TITLE: EXPERIENCE RATES AS ESTIMATORS: A SIMULATION OF THEIR BIAS AND VARIANCE

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Using an individual insured's own past loss experience to arrive at its rate is a procedure that is used in many different areas of insurance. In addition to the formal individual risk rating plans, ad hoc procedures of this type are used in large risk departments of primary companies, excess and surplus lines companies, treaty and facultative reinsurers, and by various types of insurance consultants.

The purpose of this paper is to discuss the concepts of bias and variance of experience rating procedures[1], and illustrate these concepts by using a computer simulation model to examine the properties of some simple experience rating techniques. We will also discuss the effect that the misestimation of an insured's true loss potential has on the "risk" that the insurer faces. The rating techniques used are not represented as being the best available -- however, the paper presents some useful results concerning the superiority of certain types of techniques.

EXPERIENCE RATES AS ESTIMATORS

View the loss process as follows: a given insured's losses during an accident year "a" are random variables drawn from some probability distribution determined by a vector of parameters θ_a. Let θ represent a vector containing all the parameters from the first accident year of the experience period thru the year to be rated (denoted y). So

$$\theta = (\theta_1, \ldots, \theta_y)$$

1. For the purposes of this paper, define "experience rate" as a rate quoted to a given insured where the expected losses portion of the rate is wholly or predominantly determined by the insured's own loss experience over the past several years. Note that the term insured here could refer to anything from an individual auto to an entire insurance company (under a treaty reinsurance agreement).

Let X be a vector representing the insured's known loss experience during the experience period. X is a random sample drawn from the distributions determined by θ.

Let the ultimate losses that a particular insured will have for the policy period to be rated be a random variable "L". The purpose of the experience rate is to give the "best" estimate of E(L) [2]. E(L) is some function of the θ_y, whereas the experience X was drawn from distributions determined by $\theta_1, \ldots, \theta_{y-1}$. In order for X to be useful in estimating E(L), there must be some relationship between $\theta_1, \ldots, \theta_{y-1}$ and θ_y.

The simplest assumption would be that $\theta_1 = \ldots = \theta_y$, that is that an insured's loss potential is constant over the experience period. A more refined model would be that the severity and frequency componants of the θ_i's would be influenced by inflationary trends and by changes in a measurable exposure base[3], and that, after proper adjustments for these, the parameters would be stable over time.

The experience rating procedure is an estimator[4] of E(L); it is some function "R" of the insured's past known loss and exposure information X [5]. A perfect experience rating system would be a function R such that R(X) = E(L). However, X is also a random variable, so ful-

2. This paper will only consider estimates of E(L). In real life cases, we might want estimates of other attributes of the distribution of L, such as Var(L) or 95% percentile of L.
3. Such as number of cars in a commercial fleet, or subject premium in a reinsurance treaty.
4. An estimator is a function of a random sample and is therefore a random variable; an estimate is the result of the estimator function applied to a particular realization of the random variable, and is therefore itself a particular number.
5. Consider X to be a vector containing all pertinant rating information.

filling this condition is not possible, except by chance. We can, however, hope that $R(X)$ is an unbiased estimator of $E(L)$, that is, that $E(R(X)) = E(L)$.

We would also like $R(X)$ to be close of $E(L)$, on the average. One common way of expressing this is to minimize $E(R(X) - E(L))^2)$, the mean square error (MSE), which for an unbiased estimator is equivalent to minimizing $Var(R(X))$. For many simple statistical models, the form of estimator R that satisfies these criteria can be explicitly calculated. This is referred to as a UMVU[6] estimator.

For large samples, the Maximum Likelihood Estimator (MLE) usually satisfies these properties (asymptotically). However, there are reasons why we cannot always use the MLE, the main one being that in order to calculate it we must explicitly know the forms of the probability distributions that generate X. Of course, we can specify a model of the process that we believe is "reasonable" (as is done later in this paper), but there still are several problems. First, the MLE can be very difficult to calculate; second, although it is known to have good properties for large samples, it may be a bad estimator for smaller samples (it is usually biased); third, while it may be a good estimator if the model we assume is in fact the true one, it may be a bad estimator for a different model -- that is, it may not be robust.

