A MODEL FOR CALCULATING MINIMUM SURPLUS REQUIREMENTS

By Robert J. Finger

Over the past several years a number of insurance companies have been formed by doctors, hospitals and lawyers to insure professional liability coverage. These companies typically operate in one state and write primarily one subline of insurance (i.e., professional liability). In addition, these companies may insure only a few thousand doctors, lawyers or hospital beds. A significant consideration in the formation of these companies is the amount of surplus required to establish a viable insurance operation. This paper describes a method of calculating a minimum amount of required surplus. In this paper the term "surplus" includes both capital and surplus, as does the common term "policyholder's surplus." The approach is adapted to insurers that write one line of business. For multiple-line carriers there are additional important considerations (such as covariation between the different lines) and also aspects that have much less significance (such as skewness and uncertainty in the adequacy of the rate level). This method may also be useful for estimating surplus requirements for captive insurers and analyzing the fluctuation in the underwriting experience of self-insurers.

FORMULATION OF THE MODEL

Insurance serves a useful social purpose by reducing uncertainty. For the insured a (normally) fixed premium is exchanged for the reimbursement of contingent future claims. For the insurer, the aggregate risk or uncertainty in the total potential claims payments is a much lower percentage of the expected losses than for the individual insured. This result can be explained by several theorems of mathematical statistics. First, the law of large numbers holds that the mean of a sample of independent random variables will converge to the mean of the means of the random variables, for an arbitrarily large number of variables.[1] The basic assumptions are that the variables are independent and that they have finite variances. Second, the central limit theorem holds that the distribution of the sum of a series of independent random variables converges to a normal distribution, for an arbitrarily large number of variables.[2] Finally, the variance of the sum of uncorrelated random variables is equal to the sum of the variances of the individual variables.[3] In other words, the variances of a series of random variables are additive.

For most practical insurance situations, the necessary assumptions of independence and finite variance are not too limiting. In terms of professional liability, a claim

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[1] Wilks, Samuel S., Mathematical Statistics. New York: John Wiley & Sons, Inc., 1962, pp. 99 and 108.

[2] Ibid., p. 257.

[3] Ibid., p. 83.

generally occurs through some negligent act or omission of a covered insured toward a client. Claims are relatively infrequent. For most states, claim frequency runs 10% to 20% or less per year for doctors and lawyers, and per occupied hospital bed. Since both claimants and insureds are virtually always different from claim to claim and since each claim generally arises out of unique circumstances, there would appear to be a very low correlation between different claims. The assumption of a finite variance is certainly reasonable since, even with recently reported jury verdicts, every individual claim size is finite.

The above three theorems form the basis for the analysis of aggregate claim fluctuations. The central limit theorem indicates that the distribution of the sum of individual claim sizes will be approximately normal. Numerical approximations, such as the Cornish-Fisher expansion, are available to adjust for departures from normality. The law of large numbers indicates that the mean of the sum is the sum of the means of individual claims. The additivity property of the variances of independent claims allows the calculation of the variance of the aggregate sum of individual claims.

Insurer surplus is required to offset adverse fluctuations and business decisions, so as to maintain the insurer's solvency. A fairly common formulation is that surplus is required to offset fluctuations in asset values, deficient

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loss reserves and adverse underwriting results.[4] For professional liability insurance, with its typically delayed reporting and settlement patterns, errors in ratemaking tend to be highly correlated with errors in reserving (particularly when including I.B.N.R. reserves in the latter). In other words, it is simpler and possibly more accurate for analysis to combine fluctuations in both loss reserves and underwriting results. The thrust of this paper deals with fluctuations in reserves and underwriting. This paper offers no particular theory for handling fluctuations in asset values. For a practical solution, however, reserves are discounted at a risk-free rate of return. This approach is accurate when the insurer invests its reserves in risk-free securities. If the insurer chooses riskier investment alternatives, there will be wider variations in the insurer's surplus from time to time. The insurer, however, will earn a greater investment return and will consequently have a larger surplus so that the probability of insolvency may not be any higher than with risk-free investments.

Mathematical risk theory has devised methods for analyzing and calculating the probability of ruin. The problem of minimum surplus requirements can be made equivalent to a probability of ruin formulation. That is, the minimum

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[4] See, for example, Hofflander, Alfred E., "Minimum Capital and Surplus Requirements for Multiple Line Insurance Companies: A New Approach." Printed in Insurance, Government and Social Policy, Kimball, Spencer L., and Herbert S. Denenberg, eds., Homewood, Illinois: Richard D. Irwin, 1969.

surplus can be set at a level which allows for less than a certain probability (e.g., 1%) that the insurer will become insolvent. The traditional ruin theory tends to look at an on-going insurance operation and calculates the probability that the insurer will become insolvent at any future time.[5] Two practical problems with this approach in professional liability insurance are that claims are not reported very promptly and once reported, are not settled for several years. Thus the insurer must establish fairly sizeable I.B.N.R. (on occurrence coverage) and case reserves. The insurer could thus be technically insolvent at some point on an accrual basis, but unaware of this, could generate sufficient surplus (from underwriting or investment profits or surplus contributions) to pay all claims in perpetuity. Since the insurer cannot feasibly determine its surplus at any point in time with great accuracy, another approach to ruin probability is taken.

Generally, an insurer becomes insolvent when its claim payments or liabilities plus expenses are larger than premiums plus surplus. (This discussion omits consideration of asset value fluctuations.) This point of view suggests defining surplus requirements as some fixed relationship to premiums. For professional liability, however, reserves will normally be larger than premiums. Reserves are made up of an I.B.N.R.

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[5]See, for example, Seal, Hilary L., Stochastic Theory of a Risk Business. New York: John Wiley & Sons, Inc., 1969, pp. 90-134.

provision, case reserves and the unearned premium reserve. The first and last are usually determined by some sort of formula and the case reserves are often inaccurate due to the several-year lag between reporting and payment. For none of the three reserve categories is the liability estimate particularly accurate. It is thus reasonable to assume that the aggregate reserves may fluctuate about a mean value much as the sum of the individual claims would fluctuate about the sum of the individual means. It thus becomes apparent that the surplus is required to offset fluctuations in the aggregate reserve. If the surplus is sufficient to offset adverse fluctuations, except with probability 1%, it can be said that the probability of ruin is 1%.

