A CONSTRAINED PROFIT MAXIMIZATION MODEL
FOR A MULTI-LINE PROPERTY/LIABILITY COMPANY

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One of the decisions to be made by a multiline property/liability insurance company is the amount of business to be written in each of its product lines. This decision should reflect the profitability of each line, but should also be based on the relative capital requirements for each line mix alternative, as it is likely that the company has only a limited amount of capital to be used in support of insurance operations. The product mix choice then is one of constrained optimization, intended to achieve maximum expected profit given that the product mix must be allowable by the limited available capital. The traditional economic theory of the firm includes a model that determines the profit maximizing mix of outputs for a firm that has only one input to production (and a fixed amount thereof), but several possible outputs. This model can be adapted to the product mix question for a multiline property-liability company, by assuming that capital is the input, and that the earned premiums of the product lines are the outputs.

MICROECONOMIC THEORY OF THE FIRM--ONE INPUT, SEVERAL OUTPUTS

Before developing the insurance application of this microeconomic model, it will be useful to review briefly the model in its general form. This will be done for a firm with two outputs. Generalization to several outputs can be found in text books on Microeconomics.[1)]

If a firm generates profit at fixed rates r_1, and r_2 per unit of outputs, q_1, and q_2, its total profit in a time period can be expressed as

(1) $$P = r_1 q_1 + r_2 q_2$$

1) One Such text is Henderson & Quandt, Microeconomic Theory, New York, McGraw-Hill, 1958, pp. 67-75.

If the input necessary to produce the quantities q_1 and q_2 can be expressed as:

(2) $$x = f(q_1, q_2)$$

(where x denotes units of input); the profit maximizing combination of q_1, q_2 for a given amount of x can be found from the expression

(3) $$P = r_1 q_1 + r_2 q_2 + \lambda \left[x_0 - f(q_1, q_2) \right]$$

In the above expression, λ is a Lagrange multiplier, and x_0 is the fixed amount of input.

To determine the outputs q_1 and q_2 that maximize profit, we set the partial derivatives with respect to q_1, q_2, and λ equal to zero:[2)]

(4) $$\frac{\partial P}{\partial q_1} = r_1 - \lambda \frac{\partial f}{\partial q_1} = 0$$

(5) $$\frac{\partial P}{\partial q_2} = r_2 - \lambda \frac{\partial f}{\partial q_2} = 0$$

(6) $$\frac{\partial P}{\partial \lambda} = x_0 - f(q_1, q_2) = 0$$

2) The profit maximizing conditions also include second order conditions that are explained in Henderson & Quandt. See footnote 1).

Now equations (4) (5), and (6) must be solved jointly for q_1, q_2, and λ. This is generally very difficult or impossible, however, so we turn to the following relation which is derived from equations (4) and (5).

$$\frac{r_1}{\partial f/\partial q_1} = \frac{r_2}{\partial f/\partial q_2}. \tag{7}$$

Equation (7) indicates that the ratios of marginal profit per unit of output to marginal input requirement per unit of output are equal between the two outputs. In the general case of a firm with several outputs, this condition must be true among all outputs. If it is not, then all outputs are not equally profitable users of input at the margin, and profit can be increased by switching some of the input from production of less profitable outputs to more profitable outputs. This relationship between marginal profits per unit of input will be used to identify the optimal product line mix in the application to insurance developed in this paper.

APPLICATION OF THE MODEL TO A MULTILINE INSURANCE COMPANY

In application of the model to a Multiline Insurance Company, several aspects of the company and its operations must be clarified. First of all, it must be stated how the company determines the feasible line mix alternatives that it can write with a given amount of capital. The method assumed in this paper is that the probability of insolvency or impairment (whichever is considered most relevant by management) must be less than or equal to a specified amount for the time period for which profit is to be maximized. Only line mixes that meet this condition may be written. This type of constraint is currently beyond the capability of most company managements, but is used here recognizing that this is conceptually the means by which underwriting capacity should be determined, and that further efforts to refine capital requirements may lead in this direction. In this case, the examples to be developed below will show that the marginal input requirements per unit of output among the product lines vary with respect to the expected profit of each line, the variability of the expected profit of each line, the degree of direct or inverse correlation of results with other product lines, and the absolute amount of premium to be written in each line. The ratio of marginal profit to marginal capital requirement becomes a meaningful and important profitability measurement for the lines, indicating whether or not the current line mix is optimal, and which lines should be emphasized or de-emphasized in the company's book of business.

An alternative to the probability of insolvency/impairment constraint is that the company's writings for all lines cannot exceed a certain premium/surplus ratio. A strict application of this constraint would imply that the optimal solution for the company is to write only the product line with the highest expected profit as a percentage of earned premium, up to the amount allowed by the premium/surplus ratio. Other factors, such as business requirements to provide a market for other lines, would of course require certain amounts of other product lines to be written. This method is not developed further in this paper, but, as will be shown below, determination of the optimal line mix by use of a probability of insolvency/impairment constraint is much more powerful in its recognition of varying expected profit, variability of results, etc., among lines than is the premium/surplus ratio.

