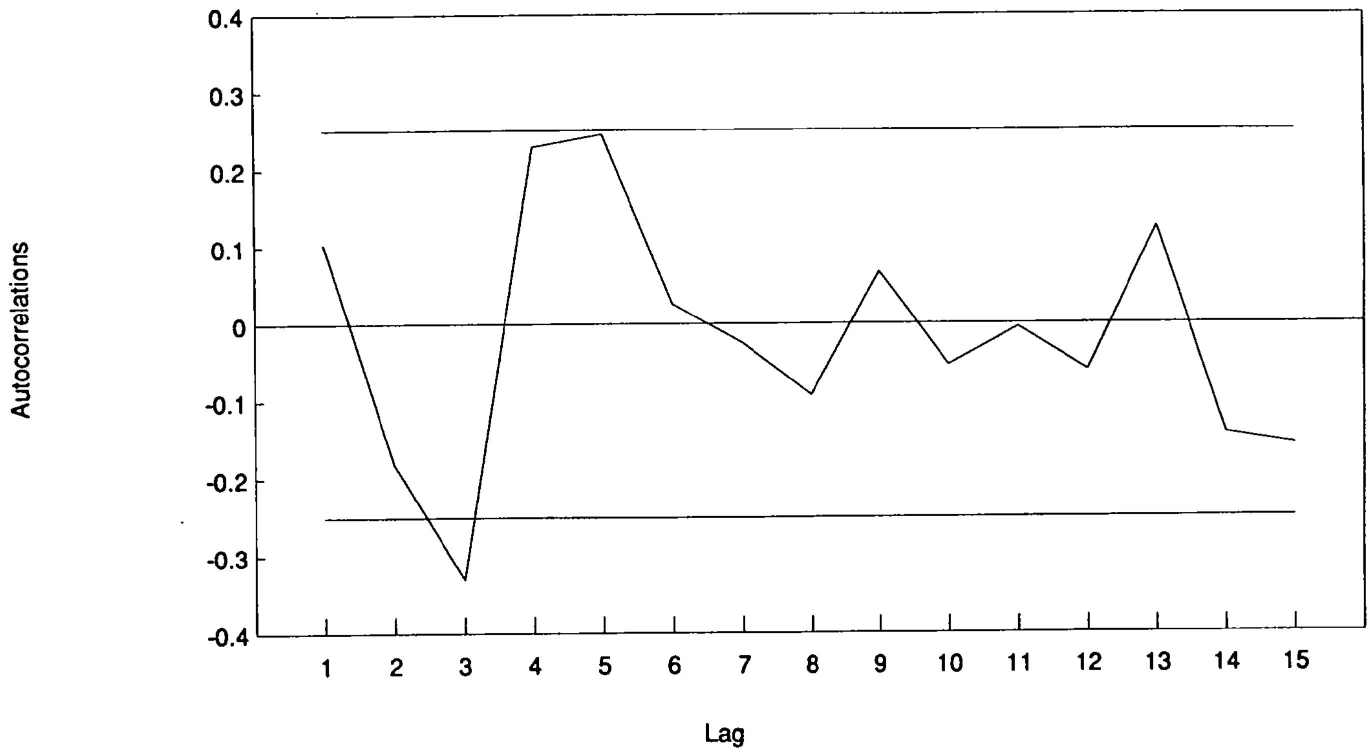
For this test h, the number of autocorrelations tested is selected judgementally, but is frequently equal to 20. The statistic Q has a Chi Square Distribution with h - p - q degrees of freedom, where p is the number of autoregressive parameters and q is the number of moving average parameters in the model.

Figure 5 presents the autocorrelation coefficients for the residuals of the inflation rate model and Figure 6 presents the autocorrelation coefficients for the residuals of the stock market index model. These charts suggests that the residuals of the inflation rate may not be independent of prior residuals but the residual of the stock series appear to be random. The statistic Q computed from these residuals is not significant at the 95th percentile for the inflation or stock series. Thus the Portmanteau test did not indicate that the inflation residual series was autocorrelated.

Several other tests are often used to test the randomness of the residuals. These are the turning points test, the difference sign test and the rank test. A turning point occurs when a sequence changes from an increasing series to a decreasing series or from a decreasing to

Figure 5

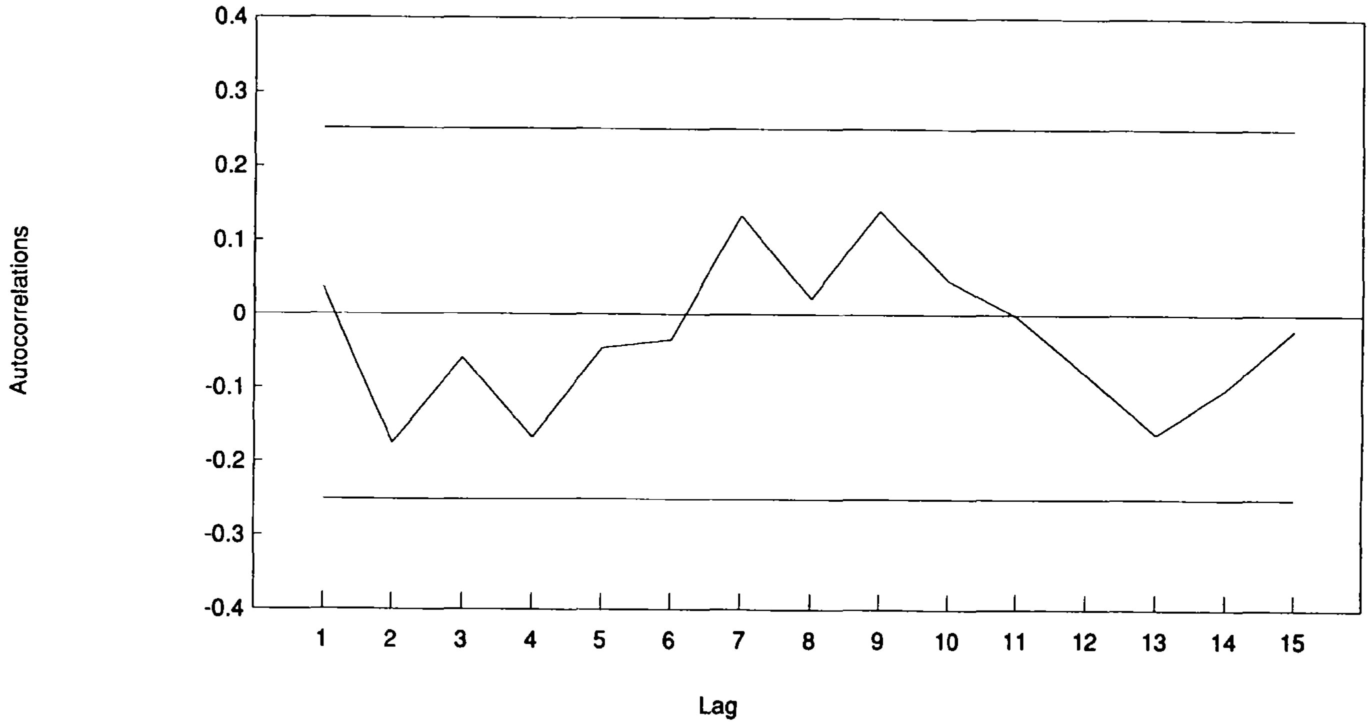
Autocorrelations
Inflation Residuals



Note: Based on inflation rates from 1926 - 1987. Rates are from Ibbotson and Sinquefield, Stocks, Bonds, Bills and Inflation, 1989.

