Risk Theoretic Issues in Loss Reserving: The Case of Workers' Compensation Pension Reserves

by

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Abstract

Opposition to the discounting of loss reserves is based on the premise that loss reserves are uncertain and insurance companies must retain additional funds in order to reduce the chance of insolvency. This paper explores the explicit calculation of a risk load for discounted loss reserves. Underlying considerations include: (1) the random nature of the claim settlements; (2) our ability to describe the distribution of actual results; and (3) how the risk load we use for loss reserves compares to the profit load we use for pricing insurance. These ideas are expressed in terms of an example: ··orkers' compensation pension reserves.

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1. Introduction

Should loss reserves be determined with an explicit recognition of risk? This question was posed by the Casualty Actuarial Society's Committee on the Theory of Risk at the November, 1984 CAS annual meeting[1]. For the sake of discussion, the committee assumed that the answer was yes, and then proceeded to outline several points that should be considered in setting a risk load for loss reserves.

The issue of discounting reserves is linked to the issue of risk loading. It could be argued that carrying reserves at the nominal value rather than the present (or discounted) value represents an implicit risk load. The long tailed lines have the most uncertain reserves and the largest difference between the nominal and present values.

The discounting of loss reserves has received a lot of recent attention. The 1986 tax law requires that property and casualty insurers calculate their taxes using discounted reserves. However, the Loss Reserve Discounting Study Group of the National Association of Insurance Commissioners declared that "... discounting of loss reserves in not a generally accepted statutory accounting practice, except in regards to fixed and determinable payments, such as those emanating from workers'

[1]The Committee on the Theory of Risk made similar presentations in 1985 at meetings of the Midwest Actuarial Forum and the Casualty Actuaries of Greater New York. Copies of the presentation, titled "Risk Theoretic Issues in Loss Reserving," are available from the Casualty Actuarial Society.

compensation and long-term disability claims."[2] Very recently, a prominent actuary declared himself to be "solidly in favor of reserve discounting, unless the change would take place without concomitant recognition of the need for contingency reserves."[3]

While the Committee on the Theory of Risk discussed several important principles on risk loading in their presentation, they did not provide a unified example applying these principles. This paper will give such an example.

Our goal is to calculate risk loads for workers' compensation pension reserves. The author considers this to be a good place to start for two reasons: (1) it is a line with perhaps the longest tail of reserves; and (2) much of the necessary mathematical work has already been done. The new textbook *Actuarial Mathematics*[4] views the future lifetime of an individual as a random variable. Formulas are provided which enable one to quantify uncertainty in the loss reserves.

While we are focusing on workers' compensation pension reserves, it is hoped that this example will be rich enough to highlight issues that are relevant to other lines of insurance.

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[2]Simms, Gary D., "NAIC Report," *The Actuarial Update*, August 1987.

[3]Kilbourne, Frederick W., *The Actuarial Review*, November 1987, p. 11.

[4]Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.L. and Nesbitt, C.J., *Actuarial Mathematics*, Society of Actuaries, 1986.

This paper is being written in the spirit of the Committee on the Theory of Risk's presentation, that is to provoke discussion. The reader should be warned in advance that a number of debatable assumptions will be made. It is hoped that the state of the art of loss reserving will be advanced by this debate.

2. Underlying Considerations

It should be clear that the risk load becomes more important when reserves are discounted. Thus we assume that reserves are discounted. We shall also assume that the interest rate is known and fixed. While this is clearly not the case, there are a number of strategies available to the insurer which minimize the effect of varying interest rates. In addition, Woll[5] argues that the insurance operation of an insurance company should get "credited for funds it provides to the investment department at risk free rates" and that the "difference between the amount of investment income and its cost of funds" is the profit earned by the investment department.

We define the expected reserve as an estimate of the expectation of the present value of future payments to be made.

Let n be the total number of claims which are open. Let P_{it} be a random variable denoting the payment made for the i^{th} claim at

[5] Woll, Richard G., "Insurance Underwriting Profits: Keeping Score," *Financial Analysis of Insurance Companies*, Casualty Actuarial Society, 1987.

the t^{th} time period. Let $\bar{P}_{it}$ be an estimate of the expected value of P_{it} and let δ be the force of interest. In this case the expected reserve, $\bar{R}$, is:

$$\bar{R} = \sum_{i=1}^{n}\sum_{t=1}^{\infty} \bar{P}_{it}e^{-\delta t} \qquad (2.1)$$

Since P_{it} is a random variable, the expected reserve may be different from the amount, R, necessary to pay the claims. If the distribution of each P_{it} is known, it is possible to calculate the distribution of the amount necessary to pay the claims. We shall refer to the risk created by the randomness of P_{it} as the <u>process risk</u>. Bowers, *et.al.*[6], describes the distribution of R for the case of life annuities (i.e. workers' compensation pension reserves).

