ANALYSIS OF LOSS DEVELOPMENT PATTERNS USING INFINITELY DECOMPOSABLE PERCENT OF ULTIMATE CURVES

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The opinions expressed in this paper are solely those of the author(s) and are not presented as the position of CIGNA Corporation.

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Abstract

We model loss development by starting with percent of ultimate curves which explicitly depend on the underlying pattern of exposure accumulation. We express such curves in general as a convolution of a "generating" function with an "exposure" density function. Such curves are decomposable with respect to exposures, where decomposability means, for instance, that an accident year curve is expressible as the weighted average of four appropriately shifted translates of the related accident quarter curve.

This approach is theoretically attractive and leads to several useful results. The most important area of practical application is fitting, interpolating, and extrapolating age-to-age factors. An essential point to note is that it is trivial to start with a parametric percent of ultimate curve and use it to calculate age-to-age factors, but the opposite derivation is not so simple.

Another area of application is in converting development patterns from one type of exposure period to another. As a natural consequence of our formulation, we are able to derive error terms for the usual "average date of loss" approximation and to generalize the approximation so that it is valid even at immature ages.

ANALYSIS OF LOSS DEVELOPMENT PATTERNS USING INFINITELY DECOMPOSABLE PERCENT OF ULTIMATE CURVES

I. INTRODUCTION

Though loss development patterns can be described in terms of age-to-age factors or percent of ultimate values, we begin modelling loss development patterns by starting with the percent of ultimate representation. The reason for this is simple enough. Though development patterns can be expressed in terms of percent of ultimate values or age-to-age factors, it is easier to derive the latter from the former than vice versa. For example, if the age-to-age factor from age t to t+1 is given by the formula, $AA_t = 1+t^{-2}$, it is not immediately obvious that the percent of ultimate at age 1 is roughly 27%. In contrast, if the percent of ultimate at age t is given by the formula, $P_t = 1-(1+t)^{-2}$, it is trivial to calculate that the first age-to-age factor is 1.185.

We further refine our model of loss development patterns by explicitly considering underlying exposures. In particular, we demand that our percent of ultimate curves be decomposable with respect to underlying exposures. What this means, for example, is that an accident year curve ought to be expressible as the average of four translates of an accident quarter curve. By continuing on with the process of decomposition, we arrive at the notion of an infinitely decomposable curve. Thus, as our accident year curve was a weighting of shifted versions of our accident quarter curve, so should our accident quarter curve be obtained by composition of translates of an accident month curve, and so on. We end up talking about the percent of ultimate curve for an exposure of "infinitesimal" duration. This is the same notion described by Philbrick in his 1986 Discussion Paper, "Reserve Review of a Reinsurance Company", when he writes about development with respect to a "single exposure point". In our formulation, we aim to express the percent of ultimate curve for the exposures in question as an exposure weighted sum of translates of a "generating" curve. This approach is mathematically equivalent to Philbrick's, even though his equations contain shifted "exposure weight" terms.

The mathematically sophisticated reader will observe that the "weighted sum of translates" idea is nothing more than convolution. Thus a convenient way to express an infinitely decomposable curve is as a convolution integral. The fundamental percent of ultimate convolution formula is given in II. The formula represents percent of ultimate development as a convolution of two functions. One function describes the exposures under consideration. The other function, called the generating pattern, describes how losses develop on an exposure of "infinitesimal" duration. In words, the percent of ultimate at time t is expressed as a sum, over all times s, of the amount of incremental exposure at time s multiplied by the development of that exposure, aged a duration of length, t-s.

The most important practical use of the convolution representation of percent of ultimate is in the fitting, interpolation and extrapolation of loss development factors. Starting with a percent of ultimate pattern, one can easily calculate associated age-to-age and age-to-ultimate factors. The generating pattern may be chosen from a family of parametric functions to thus obtain parametrically dependent loss development factors.

In section III, a methodology is presented for fitting given age-to-age factors. The procedure involves calculating the empirical percent of loss in each time interval relative to the percentage reported at the most mature age available. These interval percentages of the truncated percent of ultimate curve are then fitted by minimizing a weighted chi-square statistic.