Figure 6

Autocorrelations
Stock Residuals



Note: Based on inflation rates from 1926 - 1987. Rates are from Ibbotson and Sinquefield, Stocks, Bonds, Bills and Inflation, 1989.

an increasing series[10]. The number of turning points T has an asymptotic normal distribution with a mean of 2 (n-2)/3 and a variance of (16n-29)/90[11], where n is the number of residuals in the sample. The difference sign test is a test of the first differences of the residuals, $e_t - e_{t-1}$. The statistic S is the number of times the differenced series is positive. This statistic is normally distributed with a mean of .5(n-1) and a variance of (n+1)/12[12]. The rank test is used to test for linear trends in the residuals. The rank test is based on examining pairs of residuals, e_t and e_{t+1}. The statistic R is incremented by 1 if e_{t+1} is greater than e_t. R is asymptotically normal with a mean of .25 n (n-1) and variance .125 n (n-1) (2n + 5).

The turning point, difference sign and rank test were applied to the residuals of the inflation rate series and indicated that the residuals were not random. Thus, though the autoregressive model for inflation is used by both Wilkie and the Finnish working party, other models may provide a better fit to the inflation data. These tests were also applied to all the other models described in this paper. The tests gave no indication that residuals other than the inflation residual were not random.

Since distributional assumptions are made concerning the residual of the investment models it is useful to test the appropriateness of these assumptions. If a random variable follows the normal distribution, it has a coefficient skewness of zero and a kurtosis of three. The residuals of all models presented in this paper had nonzero skewness and a kurtosis greater than or less than three. Wilkie also noted that the series he investigated had nonzero skewness and kurtosis greater than that of the normal distribution[13]. Wilkie attributed the negative skewness and nonnormal kurtosis of his data to unexpected changes in prices in 1920 and 1921. Wilkie concluded that it would be awkward to use a distribution other than the normal distribution for the investment model residuals. He suggested the departure of the observed data from normality could be compensated for by selecting a standard deviation larger than the sample standard deviation. These nonnormal characteristics observed in the inflation rate series have been observed in many financial time series.

Using the gamma approximation, the skewness of the fitted model exactly matches the sample skewness. However, the gamma approximation has a smaller kurtosis than the sample series had. In addition, the gamma approximation cannot be used to model residuals with negative skewness and alternative should be used when the residual is negatively skewed.

Another test of the reasonableness of a fitted model is to compare the distribution of simulated investment returns to actual returns. Simulation testing of several inflation models was performed. The inflation models tested were the Wilkie model, the Finnish Working Party model and an autoregressive model with the residual simulated using the Cornish-Fisher approximation. The simulated results were compared to the actual inflation rates from 1926 - 1987 using the Kolomogorov-Smirnov test. This K-S statistic was not significant for any of the simulated series, indicating that the hypothesis that inflation rates residuals were from the assumed distribution could not be rejected. It should be noted that if inflation rates are simulated using the parameters selected by the

Finnish Working Party (a=4 and B=2), the K-S statistic is significant. It should also be noted that the gamma distribution cannot be used to approximate a residual with negative skewness. Since the residual of the stock market index series had negative skewness, it can't be modelled using the gamma distribution. Other residual series also had negative skewness. In the simulation testing of inflation rates all approximations had approximately the same K-S values, indicating that each performed equally well as an approximation to the inflation residual.

Based on the evidence presented and the work of Wilkie and Pentikainen et. al., it appears that the distributions assumed for the investment models are useful for simulation purposes despite differences between the theoretical models and the actual data. However, further research is needed to find models which provide a better approximation to the observed data.

Since tests did not indicate that any model is clearly superior, models similar to those developed by Wilkie are used in the simulations described later in this paper.

Simulation Studies of Asset Variability and Solvency

The purpose of this paper is to introduce techniques which can be used to simulate asset values and asset income as part of a comprehensive simulation model of variables influencing the solvency of insurance companies. The simulation can then be used to study the impact of asset variability on insurance company solvency. To illustrate the application of the techniques described, several simulations were performed using very simple scenarios. For these scenarios, the only stochastic variables were investment returns and asset value changes. Premiums, reserves and claim payments were assumed to be deterministic. Four categories of assets were simulated: short term investments, intermediate government bonds, long term government bonds and large company stocks. The simulation used a going concern basis, rather than a runoff basis to simulate the future surplus of a property casualty company. That is, the simulated company was assumed to continue writing future business. Taxes were not taken into consideration in the simulations. Outstanding losses and claim payments were assumed to vary with inflation. The simulated company wrote long tail business and had an initial surplus ratio to written premium of two to one. For each iteration of the simulation, surplus at the end of each year is calculated as the market value of assets minus the present value of outstanding losses. Present values were calculated using the simulated intermediate bond interest rate.

Exhibits 2 through 5 present the results of the simulation for four different asset distribution scenarios. These exhibits present the distribution of surplus for the simulated insurance company. The proportion of simulations for which surplus is negative represents the probability of ruin. For the first two scenarios where the proportion of assets invested in stocks is very low (0% and 10%) and the investment in long term bonds was moderate, insolvencies occurred at the rate of less than 1% per year. When the percentage of assets invested in stocks and long term bonds is increased, the percentage of insolvencies increases. It should be noted that the simulated scenarios made no attempt to consider

Exhibit 2

Cumulative Distribution of Surplus (000's)

	Time Period (in Years)						
Percentile	1	2	3	4	5	6	7
0.1%	20	(3,148)	(4,006)	(5,784)	(14,074)	(15,207)	(17,110)
0.5%	1,133	(186)	(2,197)	(2,532)	(5,426)	(9,207)	(11,441)
1%	1,892	318	(1,823)	(1,428)	(4,264)	(7,305)	(8,366)
5%	3,851	2,781	1,596	1,537	572	(522)	(1,498)
10%	4,699	3,908	3,305	3,632	3,009	2,728	2,759
20%	5,801	5,648	5,508	5,595	6,169	6,483	7,082
30%	6,708	6,625	7,040	7,288	8,361	8,964	10,645
40%	7,460	7,688	8,296	9,138	10,477	11,466	13,204
50%	8,016	8,649	9,577	10,768	12,360	13,690	16,070
60%	8,794	9,687	11,142	12,336	14,012	15,716	18,205
70%	9,625	10,930	12,413	14,245	15,877	18,232	21,271
80%	10,426	12,242	14,237	16,059	18,456	21,265	24,361
90%	11,687	13,883	16,248	19,152	22,168	25,473	29,472

Initial Surplus = 8,700

Asset Mix:	
Stocks	0%
Long Term Bonds	40%
Intermediate Bonds	45%
Short Term Investments	15%

Exhibit 3

Cumulative Distribution of Surplus (000's)

	Time Period (in Years)						
Percentile	1	2	3	4	5	6	7
0.1%	(2)	(1,352)	(6,119)	(5,833)	(16,238)	(17,909)	(18,870)
0.5%	1,140	(340)	(2,248)	(3,749)	(3,959)	(8,900)	(9,005)
1%	2,043	359	(496)	(1,550)	(2,499)	(6,119)	(7,891)
5%	3,815	2,728	2,124	2,365	1,563	981	1,104
10%	4,901	4,376	3,907	4,296	4,647	4,767	4,771
20%	6,035	6,024	6,427	7,139	8,078	8,848	10,542
30%	6,965	7,338	8,044	9,116	10,450	11,801	14,038
40%	7,749	8,496	9,593	10,843	12,710	14,507	17,395
50%	8,601	9,666	10,895	12,299	14,648	17,159	19,983
60%	9,339	10,795	12,189	14,591	16,730	20,022	23,281
70%	9,962	11,931	13,914	16,335	19,204	23,084	26,550
80%	10,883	13,284	15,907	18,561	21,961	26,800	31,293
90%	12,236	15,053	18,465	22,409	27,184	32,072	37,671

Initial Surplus = 8,700

Asset Mix:	
Stocks	10%
Long Term Bonds	40%
Intermediate Bonds	35%
Short Term Investments	15%

Exhibit 4

Cumulative Distribution of Surplus (000's)