The ruin model for this paper is thus formulated. The model takes the aggregate reserves (I.B.N.R., case, and unearned premium) as of a point in time and treats the aggregate as being composed of individual claims. The distribution of the aggregate reserve is approximately normal, since it represents the sum of essentially independent random variables. The surplus requirement is determined by finding the 99th percentile of the aggregate distribution and subtracting away the reserve. For example, the 99th percentile of the standard normal occurs at 2.33. If the expected value of the aggregate reserve is $10 million and the standard deviation is $1 million, the 99th percentile occurs at $12.33 million. The surplus requirement would thus be $2.33 million, assuming a normal distribution of the aggregate reserves and a 1% ruin probability.

In other words, the minimum surplus requirement is defined to be the amount that, when added to aggregate reserves at a point in time, equals the 99th percentile of the aggregate reserve distribution.

In applying this approach to practical problems, two specific difficulties arise, which have not been treated with complete success in mathematical risk theory. First, insurers universally purchase reinsurance. This paper deals only with excess of loss reinsurance. Such reinsurance will significantly reduce the coefficient of variation (CV) and skewness of the aggregate reserve distribution. Indeed, the choice of the excess of loss reinsurance retention (relative to the mean of the unlimited claim size distribution) has a significant impact on the aggregate reserve distribution and thus on the required surplus. In order to produce practical results, it is assumed that the claim size distribution of individual claims is log-normal. This assumption has proved reasonable for applications in professional liability insurance.[6] A series of simulations was then produced to determine the reduction in CV and skewness for different choices of retentions for log-normal distributions with various CV's. The methodology is explained more fully in later sections. With the exhibits presented in this paper it is thus possible to calculate aggregate reserve distributions for various choices of reinsurance retentions, under a log-normal claim

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[6]See, for example, Finger, R. J., "Estimating Pure Premiums by Layer—An Approach." PCAS LXIII (1976), p. 34.

size distribution assumption. Reinsurance retentions are defined as multiples of the unlimited average claim size (also termed the unlimited severity).

The second practical difficulty in applying risk theory approaches to this problem concerns the uncertainty in the mean value of the aggregate reserve distribution, That is, traditional risk theory methods calculate fluctuations in aggregate claim levels when the mean is known. As a practical matter, the mean is not known accurately for professional liability insurance. There is uncertainty in at least four specific areas, which produce uncertainly in at least the I.B.N.R., and unearned premium reserves; these are: 1) the trend in claim frequency, 2) the trend in claim severity, 3) the appropriateness of loss development factors, and 4) consistency in the underlying mix and quality of business. Thus, in addition to fluctuations in the aggregate reserve value, given a known mean, there is also uncertainty in the mean value. Surplus is required not only for statistical fluctuations, but for mean-value uncertainty as well. For example, assume that the aggregate reserves are $10 million, as above. The 99th percentile of the aggregate reserve is thus $12.33 million, as above, when $10 million is the true mean. It is possible, in professional liability insurance, however, that the true mean is $8 million or $12.5 million. If the true mean is $12.5 million, there is clearly more than a 1% probability of the aggregate reserve exceeding $12.33 million. Thus, in a practical situation, two types of uncertainty must be considered.

In order to keep the concepts distinguished, special terminology will be used. The aggregate reserve distribution will represent the total variations due to both statistical fluctuations and uncertain mean. The conditional aggregate reserve distribution will represent fluctuations about a known mean. The surplus required for the conditional case will be termed the fluctuation reserve. In developing the mathematics there are three different distributions, which will be denoted by subscripts T, L, and LT. First there is the distribution of individual claim sizes, unlimited by any excess of loss reinsurance.$_{(T)}$ Second, there is the first distribution limited by reinsurance.$_{(L)}$ Finally, there is the aggregate distribution, due to fluctuations in both claim count and claim sizes, with claim sizes limited by reinsurance.$_{(LT)}$ This final distribution is the conditional aggregate reserve distribution.

For this paper, both types of uncertainty are combined by assuming that each is a log-normal variable. When two log-normal variables are multiplied together, the product is log-normal. Making the log-normal assumption and knowing the CV's of the two variables, one can calculate the CV of the product.

The log-normal assumption is reasonable for the uncertainty about the mean of the aggregate reserve distribution. Log-normal variables arise naturally where a large number of independent variables are multiplied together.[7] The

[7] Ibid., p. 38.

aggregate reserve may be thought of as a product of several independent variables, such as frequency trend factor, severity trend factor, loss development factor and quality of business factor. The log-normal distribution has the property that about two-thirds of the probability lies between the range determined by the mean divided by and multiplied by 1.0 plus the CV. For example, with a CV of 0.25, two-thirds of the probability lies between 0.8 and 1.25 times the mean. This property gives a practical way of estimating and utilizing the uncertainty about the mean of the aggregate reserve. For example, assume that there is a two-thirds chance that the true mean lies between 0.8 and 1.25 times the estimated mean. This implies a log-normal distribution of uncertainty about the mean with a CV of 0.25.

The log-normal assumption is not particularly appropriate for the conditional aggregate reserve distribution. The central limit theorem indicates that the conditional aggregate reserve distribution is approximately normal, not log-normal. Further, with reinsurance, the claim size distribution is truncated and no longer strictly log-normal. Nevertheless, the assumption of log-normality should be conservative (i.e., provide greater skewness) and is probably accurate enough for practical purposes.

PRACTICAL EXAMPLES

Before discussing the mathematical theory behind the model, two practical examples are illustrated. These are shown in Exhibit I. The CV of the individual claim size distribution (CV_T) is assumed to be 2.0. As will be presented later, for smaller retentions the CV does not have a very significant impact. The examples assume that the expected losses for the latest accident year (just ended) are composed of 100 claims and an unlimited average claim size of $75,000. (The average size is computed more accurately by eliminating claims closed without payment. If these claims are included, the CV will be larger, the mean lower and the fit of the log-normal, most likely, not as good.) With an expected loss ratio of 80%, the direct annual premium is about $9.4 million.