6. Uniform Minimum Variance Unbiased.

The approach taken in this paper is to take several ad hoc (but hopefully reasonable) estimation techniques and examine their properties by a computer simulation model. Briefly, for an individual insured, the computer generates several accident years of known loss experience (X_i for the ith trial) from distributions with fixed parameters. It then applies several rating techniques to this set of known losses, arriving at several different estimates of E(L). The estimates and the actual ultimate losses are stored. This whole process (generating experience, then calculating estimates) is repeated several hundred times -- using the same underlying distributions and parameters. It can then be determined how well the estimates $R(X_i)$ fared as "guesses" of E(L), and which estimator function R does the best[7].

COMPUTER MODEL

An individual insured's past experience was "rerun" several hundred times in order to see how the results of a single rating method would be distributed.

Each iteration produced a set of loss experience for six accident years -- a five year experience period to rate from and the experience for the year to be rated (denoted y = 6). Not only was the ultimate experience generated for each of these years, but also the portion of it that would be known at any point in time.

7. E(L) can in principle be calculated explicitly from θ. However, for the loss generating model that was used, the calculation is quite complex, so the actual loss outcomes L_i were used to estimate E(L). The standard errors on these estimates were small compared with the standard errors of the estimates of E(R).

A single accident year for a single iteration was generated as follows[8]:

A random number of losses, N, was drawn from a Normal[9] distribution with mean = 40, variance = 60.

For each of the N claims, the following random variables were drawn:

M_i = Date of loss within year (Uniform with minimum = 0, maximum = 1)

Q_i = Report Lag (waiting time between accident date and report date) (Exponential with mean = 1.5 years)

All experience was viewed as being analyzed as of year-end, so a claim would first become known in $\lceil(M_i + Q_i - 1)$ years after the accident year[10].

P_i = Payment Lag (waiting time between report date and payment date) (Exponential with mean = 1 year)

So $\lceil(M_i + Q_i + P_i - 1)$ is the number of years after the accident year that the claim is paid[11]. Let "a" denote the accident year, a = 1,...,y. Then $\lceil(M_i + Q_i + P_i + a - 1)$ is the year of payment of the claim, where year 1 is the first year of the experience period.

8. The computer model allows the choice of several different distributions with arbitrary parameters. The distributions and parameters specified here were the principle, but not only ones, that were used.
9. The normal distribution was chosen as an approximation for the negative binomial, which is more difficult to simulate. Also, N was restricted to be between 1 and 65.
10. The APL symbol "$\lceil$", referred to as "ceiling", means "the smallest integer greater than",
Note that if $M_i + Q_i < 1$ the claim is reported during the accident year, "zero" years after the accident year.
11. Note that the maximum value allowed was 10 years.

An inflation index $I(M_i + Q_i + P_i + a - 1)$ of 8% per year (others were tested as will be explained in the results), from year 1 until the year of payment was assumed to affect the expected value of the payment distribution.

The random payment amount of C_i was drawn from a Lognormal distribution with $\mu = 8 + \ln I(M_i + Q_i + P_i + a - 1)$, and $\sigma^2 = 2.5$. This means that the mean and standard deviation trended at 8% per year.

So far, the number of claims, and (for each of these claims) the report date, the payment date and final payment amount have been determined. The last thing to do is set the reserve on each open claim. Each reserve was set as an unbiased guess of what the claim would settle for, if it closed in the year for which the reserve was being set.

For each claim that was reported but unpaid for at least a year, a random Reserve Error, V_i, was drawn from a Lognormal distribution with mean = 1, and variance = 2. This was multiplied by the final payment amount and the result was trended backwards from the payment year to the year for which the reserve is being set. Two things are important to note:

1. The reserve error is only chosen once for each claim, regardless of how many years it remains open, so the reserve, once set, will mearly be updated each year for inflation, and 2. this system leads to under reserving -- by the amount of future inflation[12].

12. A method of setting reserves at V times the ultimate payment, which does not lead to under reserving, was tested, but it made no significant difference in the results.

The known loss amount at time "t" on i^{th} loss from accident year "a" =

$$K_i(a,t) = \begin{cases} 0 & \text{if } M_i + Q_i > t \\ C_i\ V_i\ \dfrac{I(a-1+t)}{I(a-1+M_i+R_i+P_i)} & \text{if } M_i + Q_i \leq t < M_i + Q_i + P_i \\ C_i & \text{if } M_i + Q_i + P_i \leq t \end{cases}$$

So the actual ultimate losses $L = \sum_{i=1}^{N} C_i$.