In practice, the distribution of P_{it} is not known. It must be estimated. The uncertainty in the distribution of P_{it} creates an additional risk which we refer to as <u>parameter risk</u>. There may be a number of ways to estimate the distribution of P_{it}. The amount of parameter risk will depend on how this distribution is estimated.

The risk load in the loss reserve should reflect both process risk and parameter risk.

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[6] *Actuarial Mathematics, op. cit.*, Chapter 5.

Our goal is to translate the uncertainty in the amount necessary to pay all claims into a risk load, which is expressed in dollars. We shall use utility theory as our tool to accomplish this goal. The main problem with the use of utility theory is the selection of a utility function. Reserves are less subject to market discipline than are prices for new insurance policies. There may be strong incentives such as taxes or perceived profitability, which may influence the choice of a utility function. It is our contention that the utility function should be calibrated by examining decisions that are voluntarily made. A decision that is continually being made is whether or not to write new business with a profit margin that is determined by the marketplace. One should use utility theory to link the profit margin for new business to the risk load for loss reserves.

It is possible that the estimates used in setting the loss reserve will also be used in pricing new business. For example a mortality table used in setting workers' compensation pension reserves could also be used for ratemaking. This will introduce a correlation between underwriting results and payments of existing claims.

These considerations will be addressed below. This list of considerations is not intended to be complete.

3. The Process Distribution of Pension Reserves

Throughout this paper we will illustrate our results with a mortality table based on Makeham's mortality law[7]:

$$s(x) = e^{-Ax - B(c^x - 1)/\ln(c)} \qquad (3.1)$$

where $B > 0$, $A \geq -B$ and $c \geq 1$.

In this section we assume that the mortality table is known[8] with $A = .0007$, $B = .00005$ and $c = 10^{.04}$.

Let T be a random variable representing the future lifetime of an individual aged x. The cumulative distribution function of T, F(t), is defined:

$$F(t) = 1 - \frac{s(x+t)}{s(x)}. \qquad (3.2)$$

If this individual is paid a pension continuously until death at an annual rate of 1 per year, the present value of this pension is:

$$\bar{a}_{\overline{T|}} \equiv \frac{1-e^{-\delta T}}{\delta}. \qquad (3.3)$$

The cumulative distribution function of $\bar{a}_{\overline{T|}}$ can be expressed in terms of F(t):

$$\Pr\{\ \bar{a}_{\overline{T|}} \leq \bar{a}_{\overline{t|}}\ \} = F(t) \qquad (3.4)$$

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[7]Bowers *et. al., op.cit.* p.71.

[8]We are using the illustrative life table given in Appendix 2A of *Actuarial Mathematics*.

The density functions for T and $\bar{a}_{\overline{T}|}$ are shown in Graphs 1 and 2 for an individual aged 40. We are assuming here, as we will throughout this paper, that the effective interest rate, i, is equal to 6%.

Bowers *et. al.*[9] gives formulas for the mean and variance of $\bar{a}_{\overline{T}|}$. For the sake of completeness, we repeat them here.

Let A_x denote the net single premium for a whole life insurance policy of 1 payable at the end of the year of death. Starting with $A_{110} = 0$, we calculate A_x according to the following recursion formula:

$$A_x = vq_x + vp_xA_{x+1}. \qquad (3.5)$$

By assuming that deaths are uniformly distributed between integral ages, the net single premium for a whole life insurance policy payable at the moment of death becomes:

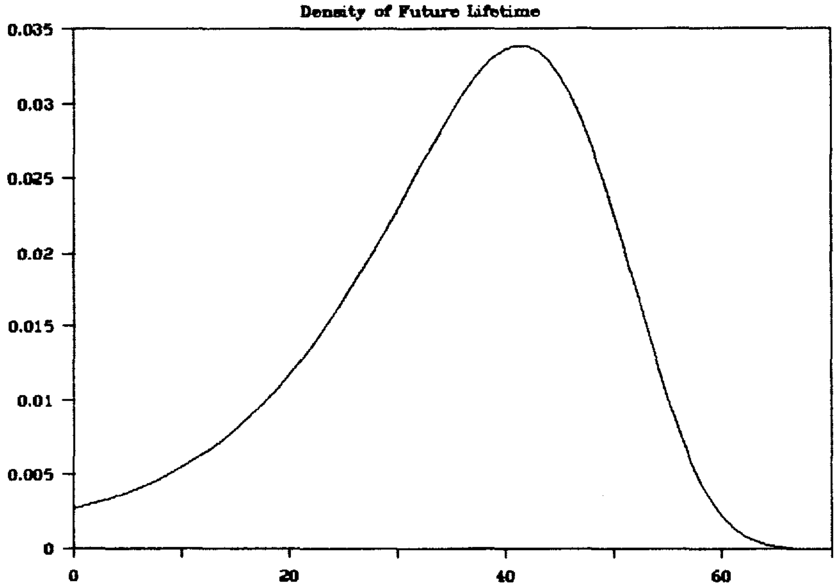
$$\bar{A}_x = \frac{i}{\delta}A_x. \qquad (3.6)$$