The advantage in such an algorithm is that, once the fit is obtained, it is a simple matter to remove the truncation point and then compute the 'tail factor' (extrapolate), or calculate age-to-ultimate factors at intermediate ages (interpolate). This follows easily because one starts with a percent of ultimate curve that can be evaluated at any age. If one does not posit this but instead begins fitting with an arbitrary parametric age-to-age factor curve, then both extrapolation and interpolation involve some complication. In effect, one must back out the implicit percent of ultimate curve after the fact.

Another area of application is in converting development patterns from one exposure period basis to another. An immediately accessible result is that one can derive error terms for the usual "average date of loss" approximation in which, for instance, policy year development is estimated as accident year development evaluated six months earlier. A generalized "average date of loss" approximation is presented in section IV. The generalization allows approximation at immature evaluation times, for which the usual approximation breaks down.

The generalization can be thought of as a two-step process. First, we must find an appropriate time at which to evaluate our approximating curve. We use an evaluation date such that the elapsed time from the conditional average date of loss to the evaluation date is the same as corresponding elapsed time differential for the exposures being approximated. The second step is to factor in the relative amount of exposure accumulation.

For example, as of the end of one year, the conditional average date of loss for a policy year is 8 months. Thus the elapsed time from the conditional average date of loss to the evaluation date is 4 months. Now an accident year at 8 months has a conditional average date of loss which is 4 months; also producing an elapsed time differential of 4 months. As of the end of one year, 50% of the policy year exposure has accumulated, while 2/3 of the accident year exposure has accumulated as of 8 months. Thus we approximate policy year as of one year by an accident year as of 8 months, times 75% (.5/(2/3)).

Our underlying conceptual approach is in accord with Philbrick's, and we feel it provides an intuitively sound basis for modelling development patterns. We believe the applications presented demonstrate the practical value of the approach. Improvement in application techniques is likely in the future.

II. CONVOLUTION EXPRESSION OF PERCENT OF ULTIMATE

We will describe loss development here in terms of percent of ultimate curves. We will first need notation to describe exposure accumulation and the dependence of loss development on the exposures. We define $G(t)$ as the fraction of ultimate exposure that has accumulated as of time t. Assuming that the exposures begin at time zero and terminate at time, U, we have $G(0) = 0$, $G(U) = 1$, and $G(t)$ is positive but less than unity for any positive t less that U. We define $g(t) = G'(t)$, here assuming that G is piecewise differentiable. We call g the density of ultimate loss exposure and assume that g is piecewise continuous. We also define the exposure date random variable, T, by regarding G as its cumulative distribution function.

The "g" functions for common accident and policy periods may be readily derived under the usual uniformity assumptions.

II.1. Exposure Densities Under Uniformity Assumptions

° Accident Year	$g(t) = 1$	for	$0 < t < 1$
° Accident Quarter	$g(t) = 4$	for	$0 < t < 1/4$
° Policy Year	$g(t) = t$	for	$0 < t < 1$
	$g(t) = 2-t$	for	$1 < t < 2$
° Policy Quarter	$g(t) = 4t$	for	$0 < t < 1/4$
	$g(t) = 1$	for	$1/4 < t < 1$
	$g(t) = 5-4t$	for	$1 < t < 5/4$

(Here it is assumed that t is measured in years.)

Now that we have shown that exposure periods of interest may be designated by ultimate loss accumulation functions, we proceed to define development patterns with respect to such functions. Let $F_G(t)$ be the expected fraction of ultimate loss <u>developed</u> as of time t on ultimate loss <u>exposures</u> which accumulate as specified by G.

Now we assume the existence of a continuous function, F, such that one unit of ultimate loss incurred at exactly time, s, will develop, as of time, t, to a value, F(t-s), dependent only on the elapsed time from exposure, t-s. Then we may write:

II.2. Convolution Representation Formula

$$F_G(t) = \int_0^t F(t-s)\, g(s)ds$$

This is the fundamental convolution formula representation of percent of ultimate loss development. Observe that we explicitly denote the dependence of a development patern on the exposures by using the subscript "G". This distinguishes the resulting pattern, F_G, from the "generating" pattern, F, which carries no subscript. "F" may be interpreted as the development pattern for an exposure of infinitesimal duration. F is somewhat more general than a cumulative distribution. In particular, we could allow "F" that go above unity, since loss development patterns sometimes exhibit such behavior.