The full experience matrix known at the beginning of year y for an insured would be

$$\begin{pmatrix} \sum_{i=1}^{N_1} K_i(1,1) & \cdots & \sum_{i=1}^{N_1} K_i(1,y-a) \\ \vdots & \iddots & 0 \\ & \iddots & \vdots \\ \sum_{i=1}^{N_{y-1}} K_i(y-1,1) & 0 \cdots & 0 \end{pmatrix}$$

This represents the familiar "loss development triangle". We will denote such an experience matrix by "$" and the triangle of claim counts by "#"[13].

Once the experience matrices $ and # have been calculated for one iteration, they are used as input for several different rating techniques (estimators of E(L)). These will be described in the "RATING METHODS" section.

13. The results to date are based on rating methods that use $ and/ or # as their input statistics. Of more interest are techniques that use triangles of some function of each known loss (such as losses truncated at basic limits).

CREDIBILITY

Often in experience rating we wish to use some outside experience that we believe is "related" to the insured in question. For example, we may use an insured's own basic limits experience, but rely on outside information for loss development factors, trend factors and expected excess losses.

The model underlying the use of this outside data is that the particular insured being rated was randomly selected from the group of all potential insureds of the same type. Therefore, the θ that we are trying to estimate is a realization of a random variable. θ's probability distribution is referred to as a structure function $U(\theta)$ [14]. If we have statistics available for many other insureds we can estimate certain properties of the group of all potential insureds (referred to as the collective). This then gives us valid information to use in estimating $E(L)$ for a particular insured. Credibility theory addresses the question of how to combine data from the collective with data from the individual insured to arrive at the estimator of $E(L)$ with the best properties[15]. The MSE's for certain credibility systems have been explicitly calculated[16] however, trend has seldom been[17] and loss development has not been addressed.

14. H. Bühlmann, Mathematical Methods in Risk Theory, Springer-Verlag, 1970
15. More precisely credibility theory restricts itself to linear combinations of collective data and individual risk data.
16. Fl. DeVylder, Introduction to the Actuarial Theory of Credibility.
17. C. Hachemeister, "Credibility for Regression Models with Application to Trend" in Credibility Theory and Applications ed. P. Kahn, Academic Press, 1975.

Several rating techniques that use outside information, in particular trend factors, were tested in the simulation. These tests are not strictly valid within the framework of credibility theory because the trend factors have not been estimated from a collective -- they have simply been postulated[18]. However, in all cases several different trend factors have been tested, including ones known to be wrong, in order to test the sensitivity of the rating method to incorrect assumptions about trend.

RATING METHODS

The following methods that used only the information contained in $ were tested:[19]

Method #1: Loss dollars are projected to ultimate by age-to-age factors. A least squares line[20] (restricted to a slope ≥ 0) is fitted to the five ultimate results to project the sixth year.

Method #2a, b, c, d, e:

Loss dollars are projected to ultimate using age-to-age factors. The ultimate result for each accident year is trended to the current year by multiplying it by an inflation factor raised to the appropriate power. The 5 trended results are then averaged to predict the current year. Cases, a, b, c, d and e refer to trend factors of 0%, 5%, 8%, 12%, and 15%, respectively.

18. To the extent that trend factors serve to project the effects of future inflation rather than adjust experience for the effects of inflation during the experience period, one would probably not want to estimate inflation from the data anyway, but rather use an exogenuous factor based on economic considerations.
19. A numerical example of each rating technique is contained in Appendix B.
20. Unrestricted linear and exponential fits were tested and gave similar results.

Method #3a, b, c, d, e:

"Adjustment to Total Known Losses method"

Estimated expected ultimate losses $= \left(\sum_{j=1}^{5} K_j \cdot (1+i)^{6-j} \right) \div \left(\sum_{j=1}^{5} \frac{1}{f_j} \right)$

Where K_j is known losses at current year for accident year j

f_j is the age-to-ultimate factor for accident year j

i is a trend factor which was set at 0%, 5%, 8%, 12% and 15% for 3a thru 3e, respectively.

The derivation of this formula is given in Appendix A.

The following rating methods using both $ and # were tested:

Method #4a, b, c, d, e:

Claim counts are projected to ultimate using age-to-age factors. The estimate of the 6th year is the average of these five results. This is multiplied by actual average known claim size, trended to the current year as in Method 2.

Method #5a, b, c, d, e:

Same as Method 4 except ultimate claim counts are projected by the Adjustment to Total Known Losses method.

Method #6: Ultimate claim counts by year are projected by the Adjustment to Total Known Losses method. For each accident year these are multiplied by actual average claim size. The results are trended by linear least squares (restricted to a slope ≥ 0) to project the sixth year.