We then have: $$E\left[\bar{a}_{\overline{T}|}\right] = \frac{1 - \bar{A}_x}{\delta}. \qquad (3.7)$$

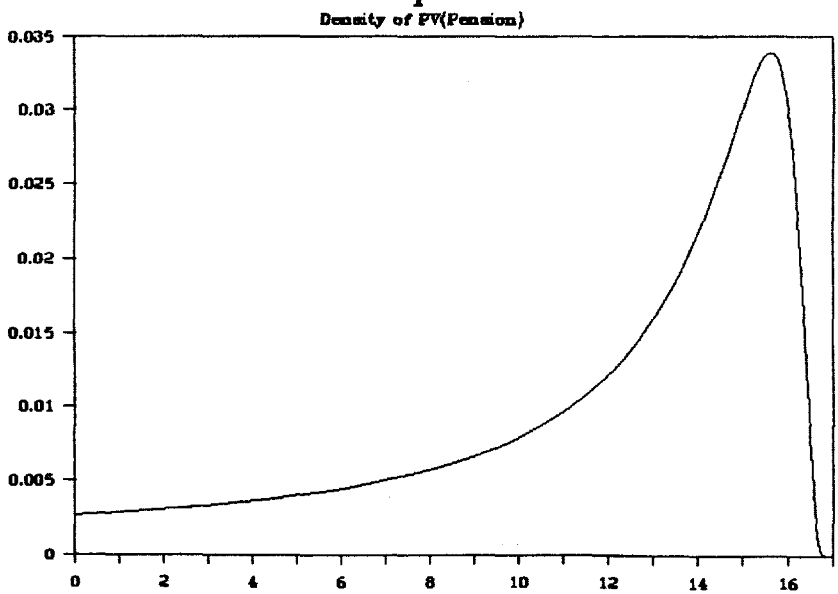
Let ${}^2\bar{A}_x$ denote the net single premium of a whole life insurance policy of 1, payable at the moment of death, and calculated with the force of interest 2δ. We then have:

[9] *i.b.i.d.*, Chapter 5.

Graph 1



Graph 2



$$\mathrm{Var}\left[\bar{a}_{\overline{T}|}\right] = \frac{{}^2\bar{A}_x - \bar{A}_x^2}{\delta^2}. \qquad (3.8)$$

In our example, we will be considering two groups of lives. Group A will consist of lives for which reserves are currently being held. Group A is described in the following table.

Table 3.1

Age	Annual Pension	# Lives
30	$10,000	24
40	12,500	36
50	15,000	48
60	17,500	60

Group B will consist of lives which are currently being insured in addition to those lives for which reserves are currently being held. The lives which are currently being insured are described in the following table.

Table 3.2

Age	Annual Pension	# Lives	Pr{Claim}
30	$10,000	1500	.002
40	12,500	1500	.003
50	15,000	1500	.006
60	17,500	1500	.014

Using Equations 3.7 and 3.8 we calculate 30,482,413 and 630,686 as the expected value and standard deviation of the loss reserve for Group A. We also calculate 6,897,916 and 1,170,220 as the mean and standard deviation of the incurred loss for new business described in Table 3.2. Since losses for the lives described in the two tables are independent, the means and variances in the two tables can be summed to obtain the mean and the variance for Group B. The resulting mean is 37,380,329 and the resulting standard deviation is 1,329,353.

Bowers *et. al.*[10] uses the normal approximation to describe the distribution of the total loss reserve. These distributions can be calculated numerically by use of the Heckman-Meyers algorithm. Graphs 3 and 4 show the numerically calculated density functions for Groups A and B. The "+" marks on the graphs show the density functions for normal distributions with the same mean and variance. The normal approximation is apparently a good one, and thus we use it to describe the process distribution.

4. Maximum Likelihood Estimation of the Mortality Table

The distribution of the loss reserves derived above assumed that the distribution was known. This is clearly not the case. The distribution must be estimated by mortality studies. One should consider the method of estimation when examining the risk in loss reserves. For example, one should expect a different precision in the estimates if the fitting of the mortality table was done by the method of moments rather than by maximum likelihood estimation. Also, one should expect greater precision when the sample size is increased.