Before presenting a few particular examples, we will show that the convolution formula automatically yields decomposable percent of ultimate curves. First we must mathematically define decomposition.

II.3. Decomposition

A decomposition of a random variable, T, is a set of mutually exclusive random variables $[T_1, T_2, \ldots, T_n]$ along with a set of weights $[w_1, w_2, \ldots, w_n]$ summing to unity such that $T = \sum w_i T_i$.

If G is the cumulative distribution of T and if G_i is the cumulative distribution of T_i, it follows that $G = \sum w_i G_i$. From this expression for the cumulative distribution it can be immediately proved that the convolution formula yields decomposable curves.

II.4. The Convolution Formula Yields Decomposable Curves

Let T be an exposure date random variable with exposure beginning at time, zero, and terminating at time, U. If

$T_1, \ldots, T_n$ is any collection of positive, mutually exclusive, random variables so that $T = \sum w_i T_i$ for some positive weights, w_i; then

$$(\text{ i}) \quad F_G(t) = \sum w_i \, F_{G_i}(t)$$

In particular, if $G_i(t) = H_i(t - (i-1)U/n)$ where the H_i are cumulative distributions corresponding to exposures beginning at zero and terminating at U/n, then

$$(\text{ ii}) \quad F_G(t) = \sum w_i \, F_{H_i}(t - (i-1)U/n)$$

Observe that in II.4.(ii), we have achieved an expression for percent of ultimate as the weighted sum of translates.

Turning now to particular examples, consider an exposure density which is uniform over a period of duration k. Percent of ultimate formulas for exponential and Pareto curves may then be derived.

II.5. Examples

Exponential and Pareto Development for Accident Period of Duration k.

(i) Exponential

$$F(t) = 1 - e^{-bt}$$

For $t \leq k$

$$F_G(t) = [t - (1-e^{-bt})/b] / k$$

For $t > k$

$$F_G(t) = 1 - e^{-bt}(e^{kb}-1)/(b \cdot k)$$

(ii) Pareto

$$F(t) = 1 - (B/t+B)^Q$$

For $t \leq k$

$$F_G(t) = [t - \frac{B}{Q-1} (1 - \left(\frac{B}{t+B}\right)^{Q-1})] / k$$

For $t > k$

$$F_G(t) = 1 - \frac{B}{k \cdot (Q-1)} \left(\left(\frac{B}{B+t-k}\right)^{Q-1} - \left(\frac{B}{B+t}\right)^{Q-1} \right)$$

These formulas may be readily verified by working out the appropriate integrals using $g(t) = 1/k$ for $0 < t < k$.

Turning to another example, we had mentioned that the percent of ultimate curve could go above unity. This could be modelled by constructing a generating curve as a weighted sum of two tractable generating curves if we allow one of the weights to be negative. For example, with two related exponentials, we could fashion such a curve.

II.6. Example

- Weighting Two Related Exponentials to Form a Curve that Could Exceed Unity.

$$F(t) = (1+a)\, F_1(t) - a\, F_2(t)$$

where

$$F_1(t) = 1 - e^{-bt}$$
$$F_2(t) = 1 - e^{-rbt}$$

and

"a" and "b" are non-negative parameters and "r" is between zero and one.

In the case of uniform accident year exposure, we would have

For $t \leq 1$

$$F_G(t) = (1+a)\,(t-(1-e^{-bt})/b) - a(t - (1-e^{-brt})/br)$$

For $t > 1$

$$F_G(t) = 1 - (1+a)\, e^{-bt}\, (e^{b}-1)/br + a\, e^{-brt}(e^{br}-1)/br$$

Finally, we give another example using the exponential generating function to construct policy year percent of ultimate curves.