RESULTS

The computer simulation model was written in APL and run on an IBM 5110 mini-computer. Creating six year's experience (five years to rate from, and one year as the policy's experience) for an average of forty losses per year, then applying twenty different rating techniques to the known losses took about 5 minutes, so 500 iterations took about 42 hours to run.

The simulation was run under four different sets of parameters. The first set were the ones given in the previous section. The second set were the same except that the severity trend was 8% the first four years (during the experience period) and 12% thereafter. The third set was the same as the first except the expected value and standard deviation of number of claims (N) increased by 5% each accident year starting with an expect number of 25 the first year. This could be used to reflect either an increase in exposure units not reflected in the rating method, or an unsuspected frequency trend. For the fourth run, the distributions were set as uniform to test the robustness of the previous results to wild departures in the form of the distributions. Exhibit 1 gives a summary of the parameters in each of the above cases. It also shows true[21] E(L) for each case -- this is the value we wish the rating techniques to be close to most of the time.

Exhibit 2 shows the simulation results of the distribution of R (the experience rated estimate of E(L)) for the first set of parameters.

21. Actually this value is also an estimated one, see note 7.

EXHIBIT 1

The four sets of parameters against which the rating methods were tested.

	#1	#2	#3	#4
E(L) for year 6	\$731,000	\$837,000	\$575,000	\$1,664,000
Standard Error of Estimate of E(L)	\$8,000 based on 2025 iterations	\$13,000 based on 1000 iterations	\$15,000 based on 500 iterations	\$29,000 based on 1000 iterations
Number of Losses N	Normal $\mu = 40$ $\sigma^2 = 60$		Normal $\mu = 25 \times (1.05)^j$ $\sigma^2 = 40 \times (1.05)^{2j}$ $j = 0, \ldots, 5$	Uniform max = 30 min = 1
Date of Loss within year M_i	Uniform max = 1 min = 0			
Report Lag Q_i	Exponential mean = 1.5			Uniform max = 4 min = 0
Payment Lag P_i	Exponential mean = 1			Uniform max = 4 min = 0
Payment Amount C_i	Lognormal $\mu = 8 + \ln I(a + M_i + Q_i + P_i - 1)$, $\sigma^2 = 2.5$ (mean = 10,405 x I, standard deviation = 34,793 x I)			Uniform max = 100,000 x $I(a + M_i + Q_i + P_i - 1)$ min = 1
	$I(t) = 1.08^{t-1}$	$I(t) = \begin{cases} 1.08^{t-1} & t \leq 5 \\ 1.08^4 \times 1.12^{t-5} & t > 5 \end{cases}$	$I(t) = 1.08^{t-1}$	$I(t) = \prod_{j=1}^{t} (1+r_j)$ where r_j generated randomly uniform (.2, 0)
Reserve Error V_i	Lognormal = -.549 = 1.099 (mean = 1, variance = 2)			Uniform max = 2 min = 0

EXHIBIT 2

ACTUAL INFLATION 8% PER YEAR
TRUE E(L) = $731,000

Rating Method	Information Used	Chosen Trend Factor	Distribution of R	
			Bias	Standard Deviation
1	$	fit	$240,000	$870,000
2c′	$	8%	50,000	370,000
3c′	$	8%	- 30,000	250,000
4c′	$, #	8%	- 70,000	220,000
5c′	$, #	8%	- 90,000	180,000
6	$, #	fit	- 50,000	540,000

The bias (E(R-L))column shows whether each method will produce too much or not enough premium on the average. The standard deviation of R measures of how wide a range of results the various methods will give. Because there is only one right answer ($731,000), the smallest possible spread of estimates is the most desirable.

The six rating methods can be sorted into two groups depending on how they handle trend. Methods 1 and 6 fit a least squares line thru the estimated ultimate results for the past five years to project the sixth year. They, therefore, try to estimate the underlying trend based solely on the insured's experience. Both of these perform poorly in terms of standard deviation, and method 1 is highly biased.

Methods 2 thru 5 use a postulated trend factor that adjusts each accident year to current level. Of course, the bias for the version using an 8% trend factor (Methods 2c thru 5c) should be low because that is the true inflation rate underlying the model. The bias need not be zero because the rating technique may not take inflation into account exactly the way the loss generating model does (the rating techniques all trend past accident years to the current year whereas in the loss model inflation acts on all open claims across calendar years).