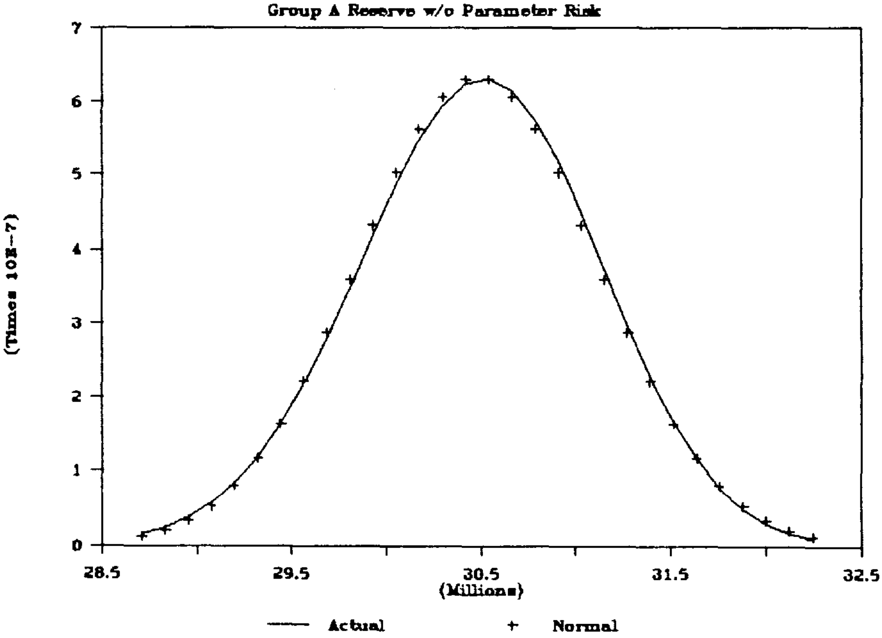
In our example, we assume that the parameters of Makeham's law were estimated by maximum likelihood. The study was assumed to observe n = 1000 people starting at age $t_0 = 25$ and observing

[10] *i.b.i.d.*, Chapter 5.

[11] Heckman, Philip E., and Meyers, Glenn G., "The Calculation of Aggregate Loss Distributions from Claim Count and Claim Severity Distributions," *Proceedings of The Casualty Actuarial Society*, 1983. The companion program, CRIMCALC, written by Glenn Meyers, was used to perform the calculations.

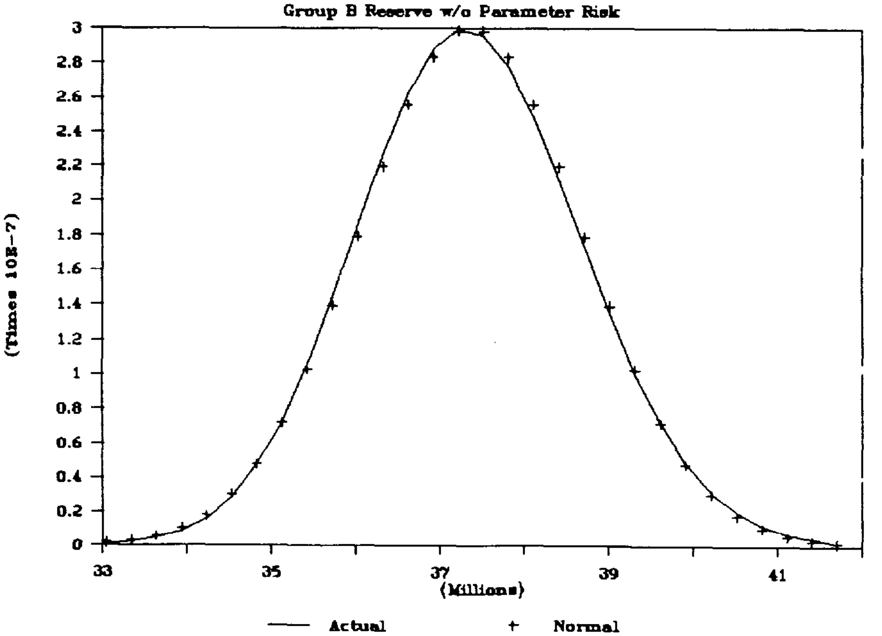
Graph 3

Group A Reserve w/o Parameter Risk



Graph 4

Group B Reserve w/o Parameter Risk



their (integral) age of death. It is assumed that everybody dies by age $\omega = 110$.

Now there are many methods of fitting mortality tables. By the choice of maximum likelihood as our method to estimate parameters, we do not necessarily mean to imply that this is the best way to fit mortality tables. This choice was made in order to take advantage of some very powerful mathematical tools which measure the uncertainty of our estimates.

Let $\vec{\theta} = (A,B,c)'$ be a vector consisting of the parameters for Makeham's law. The maximum likelihood estimate, $\vec{\theta}_M$, of $\vec{\theta}$ is the vector which maximizes:

$$L(\vec{\theta}) = \prod_{t=t_0}^{\omega-1} \left[s(t;\vec{\theta})-s(t+1;\vec{\theta})\right]^{n_t} \qquad (4.1)$$

where n_t is the number of deaths observed in the interval $[t,t+1)$. Hogg and Klugman[12] provide methods of calculating $\vec{\theta}_M$. Now $\vec{\theta}_M$ is a statistic. <u>For given $\vec{\theta}$</u>, the sampling distribution of $\vec{\theta}_M$ has an approximate trivariate normal with mean $\vec{\theta}$ and covariance matrix Σ^2. The probability density function, $f(\theta_M|\vec{\theta})$, is given by:

$$f(\vec{\theta}_M|\vec{\theta}) = \frac{1}{(2\pi)^{3/2}|\Sigma|}\cdot e^{-(\vec{\theta}_M-\vec{\theta})'\Sigma^{-2}(\vec{\theta}_M-\vec{\theta})/2} \qquad (4.2)$$

[12]Hogg, Robert V., and Klugman, Stuart A., *Loss Distributions*, John Wiley & Sons, 1984, Chapter 4.3.