There are several additional assumptions that are made in the model developed in this paper, and these are listed below.

1) Profit as a percentage of earned premium includes both underwriting and investment income, and is an expected value. The quantity to be maximized will be the expected future profit.

2) Investment income from capital funds is not included in the definition of profit. Inclusion of this quantity in the profits of the product lines would not help in the selection of the optimal product mix, since it is a fixed amount independent of the product mix to be selected. Also, investment income from capital funds can be incorporated into the model by assuming that the fixed amount of capital available includes the investment income earned on capital in the period for which profit is to be maximized.

3) The variance of actual profit as a percentage of earned premium for each product line is independent of the amount of earned premium. More specifically, if a Standard Deviation or Variance for the results of each line were known, it would be a percentage of earned premium independent of the amount of earned premium or number of risks to be written in the product line. This ignores the law of large numbers used for credibility purposes, but is acceptable if the factors causing variability in product line results are primarily changes in claim cost inflation rates, nationwide claim frequency, or underwriting pricing cycles.

A final assumption is that the amount of capital is a constraining factor on the company's operations. If capital is more than sufficient to write all the profitable business that a company can service, then obviously the optimal product mix results if the company writes all the profitable business that it can obtain in all lines The marginal profit to marginal input ratios are irrelevant in this case

APPLICATION OF THE MODEL--USE OF PROBABILITY OF INSOLVENCY/ IMPAIRMENT FROM MULTIVARIATE NORMAL DISTRIBUTION

As indicated in the general explanation on the Microeconomic Theory of the Firm above, the ratios of marginal profits to marginal input will be used to identify the optimal line mix We have the marginal profits by assumption, and if an explicit probability distribution for the results of any given line mix is available, then it is possible to calculate finite approximations of the marginal input requirements, and to determine if the conditions

$$\text{(8)} \qquad \frac{r_1}{\partial f/\partial q_1} = \qquad = \frac{r_n}{\partial f/\partial q_n}$$

are substantially met If they are not, then increments of output can be added/subtracted to the appropriate product lines By use of systematic iterations, an optimal solution can be approached

For the examples to be developed here, it will be assumed that the results of the product lines conform to a multivariate normal distribution, and that the mean, standard deviation, and correlation with other product lines are known for each individual product line These assumptions are listed on Exhibit I If this is the case, then the distribution of the profit/loss for the whole company has a standard normal distribution with mean equal to the sum of the means of the individual product lines, and variance equal to the sum of the variances and covariances of the product lines[3]

For the first case, it is assumed that the firm writes only one product line and that the product line has an expected profit of 5% Earned Premium, and a standard deviation of profit/loss of 7 1/2% of Earned Premium. The firm has $300 million of capital, and does not wish to expose itself to more than a 1/10% chance of losing as much as one-half of its capital. This might be a realistic scenario for a company management, which might feel that a loss of one-half of capital would result in impairment of operations due to internal or regulatory restrictions. In this case the Earned Premium to be written can be solved for in the equation below

(9) $$3.1(.075)EP = .05\ EP + \$150 \text{ million}$$

Note that 3.1 is the appropriate number of standard deviations for a 1/10% probability for a standard normal distribution. The answer in this case is Earned Premium of $822 million, and Net Income of $41.1 million. The answers for this case and all others are summarized on Exhibit I. (The first case does not illustrate the application of the Microeconomic model of the firm, but is presented as a contrast to the other examples with several product lines.)

For the second case, it is assumed that the firm has three product lines, all with mean = 5% of Earned Premium, and Standard Deviation = 7 1/2% of Earned Premium. It is also assumed that the results of each product line are independent of the results of the other product lines. The Earned

3) Graybill, Franklin A., An Introduction to Linear Statistical Models, McGraw-Hill, 1961, pp. 56-57

Premium for each of the product lines is found from the equation

$$3\ 1\left[(\ 075EP)^2 + (\ 075EP)^2 + (\ 075EP)^2\right]^{1/2} = 3\ (\ 05EP) + \$150 \text{ million}$$

The answer in this case is $593 million Earned Premium for each product line, with total Earned Premium and expected profit of $1,779 million and $89 million respectively Since all product lines are identical they all have equal ratios of marginal profit to marginal input, and the conditions for optimality are met Incidentally, if the lines were not independent but were exactly correlated, i e , correlation equal to 1 0 or covariances equal to variances, then writing the three lines would be the same as writing one line and the total Earned Premium and profit would be the same as for the first case

In the third case, the firm is again assumed to have three lines with standard deviation = 7 1/2% of Earned Premium, but one line has a mean profit expection of 10% of Earned Premium while the others have means of 5% Descriptions of cases 4 through 7 are on Exhibit I, and are self-explanatory, except to say that cases 6 and 7 introduce correlations of +0 5 and -0 5 among lines and that management would obviously require knowledge of each correlation, either empirically or subjectively, to use them Cases 3 through 7 introduce use of the marginal profits to marginal input ratios of the product lines to find an optimal procedure Exhibit II shows examples of a single iteration for each of Cases 3 and 6, and Exhibit III shows the interim results after each iteration for Cases 3 through 7, up to and including the final iteration and optimal product mix