Percentile	Time Period (in Years) 1	2	3	4	5	6	7
0.1%	(2,692)	(6,391)	(11,975)	(14,840)	(20,775)	(27,652)	(30,232)
0.5%	(727)	(5,119)	(8,248)	(10,585)	(14,692)	(20,734)	(25,281)
1%	158	(3,592)	(6,791)	(7,750)	(11,419)	(18,170)	(22,955)
5%	2,350	6	(2,516)	(3,847)	(6,387)	(9,638)	(12,196)
10%	3,391	1,352	(241)	(1,265)	(2,933)	(4,801)	(5,714)
20%	4,999	3,678	2,727	2,201	1,638	788	985
30%	6,056	5,445	5,123	5,221	4,591	4,790	5,305
40%	6,940	6,833	7,095	7,409	7,964	8,687	10,431
50%	7,762	8,373	8,981	9,681	11,174	12,670	14,797
60%	8,693	9,925	11,010	12,708	14,541	16,380	18,690
70%	9,808	11,502	13,515	15,354	18,300	20,402	23,760
80%	11,003	13,172	15,719	18,469	21,765	25,387	30,580
90%	12,993	15,673	19,384	23,177	27,459	33,063	38,574

Initial Surplus = 8,700

Asset Mix:	
Stocks	0%
Long Term Bonds	75%
Intermediate Bonds	10%
Short Term Investments	15%

Exhibit 5

Cumulative Distribution of Surplus (000's)

	Time Period (in Years)						
Percentile	1	2	3	4	5	6	7
0.1%	(6,166)	(13,123)	(15,562)	(22,555)	(23,364)	(29,109)	(32,492)
0.5%	(4,565)	(7,343)	(11,762)	(14,125)	(18,273)	(23,118)	(23,908)
1%	(3,728)	(6,730)	(10,616)	(11,939)	(14,703)	(20,029)	(21,146)
5%	12	(2,482)	(4,316)	(5,690)	(6,156)	(7,890)	(9,363)
10%	1,678	480	(467)	(1,437)	(1,198)	(1,737)	(1,562)
20%	4,035	3,748	3,874	4,853	5,494	6,846	9,706
30%	6,309	6,854	7,561	9,096	11,474	14,803	17,979
40%	8,238	9,297	10,967	13,610	16,790	21,548	25,414
50%	9,757	12,039	14,476	18,208	23,232	27,914	34,169
60%	11,652	14,633	18,009	22,927	29,656	34,590	43,221
70%	13,486	17,602	21,809	28,796	35,673	44,856	51,998
80%	15,654	21,963	28,142	34,385	44,039	56,122	65,373
90%	19,032	27,452	37,760	47,040	58,714	75,267	89,424

Initial Surplus = 8,700

Asset Mix:	
Stocks	50%
Long Term Bonds	20%
Intermediate Bonds	15%
Short Term Investments	15%

asset liability management. If a company attempts to match assets and liabilities, it should be less subject to risk resulting from the variability of the market value of assets.

More comprehensive studies of the impact of asset variability on insurance company solvency were performed by the Finnish Working Party and the British Working Party. In the research performed by the Finnish Working Party[14], asset variability was studied by allowing investment variables to be vary while all other variables were treated as deterministic. Pentakainen et. al. concluded that the impact of investment variability on solvency was approximately equal to that of claim variability. To study the effect of asset mix on insolvency, all variables were simulated stochastically, and the mix of assets was changed while the parameters for the other variables remained constant. Pentakainen et. al. observed that insolvencies increased from .5% per year to 1% per year when the percentage of assets invested in equities was raised from 30% to 50%.

The British Working Party study evaluated insurer financial strength on an "emerging cost" basis. That is, the cash flow of a company is modelled and a company is considered insolvent only if the income from premium and investments is less than the amount needed to pay losses and expenses. Thus, asset variability will not affect the company's claims paying ability as long as assets are not sold at a loss before maturity in order to meet loss and expense payments.

In the British Working Party study[15], the number of insolvencies due to different asset mixes varied between 7 per 1000 simulations and 45 per 1000 simulations when the analysis was performed on a runoff basis. Insolvencies ranged between 30 per 1000 simulations and 76 per 1000 simulations when future business was incorporated into the analysis. A low percentage of insolvencies was associated with a low percentage investment in stocks. The high percentage of insolvencies occurred when 100% of assets were invested in stocks.

The simulation results seem to indicate that companies which invest conservatively, i.e. in high quality bonds and stocks, with a relatively low percentage of company assets invested in equities and a moderate percentage invested in long term bonds, face a small but significant probability of insolvency due to asset variability. Increasing investments in risky assets increases the probability of insolvency.

Contemporary insurance company managements invest in many assets besides the conventional kinds of assets modelled in this and other studies. The risks associated with these assets merit evaluation in future studies of the impact of asset risk on insurance company solvency. The highly publicized insolvency of Executive Life and the downgrading of Prudential Insurance Company's Moody's ratings illustrate that asset risk is a serious risk to insurance companies. In these two cases investments in junk bonds and real estate exposed the companies to significant decline in asset values. While the published research has not examined the impact of junk bond and real estate variability on insurance company financial strength, techniques such as those described in this paper could be used for such an investigation.

Risk Based Capital

The National Association of Insurance Commissioners is drafting regulations which will replace current capital requirements for property and casualty insurance companies which are based on ratios of premium to surplus with capital requirements based on the underwriting and investment risks of each company. A significant percentage of the new capital requirements will be allocated to investment risk.

For purposes of determining the new capital requirements, the assets are classified into different categories depending on the kind of investment and the riskiness of each investment. The new capital requirement will be a percentage of the value of assets in each category. Capital will be required for all asset categories, including cash. For low risk assets such as high quality bonds, the proposed capital requirement is small (.3%). For high risk assets such as common stocks, the proposed capital requirement is large(30%).

While the proposed requirements appear to be an improvement over capital requirements based on ad hoc rules, the capital requirements for each asset category were arrived at judgementally. Simulation studies such as those describes in this paper could be used to evaluate the proposed regulations and determine appropriate capital requirements for each asset. This could be accomplished by finding the amount of surplus for each asset required to reduce the probability of ruin to a very low level. In addition, the proposed regulations ignore asset liability management in the determination of capital requirements.

A surplus requirement for the risk of asset liability mismatch seems appropriate and could be determined through simulation research.

CONCLUSION

Procedures have been presented in this paper which can be used to model changes in the values of assets and in the yield of investments. Wilkie developed a group of interrelated models for approximating bond and stock market performance. The Finnish Working Party developed a more general model which relates the returns of a given asset category to inflation and to a random component independent of inflation. Using procedures similar to those of the British Working Party and the Finnish Working Party, asset models can be incorporated into a more comprehensive model of insurance company income flows, to quantify probabilities of insolvency and risk margins needed to prevent insolvency.

The models presented in this paper appear to violate some of the technical assumptions of the underlying assumed theoretical distributions. However, these models are intended to provide a reasonable approximation of asset variability in a simulation and are not intended for use in forecasting. The British Working Party and the Finnish Working Party have found models similar to those presented in this paper to be useful for simulation.

The simulation models discussed in this paper indicate that asset risk is a significant risk to insurance companies and that the mix of assets affects the company's probability of ruin. Thus, a company which invests in risky assets requires a larger amount of surplus to avoid insolvency. The risk based capital regulations proposed by the NAIC would relate the capital requirements of property and casualty insurance companies to the riskiness of assets purchased by the company. Simulation models such as those described in this paper could be used to quantify surplus requirements for different classes of assets.

Future research in the evaluation of the impact of investment risk on insurance company solvency should incorporate an analysis of a broader spectrum of assets, including junk bonds and real estate. In addition, a more complete model, in which premiums, losses and expenses are modelled stochastically, along with investment variables, would be a useful tool in the evaluation of investment risk.

APPENDIX I - Time Series Models

The techniques which are used to derive parameters for the models described in this paper are presented in this appendix.