The two examples differ in the retention. Case I assumes a retention of $150,000, or 2.0 times the unlimited severity. Case II assumes a retention of $375,000, or 5.0 times the unlimited severity. Utilizing log-normal tables for CV=2.0 (Figure 1) it is determined that 30% of the losses will be ceded in Case I and 12% in Case II. Typical reinsurance premiums might be 35% in Case I and 15% in Case II. From these reinsurance factors, one calculates net expected losses and net premiums.

In a practical situation, one could begin with the actual reserves at year-end. For this paper a steady-state assumption is made. That is, with the payment pattern shown in Exhibit II and a 15% annual trend in pure premiums, the steady-state reserves will be 2.28 times the expected (undiscounted) losses in the latest year. The reserves are discounted at 5% interest to reflect the risk-free investments. In a steady-state situation, then, the reserves will be 2.28 times the latest year's expected losses. It is also assumed that the average claim size in the reserve equals the average incurred claim size in the most recent accident year. (In a practical situation this assumption might be adjusted appropriately.) For both cases, then, there will be an expected value of 228 claims in the aggregate reserve.

The 99th percentile of the conditional aggregate reserve distribution (i.e., given a known mean) may be found from the following formula:

$$\text{deviation} = CV_{LT} \left[Z_{.99} + \frac{\gamma_{LT}}{6} (Z^2_{.99} - 1) \right] \qquad (1)$$

The above deviation is expressed as a fraction of expected losses (i.e., aggregate reserves). This formula incorporates the Cornish-Fisher expansion to the skewness term.[8] $Z_{.99}=2.33$. CV_{LT} is the CV of the aggregate distribution, limited by excess of loss reinsurance. γ_{LT} is the skewness coefficient (i.e., the third central moment divided by the standard deviation cubed) of the aggregate distribution, limited by reinsurance. Both CV_{LT} and γ_{LT} vary inversely with the square root of the expected number of claims, λ, in the reserve. Figure 2 depicts $\sqrt{\lambda}\, CV_{LT}$ and Figure 3 depicts $\sqrt{\lambda}\, \gamma_{LT}$. Both of these factors vary by CV of the individual claim size distribution (which is assumed to be 2.0 in these examples) and the reinsurance retention. The fluctuation reserve (i.e., surplus requirement given a known mean) is .236 times the reserve in Case I and .287 in

[8]Mayerson, Allen L., Donald A. Jones, and Newton L. Bowers, Jr., "On the Credibility of the Pure Premium." PCAS LV (1968), p. 178.

Case II.

The next step is to adjust for uncertainty in the mean aggregate reserve value. It is assumed that there is a two-thirds chance that the aggregate reserve falls between .83 and 1.20 its stated value. That is, that the CV of the uncertainty in the mean value is .20. Using Exhibit III one can calculate the additional surplus needed to cover uncertainty in the expected aggregate reserve. The percentage figures in Exhibit III are calculated as a ratio to the larger CV. Adding both surplus provisions together yields minimum surplus requirements of .316 (Case I) and .356 (Case II) of the stated reserves. (It is assumed that stated reserves are equal to their expected value.) Equivalently, the maximum net premium to surplus ratios are 1.61 (Case I) and 1.47 (Case II).

DERIVATION OF FORMULAS

The basic formula for calculating the variance of the conditional aggregate reserve distribution is:[9]

$$\text{var}_T = \mu_2\lambda + \mu^2\lambda_2$$

where μ_2 is the second central moment of the claim size distribution, μ is the mean of the claim size distribution and λ_2 and λ similarly describe the claim frequency distribution. The above formula holds when the frequency and severity are independent. This assumption may not strictly hold given deductibles, nuisance claims and inconsistent handling of claims closed without payment. It should nevertheless prove adequate

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[9] *Ibid.*, p. 179.

to estimate parameters based on observed values and assume that subsequent statistics are maintained on a consistent basis.

The second central moment of the claim frequency distribution is not easy to measure. In many situations it is assumed that the claim count process is a Poisson process, where $\lambda_2=\lambda$. This assumption will only be true when each exposure unit has the same claim frequency. In practice this is probably true only in unusual circumstances. Surgeons, for example, have frequencies about triple physicians. In automobile insurance the negative binomial distribution has proven a better fit to claim frequencies than the Poisson.[10] The negative binomial distribution derives naturally from a situation where individual exposure units sustain claims according to a Poisson process, but individual frequencies are gamma-distributed. In general, λ_2 is larger than λ. Hansen[11] has shown that λ_2 can be bounded above by an exponential structure function. In fact, λ_2 is bounded by $\lambda(1+f)$, where f is the average frequency in the exposed population. For the Figures and calculations in this paper $\lambda_2=1.2\lambda$ was chosen. It will be shown that this assumption has relatively little significance to the final result.

The CV of the aggregate limited distribution is now

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[10] Dropkin, Lester B., "Some Considerations on Automobile Rating Systems Utilizing Individual Driving Records." PCAS XLVI (1959), p. 165.

[11] Hansen, Ernest J., "A Note on Full Credibility For Estimating Claim Frequency." PCAS LIX (1972), p. 51.

calculated. Figure 4 depicts the effect on the CV of excess of loss reinsurance for various retentions. These graphs were developed by randomly-generated log-normal variables. A sample of variables (e.g., 1,000) was generated. For selected retentions (e.g., 1, 2, 3, 4, 5, . . .) the individual claim values were limited and the limited CV was calculated. Figure 4 depicts α, the ratio of the limited CV (CV_L) to the unlimited CV (CV_T). For example, for CV_T=2.0 and a retention of 10 times the unlimited mean CV_L=(.81)2.0=1.62. The graphs are approximate, since it was very difficult to generate samples with actual CV close to the theoretical CV.