A way of reflecting trend in Methods 2 thru 5 that appears to be superior[22] to trending each accident year separately and averaging the results (as is done in methods 2 thru 5,b thru e) is to adjust

22. The conditions under which each of the two methods are superior are discussed in Appendix C. The simulation did not provide conclusive results either way.

the untrended result for three years trend; in other words, trend the average result rather than average the trended results. The methods labled 2c′ thru 5c′ are ones for which the untrended results (2a thru 5a) were adjusted by (1.08^3). [23] The bias and standard deviations shown in Exhibit 2 for methods 2c′ thru 5c′ were not arrived at by simulation, but rather a straight adjustment of the simulated results for methods 2a thru 5a (the untrended versions).

Methods that use both $ and # (methods 4 thru 6) have a smaller variance than those that use $ alone (methods 1 thru 3). [24] However, all the ones using $ and # tested here suffer from a serious defect. That is that they have no way of detecting reserve deficiencies from the data. In this model, (the expected value of) reserves are deficient to the extent of future inflation, so this leads to a downward bias in the techniques. Methods that analyze loss development from $ can attempt to detect such under reserving (at least to the extent that the earliest experience year is truly fully mature).

One obvious conclusion is that the more things we try to estimate from the data (e.g., trend, reserve deficiency) the higher the variance of the estimator will be. This suggests that for a given set of data we should be realistic about what effects we can estimate from it. This is, of course, the "full credibility" question: "How much data do you need to give your estimator satisfactory variance?" In the case of the risk sizes used in this simulation it seems that one should not

23. Actually the unbiased adjustment is $5/\sum_{i=1}^{5} \frac{1}{1.08^i}$ which is very close to 1.08^3.
24. This is plausable result, which should be true in all but very unusual cases. However, it should be noted that the loss model further tilts results in this direction because it uses constant frequency parameters.

try to estimate trend (methods 1 and 6) but one can use a method that is sensitive to reserve deficiency (method 3).

Method 3c′ gives the best overall result with a variance slightly higher than methods 4c′ and 5c′, but the smallest (absolute value of) bias of any method. It is interesting that the Adjustment to Total Known Losses method (methods 3 and 5), which takes the total known losses for all years and divides that sum by an overall adjustment factor for loss development, has a smaller variance than simply projecting accident years to ultimate and averaging the results (methods 2 and 4). This is analoguous to the earlier comment about more efficient trend adjustment. Appendix C shows that under some conditions this is a Best Linear Unbiased Method.

The calculation of the bias and standard deviation for any of the methods 2 thru 5 where a trend factor different from 8% was (incorrectly) selected is straight forward:

$$\text{bias for trend } r = (E(L) + \text{bias for } 8\%)\left(\frac{1 + r}{1.08}\right)^3$$

$$\text{std. dev. for trend } r = (\text{std dev for } 8\%)\left(\frac{1 + r}{1.08}\right)^3$$

A 50% error in selecting r (i.e., 12% or 4% instead of 8%) will introduce a bias of about + 12% to an otherwise unbiased technique.

Exhibit 3 shows how well each method performed under parameter sets 2, 3, and 4. Remember set 2 has an accelerating severity trend, set 3 has a frequency trend and set 4 uses all uniform distributions.

EXHIBIT 3

Rating Method	Parameters #2 True E(L) = $837,000		Parameters #3 True E(L) = $575,000		Parameters #4 True E(L) = $1,664,000	
	Bias R	Std. Dev. R	Bias	Std. Dev. R	Bias	Std. Dev. R
1	$170,000	$1,140,000	$100,000	$780,000	$800,000	$5,120,000
2c′	- 30,000	460,000	- 60,000	290,000	130,000	1,870,000
3c′	- 120,000	260,000	-120,000	210,000	-280,000	380,000
4c′	- 170,000	250,000	-140,000	170,000	-330,000	470,000
5c′	- 190,000	230,000	-130,000	160,000	-370,000	330,000
6	- 140,000	540,000	-100,000	360,000	-460,000	570,000

Method 2c′ does the best in terms of bias, however, has a high standard deviation. This exhibit shows that the ranking of methods from low to high standard deviation and from low to high bias seems to be fairly insensitive to changes in parameters. However, performance in terms of absolute value of bias depends on how the trend underlying the model compares with the trend chosen in the rating method.

VALIDITY OF THE RESULTS

Two issues should be considered when assessing the validity of the results.