Univariate ARMA Models

ARMA (Auto Regressive Moving Average) models have been applied perhaps more frequently than any other procedure to the modelling of investment returns.

The ARMA model is denoted:

$$(X_t - \mu) - \phi_1 (X_{t-1} - \mu) - \dots \phi_n (X_{t-n} - \mu) = \theta_1 \epsilon_{t-1} + \dots \theta_n \epsilon_{t-1} + \epsilon_t$$

Where X_t represents an investment return at time t. The ARMA models are linear models which relate the investment return at time t to the investment return for prior time periods plus random shock terms for prior time periods. The random shock terms are assumed to be independent and identically distributed. The investment return series is also assumed to be stationary. A stationary series is series for which the first two moments are constant over all time periods. That is, neither the mean nor the variance of the series changes over time.

Series for which the mean value shifts over time can frequently be transformed into stationary series by differencing. ARMA modelling procedures can then be applied to the differenced series. For the remainder of this discussion, it will be assumed that all data has been suitably differenced.

One of the most tractable of the ARMA models is the autoregressive model. For this model the returns at time t are dependant upon the returns at prior time periods. The AR(n) model is denoted as follows:

$$X_t - \mu = \phi_1 (X_{t-1} - \mu) + \phi_2 (X_{t-2} - \mu) / + \dots \phi_n (X_{t-n} - \mu) + e_t$$

The parameter μ is the mean of the series and is usually estimated by the mean of the sample.

$$\hat{\mu} = \frac{\sum_{i=1}^{n} X_i}{n}$$

The simplest autoregressive model is an AR(1) model or an autoregressive model of order 1. The value of the variable at time t depends only on its value for the prior period. This model is denoted:

$$X_t - \mu = \phi(X_{t-1} - \mu) + e_t$$

The parameter ϕ for the AR(1) model is most readily estimated using the autocorrelation coefficient. That is[16]:

$$\hat{\rho} = \hat{\phi} = \frac{\sum_{t=1}^{N-1} (X_t - \hat{\mu})(X_{t-1} - \hat{\mu})}{\sum_{t=1}^{N-1} (X_t - \hat{\mu})^2}$$

where ρ, the autocorrelation coefficient measures the linear correlation between X_t and X_{t-1}. This estimate of ϕ is known as the conditional least squares estimate. It can also be obtained by regressing $(X_t - \hat{\mu})$ on $(X_{t-1} - \hat{\mu})$.

A slightly different estimate for ϕ is obtained using the unconditional least squares estimate[17]:

$$\hat{\phi} = \frac{\sum_{t=1}^{N-1} (X_t - \mu)(X_{t-1} - \mu)}{\sum_{t=2}^{N-1} (X_t - \mu)^2}$$

An illustration of both estimation procedures is displayed on Exhibit A-1. The parameter ϕ is estimated both for inflation rates and the log of (1 + inflation rate).

The parameters of higher order autoregressive models can be estimated using linear regression. This is accomplished by regressing $(X_t - \hat{\mu})$ on $(X_{t-1} - \hat{\mu})$, $(X_{t-2} - \hat{\mu})$ etc. The number of lagged values of X to include in the model can be determined by evaluating the standard error of the regression (or alternatively the R^2). When the decrease in the error (or increase in R^2) which occurs after adding an additional term is not significant, no further additional autoregressive parameters are added.

Exhibit A-1

Estimation of Autoregressive Parameter

Conditional Least Squares		Unconditional Least Squares	
Phi (i) =	0.7347	Phi (i) =	0.7626
Phi (ln(1+i)) =	0.7373	Phi (ln(1+i)) =	0.7662

Year	Inflation Rate	(i(t)-mu) ^2	(i(t)-mu)* (i(t+1)-mu)	Natural Log (1 + infl rate)	(i(t)-mu) ^2	(i(t)-mu)* (i(t+1)-mu)
1960	0.0148	0.0013	0.0015	0.0147	0.0011	0.0014
1961	0.0067	0.0019	0.0017	0.0067	0.0018	0.0015
1962	0.0122	0.0014	0.0013	0.0121	0.0013	0.0012
1963	0.0165	0.0011	0.0013	0.0164	0.0010	0.0012
1964	0.0119	0.0015	0.0012	0.0118	0.0013	0.0011
1965	0.0192	0.0010	0.0005	0.0190	0.0009	0.0005
1966	0.0335	0.0003	0.0003	0.0330	0.0002	0.0003
1967	0.0304	0.0004	0.0001	0.0299	0.0003	0.0000
1968	0.0472	0.0000	-0.0000	0.0461	0.0000	-0.0000
1969	0.0611	0.0001	0.0001	0.0593	0.0001	0.0001
1970	0.0549	0.0000	-0.0001	0.0534	0.0000	-0.0001
1971	0.0336	0.0003	0.0003	0.0330	0.0002	0.0002
1972	0.0341	0.0003	-0.0006	0.0335	0.0002	-0.0005
1973	0.0880	0.0014	0.0027	0.0843	0.0013	0.0024
1974	0.1220	0.0051	0.0014	0.1151	0.0044	0.0013
1975	0.0701	0.0004	-0.0000	0.0678	0.0004	-0.0000
1976	0.0481	0.0000	-0.0000	0.0470	0.0000	-0.0000
1977	0.0677	0.0003	0.0007	0.0655	0.0003	0.0006
1978	0.0903	0.0016	0.0033	0.0865	0.0014	0.0029
1979	0.1331	0.0069	0.0061	0.1250	0.0058	0.0052
1980	0.1240	0.0054	0.0029	0.1169	0.0047	0.0025
1981	0.0894	0.0015	-0.0005	0.0856	0.0014	-0.0004
1982	0.0387	0.0001	0.0001	0.0380	0.0001	0.0001
1983	0.0380	0.0002	0.0001	0.0373	0.0001	0.0001
1984	0.0395	0.0001	0.0001	0.0387	0.0001	0.0001
1985	0.0377	0.0002	0.0005	0.0370	0.0001	0.0004
1986	0.0113	0.0015	0.0002	0.0112	0.0014	0.0002
1987	0.0441	0.0000	0.0001	0.0432	0.0000	0.0000
1988	0.0414	0.0001	0.0000	0.0406	0.0001	0.0000
1989	0.0482			0.0471		
Sum	1.5077	0.0344	0.0253	1.4557	0.0303	0.0224
Sum*		0.0332			0.0292	

Note:

Inflation rates from Federal Reserve Bulletin

*Summation is for uncomditional Least Squares Estimate

An ARMA processes with no autoregressive terms is a MA (Moving Average) process. This process is denoted.

$$X_t - \mu = e_t - \theta_1 e_{t-1} \ \dots \ -\theta_n e_{t-n}$$

Investment returns are from a moving average process if returns at time t are related to unexpected changes in returns during prior periods. The estimation of the parameters of a moving average process requires the use of iterative techniques. Appendix III describes a straightforward procedure for estimating the parameters of an MA(1) model. The procedure illustrated uses a grid search technique to find the value of the parameter θ which minimizes the deviation between the sample and fitted values.

Iterative techniques must also be used to estimate the parameters of ARMA models which include both autoregressive and moving average terms. In Appendix III the use of a grid search procedure to estimate the parameters of an ARMA(1,1) model is illustrated.

Although a grid search procedure can be used to estimate the values of ARMA parameters, such an approach becomes cumbersome and time consuming as the number of parameters increases. When performing time series analyses it is customary to use specialized statistical software to identify appropriate models and compute the estimates of the ARMA parameters. The software uses numerical techniques which derive parameter estimates much more quickly than grid search techniques do. Although investment models more complicated than the ARMA(1,1) model are rarely used, specialized time series software incorporates procedures for model testing and identification. Since this software is widely available and inexpensive, it is assumed that individuals wishing to use ARMA models will purchase the appropriate software.