Returning to the basic formula $var_{LT}=\mu_{2,L}\lambda+\mu_L{}^2\lambda_2$ where L subscripts denote the limited distribution, but

$$\lambda_2=1.2\lambda$$

$$\frac{\mu_{2,L}}{\mu_L{}^2}=CV_L{}^2=(\alpha CV_T)^2$$

thus $$var_{LT}=(\alpha\mu_L CV_T)^2\lambda+\mu_L{}^2 1.2\lambda$$

$$CV_{LT}=\left(\frac{var_{LT}}{\mu_{LT}{}^2}\right)^{1/2}=\frac{var_{LT}{}^{1/2}}{\mu_L\lambda}$$

$$CV_{LT}=\frac{\sqrt{\lambda}\mu_L[(\alpha CV_T)^2+1.2]^{1/2}}{\lambda\mu_L}$$

$$CV_{LT}=\frac{[(\alpha CV_T)^2+1.2]^{1/2}}{\sqrt{\lambda}}$$

The λ_2 assumption has relatively little significance. Assume, for example, that α=.81 and CV_T=2.0. For $\lambda_2=1.2\lambda$, $\sqrt{\lambda}CV_{LT}$=1.96. For $\lambda_2=1.3\lambda$, $\sqrt{\lambda}CV_{LT}$=1.98. For $\lambda_2=1.1\lambda$, $\sqrt{\lambda}CV_{LT}$= 1.93. Thus, for these examples a 10% change in λ_2 has only about a 1% effect on CV_{LT}.

The skewness of the aggregate limited distribution is calculated in a similar manner. Figure 5 shows the reduction in the skewness coefficient, β, for various CV and reinsurance retentions. If γ_T is the skewness of the unlimited claim size distribution and γ_L is the skewness of the limited claim size distribution, then $\beta=\frac{\gamma_L}{\gamma_T}$. A derivation for the skewness of the aggregate limited distribution is shown in Appendix A.3.

$$\gamma_{LT}=\frac{1}{\sqrt{\lambda}}\ \frac{\beta(CV_T^{2}+3)CV_T(\alpha CV_T)^3+3(\alpha CV_T)^2+1}{[(\alpha CV_T)^2+1]^{3/2}}$$

For this derivation a Poisson distribution was assumed for the claim frequency. That is, $\lambda=\lambda_2=\lambda_3$. Little is gained by making λ_2 larger than λ since λ_2 appears in both numerator and denominator.

At this point the fluctuation reserve can be calculated from the limited CV and limited skewness, as given by equation (1). Exhibit IV shows the fluctuation reserve for probability of 1% (i.e., Z=2.33), for expected claim counts of 50, 100, 250, 500 and 1,000. Figure 6 graphically depicts the fluctuation reserve for reserves with 50 claims. From Exhibit IV and from Figures 2 and 3 one can see that fluctuations do not vary a great deal, at lower attachments, for different CV's of the claim size distribution. This is due to the impact of reinsurance in cutting down on large claims. It is thus clear that the retention is a far more crucial variable than is the CV of the individual claim size distribution in determining surplus requirements.

The final theoretical step is to adjust for uncertainty in the mean value of the expected losses (or equivalently in the mean value of the aggregate reserves). Appendix A.2 demonstrates that the multiplication of two log-normal variables yields a log-normal variable. There is thus a relatively simple formula which provides the CV of the product of two log-normals, given the CV's of the individual variables. For current purposes, one CV is the CV of the conditional aggregate reserve distribution. The other represents the uncertainty in the mean value of the aggregate reserve. Exhibit III shows the CV of the product of two log-normal variables as the increase in the larger of the two CV's. For example, assume both CV's are 0.30. According to Exhibit III the CV of the product will be (0.30)(1.45)=.44. As might be expected, when one CV is significantly larger than the other, the CV of the product is not much larger than the larger CV.

To determine deviations in the aggregate reserve, for both statistical fluctuations and uncertainty in mean value, the CV of the product of two CV's, representing these items, may be substituted in equation (1). In effect, the percentage increase in surplus requirements due to uncertainty in the mean, is the percentage increase shown in Exhibit III for the product of two log-normal variables. (Note, however, that values in Exhibit III apply against the larger CV. In some cases [e.g., large numbers of claims] the statistical fluctuation may be less than the uncertainty in the mean.)

CONCLUSION

This paper has presented a method for calculating minimum surplus requirements. The method is tailored to a monoline insurer, captive or self-insurer. The minimum surplus requirements is defined to be the amount which, when added to aggregate reserves (case, I.B.N.R. and unearned premium), equals the 99th percentile of the aggregate reserve distribution. That is, the aggregate reserve is treated as a random variable, a sum of its individual claims. There are two types of variations in the aggregate reserve: 1) statistical fluctuations in the number and size of claims about a given mean value and 2) uncertainty in the mean (or stated) reserve. Variations about a known mean are calculated by assuming: 1) a log-normal distribution for individual claim sizes and 2) independence between the number of claims and the individual claim size distribution. Variations are calculated for various excess of loss reinsurance retentions. Reinsurance has a significant impact on surplus requirements, since it can greatly affect the CV and skewness of the claim size distribution. Conversely, different assumptions for the CV of the claim size distribution have relatively little impact on surplus requirements, for smaller retentions. Surplus requirements are roughly proportionate to the reciprocal of the square root of the number of claims in the aggregate reserve. Uncertainty about the mean of the aggregate reserve value can be combined with the fluctuations about a known mean by assuming that both are log-normal variables. The combined variation is a log-normal variable, since a product of two log-normal variables is also log-normal. The combined variation is then used in a formula, which also incorporates the Cornish-Fisher expansion, to derive the 99th percentile of the aggregate reserve distribution. The result is the minimum surplus requirement, which is expressed as a fraction of the aggregate reserve.