II.7. Policy Year Percent of Ultimate From Exponential Generating Pattern

For $t \leq 1$

$$F_G(t) = K(t)$$

For $1 < t \leq 2$

$$F_G(t) = K(t) - 2\,K(t-1)$$

For $t > 2$

$$F_G(t) = K(t) - 2\,K(t-1) + K(t-2)$$

where

$$K(t) = \frac{1}{b^2} - \frac{t}{b} + \frac{t^2}{2} - \frac{e^{-bt}}{b^2}$$

To provide some feel for these curves, we have computed percent of ultimate values using various parameters. These figures are shown in Exhibit I.

III. SMOOTHING, EXTRAPOLATION, AND INTERPOLATION OF LOSS DEVELOPMENT PATTERNS

Typically, the actuary is given a series of age-to-age factors obtained from a triangle of data. From these "empirical" factors, the actuary would like to derive age-to-ultimate factors at the given ages and sometimes also at intermediate ages. In this section, we discuss how the convolution representation approach can be applied to smooth, extrapolate, and interpolate loss development patterns.

We begin by observing that convolution representation provides a formula for evaluating percent of ultimate at any desired age; and, by taking multiplicative inverses, age-to-ultimate factors at any desired age can readily be obtained. Age-to-age factors for given ages are likewise easy to compute starting with a percent of ultimate pattern.

If we allow parametrically dependent generating patterns, we would thus have at our disposal parametrically dependent age-to-age and age-to-ultimate factors. Thus, if had a fitting procedure to select the parameter which best describes the data, there would be no problem in computing age-to-ultimate factors at any desired age.

Before discussing the critical details involved in fitting the data, it is worth noting how this strategy contrasts with other procedures. Under the procedure described by Sherman ('Extrapolating Smoothing, and Interpolating Development Factors', Richard E. Sherman, PCAS, 1985), one fits a parametric age-to-age factor curve against the given age-to-age data. While excellent fits are obtained in many cases, the extrapolation of a "tail factor" and the interpolation of factors at intermediate ages is not trivial. Though extrapolation may be tractable with particular curves and interpolation accomplished by resort to iterative schemes, it is much more convenient to eliminate these additional complications altogether.

Turning now to the mathematical exposition of our strategy, we suppose that age-to-age factors, AA_i, are given where $i = 1,2,..,k$ and AA_i applies to loss development from age i to $i+1$. We do not assume AA_k is unity. In other words, losses are not necessarily at ultimate at age $k+1$.

From these empirical age-to-age factors, we calculate age-to-most-mature-age factors, percent of truncated ultimate values, and resultant interval percentage values.

III.1. Age-to-Most-Mature-Age Factors
Percent of Truncated Ultimates
Interval Percentages of Truncated Ultimate
- Derived From Given Age-to-Age Factors

(i) AM_i Age-to-Most-Mature-Age Factor

$$AM_i = AA_i AA_{i+1} \ldots AA_k \qquad i=1,2,\ldots,k$$

(ii) P_i Percent of Truncated Ultimate

$$P_i = 1/AM_i \qquad l=1,2,\ldots,k$$

$$P_0 = 0$$

$$P_{k+1} = 1$$

(iii) R_i Interval Percentage

$$R_i = P_i - P_{i-1} \qquad i=1,2,\ldots,k+1$$

We will next fit the interval percentage numbers. First we suppose that a parametrically dependent generating curve has been chosen. We write it as $F(t|\theta)$. From it we derive the percent of ultimate curve $F_G(t|\theta)$ using the convolution formula.

We define the percent of truncated ultimate curve:

III.2. Percent of Truncated Ultimate from Convolution Formula

$$F_G(t|\theta;k+1) = F_G(t|\theta)/F_G(k+1|\theta)$$

$$F_G(t|\theta;k+1) = 1 \text{ for } t \geq k+1$$

Finally we calculate interval percentages from the convolution formula.

III.3. Interval Percentages from Convolution Formula

$$H(i|\theta) = F_G(i|\theta;k+1) - F_G(i-1|\theta;k+1)$$

Note that in this defintion we have omitted the explicit dependence of H on the exposures G and the truncation point k + 1. This is done in the interests of notational brevity.