Another procedure which has been found useful in modelling time series data is exponential smoothing. This model is denoted:

$$X_t = \alpha X_{t-1} + \alpha(1-\alpha) X_{t-2} +.. \alpha(1-\alpha)^{n-1} X_{t-n}$$ [18]

To be modelled by exponential smoothing, the forecast, or expected value of a variable at time t is an average of all prior observations of the variable, with the weight assigned to prior observations decaying exponentially as the distance (in time) between the forecast value and the observation increases. Thus more recent observations are given more weight. The exponential smoothing model can also be expressed as a time series model given by:

$$X_t - \mu = \alpha \sum_{i<t} e_i + e_t$$ [19]

This model is an infinite moving average model which is nonstationary. If the series is differenced, the resulting stationary series follows MA(1) process:

$$Z_t = (\alpha-1)\, e_{t-1} + e_t$$

where $Z_t = X_t - X_{t-1}$

In this paper, models are presented in which exponential smoothing is applied to the inflation rate. Many financial investigations use the expected inflation rate, rather than the actual inflation rate, as an explanatory variable in a model. The expected inflation rate might reasonably be assumed to be an exponentially smoothed average of past inflation rates.

The exponential smoothing parameter is derived using a procedure similar to the derivation of the MA(1) parameter. The procedure is illustrated in Appendix IV.

Multivariate ARMA Models

The multivariate ARMA model is an extension of the univariate ARMA model to an array of time series variables. This model can be denoted

$$X_t - \mu = \Phi_1(X_{t-1} - \mu) \ldots + \Phi_n (X_{t-n} - \mu) + V_t - \Theta_1 V_{t-1} - \Theta_n V_{t-n}$$

Where **X** is a vector of variables observed over a number of time periods, Φ and Θ are arrays of parameters and **V** is a matrix of variances and covariances.

Because of the complexity involved in the estimation of multivariate ARMA parameters, only special cases will be considered. The first special case to be considered will be that of the bivariate ARMA model which is denoted.

$$Y_t - \mu_y - \phi_1 (Y_{t-1} - \mu_y) - \ldots \phi_n (Y_{t-n} - \mu_y) =$$

$$v_1 (X_{t-p} - \mu_x) + \ldots v_n (X_{t-p-n} - \mu_x)$$

$$+ \theta_1 e_{t-1} + \ldots \theta_n e_{t-n} + e_t$$

The random variable Y (known as the output series) is related to prior values of itself, prior values of another independent variable, X (also known as the input series) and prior shock terms. The parameter p determines the lag between Y_t and the impact of X.

Two statistical tools have been found to be useful in the preliminary estimation of parameter values of a bivariate ARMA model. These are the cross correlation function and the transfer function.

The cross covariance between two variables Y and X at lag t is given by[20]:

$\gamma_{yx}\ (k) = E(Y_t - \mu_y)\ (X_{t+k} - \mu)$ and

$\gamma_{xy}\ (k) = E(X_t - \mu_x)\ (Y_{t+k} - \mu_y) = E(Y_{t+k} - \mu)\ (X_t - \mu) = \gamma_{yx}(-k)$

The cross covariance function is not symmetric, since γ_{yx} (k) is not equal to $\gamma_{xy}(k)$ but to $\gamma_{xy}(-k)$. The cross correlation function between Y and X at lag k is defined as:

$$\rho_{YX}(k) = \frac{\gamma_{YX}(k)}{\sigma_Y\ \sigma_X}$$

An estimate of ρ_{yx} can be derived using the sample cross correlation for lag k, C_{yx} where[21]:

$$C_{YX} = \frac{1}{n-k}\frac{\sum_{t=1}^{n-k}(Y_t - \mu_Y)(X_{t+k} - \mu_X)}{\sqrt{\sum(Y_t - \mu_y)^2(X_t - \mu_x)^2}}$$

Values of σ_x and σ_y can be estimated using the sample standard deviations.

Although the cross correlation function may give an indication of the relationship between Y and X, it is not typically used directly to estimate the parameters of coefficients relating

X_t, X_{t-1}, etc. to Y_t. Because of autocorrelations between Y_t and its prior values and between X_t and prior values of X_t the properties of the cross correlation function are complex and difficult to determine. Therefore a process called "prewhitening" is performed on both series to remove the effect of the correlation between the variables and their prior values. This procedure was applied to all series used to develop the models in this paper. The "prewhitening" process is performed as follows[22]:

1) Fit a univariate ARMA model to the input series X.

2) Using the estimated ARMA parameters, compute the residuals of the input series. Denote the residual series as r.

3) Using the same ARMA parameters as used in step 2) compute the residuals for the output series Y. Denote this residual series as z.

Then the coefficients of the function

$$z_t = v_0 r_t + v_1 r_{t-1} / +\dots v_n r_{t-1}$$

are calculated. The coefficients relating the series r to z_t are known as the transfer function.

The transfer function coefficients are related to the cross correlation coefficient between z and r. This relationship is expressed as[23]:

$$v_k = \frac{\gamma_{yx}(k)}{\sigma_y^2} = \frac{\sigma_x}{\sigma_y}\,\rho_{yx}(k)$$

The sample standard deviations and cross correlations can be used as estimates of σ_x, σ_y and ρ_{yx} (k) to derive the v_k.

To illustrate the transfer function approach, the relationship between inflation and the change in market value (capital appreciation) on long term government bonds is modelled. First, both series are prewhitened. To prewhiten these series, an ARMA model is fit to annual inflation rates from 1926 through 1987. The model fit is the AR(1) model.

$$X_t - \hat{\mu}_x = .66\ (X_{t-1} - \hat{\mu}_x) + e_t$$

The residual, or, prewhitened inflation series is computed using:

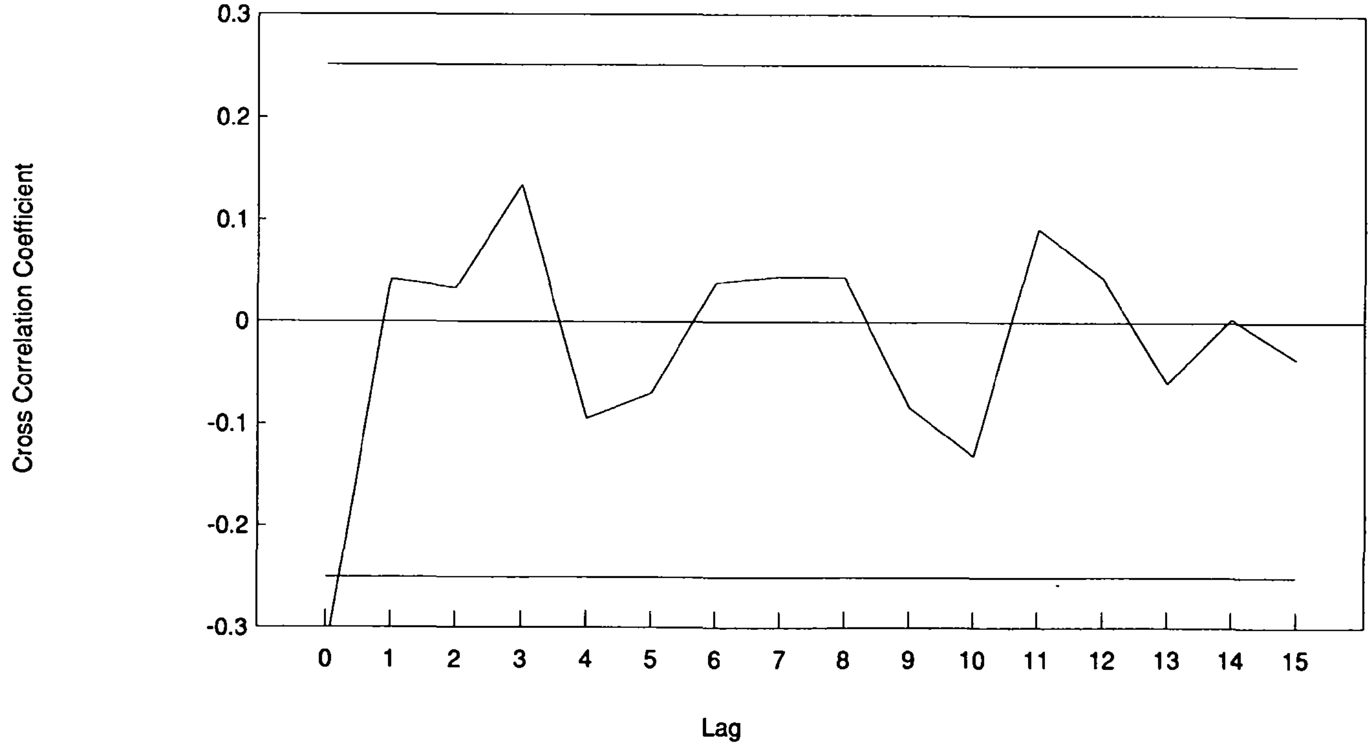
$$e_t = (X_t - \hat{\mu}_x) - .66\ (X_{t-1} - \hat{\mu}_x)$$