APPENDIX A

A.1 THE LOG-NORMAL DISTRIBUTION

Density Function $\frac{1}{\sqrt{2\pi}\sigma x} e^{\frac{-(\log x-\mu)^2}{2\sigma^2}}$ $0 \leq x < \infty$

Mean $\mu' = e^{\mu+\sigma^2/2}$

Variance $\sigma'^2 = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$

Coefficient of Variation $CV = \frac{\sigma'}{\mu'} = \sqrt{e^{\sigma^2}-1}$

Skewness $\gamma = \frac{E(x-\mu)^3}{\sigma^3} = (e^{\sigma^2}+2)\sqrt{e^{\sigma^2}-1}$

$= (CV^2+3)CV$

A.2 MULTIPLICATION OF TWO LOG-NORMAL VARIABLES

Let y_1 and y_2 be independent log-normal variables

Let $X = (y_1)(y_2)$

Then $\ln X = \ln y_1 + \ln y_2$

Sums of independent normal random variables are normally-distributed with mean equal to the sum of the individual means and variance equal to the sum of the individual variances.[1]

Since $\ln y_1$ and $\ln y_2$ are normally-distributed, $\ln X$ is also normally distributed. Thus X is log-normally distributed.

For a log-normal variable, if the log is distributed with mean μ and variance σ^2, the log-normal variable is distributed with mean, $M=e^{\mu+\frac{1}{2}\sigma^2}$ and $CV= (e^{\sigma^2}-1)^{\frac{1}{2}}$.[2] The CV is independent of the mean.

Let CV_1 be the CV for y_1 and CV_2 similarly for y_2.

$CV_1 = (e^{\sigma_1^2}-1)^{\frac{1}{2}}$ (for log-normal variables)

$\sigma_1^2 = \ln(CV_1^2+1)$ (on rearrangement) .

Thus $\sigma^2_{\ln X} = \ln(CV_1^2+1) + \ln(CV_2^2+1)$

(since the variances add and ln X is normal)

Thus $CV_X = (e^{\ln(CV_1^2+1) + \ln(CV_2^2+1)}-1)^{\frac{1}{2}}$

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[2]See Appendix A.1.

[1]Parzen, E. Modern Probability Theory and Its Applications. New York: John Wiley & Sons, Inc., 1960. Page 406.

A.3 SKEWNESS OF THE LIMITED DISTRIBUTION

Basic Formula: $\gamma = \dfrac{E[T-E(T)]^3}{(E[T-E(T)]^2)^{3/2}}$

Given: $\beta = \dfrac{\gamma_L}{\gamma_T}$ reduction in skewness coefficient of limited claim size distribution

$\alpha = \dfrac{CV_L}{CV_T}$ reduction in CV of claim size distribution

$\Gamma = \dfrac{\mu_L}{\mu_T}$ reduction in mean of claim size distribution

Other Formulas:

$$\begin{cases} E[T-E(T)]^2 = \mu_2\lambda + \mu^2\lambda_2 \\ E[T-E(T)]^3 = \mu_3\lambda + 3\mu_2\mu\lambda_2 + \mu^3\lambda_3 \\ CV = \dfrac{\sigma}{\mu} \\ \gamma_T = (CV_T^2+3)CV_T \qquad \text{log-normal distribution} \end{cases}$$

Assumptions: $\lambda_1 = \lambda_2 = \lambda_3$ (true for Poisson process)

Derivation: 1. $E_L[T-E(T)] = \mu_{3,L}\lambda + 3\mu_{2,L}\mu_L + \mu_L{}^3\lambda$

$$\gamma_L = \frac{\mu_{3,L}}{\sigma_L{}^3}$$

$$\begin{aligned} \mu_{3,L} &= \gamma_L\sigma_L{}^3 \\ &= \beta\gamma_T\sigma_L{}^3 \\ &= \beta(CV_T{}^2+3)CV_T(CV_L\mu_L)^3 \\ &= \beta(CV_T{}^2+3)CV_T(\alpha CV_T\Gamma\mu)^3 \end{aligned}$$

$$\begin{aligned} \mu_{2,L} &= (CV_L\mu_L)^2 \qquad (\text{note: } \sigma = \mu_2) \\ &= (\alpha CV_T\Gamma\mu)^2 \end{aligned}$$

$$E_L[T-E(T)]^3 = \lambda[\beta(CV_T{}^2+3)CV_T(\alpha CV_T\Gamma\mu)^3 + 3(\alpha CV_T\Gamma\mu)^2\Gamma\mu + (\Gamma\mu)^3]$$

2. $E_L[T-E(T)]^2 = \mu_{2,L}\lambda + \mu_L{}^2\lambda$

$$= \lambda[(CV_L\mu_L)^2 + \mu_L{}^2]$$

$$= \lambda(\Gamma\mu)^2[(\alpha CV_T)^2+1]$$

3. $$\gamma_{LT} = \frac{\lambda(\Gamma\mu)^3[\beta(CV_T{}^2+3)CV_T(\alpha CV_T)^3+3(\alpha CV_T)^2+1]}{(\lambda(\Gamma\mu)^2[(\alpha CV_T)^2+1])^{3/2}}$$

$$= \frac{1}{\sqrt{\lambda}}\,\frac{\beta(CV_T{}^2+3)CV_T(\alpha CV_T)^3+3(\alpha CV_T)^2+1}{[(\alpha CV_T)^2+1]^{3/2}}$$

Exhibit I

CALCULATION OF SURPLUS REQUIREMENTS
(Assuming CV_T=2.0)

1.	CURRENT YEAR PREMIUMS			
	a.	Claim Count	100	
	b.	Direct Average Claim Size	$75,000	
	c.	Direct Expected Losses (1a x 1b)	7.50 million	
	d.	Expected Loss Ratio	80%	
	e.	Direct Premiums (1c ÷ 1d	9.38 million	
2.	REINSURANCE		CASE I	CASE II
	a.	Retention	$150,000	$375,000
	b.	Times Unlimited Mean (2a ÷ 1b)	2.0	5.0
	c.	Net Expected Losses (Figure 1)	.70	.88
	d.	(1c x 2c)	5.25 million	6.60 million
	e.	Net Premiums	.70	.90
	f.	(1e x 2e)	6.57 million	8.44 million
3.	FLUCTUATION RESERVE			
	a.	Reserves to latest year (Exhibit 2) Expected Losses	2.28	2.28
	b.	Net Reserves (2d x 3a)	11.97 million	15.05 million
	c.	Reserve Claim Count (1a x 3a)	228	228
	d.	$\sqrt{\lambda}CV_{LT}$ (Figure 2)	1.46	1.72
	e.	$\sqrt{\lambda}\gamma_{LT}$ (Figure 3)	2.16	3.80
	f.	CV_{LT} (3d ÷ $\sqrt{3c}$)	.097	.114
	g.	1% Level (2.33 + [3e ÷ $\sqrt{3c}$] .738)	2.44	2.52
	h.	Deviation (3f x 3g)	.236	.287