The fit of the $H(i|\theta)$ against the R_i could be done in various ways. While a maximum likelihood approach would be most satisfying, we were not able to implement such an algorithm at this time. Instead, we were able to achieve fairly good results by minimizing a weighted Chi-square statistic.

III.4. Weighted Chi-Square

$$X^2 = \sum_{i=1}^{k+1} w_i \; [(R_i - H(i|\theta))^2/H(i|\theta)]$$

Generalized least squares fitting routines, such as the Marquardt algorithm, can be easily adapted to minimizing this statistic. For the weights, we have found that a decreasing series slightly improves fit to the earliest factors without appreciably harming fit to the later factors. The weighting can be regarded as an embellishment since even uniform weights yield reasonable results.

The following example should clarify the procedure we have described.

III.5. Example

Accident Year

Age i	Age-to-Age Factor (Data) AA_i	Age-to-Most Mature Age Factor AM_i	Truncated Percent of Ultimate P_i	Truncated Interval Percentage R_i
1	2.22	3.025	33.1	33.1
2	1.25	1.363	73.4	40.3
3	1.09	1.090	91.7	18.3
4	--	1.000	100.0	8.3

Exponential Generating Curve - Parameter b

For i = 1, 2, 3

$$F_G(i|b;4) = (1 - e^{-ib}(e^b-1)/b)/D$$

where

$$D = (1 - e^{-4b}(e^b-1)/b)$$

Thus

$$H(1|b) = (1 - e^{-b}(e^b-1)/b)/D$$

$$H(2|b) = (e^{-b} - e^{-2b})(e^b - 1)/(b\cdot D)$$

$$H(3|b) = (e^{-2b} - e^{-3b})(e^b - 1)/(b\cdot D)$$

$$H(4|b) = (e^{-3b} - e^{-4b})(e^b - 1)/(b\cdot D)$$

We use a numerical routine to find that the Chi-square error is minimized when $b = .198$

We applied these procedures to the Workers' Compensation data in Sherman's paper using a two parameter pareto generating curve. Results are shown in Exhibit 2. Fits somewhat inferior to those with Sherman's three parameter inverse power curve would be expected because our curve has one less parameter. Thus the quality of our fits appears to be relatively good. Note that we are immediately able to obtain a "tail factor" and age-to-ultimate factors by quarter.

IV. AVERAGE DATE OF LOSS APPROXIMATION METHOD

The average date of loss transformation method is a well known method for approximating the loss development pattern for one type of exposure given the loss development pattern for another type of exposure. An example in common practice is the use of accident year development to estimate policy year development. We will derive a generalization of this method and use the convolution integral to put precise boundaries on the error.

To begin the mathematical development, we use the exposure date random variable, T, to define:

IV.1. Definition of Conditional Average Date of Loss and Conditional Variance of Date of Loss

Let T be the exposure date random variable corresponding to G. For positive t define

(i) Conditional Mean

$$m(t) = E[T|T \leq t] = [\int_0^t sg(s)ds] \div G(t)$$

(ii) Conditional Variance

$$v(t) = E[(T-m(t))^2 | T \leq t] = [\int_0^t (s-m(t))^2 g(s)ds] \div G(t)$$

Note that for t greater than U, we have $m(t) = m(U)$ and $v(t) = v(U)$. Since $m(U)$ is the mean and $v(U)$ is the variance of the distribution defined by G, we often write simply "m" and "v" to condense notation.

To describe the usual average date of loss approximation, suppose there are two types of exposures described by the cumulative distributions, G and G*, respectively. Let $F(s)$ be the underlying generating pattern and let $F_G(s)$ and $F_{G*}(s)$ be the resulting percent-of-ultimate functions. Other quantities derived from G* will also carry a superscript "*". The average date of loss approximation can be expressed as:

IV.2. Usual Average Date of Loss Approximation

$$F_G(t) = F_{G*}(t + m* - m) + h$$

where h is an error term.