Residuals for the long term bond series are also calculated using the same formula for e_t as is used for the inflation series. The cross correlations between these two residual series is then calculated.

Figure 7 presents the cross correlation function or the cross correlation coefficients, C_{yx} (k) for the first 15 lags for the prewhitened long term bond and inflation series. The comparison of $C_{yx}(k)$ to $\frac{1.96}{\sqrt{n}}$ gives an indication of whether the correlation at a given lag

Figure 7

Long Term Bond Capital Appreciation



Note: Based on bond and inflation rates from 1926 - 1987. Rates are from Ibbotson and Sinquefield, Stocks, Bonds, Bills and Inflation, 1989.

Transfer Function
Long Term Bond Capital Appreciation

Lag	Transfer Function Coefficients
0	-0.7947
1	0.1025
2	0.0772
3	0.3369
4	-0.2277
5	-0.1616
6	0.0953
7	0.1054
8	0.1284
9	-0.1963
10	-0.3143
11	0.2149
12	0.1281
13	-0.1453
14	0.0165
15	-0.0997

is significant. Note that only the lag 0 coefficient appears to be significant. The result is not surprising, since an increase in inflation will cause the market value of bonds to drop.

The transfer function coefficients relating bond capital appreciation to inflation are shown on Exhibit A-2. Based on the observed coefficients it is hypothesized that the transfer function is $v_0 = -.8$. or

$$Y_t - \mu_y = -.8\ (X_t - \mu_x) + N_t$$

N_t is a residual series independent of inflation which can be modeled as an ARMA process. Using the model above, a residual series is computed for the long term bond returns, by adding .8 of $X_t - \hat{\mu}_x$ to each Y_t - *mle*$_y$. The following white noise model was fit to the residual series:

$$N_t = .07z_t$$

where z is a standard normal variable and the quantity $.07z_t$ is the random shock which for each period.

Multivariate Autoregression

An alternative to the transfer function procedure for modeling multivariate time series processes is multiple regression. For many time series processes, the usual assumptions for multiple regression are violated. In particular, the assumption that the error at time t is uncorrelated with the error of any prior time period is frequently violated. The Durbin Watson statistic is commonly employed to test for the violation of this assumption.

If the errors follow a first order autoregressive process, there are established procedures for estimating the regression parameters. One of these procedures is the Cochrane-Orcutt procedure[24].

Suppose: $Y_t = a + b_1 X1_t + b_2 X2_t + e_t$

$$e_t = \rho e_{t-1} + n_t$$

Where n is an independent identically distributed noise term. The Cochrane-Orcutt procedure applies a transformation to the variables which eliminates the serial correlation. The procedure is performed as follows:

(1) Use multiple regression to derive a preliminary estimate of a, b_1 and b_2

(2) Compute $e_t = Y_t - \hat{Y}_t = Y_t - (a+b_1X1_t+b_2X2_t)$

(3) Use the sample autocorrelation coefficient of e_t at lag 1 as a preliminary estimate of ρ

$$\hat{\rho} = r = \sum_{t=1}^{n-1} e_t\, e_{t+1}$$

(4) Compute
$$Y_t' = Y_t - rY_{t-1}$$
$$X1_t' = X1_t - rX1_{t-1}$$
$$X2_t' = X2_t - rX2_{t-1}$$

(5) Using multiple regression, regress Y' on X1' and X2'. The new estimated parameters are a', b_1' and b_2'.

(6) Recompute e_t using $a" = a'/(1-r)$, b_1' and b_2' instead of a, b_1 and b_2. That is

$$e^{new} = Y_t - (a" + b_1'X1_t + b_2'X2_t)$$

(7) Perform steps (2) - (6) until the parameter estimates converge.

To illustrate an application of the Cochrane-Orcutt procedure, stock dividend income is regressed on expected inflation. Figure 8 shows the cross correlation function between the log of stock dividend income and inflation. Beginning at lag 1 the cross correlations are relatively large and seem to persist, suggesting that dividend income may be related to a weighted average of past inflation rates. If expected inflation is modelled as an exponentially smoothed average of past inflation, then stock income may be related to expected inflation. Because of the random variability of the cross-correlation coefficients, it is difficult to determine appropriate smoothing parameters using the transfer function approach. An alternative approach is to perform a regression in which expected inflation is an independent variable.

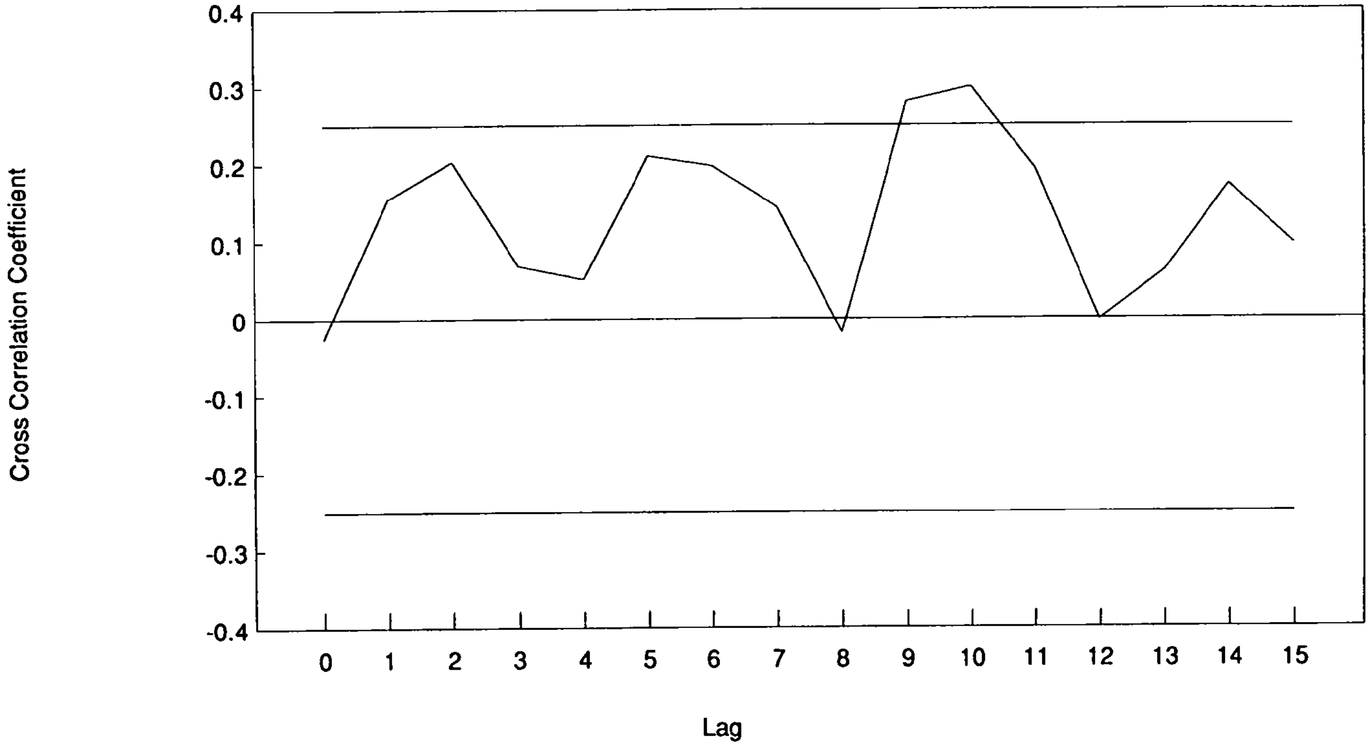
The following regression was fit to stock and inflation data.