The elements $a_{ij}(\vec{\theta})$ of the information matrix, $\mathcal{A} \equiv \Sigma^{-2}$, are given by the following formulas[13].

$$P_t(\vec{\theta}) = \frac{s(t;\vec{\theta})-s(t+1;\vec{\theta})}{s(t_0;\vec{\theta})} \tag{4.3}$$

$$a_{ij}(\vec{\theta}) = n\sum_{t=t_0}^{\omega-1} \frac{\partial P_t(\vec{\theta})}{\partial\theta_i}\cdot\frac{\partial P_t(\vec{\theta})}{\partial\theta_j}\cdot\frac{1}{P_t(\vec{\theta})} \tag{4.4}$$

5. The Predictive Distribution of Pension Reserves

To summarize the previous section, we have given formulas for the distribution of the estimator, $\vec{\theta}_M$, of our mortality table parameter in terms of the given parameter $\vec{\theta}$. This distribution depends upon the size of the sample, and the method of parameter estimation.

What we need, however, is just the opposite, i.e. the distribution of $\vec{\theta}$ in terms of $\vec{\theta}_M$.

A historical comment may be in order here. Our problem is very similar to the one addressed by the Rev. Thomas Bayes for the binomial distribution. Stigler[14] attributes the following statement to Bayes.

"*Given* the number of times in which an unknown event has happened and failed: *Required* the chance that the probability of its happening in a single trial lies between any two degrees of probability that can be named."

We must go one step further. What we really need is the distribution of the loss reserve, R, in terms of $\vec{\theta}_M$. To get this, we begin with the density function for the joint distribution of R, $\vec{\theta}$, and $\vec{\theta}_M$:

$$f(r,\vec{\theta},\vec{\theta}_M) = f(r|\vec{\theta},\vec{\theta}_M)\cdot f(\vec{\theta},\vec{\theta}_M). \tag{5.1}$$

Now R is independent of $\vec{\theta}_M$, and $f(\vec{\theta},\vec{\theta}_M) = f(\vec{\theta}_M|\vec{\theta})\cdot f(\vec{\theta})$. Thus:

$$f(r,\vec{\theta},\vec{\theta}_M) = f(r|\vec{\theta})\cdot f(\vec{\theta}_M|\vec{\theta})\cdot f(\vec{\theta}). \tag{5.2}$$

The process distribution, $f(r|\vec{\theta})$, is assumed to be normal with the mean and variance calculated from Equations 3.7, 3.8 and the information provided by Tables 3.1 and 3.2.

The sampling distribution of the maximum likelihood estimator,

[14] Stigler, Stephen M., *The History of Statistics - The Measurement of Uncertainty before 1900*, The Belknap Press of Harvard University Press, 1987, p.123.

$f(\vec{\theta}_M|\vec{\theta})$ is given by Equation 4.2.

Our version of "Bayes' Postulate"[15] is to assume that the prior distribution is uniform, i.e. $f(\vec{\theta})\equiv 1$. This reflects the view that one should not favor one value of $\vec{\theta}$ over another. The author concedes that this choice is debatable. Our purpose in this paper is merely to illustrate an example.

The joint distribution of R and $\vec{\theta}_M$ is obtained by integrating out $\vec{\theta}$.

$$f(r,\vec{\theta}_M) = \int f(r|\vec{\theta})\cdot f(\vec{\theta}_M|\vec{\theta})\cdot f(\vec{\theta})\,d\vec{\theta} \qquad (5.3)$$

Then: $$f(\vec{\theta}_M) = \int_0^\infty f(r,\vec{\theta}_M)\,dr \qquad (5.4)$$

and the predictive density of r is given by:

$$f(r|\vec{\theta}_M) = \frac{f(\cdot,\vec{\theta}_M)}{f(\vec{\theta}_M)}. \qquad (5.5)$$

The integrals in Equations 5.3 and 5.4 are done numerically. Equation 5.4 is particularly difficult since it involves a triple integral over an infinite region. Recall that $\vec{\theta} = (A,B,c)'$. The method used, which is best described as "brute force", is outlined in the Appendix.

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[15]While our use of the term "Bayes' Postulate" may correspond to common usage, it may not be what Bayes himself actually assumed. See Stigler *op.cit.*, p. 127.