1. Are 500 iterations a sufficient number to give stable estimates of the mean and variance of the distribution of R? The standard errors of the bias can be estimated as $\left(\frac{\text{Var}(R-L)}{500}\right)^{\frac{1}{2}}$

   A 95% confidence interval around the estimates of the bias shown in exhibits 2 and 3 should be roughly two standard errors on either side of the estimated value.

   Taking Var(R-L) to equal $\hat{\text{Var}}(R) + \hat{\text{Var}}(L)$ where these are the variances estimated by the simulation, give standard errors of the bias estimates ranging from \$15,000 to \$30,000 (the rating methods with larger Var(R) having larger standard errors). This means that a rating method that is actually unbiased could show a bias of roughly + \$50,000 based on 500 simulations.

   The stability of the estimates of Var(R) are not known.

Note that because several (but not all) rating methods were tested during the same computer run (the same set of 500 simulated experience periods) there is a positive covariance between the estimates of E(R-L) (and also Var(R)) for rating methods 1 thru 4, and 5 and 6, but the estimates between these two groups of methods are independent.

2. Are the results specific to the form of the loss generating model that was used; how different would the ranking of efficiency of the rating methods been under a somewhat different model?[25] Many possibilities suggest themselves: inflation may affect different sizes of losses differently, reserves may be set in a different fashion with strengthenings occurring during a calendar year across all accident years, frequency and severity may not be independent. At least the model has shown that an extreme change in parameters (set 4) does not affect the conclusions greatly.

At the time of the writing of this paper, the computer model was not sufficiently sophisticated to test rating techniques of real interest, such as ones that adjust losses for changes in exposure during the experience period, ones that truncate losses at various levels, credibility weighing techniques and excess of loss experience rating techniques. Hopefully simulation results on some of these types of techniques will be available for presentation at the Spring meeting.

25. One error in the current model is that the severity distribution should allow for claims closed without payment.

RISK

Viewing premium as a random variable raises some new issues in the calculation of profit loading.

The random variable of ultimate interest to an insurer is its profit[26] on a given insured or group of insureds.

Let U be the random variable underwriting profit on the individual insured.

Let π be a fixed profit loading[27]

Let R be the experience rated estimator of E(L)

So U = Experience rated premium - L = $(\pi + R) - L$

The variance of profit on a single insured is

$$\begin{aligned} \mathrm{Var}(U) &= \mathrm{Var}(\pi + R - L) \\ &= \mathrm{Var}(R) + \mathrm{Var}(L) - 2\,\mathrm{Cov}(R,L) \end{aligned}$$

R is based on known losses for prior years whereas L is losses for the period to be rated. We have been assuming that loss occurrences are independent, so Cov(R,L) = 0.

26. For simplicity's sake we are ignoring investment income considerations here.
27. Of course, this term should depend on the "riskiness" of insured.

If U were not random, the insurer would face no risk or variability of results. The insurer's risk[28] arises from the variability of U, which in turn arises from the variability of both R and L.

The insurer is frequently in a situation of being one of several companies quoting prices from which the insured will pick the lowest. This means that E(U) no longer equals $E(R) - E(L) + \pi$ (or bias plus loading) but rather

$$E(R \mid R+\pi < k) - E(L) + \pi$$

where k is the minimum of the other quoted prices for the insured.

Consider an unbiased rating technique R . Assume that the "proper" expected profit margin (based on risk considerations) has been determined to be π'. That is, we wish

$$E(U) = \pi'$$

$$\begin{aligned} E(U) &= E(R \mid R+\pi < k) - E(L) + \pi \\ &= E(R) - E(L) - (E(R) - E(R \mid R+\pi < k)) + \pi \end{aligned}$$

$$\text{So } \pi = \pi' + (E(R) - E(R \mid R+\pi < k))$$

This says that the profit margin added to an unbiased estimate of expected losses should contain two pieces, 1. a risk loading (π') and 2. a factor to load for the antiselection you expect to suffer

28. The proper measure of "risk" for an insurer (or in fact for any financial transaction) is a much debated topic. Two the leading candidates are Var(U), which seems to be favored by actuaries, and Cov(U,M) where M is the return the entire market of assets, which arises from the CAPM. The CAPM unfortunately implies that insurance underwriting is almost riskless, because Cov(U,M) = Cov(R,M) + Cov(L,M) (with our independence assumptions) and both of the terms on the right should be near zero.

in a competative bidding situation (if your quote is accepted, it is more likely that you underestimated expected losses).[29] Notice that an estimator R with a smaller variance will be desirable because it will decrease both components of the loading.