$$\ln (SI_t) = a + b_1 E(i_{t-1})$$

where SI is stock dividend income at time t, and $E(i_t)$ is the expected inflation rate at time t. Several values of expected inflation are computed using different values for the smoothing parameter. It was found that ln(SI) was most strongly correlated with a lagged value of expected inflation computed using a smoothing parameter of .05. The Durbin Watson statistic of .175 gave evidence of significant positive autocorrelation, therefore the Cochrane-Orcutt procedure was used to estimate the parameters of the model.

Figure 8

ln(Stock Dividend Income)



Note: Based on dividend and inflation rates from 1926 - 1987. Rates are from Ibbotson and Sinquefield, Stocks, Bonds, Bills and Inflation, 1989.

Exhibit A-3 compares the initial fitted parameters and the parameters fit after the Cochrane-Orcutt procedure was applied.

Regression Results
Stock Dividend INcome

Original Regression		
Variable	Coefficient	T Statistic
Constant	-0.187	-0.263
E(inflation(t-1))	25.133	8.388
R^2 =	0.536	
Significance	0.001	

Cochrane-Orcutt Results		
Variable	Coefficient	T Statistic
Constant	-3.630	-9.804
E(inflation(t-1))	22.869	2.378
R^2 =	0.088	
Significance	0.021	

APPENDIX III: Estimation of Parameters of ARMA Models

The procedure for of estimating parameters for a moving average process is illustrated for the MA(1) model:

$$X_t - \mu = e_t - \theta\, e_{t-1}$$

Exhibit A-4 presents the derivation of Moving Average parameters for the inflation rate series[25]. First, the sample mean is used as the estimate of μ. An initial estimate of θ between 0 an 1 is selected. Then, assuming e_0, the shock term for the first inflation rate in the series to be 0, an estimate of, e_1, the shock term for the second term in the series is derived using:

$$e_1 = X_1 - \mu + \theta e_0 = X_1 - \mu$$

Subsequent residuals are estimated using the recursive equation:

$$e_t = X_t - \mu + \theta\, e_{t-1}$$ [26]

The square of each residual is calculated and then the sum of the squared residuals is calculated. A grid search technique is used to find the value of θ which minimizes the sum of the squared errors (SSE). The result of such a grid search is presented on Exhibit A-4. The Lotus data table command was used to compute the SSE for various values of θ. The minimum SSE occurs when a value of -.735 is used for θ.

Derivation of Moving Average Parameters

Year	Inflation Rate	Residual e	Squared Residual e^2
1960	0.0147	-0.0338	0.0011
1961	0.0067	-0.0170	0.0003
1962	0.0121	-0.0239	0.0006
1963	0.0164	-0.0146	0.0002
1964	0.0118	-0.0260	0.0007
1965	0.0190	-0.0104	0.0001
1966	0.0330	-0.0079	0.0001
1967	0.0299	-0.0128	0.0002
1968	0.0461	0.0070	0.0000
1969	0.0593	0.0057	0.0000
1970	0.0534	0.0008	0.0000
1971	0.0330	-0.0160	0.0003
1972	0.0335	-0.0032	0.0000
1973	0.0843	0.0382	0.0015
1974	0.1151	0.0385	0.0015
1975	0.0678	-0.0091	0.0001
1976	0.0470	0.0051	0.0000
1977	0.0655	0.0132	0.0002
1978	0.0865	0.0282	0.0008
1979	0.1250	0.0557	0.0031
1980	0.1169	0.0274	0.0008
1981	0.0856	0.0169	0.0003
1982	0.0380	-0.0230	0.0005
1983	0.0373	0.0057	0.0000
1984	0.0387	-0.0140	0.0002
1985	0.0370	-0.0013	0.0000
1986	0.0112	-0.0364	0.0013
1987	0.0432	0.0214	0.0005
1988	0.0406	-0.0237	0.0006
1989	0.0471	0.0159	0.0003
Mean	0.0485	0.0002	
S.D.	0.0318	0.0224	

Data Table

Selected Theta:	-0.735

Theta	SSE
-0.740	0.0150917
-0.739	0.0150907
-0.738	0.0150897
-0.737	0.0150889
-0.736	0.0150882
-0.735	***0.0150875***
-0.734	0.0150870
-0.733	0.0150865
-0.732	0.0150862
-0.731	0.0150859
-0.730	0.0150857

Minimum SSE 0.0150857

Note:
Inflation rates from Federal Reserve Bulletin

In Exhibits A-5 and A-6 the estimation of the parameters of an ARMA (1,1) model is illustrated using the inflation rate data. The following recursive equation was used to compute the values of e_t.

$$e_t = (X_t - u) - \phi (X_{t-1} - u) + \theta e_{t-1}$$

A grid search is used to find the value of ϕ and θ which minimize the SSE.

Derivation of ARMA (1,1) Parameters

Year	Inflation Rate	Residual e	Residual Squared e^2
1960	0.0147		
1961	0.0067	-0.0215	0.0005
1962	0.0121	-0.0018	0.0000
1963	0.0164	-0.0095	0.0001
1964	0.0118	-0.0132	0.0002
1965	0.0190	-0.0017	0.0000
1966	0.0330	0.0029	0.0000
1967	0.0299	-0.0105	0.0001
1968	0.0461	0.0134	0.0002
1969	0.0593	0.0063	0.0000
1970	0.0534	-0.0043	0.0000
1971	0.0330	-0.0165	0.0003
1972	0.0335	0.0016	0.0000
1973	0.0843	0.0441	0.0019
1974	0.1151	0.0257	0.0007
1975	0.0678	-0.0320	0.0010
1976	0.0470	0.0010	0.0000
1977	0.0655	0.0175	0.0003
1978	0.0865	0.0201	0.0004
1979	0.1250	0.0449	0.0020
1980	0.1169	0.0028	0.0000
1981	0.0856	-0.0051	0.0000
1982	0.0380	-0.0306	0.0009
1983	0.0373	0.0085	0.0001
1984	0.0387	-0.0068	0.0000
1985	0.0370	-0.0027	0.0000
1986	0.0112	-0.0292	0.0009
1987	0.0432	0.0299	0.0009
1988	0.0406	-0.0179	0.0003
1989	0.0471	0.0112	0.0001
Mean	0.0485	0.0009	
S.D.	0.0318	0.0195	