4.	**ADDITIONAL UNCERTAINTY DUE TO UNKNOWN MEAN**		
a.	Uncertainty about mean	.20	.20
b.	Increase in CV_{LT} (Fxhibit III)	34%	24%
c.	Total Deviation (3h x [1+4b])	.316	.356
5.	**REQUIRED SURPLUS**		
a.	Aggregate Amount (3b x 4c)	3.78 million	5.36 million
b.	Net Premium to Surplus Ratio (2f ÷ 5a)	1.74	1.58
c.	Surplus to Net Premium Ratio (5a ÷ 2f)	.58	.63
d.	Surplus to Net Reserves Ratio (4c)	.316	.356

Exhibit II

RELATIONSHIP OF RESERVES AND EXPECTED LOSSES AT STEADY STATE

Year	Payment Pattern (once reported)	Unpaid (year end)	15% Annual Trend: Relative Value	15% Annual Trend: Discount Factor (5% interest)	15% Annual Trend: Reserve	25% Annual Trend: Relative Value	25% Annual Trend: Reserve
1	5	95	1.000	.882	83.8	1.000	83.8
2	10	85	.870	.914	67.6	.800	62.2
3	20	65	.756	.941	46.2	.640	39.1
4	30	35	.658	.956	22.0	.512	17.1
5	20	15	.572	.976	8.4	.410	6.0
6	15	0	-	-	-	-	-
Total	100	295	.846	.913	228.0	.775	208.2

Exhibit III

CV OF PRODUCT OF TWO LOG-NORMAL VARIABLES
(AS PERCENT INCREASE IN LARGER CV)

	COEFFICIENT OF VARIATION														
CV	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50	.60	.70	.80	.90	1.00
.05	42	12	5	3	2	2	1	1	1	1	1	-	-	-	-
.10		42	21	13	8	6	5	4	3	2	2	2	1	1	-
.15			43	26	18	13	10	8	6	5	4	3	3	2	2
.20				43	30	22	17	14	11	10	7	6	5	4	4
.25					44	33	25	21	17	15	11	9	8	7	6
.30						45	35	29	24	20	16	13	11	10	9
.35							46	38	32	27	21	17	15	13	12
.40								47	40	34	27	22	19	17	15
.45									48	42	33	27	23	21	19
.50										50	40	33	28	25	23
.60											54	45	39	34	31
.70												58	50	45	41
.80													63	56	51
.90														68	62
1.00															73

FLUCTUATION RESERVE
(SURPLUS REQUIREMENT WITH KNOWN MEAN RESERVE)

P( ) = 1%
Z = 2.33

$$\text{deviation} = CV_{LT}\left[Z_{.99} + \frac{\gamma_{LT}}{6}\left(Z_{.99}^{2}-1\right)\right]$$

Retention (times unlimited mean) CV_T=	CV_{LT} 1.5	2.0	2.5	3.0	γ_{LT} 1.5	2.0	2.5	3.0	Deviation 1.5	2.0	2.5	3.0
I. 50 CLAIMS												
1	.184	.190	.199	.208	.205	.223	.247	.269	.456	.473	.501	.526
2	.202	.206	.215	.223	.288	.305	.361	.415	.514	.528	.558	.589
5	.236	.243	.249	.256	.517	.537	.642	.782	.640	.663	.698	.744
10	.263	.276	.286	.291	.764	.792	.995	1.238	.761	.804	.875	.945
Ult.*	.263	.322	.386	.451	.829	1.581	2.761	4.472	.774	1.128	1.686	2.540
II. 100 CLAIMS												
1	.130	.134	.141	.147	.145	.158	.175	.190	.317	.328	.347	.363
2	.143	.146	.152	.158	.204	.216	.255	.293	.355	.363	.383	.402
5	.167	.172	.176	.181	.366	.380	.454	.553	.434	.449	.469	.496
10	.186	.195	.202	.206	.541	.560	.704	.876	.508	.531	.576	.613
Ult.*	.186	.228	.273	.319	.586	1.118	1.952	3.162	.514	.719	1.029	1.488
III 250 CLAIMS												
1	.082	.085	.089	.093	.092	.100	.110	.120	.197	.204	.215	.225
2	.090	.092	.096	.100	.129	.136	.162	.186	.219	.224	.235	.247
5	.106	.109	.111	.114	.231	.240	.287	.350	.264	.273	.283	.296
10	.118	.123	.128	.130	.342	.354	.445	.554	.304	.320	.340	.357
Ult.*	.118	.144	.173	.202	.371	.707	1.235	2.000	.306	.411	.560	.768

Retention (times unlimited claim size) CV_T=	CV_{LT} 1.5	2.0	2.5	3.0	γ_{LT} 1.5	2.0	2.5	3.0	Deviation 1.5	2.0	2.5	3.0
IV. 500 CLAIMS												
1	.058	.060	.063	.066	.065	.071	.078	.085	.138	.143	.151	.157
2	.064	.065	.068	.071	.091	.096	.114	.131	.153	.157	.164	.171
5	.075	.077	.079	.081	.163	.170	.203	.247	.183	.189	.195	.203
10	.083	.087	.090	.092	.242	.250	.315	.392	.209	.219	.231	.241
Ult.*	.083	.102	.122	.143	.262	.500	.873	1.414	.210	.275	.363	.481
V. 1,000 CLAIMS												
1	.041	.042	.045	.046	.046	.050	.055	.060	.097	.100	.106	.110
2	.045	.046	.048	.050	.065	.068	.081	.093	.108	.110	.115	.120
5	.053	.054	.056	.057	.116	.120	.144	.175	.128	.132	.136	.141
10	.059	.062	.064	.065	.171	.177	.222	.277	.144	.152	.159	.165
Ult.*	.059	.072	.086	.101	.185	.354	.344	1.000	.145	.186	.223	.309