This formula is an attempt to equate development between different exposure types by adjusting the time of evaluation. Accordingly policy year development is estimated by accident year development from a half year earlier ($m* - m = 1/2 - 1 = -1/2$). The adjustment works well for sufficiently large evaluation times though the error bounds are generally not known. Furthermore, the approximation breaks down for immature evaluations. We propose to refine the approximation by generalizing the evaluation time adjustment and by factoring in the relative accumulation of exposure.

Before presenting our generalized approximation formula we will define the needed evaluation time adjustment.

IV.3. Evaluation Time Adjustment Function

Let j(t) (if it exists) be the function which solves

$$j(t) - m*\,(j(t)) = t - m(t)$$

Since the function, $r^*(j) = j - m^*(j)$, is continuous, has a minimum of zero, and is unbounded, it follows that at least one "j" exists to solve the defining equation for each particular t. Unfortunately, this "j" need not always be unique and thus $j(t)$ is well-defined when $G^*(t)$ exceeds $g^*(t) \cdot r^*(t)$. This happens in most situations encountered in practice. In certain cases $j(t)$ may be determined explicitly. In other cases it may be found using fixed point techniques.

IV.4. Example: j(t) for Policy Year Approximation by Accident Year.

When approximating policy year development with accident year development; $G = G_{PY}$, $G^* = G_{AY}$, it can be shown that:

$$j(t) = \begin{cases} 2\,(t - m(t)) & t < m(t) + 1/2 \\ (t + \frac{1}{2} - m(t)) & t > m(t) + 1/2 \end{cases}$$

Note that when t   2, we have $j(t) = t - 1/2$, which is the adjustment used in the usual approximation.

To facilitate our derivation of the generalized approximation we expand $F_G(t)$ using a Taylor series.

IV.5. Taylor Expansion of $F_G(t)$

<u>Lemma</u>

Suppose the generating pattern, $F(t)$, is second order differentiable. Then

$$F_G(t) = G(t) \left(F(t - m(t)) + \frac{F''(c) \quad v(t)}{2} \right)$$

where

$$\max(0, t-U) \leq c \leq t$$

<u>Proof</u>

By definition

$$\text{(i)} \quad F_G(t) = \int_0^t g(s)\, F(t-s)ds$$

Using a Taylor series, expand $F(t-s)$ around $t-m(t)$ to three terms to obtain

$$\text{(ii)} \quad F(t-s)=F(t-m(t)) + F'(t-m(t))(m(t)-s) + \frac{F''(c')(m(t)-s)^2}{2}$$

where c' is between $t-s$ and $t-a$. The Lemma follows directly by substituting (ii) into (i). The existence and boundaries on c are obtained via the Mean Value Theorem. $\square$

Armed with this expansion and our earlier definitions, we present our generalized average date of loss approximation.

IV.6. Generalized Average Date of Loss Approximation

$$F_G(t) = F_{G*}(j(t)) \frac{G(t)}{G*(j(t))} + k$$

where the error term,

$$k = \frac{G(t)}{2} \left(F''(c)\ v(t) - F''(c*)\ v*(j(t))\right)$$

Proof

Using Lemma IV.5. we rewrite $F_G(t)$ and $F_G(j(t))$

$$(\ i)\quad F_G(t) = G(t)\ \left(F(t-m(t)) - \frac{F''(c)\ v(t)}{2}\right)$$

$$(\ ii)\ F_{G*}(j(t))=G*(j(t)) \left(F(j(t)-m*(j(t))) + \frac{F''(c*)\ v*(j(t))}{2}\right)$$

Multiplying (ii) by $G(t)/G*(j(t))$ and subtracting from (i) we obtain the generalized approximation with error term k. $\Box$

An example of the approximation technique is shown in Exhibit 3. Policy year development is approximated by accident year development using the generalized average date of loss technique. The policy year and accident year are both derived from an underlying exponential development pattern. We can thus compare the approximation against the true answer. Such comparison demonstrates that the generalization works rather well. Also shown is the usual approximation and its breakdown at immature ages.