Derivation of ARMA (1,1) Parameters

Data Table

Selected Phi:	0.6
Selected Theta:	-0.44

Phi	Theta						Minimum
	-0.48	-0.46	-0.44	-0.42	-0.4	-0.38	SSE
0.500	0.011162	0.011159	0.011192275	0.0112561	0.0113481	0.0114660	0.0111598
0.520	0.011118	0.011104	0.011125906	0.0111786	0.0112596	0.0113662	0.0111045
0.540	0.011088	0.011063	0.011073755	0.0111158	0.0111861	0.0112820	0.0110631
0.560	0.011071	0.011035	0.011035823	0.0110677	0.0111278	0.0112133	0.0110355
0.580	0.011067	0.011021	0.011012109	0.0110342	0.0110845	0.0111602	0.0110121
0.600	0.011077	0.011021	0.011002613	0.0110153	0.0110563	0.0111227	0.0110026
0.620	0.011101	0.011035	0.011007336	0.0110110	0.0110432	0.0111007	0.0110073
0.640	0.011138	0.011063	0.011026276	0.0110214	0.0110452	0.0110943	0.0110214
0.660	0.011188	0.011105	0.011059436	0.0110464	0.0110622	0.0111035	0.0110464
0.680	0.011252	0.011160	0.011106813	0.0110861	0.0110944	0.0111282	0.0110861

Minimum SSE

0.011067	0.011021	0.011002613	0.0110110	0.0110432	0.0110943	0.0110026

APPENDIX IV: DERIVATION OF EXPONENTIAL SMOOTHING PARAMETER

The exponential smoothing parameter is derived using a procedure similar to the derivation of the MA(1) parameter. First, an initial estimate for the forecast value for the first observation is needed. In the example shown on Exhibit A-7, the average of the first five observations is use for X_0. Next, recursively compute the forecast value for all subsequent observations using the following formula.

$$e_{t-1} = X_{t-1} - \hat{X}_{t-1}$$

$$\hat{X}_t = X_{t-1} + \alpha e_{t-1}$$

Compute e_t^2 for each observation and sum the result. Then use a grid search procedure to find the value of X which minimizes the sum of the squared errors. As seen on Exhibit A-4, this procedure produces an estimate of α of .92 for the inflation series.

Exponential Smoothing of Inflation Rate

	Inflation Rate	Forecast Value	Error	Error Squared
	0.006677	0.0132	-0.0065	0.00004
	0.012126	0.0072	0.0049	0.00002
	0.016365	0.0117	0.0046	0.00002
	0.011829	0.0160	-0.0042	0.00002
	0.019018	0.0122	0.0069	0.00005
	0.032951	0.0185	0.0145	0.00021
	0.029947	0.0318	-0.0018	0.00000
	0.046119	0.0301	0.0160	0.00026
	0.059306	0.0448	0.0145	0.00021
	0.053445	0.0581	-0.0047	0.00002
	0.033047	0.0538	-0.0208	0.00043
	0.033531	0.0347	-0.0012	0.00000
	0.084341	0.0336	0.0507	0.00257
	0.115112	0.0803	0.0348	0.00121
	0.067752	0.1123	-0.0446	0.00199
	0.046979	0.0713	-0.0243	0.00059
	0.065506	0.0489	0.0166	0.00027
	0.086452	0.0642	0.0223	0.00050
	0.124957	0.0847	0.0403	0.00162
	0.116893	0.1217	-0.0048	0.00002
	0.085627	0.1173	-0.0317	0.00100
	0.037969	0.0882	-0.0502	0.00252
	0.037295	0.0420	-0.0047	0.00002
	0.038739	0.0377	0.0011	0.00000
	0.037006	0.0387	-0.0016	0.00000
	0.011236	0.0371	-0.0259	0.00067
	0.043155	0.0133	0.0298	0.00089
	0.042336	0.0408	0.0016	0.00000
	0.044423	0.0422	0.0022	0.00000
	0.051903	0.0442	0.0077	0.00006
Mean	0.0497	0.0484	0.0014	0.0005
S.D.	0.0312	0.0313	0.0225	

Data Table

Selected Alpha:	0.92

Alpha	SSE
0.85	0.09098520
0.86	0.09097036
0.87	0.09095715
0.88	0.09094589
0.89	0.09093690
0.90	0.09093052
0.91	0.09092711
0.92	0.09092704
0.93	0.09093070
0.94	0.09093848
0.95	0.09095082
0.96	0.09096812
0.97	0.09099085
0.98	0.09101947
0.99	0.09105445
1.00	0.09109630

Minimum SSE 0.09092704

Notes:

Inflation rates from Federal Reserve Bulletin and Bureau of Labor Statistics

Parameter Fit shown is for inflation rates from 1926 - 1990

APPENDIX V: ESTIMATION OF GAMMA PARAMETERS

When using a three parameter gamma to model the random shock term of a financial series, the method of moments can be used to derive parameters. The moments of the three paramrter gamma are:

$$E(x) = s + \frac{a}{B}$$

$$Var(X) = \frac{a}{B^2}$$

$$\text{coefficient of skewness} = \frac{E(X - \mu)^3)}{Var(x)^{\frac{3}{2}}} = \frac{2}{a^{\frac{1}{2}}}$$

The coefficient of skewness can be used to determine a. Then, B can be determined by equating the sample variance and the theoretical variance. Finally the sample mean is used to derive s. The coefficient of skewness of the residual of the inflation rate data used in this study was 1.246, therefore the fitted a was 2.576. Using the mean and variance of 0 and .0013, B and s were estimated to be 1981 and -.001. It should be noted that the three parameter gamma random variable has positive probability only for values greater than s. Values less than the fitted s did occur in the actual data.

As an alternative to the method of moments, s can be set equal to the minimum observed residual with the parameter a estimated from the coefficient of skewness. Then B is estimated by equating the theoretical and sample mean.

REFERENCES

1. Beard, R.E., Pentakainen, T. and Pesonen, E., Risk Theory: The Stochastic Basis of Insurance, Chapman and Hall, 1984, p. 129.

2. Pentikainen, T., Bonsdorf, H., Pesonen, M., Rantala, J., Ruohonen, M., Insurance Solvency and Financial Strength, Finnish Insurance Training and Publishing Company Ltd., 1989

3. Wilkie, A.D., "A Stochastic Investment Model for Actuarial Use," Transactions of The Faculty of Actuaries, 1984.

4. Daykin, C., Bernstein, G., Coutts, S., Devitt, E., Hey, G., Reynolds, D., Smith, P., "Assessing the Solvency and Financial Strength of a General Insurance Company", Journal of the Institute of Actuaries, Vol 114, Part 2, No. 457, Sept. 1987, pp. 227 - 325.

5. of. cit. Pentakeinen et. al. 1989, p. 251

6. op. cit., Pentikainen et. al., 1989, p. 119.

7. op. cit., Wilkie, p. 350.

8. Press, William, Flannery, Brian, Teukolsky, Saul, Vetterling, William, Numerical Recipes, Cambridge University Press, 1989, p. 206.

9. Brockwell, Peter and Davis, Richard, ITSM: An Interactive Time Series Modelling Package for the PC, Springer Verlag, 1991, p. 35.

10. Brockwell, Peter and Davis, Richard, Time Series: Theory and Methods, Springer Verlag, 1991, p. 312.

11. ibid., p. 36.

12. ibid., p. 36.

13. Wilkie, A., A Stochastic Investment Model, Institute Of Actuaries Discussion Paper No. 36, 1984, p. 52.

14. op. cit., Pentikainen et. al., p. 196

15. op. cit., Daykin et. al., p. 293.

16. Abraham, Bovas and Ledolter, Johannes, Statistical Methods for Forecasting, John Wiley and Sons, p. 257.

17. ibid., p. 258.

18. ibid., p. 85.

19. ibid., p. 86.

20. op. cit., Abraham and Ledolter, p. 339.

21. op. cit., Abraham and Ledolter, p. 339.

22. Brockwell, Peter and Davis, Richard, Time Series: Theory and Methods, Springer Verlag, 1991, pp. 508-511.

23. ibid., p. 341.

24. Ostrom, Charles, Time Series Analysis: Regression Techniques, SAGE Publications, 1990, pp. 53-56.

25. Chatfield, C., The Analysis of Time Series, Chapman and Hall, 1989, pp. 56-58.

26. op. cit., Chatfield, p. 71.