*Ult. = Unlimited

Retention
(times unlimited mean)

REDUCTION IN MEAN (r)
(For different CV_T)

Figure 1

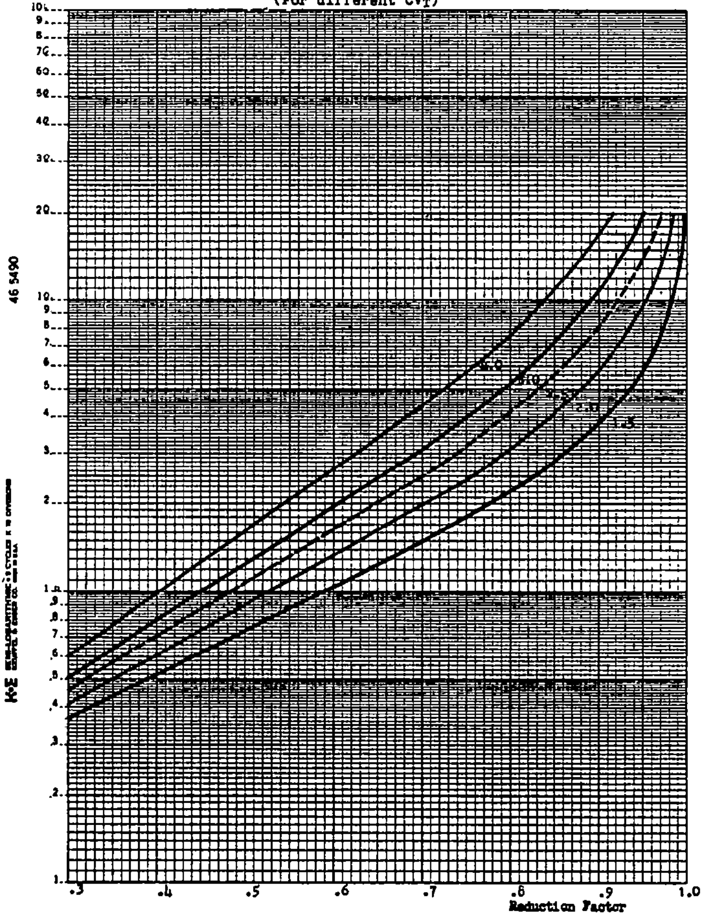
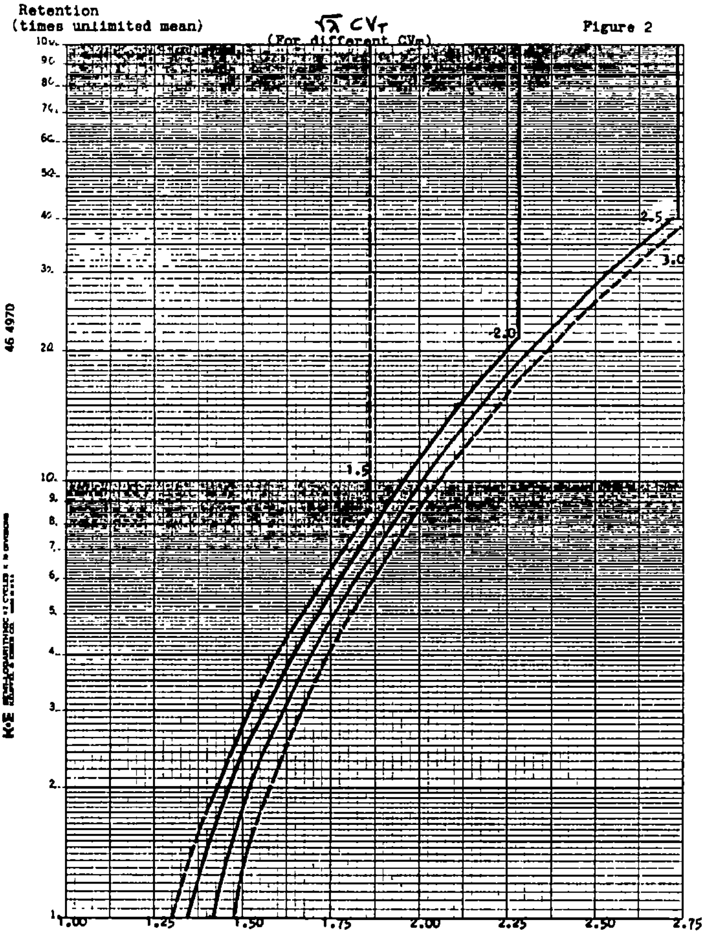
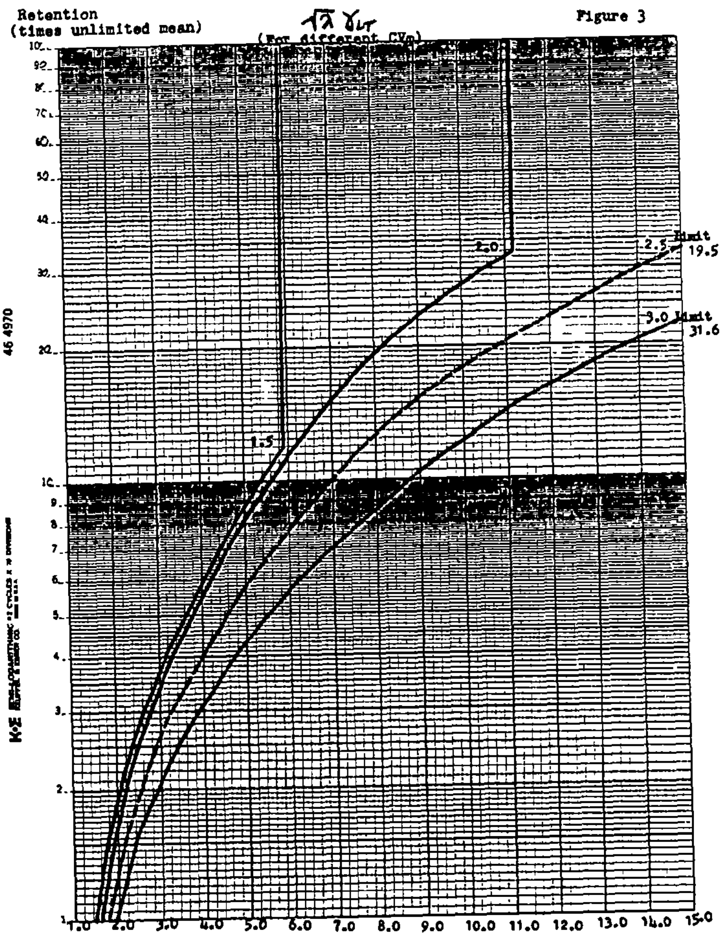