29. Two implications of this:
    1. in a renewal situation with no outside quotes, an insurer should be able to quote a lower price than otherwise because he will not need this loading
    2. the more companies quoting, the higher this loading should be

APPENDIX A

Derivation of "Adjustment to Total Known Losses method"

For Accident Year j

k_j = Known losses (thru current year)

u_j = Actual ultimate losses

$IBNR_j$ = IBNR

f_j = Age-to-ultimate factor

i = Trend factor

u = True expected losses for year 6

Assume 1) $u = \frac{1}{5} \sum_{j=1}^{5} u_j (1+i)^{6-j}$

2) $u_j = k_j + IBNR_j$

So $5 \cdot u = \sum_{j=1}^{5} K_j (1+i)^{6-j} + \sum_{j=1}^{5} IBNR_j (1+i)^{6-j}$ (*)

Assume 3) $IBNR_j = (1-\frac{1}{f_j}) \frac{u}{(1+i)^{6-j}}$

So $\sum_{j=1}^{5} IBNR_j (1+i)^{6-j} = u(5-\sum_{j=1}^{5} \frac{1}{f_j})$

substituting this into (*) gives

$$u = \left(\sum_{j-1}^{5} k_j (1+i)^{6-j}\right) \div \left(\sum_{j=1}^{5} \frac{1}{f_j}\right)$$

Numerical Examples of Rating Methods

Let

$$
\$ = \begin{pmatrix}
\$242,744 & \$202,907 & \$216,946 & \$223,772 & \$243,633 \\
135,700 & 536,598 & 608,794 & 636,252 & 0 \\
70,734 & 535,107 & 733,341 & 0 & 0 \\
42,031 & 222,841 & 0 & 0 & 0 \\
185,689 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

(1) Accident Year	(2) Age to Age Factor[30]	(3) Age to Ultimate Factor	(4) Known Losses	(5) Ultimate Losses	(6) Reciprocal of Age to Ultimate Factor
1	1	1	\$243,633	\$243,633	1
2	1.0888	1.0888	636,252	692,751	.9184
3	1.0415	1.1340	733,341	831,609	.8818
4	1.2232	1.3871	222,841	309,103	.7209
5	3.0485	4.2285	185,689	785,186	.2365
Total			\$2,021,756	\$2,862,282	3.7576

Method #1 Column 5 projected to accident year 6 by linear least squares = \$782,294

Method #2c′ $((\text{Sum of column 5}) \div 5) \times 5/\sum_{i=1}^{5} \frac{1}{1.08^i} = \$716,877$

Method #2c $(\$243,633 \times 1.08^5$

$+ 692,751 \times 1.08^4$

$\vdots$

$+ 785,186 \times 1.08) \div 5 = \$711,317$

Method #3a ((Sum of column 4) ÷ (Sum of column 6)) = \$538,044

30. e.g., $1.0415 = \dfrac{636,252 + 223,712}{608,794 + 216,946}$

Let

$$\# = \begin{pmatrix} 10 & 21 & 28 & 28 & 29 \\ 23 & 45 & 49 & 52 & 0 \\ .14 & 44 & 54 & 0 & 0 \\ 11 & 29 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(7) Accident Year	(8) Age-to-Age Factor	(9) Age to Ultimate Factor	(10) Known Counts	(11) Ultimate Counts	(12) Reciprocal of Age to Ultimate Factor
1	1	1	29	29	1
2	1.0357	1.0357	52	53.86	.9655
3	1.0390	1.0761	54	58.11	.9293
4	1.1909	1.2815	29	37.16	.7803
5	2.3966	3.0712	11	33.78	.3256
Total			175	211.91	4.0007

Method #4a ((Sum of Column 11)÷ 5) x (($243,633 ÷ 29
+ 636,252 ÷ 52
+ 733,341 ÷ 54
+ 222,841 ÷ 29
+ 185,689 ÷ 11) ÷ 5) = $498,263

Method #5a ((Sum of Column 4) ÷ (Sum of Column 12)) = $505,351

(13) Accident Year	(14) $(1 - (12)) \times \frac{211.91}{5}$	(15) (10) + (14)	(16) (15) x (5) ÷ (10)
1	0	29	$243,633
2	1.46	53.46	654,116
3	3.00	57.00	774,082
4	9.31	38.31	294,381
5	28.58	39.58	668,143

Method #6 Column 16 projected to accident year 6 by linear least squares = $673,657.