Finally, it should be noted that real world applications would entail approximation of accident year development at intermediate ages (8 months, for example). If one had a generating curve consistent with the given accident year development, such interpolation would be trivial. However, it would also be unnecessary since one could just as easily derive policy year development from the convolution formula. Thus the generalized formula is probably useful in solving practical problems only when one wants a quick estimate without going through the trouble of finding an appropriate underlying generating pattern. Nonetheless, the generalized formula aids our intuitive understanding of what is happening at immature ages. Further the error bounds are useful if one is willing to make a few assumptions, and can likely be refined with further research.

V. CONCLUSION

While both the generalized average date of loss approximation and the loss development factor analysis technique could be improved, they are nonetheless useful even at this stage. The relative ease with which they were derived is evidence of the value of the convolution representation. Perhaps the most important point is that models based on a percent of ultimate curve and which incorporate exposures are models with a solid and intuitively reasonable conceptual foundation. Research using models of this sort is likely to yield additional results and refinements in the future.

Exhibit 1 Sheet 1

Example

Accident Year

Percent of Ultimate

Exponential Generating Function

Percent of Ultimate

Age (Years)	Exponential Parameter .75	.50	.25
.5	8.3	5.8	3.0
1.0	29.7	21.3	11.5
1.5	51.7	38.7	21.9
2.0	66.8	52.3	31.1
2.5	77.2	62.8	39.2
3.0	84.3	71.1	46.3
3.5	89.2	77.5	52.6
4.0	92.6	82.4	58.2
4.5	94.9	86.3	63.1
5.0	96.5	89.4	67.5
6.0	98.4	93.5	74.7
7.0	99.2	96.1	80.3
8.0	99.6	97.6	84.6
9.0	99.8	98.6	88.0
10.0	99.9	99.1	90.7

Example
Policy Year
Percent of Ultimate

Exponential Generating Function

Percent of Ultimate

Age (Years)	Exponential Parameter .75	.50	.25
.50	1.4	1.0	0.5
1.00	10.5	7.4	3.9
1.50	29.7	21.6	11.8
2.00	50.5	38.1	21.7
2.50	66.0	51.8	30.9
3.00	16.6	62.4	39.0
3.50	83.9	70.8	46.2
4.00	89.0	77.2	52.5
4.50	92.4	82.3	58.1
5.00	94.8	86.2	63.0
6.00	97.5	91.6	71.2
7.00	98.8	94.9	77.6
8.00	99.5	96.9	82.5
9.00	99.7	98.1	86.4
10.00	99.9	98.9	89.4

Exhibit 1 Sheet 3

Example

Accident Year

Percent of Ultimate

Pareto Generating Function

Percent of Ultimate

		Pareto Parameters		
Age	B	10	10	20
(Years)	Q	10	5	5
.5		10.5	5.7	3.0
1.0		36.0	20.8	11.4
1.5		60.0	37.3	21.4
2.0		74.4	49.8	30.2
2.5		83.3	59.5	37.8
3.0		88.9	67.0	44.4
3.5		92.6	72.9	50.2
4.0		94.9	77.6	55.3
4.5		96.5	81.3	59.7
5.0		97.5	84.3	63.7
6.0		98.7	88.8	70.3
7.0		99.3	91.8	75.5
8.0		99.6	93.9	79.6
9.0		99.8	95.4	83.0
10.0		99.9	96.4	85.7

Exhibit 1 Sheet 4

Example

Accident Year

Percent of Ultimate

Double Exponential Generating Function

$$F(t) = (1+a)\,(1-e^{-bt}) - a\,(1-e^{-brt})$$

Parameters

$a = .3$

$b = 2$

$r = .45$

Age (Years)	Percent of Ultimate
.5	21.0
1.0	63.6
1.5	91.9
2.0	100.4
2.5	102.3
3.0	102.2
3.5	101.7
4.0	101.2
4.5	100.8
5.0	100.5
6.0	100.2
7.0	100.1
8.0	100.04
9.0	100.01
10.0	100.00