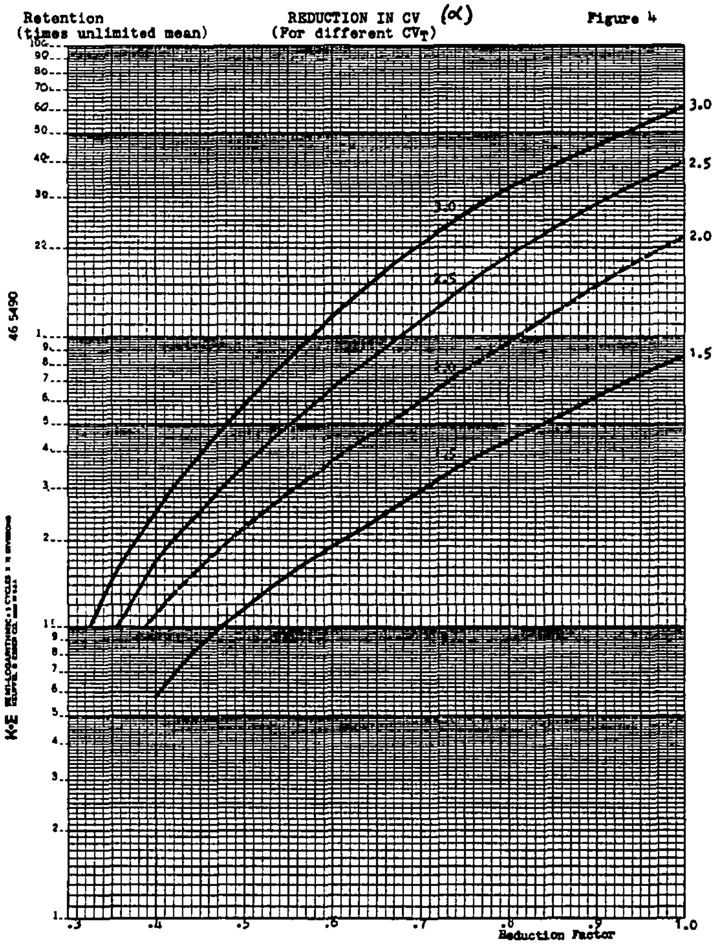
Figure 2



Retention
(times unlimited mean)
$\sqrt{\lambda}\ CV_T$
(For different CV_m)
1.5
2.0
2.5
3.0
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75



Retention
(times unlimited mean)
Figure 3
(For different CVm)
10
9
8
7
6
5
4
3
2
1
1.5
2.0
2.5
Limit
19.5
3.0
Limit
31.6
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
11.0
12.0
13.0
14.0
15.0


Retention
(times unlimited mean)
REDUCTION IN CV (α)
(For different CV_T)
Figure 4
3.0
2.5
2.0
1.5
.3
.4
.5
.6
.7
.8
.9
1.0
Reduction Factor

Retention (times unlimited mean) REDUCTION IN SKEWNESS COEFF. (β) Figure 5

(For different CV_T)

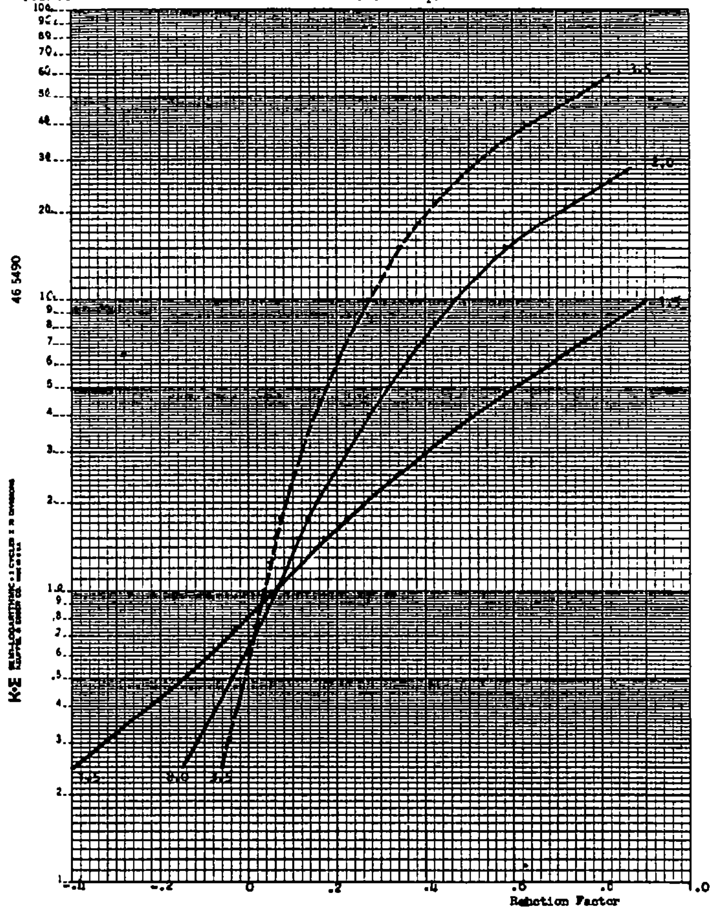


Figure 6

FLUCTUATION RESERVE FOR 50 CLAIMS

(As a percentage of expected losses; CV_T=2.0)