The mean and standard deviation for Group A is 29,903,274 and 1,700,463. The mean and standard deviation for Group B is 36,649,786 and 2,389,486. Note the marked increased in the standard deviation when parameter uncertainty is considered. It is perhaps more interesting to note that the estimates of the mean are lowered when parameter uncertainty is considered.

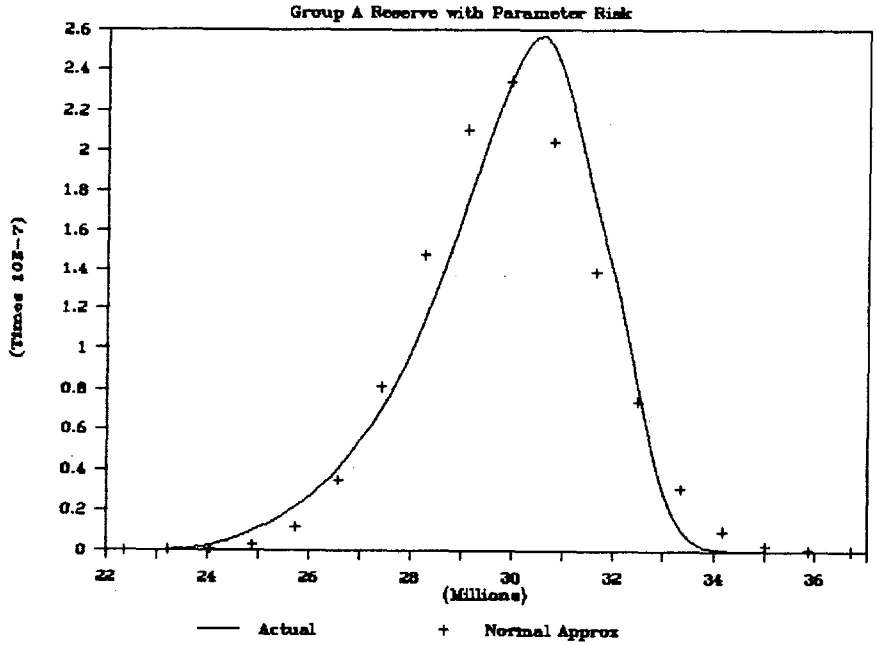
The mean and standard deviation for the group described by Table 2 only has mean 6,746,512 and standard deviation 1,223,232. If we assume independence of the reserves described by Tables 3.1 and 3.2, we calculate a standard deviation of 2,094,724 for Group B by summing variances. This falls short of the variance calculated above. This is because the same estimate of $\vec{\theta}$ is used for the groups described by Tables 3.1 and 3.2.

Plots of the predictive density of the reserve for Groups A and B are given in Graphs 5 and 6. Note that the modes are equal to the means of the reserve distributions when parameter uncertainty is not considered.

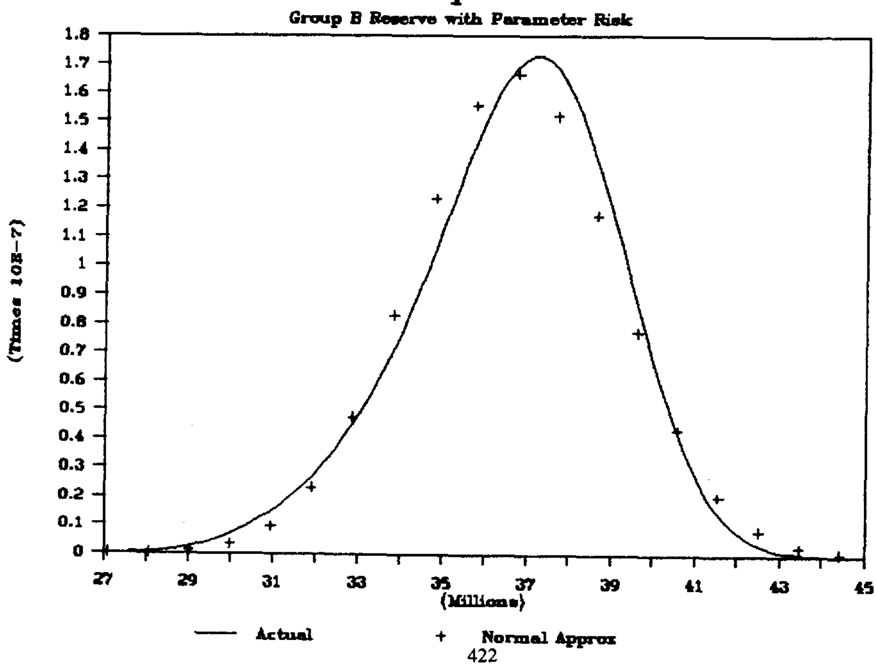
6. Calculation of the Risk Load Using Utility Theory

Let us consider an insurer who has reserves for expired polices described by Table 3.1. Assume that the insurer is considering three alternatives: (1) Sell the reserves; (2) Keep the reserves but do not write new business; and (3) Keep the reserves and write the new business described by Table 3.2. Alternative 2 contains a provision for a risk load. Alternative 3 contains a

Graph 5



Graph 6



provision for a risk load for the loss reserve plus a risk load for new business which is <u>determined by the competitive market</u>.