APPENDIX C

Comparison of "adjusting, then averaging" vs "averaging, then adjusting"

Let X_i be a random variable representing observed losses for accident year i

Assume that these losses arise from distributions with expected values that are constant over time, except for an adjustment factor. This adjustment factor can represent either a loss development factor or a trend factor or both.

So $X_i = \frac{\mu}{a_i} + \epsilon_i \quad i=1,\ldots,n$

where μ = underlying expected losses

a_i = non-random adjustment factor (≥ 1)

ϵ_i = random error $E(\epsilon_i) = 0$, $Var(\epsilon_i) = \sigma_i^2$

We wish to estimate μ

Let $\hat{\mu}_1 = \frac{1}{n} \sum_{i=1}^{n} X_i a_i$

This represents trending (and/or developing) known losses for each year and averaging the results

Let $\hat{\mu}_2 = \left(\sum_{i=1}^{n} X_i\right) \div \left(\sum_{i=1}^{n} \frac{1}{a_i}\right)$

This represents the "adjustment to total known losses method"

It is easy to see that both $\hat{\mu}_1$ and $\hat{\mu}_2$ are unbiased, ie

$E(\hat{\mu}_1) = E(\hat{\mu}_2) = \mu$

Calculate the Best Linear Unbiased Estimate (B.L.U.E.)[31] of μ

That is, find weights c_i, such that $\hat{\mu}\left(=\sum_{i=1}^{n} c_i X_i\right)$ is unbiased and has minimum variance.

So minimize $\operatorname{Var}\left(\sum_{i=1}^{n} c_i X_i\right)$ subject to $E\left[\sum_{i=1}^{n} c_i X_i\right] = \mu$

$$\operatorname{Var}\left(\sum_{i=1}^{n} c_i X_i\right) = \sum_{i=1}^{n} c_i^2 \sigma_i^2$$

$$E\left[\sum_{i=1}^{n} c_i X_i\right] = \sum_{i=1}^{n} \frac{c_i}{q_i}\mu = \mu \Rightarrow \sum_{i=1}^{n} \frac{c_i}{q_i} = 1$$

Let

$$L = \sum_{i=1}^{n} c_i^2 \sigma_i^2 + \lambda\left(1 - \sum_{i=1}^{n} \frac{c_i}{q_i}\right)$$

$$\frac{\partial L}{\partial c_i} = 2c_i\sigma_i^2 - \frac{\lambda}{q_i} = 0 \qquad i = 1, \cdots, n$$

So $c_i = \dfrac{\lambda}{2q_i\sigma_i^2}$, $\displaystyle\sum_{i=1}^{n} \frac{c_i}{q_i} = \sum_{i=1}^{n} \frac{\lambda}{2q_i^2\sigma_i^2}$

So $\lambda = \dfrac{2}{\displaystyle\sum_{i=1}^{n} \frac{1}{q_i^2\sigma_i^2}}$

So $c_i = \dfrac{1}{q_i\sigma_i^2} \dfrac{1}{\displaystyle\sum_{j=1}^{n} \frac{1}{q_j^2\sigma_j^2}}$

Now consider various possibilities for σ_i^2

1. Let $X_i q_i = \mu + e_i$ where $\operatorname{Var}(e_i) = \sigma^2 \;\; \forall i$

   This means that $\epsilon_i = \dfrac{e_i}{q_i}$ so $\sigma_i^2 = \dfrac{1}{q_i^2}\sigma^2$ so $c_i = q_i$

   Therefore $\hat{\mu}_1$, is the BLUE

2. Let $\dfrac{\operatorname{Var}(X_i)}{E[X_i]} = k \;\; \forall i$

   So

$$\frac{\sigma_i^2}{\mu/q_i} = \frac{\sigma_i^2 q_i}{\mu} \Rightarrow \sigma_i^2 q_i = k\mu$$

31. The approach of calculating the B.L.U.E. was suggested by Aaron Tenenbein, Associate Professor, Statisics and Actuarial Science.

This means that $c_i = 1 \Big/ \sum_{i=1}^{n} \frac{1}{q_i}$

Therefore $\hat{\mu}_2$ is the BLUE

As was discussed in the results section, $\hat{\mu}_2$ performed better than $\hat{\mu}_1$, in the simulation.

ADDITIONAL REFERENCES

Bicker, Peter and Doksum, Kjell, Mathematical Statistics: Basic Ideas and Selected Topics, Holden-Day, 1977

Fishman, George, Concepts and Methods in Discrete Event Digital Simulation, John Wiley & Sons, 1973