Accident Year

Age-to-Age Loss Development Factors

Year of Development	Actual	Sherman Three Parameter Inverse Power Fit	Two Parameter Pareto Fit (Even Weight)	Two Parameter Pareto Fit (Decreasing Weight)
2:1	1.920	1.889	1.983	1.973
3:2	1.228	1.224	1.210	1.208
4:3	1.098	1.100	1.096	1.095
5:4	1.051	1.056	1.055	1.055
6:5	1.036	1.036	1.036	1.036
7:6	1.025	1.025	1.025	1.025
8:7	1.019	1.018	1.019	1.019
9:8	1.014	1.014	1.014	1.015
10:9	1.011	1.011	1.011	1.012
11:10	1.009	1.009	1.009	1.009
12:11	1.008	1.008	1.008	1.008
Tail Factor		1.076	1.080	1.086

Two Parameter Pareto

$$F(t|B,Q) = 1 - (B/(t+B))^{Q}$$

Fit Parameters

	Even Weights	Decreasing Weights
B	4.75	4.49
Q	1.10	1.05

Decreasing Weights

$$w_i = c(13-i)^2 \qquad i = 1, 2, \ldots, 12$$

where c is a normalizing constant

Exhibit 2 Sheet 2

Interpolation
Estimated
Age-to-Ultimate
Factors by Quarter

Time (Years)	Pareto "Even Weights Fit	Pareto "Decreasing Weights" Fit
1.00	3.375	3.376
1.25	2.495	2.500
1.50	2.089	2.096
1.75	1.855	1.863
2.00	1.703	1.712

Accident Year and Policy Year Loss Development Comparison

Accident Year

"t" Interval	g*(t)	G*(t)	m*(t)	v*(t)
$(-\infty,0)$	0	0	-	-
$(0,1)$	1	t	$t/2$	$t^2/12$
$(1,\infty)$	0	1	$1/2$	$1/12$

Policy Year

"t" Interval	g(t)	G(t)	m(t)	v(t)
$(-\infty, 0)$	0	0	-	-
$(0,1)$	t	$t^2/2$	$2t/3$	$t^2/18$
$(1, 2)$	$2-t$	$1-((t-2)^2/2)$	$(1/3+(t-1)(1-(t-1)^2/3))/G(t)$	
$(2,\infty)$	0	1	1	$1/6$

Exhibit 3 Sheet 3

Usual Average Date of Loss Approximation

t	$F_{AY}(t)$	$F_{PY}(t)$	Usual Approximation	Actual Error
.25	.0288	.0024	.0000	.0024
.50	.1065	.0185	.0000	.0185
.75	.2224	.0589	.0288	.0301
1.00	.3679	.1321	.1065	.0256
1.25	.5077	.2398	.2224	.0175
1.50	.6166	.3649	.3679	-.0029
1.75	.7014	.4897	.5077	-.0180
2.00	.7675	.6004	.6166	-.0162
2.25	.8189	.6888	.7014	-.0126
2.50	.8590	.7576	.7675	-.0098
2.75	.8902	.8113	.8189	-.0076
3.00	.9145	.8530	.8590	-.0060

Generalized Average Date of Loss Approximation

t	Fpy(t)	j(t)	Generalized Approximation	Actual Error	Theoretical Error upper bound	Theoretical Error lower bound
.25	.0024	.1667	.0025	.0000	.0000	.0000
.50	.0185	.3333	.0187	-.0002	.0001	-.0005
.75	.0589	.5000	.0599	-.0010	.0009	-.0026
1.00	.1321	.6667	.1351	-.0029	.0041	-.0091
1.25	.2398	.8913	.2431	-.0032	.0152	-.0203
1.50	.3649	1.0952	.3721	-.0072	.0218	-.0385
1.75	.4897	1.2769	.5045	-.0148	.0181	-.0608
2.00	.6004	1.5000	.6166	-.0162	.0140	-.0740
2.25	.6888	1.7500	.7014	-.0126	.0109	-.0577
2.50	.7576	2.0000	.7675	-.0098	.0085	-.0449
2.75	.8113	2.2500	.8189	-.0076	.0066	-.0350
3.00	.8530	2.5000	.8590	-.0060	.0051	-.0272