Acceptance of this alternative indicates an acceptance of the risk load for new business.

Two of the three alternatives involve uncertain outcomes. We shall use utility theory to compare these outcomes.

Let:

S = surplus of the insurance company;

R_A = random variable for the reserve for Group A;

R_B = random variable for the reserve for Group B;

$\bar{R}_A$ = expected reserve for Group A;

$\bar{R}_B$ = expected reserve for Group B;

L_A = risk load for Group A; and

P = risk load for new business.

Let u be a utility function. The insurer is indifferent to the three alternatives if:

$$u(S) = E\left[u(S + \bar{R}_A + L_A - R_A)\right]; \text{ and} \tag{6.1}$$

$$E\left[u(S + \bar{R}_A + L_A - R_A)\right] = E\left[u(S + \bar{R}_B + L_A + P - R_B)\right]. \tag{6.2}$$

We shall consider utility functions of the form:

$$u(x) = -e^{-(x/b)^c} \qquad (b>0 \text{ and } c\leq 1). \qquad (6.3)$$

This choice of utility functions is not unique. Others could be considered. This utility function does satisfy certain criteria (e.g. risk averse and decreasing risk aversion) that are desirable for insurance companies[16].

The Committee on the Theory of Risk[17] suggests that the risk load could be obtained by solving Equation 6.1 for the risk load, L_A. Our solution is a bit more involved. Our goal is to use information provided by the decision to compete for new business in the marketplace. This information should provide us with some hints as to which utility function to use. We would like to choose the risk load, L_A, and utility function parameters, b and c, which give a simultaneous solution to Equations 6.1 and 6.2.

Since we have two equations with three unknowns, we will pick several arbitrary values of c, and solve the resulting equations for b and L_A. The solution will be iterative. We start by taking an initial guess at L_A. We then repeat the following steps until the values of b and L_A converge.

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[16]Venter, Gary, "Utility Theory with Decreasing Risk Aversion," *Proceedings of the Casualty Actuarial Society, 1983.*

[17]Committee on the Theory of Risk, *op.cit.* p.29.

Step	Description
1.	Solve Equation 6.2 for b.
2.	Solve Equation 6.1 for L_A.

Convergence is rapid. Numerical integration was used to calculate the expected values and the secant algorithm[18] was used to solve the equations.

In our example we set the surplus equal to one half of the expected loss for the new business, or 3,373,256. We set the profit equal to 12% of the surplus, or 404,791. The simultaneous solutions to Equations 6.1 and 6.2 for given values of c appear in the following table.

Table 6.1

c	b	L_A
0.5	6,380,932	348,034
0.6	4,042,686	376,560
0.7	3,238,015	402,100
0.8	2,921,056	425,562
0.9	2,783,539	447,068
1.0	2,726,577	466,740

The linking of Equations 6.1 and 6.2 severely limits the subjective element in choosing the parameters of our utility function. The risk load, L_A, is confined to a relatively narrow range. The main determinant of this range is the profit loading which is in turn determined by market pressures. The decision to compete is a real decision made by company management.

[18] Burden, Richard L., and Faires, J. Douglas, *Numerical Analysis*, 3rd Edition, Prindle, Weber & Schmidt, 1985, p. 47.

It was mentioned in the introduction that carrying reserves at their nominal value represented an implicit risk load. We now compare this implicit risk load with the explicit risk load calculated above. The amounts reported here represent the mean of the predictive distribution (Equation 5.5). The "implicit risk load" is the difference between the predictive mean and expected loss reserve, 29,903,274, for Group A. We also consider the interest rate of 3.5% which many regulators allow companies to use for discounting workers' compensation pension reserves.

Table 6.2

Interest Rate	Predictive Mean	Implicit Risk Load
3.5%	39,158,882	9,255,608
0.0	64,425,775	34,522,501

7. Discussion

This paper has presented an example of how one might approach the problem of calculating risk loads for loss reserves. This being an example, we took great latitude in our assumptions and methods. We believe that this example is illustrative of a general approach that can be taken. However there are a number of conceptual and technical problems that must be addressed.

Central to this approach is that a probabilistic model for loss reserves must be specified. In our case we assumed that the future lifetime on an individual is a random variable whose distribution is given by Makeham's mortality law[19]. It will be difficult to come up with such a model which is appropriate for other lines of insurance.

The reason for selecting a model is that the parameters of the model must be estimated from data. The design of the study and the method of estimation will determine the predictive distribution of the loss reserves. Jewell[20] demonstrates the effect of study design for predicting claims which have been incurred but not yet reported. His methods are similar to those described above.

This approach is Bayesian. Great care must be exercised in selecting the prior distribution. While our assumption that the $\vec{\theta}$'s are uniformly distributed may seem innocent enough, consider a reparameterization of Makeham's law. For example we could have $\vec{\phi}' = (\theta_1^3, \theta_2^3, \theta_3^3)'$. One could then estimate $\vec{\phi}_M$, and assume that the $\vec{\phi}$'s are uniformly distributed. Question: would this make a noticeable difference in our estimation of the expected loss reserve, or the risk load?[21]

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[19] It is not even agreed that Makeham's law is appropriate for future lifetime. See London, Dick, *Graduation*, ACTEX Publications, 1985, and *Survival Models*, ACTEX Publications, 1987, for a description of other approaches to fitting mortality tables.

[20] Jewell, William S., *Predicting IBNYR Events and Delays*, (In preparation).

[21] See Box, George E.P. and Tiao, George C., *Bayesian Inference in Statistical Analysis*, Addison-Wesley, 1973, Ch. 1, for a discussion of the use of noninformative prior distributions.

There are computational problems with this approach. The dimension of the integral is equal to the number of parameters estimated. Actuarial models tend to have many parameters. Also, the integrand can be time consuming to evaluate. This is not an overwhelming problem. With the powerful computers that are available today, the problem can be solved. It would be nice to find a better solution.

These are only a few of the problems that must be solved.

The purpose of this paper is to continue the debate on risk loading and discounting of loss reserves. It is hoped that it provides a clearer view of the issues involved and an indication of what might be possible.

Appendix

Most of the calculations in this paper can be done with elementary numerical analysis. This subject is well within the grasp of most actuaries. However, evaluating the integral in Equation 5.3 requires considerable effort. This appendix outlines the method of evaluating this integral.

The probability distributions involve several numerical constants which cancel when we form the quotient in Equation 5.5. In what follows we will indicate the omission of the numerical constants in the probability distributions by replacing the symbol "=" with "∝".

Our goal is to evaluate:

$$f(r,\vec{\theta}) = \int f(r|\vec{\theta})\cdot f(\vec{\theta}_M|\vec{\theta})\cdot f(\vec{\theta})d\vec{\theta}. \tag{5.2}$$

We have: $f(r|\vec{\theta}) \propto e^{-(r-\mu(\vec{\theta}))^2/2\sigma^2(\vec{\theta})}$

with $\mu(\vec{\theta})$ and $\sigma^2(\vec{\theta})$ determined from Equations 3.7, 3.8 and the information in Tables 3.1-3.2.

We have: $f(\vec{\theta}_M|\vec{\theta}) \propto |\Sigma|^{-1}\cdot e^{-(\vec{\theta}_M-\vec{\theta})'\Sigma^{-2}(\vec{\theta}_M-\vec{\theta})/2}$

with $\Sigma^{-2} = \mathcal{A} = (a_{ij}(\vec{\theta}))$. The formula for the a_{ij}'s is given by Equation 4.4.

The general form of the partial derivative $\frac{\partial P_t(\vec{\theta})}{\partial \theta_i}$ is given by Hogg and Klugman[22].

$|\Sigma|^{-1}$ was calculated by factoring $\mathcal{A} = LL'$ by Choleski's method[23] and multiplying the diagonal elements of L.

We chose $f(\vec{\theta}) = 1$ when the restrictions of Equation 3.1 were satisfied and $f(\vec{\theta}) = 0$ otherwise.

Equation 5.2 can now be integrated numerically over a large three dimensional rectangle. The integral could be more easily evaluated if we had some idea how large this rectangle should be. We tried the following linear transformation:

$$\vec{Z} = (\vec{\theta}_M - \vec{\theta})\Sigma_M^{-1}$$

where Σ_M^2 is the covariance matrix for $\vec{\theta} = \vec{\theta}_M$. The motivation for this transformation was that if Σ was approximately constant, than the rectangle could be contained in a region corresponding to the high density region of a normal distribution, say $-3 \le Z_i \le 3$ for i = 1, 2 and 3.

It didn't work. The region looked like a tadpole with the body in the high density region of a normal distribution, but the tail extended out quite far. After considerable trial and

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[22]Hogg and Klugman, *op.cit.*, p. 145.

[23]Burden, Richard L., and Faires, J. Douglas, *op.cit.*, p. 351.

error, we settled on the following rectangular region.

$$-6 \le Z_1 \le 3, \quad -12 \le Z_2 \le 6 \text{ and } -40 \le Z_3 \le 6.$$

The numerical integration was done by the trapezoidal rule with 9 intervals along the Z_1-axis, 19 intervals along the Z_2-axis and 45 intervals along the Z_3-axis. The author feels comfortable with the numerical results obtained in the final answers, but there ought to be a better way to do this.