Several observations concerning the results on Exhibit I follow

a) The comparison among all cases shows the lack of validity of a single rule for premium/surplus ratio, if it is admitted that probability of impairment is a valid way to determine the amount of business that may be written For the same probability of impairment, the premium/surplus ratios for the cases shown range from 2 74 to 9 26 (It is assumed here that surplus equals capital--$300 million)

b) The marked increase in premiums and profit from Case 1 to Case 2 indicates the value of writing several lines, if their results are independemt of each other (This is an application of the Law of Large Numbers) In fact, as the number of lines written increases, in this case as the number equals 22, surplus is no longer necessary, as the standard deviation for the total result becomes less than the mean divided by 3 1 This means that expected profit alone provides enough cushion to meet the 1/10/ probability condition

c) Although both the mean and standard deviation of expected results have a large effect on the amount of a product line in the optimal mix (comparing Cases 3 and 4 with Case 2), it is notable that the effect of the standard deviation is greater than the mean In Case 5 both the mean and standard deviation of product line C are twice that of the other product lines, yet the optimal amount of line C is one-half that of the other lines If companies were to select their product mix by this type of procedure, there would obviously be less incentive to write line C without a further increase in expected profit

d) Case 7 illustrates the value of trying to select a product mix with lines that are inversely correlated (bad results in one line would likely be offset by good results in another line) Case 6 illustrates the cost of having lines that are directly correlated, though the effect on profit and premium volume is not so drastic as Case 7

It should also be noted that Exhibit III, in addition to displaying the results of the iterations, also demonstrates the marginal profit to marginal input ratio as management information Any single ratio represents the return to additional capital if invested to expand operations in a single line, and the relative value of the ratios indicates which lines should be expanded or contracted, if a company is not operating at an optimal line mix

CONCLUSION

While actual application of the methods developed in this paper would require use of some relatively esoteric concepts such as probability of insolvency/impairment and means, standard deviations, and correlation of profit among lines of business, these variables do have a very large effect on the efficiency of use of capital If these concepts can be put to use, it might be found that certain companies are using much more or much less capital than they need, or that shifts in line mix may accomplish a marked increase in profit without the need for additional capital, and without greater exposure to insolvency/impairment Use of these techniques may also show that certain product lines are unattractive in any company's line mix (due to low expected profit, high standard deviation or high correlation with other lines) and that changes in profit expectation or variance are necessary for firms to provide the coverage that the insuring public requires

Exhibit I

	Product Line Assumptions			Optimal Solutions	
	Expected Profit	Standard Deviation	Correlation With Other Lines	Earned Premium	Expected Profit
	(% of Earned Premium)			(Millions)	(Millions
Case 1					
Product Line A	5%	7 1/2%	-	$ 822	$ 41 1
Case 2					
Product Line A	5%	7 1/2%	-	593	29 7
Product Line B	5%	7 1/2%	-	593	29 7
Product Line C	5%	7 1/2%	-	593	29 7
Total				$1,779	$ 89 1
Case 3					
Product Line A	5%	7 1/2%	-	$ 534	$ 26 7
Product Line B	5%	7 1/2%	-	534	26 7
Product Line C	10%	7 1/2%	-	1,135	113 5
Total				$2,203	$166 9
Case 4					
Product Line A	5%	7 1/2%	-	$ 633	31 7
Product Line B	5%	7 1/2%	-	633	31 7
Product Line C	5%	15%	-	163	8 2
Total				$1,429	$ 71 6
Case 5					
Product Line A	5%	7 1/2%	-	$ 594	$ 29 7
Product Line B	5%	7 1/2%	-	594	29 7
Product Line C	10%	15%	-	297	29 7
Total				$1,485	$ 89 1
Case 6					
Product Line A	5%	7 1/2%	+0 5 with B	$ 421	$ 21 1
Product Line B	5%	7 1/2%	+0 5 with A	421	21 1
Product Line C	5%	7 1/2%	-	626	31 3
Total				$1,468	$ 73 5
Case 7					
Product Line A	5%	7 1/2%	-0 5 with B	$1,108	$ 55 4
Product Line B	5%	7 1/2%	-0 5 with A	1,108	55 4
Product Line C	5%	7 1/2%	-	562	28 1
Total				$2,778	$138 9

Note For all cases above, profit is maximized subject to chance of loss greater than $150 million less than or equal to 1/10%

ITERATIVE STEPS TO REACH OPTIMAL SOLUTIONS

Case 3

Start with any initial product mix meeting the probability constraints In this case, initial product mix assumed all three product lines have same Earned Premium Initial Earned Premium was solved for in equation below

$$3\ 1\left[(075LP)^2 + (075EP)^2 + (075EP)^2\right]^{1/2} = 2(05EP) + (10EP) + 150$$

EP = $740 million each line , $2,220 total Profit = $148

Next step is to check ratios of marginal profit to marginal input requirement for each line, to determine which lines should have more Earned Premiums and which less

For a 5% profit line, suppose Earned Premium is increased by an increment of $10 million Increase in profit would be $0 5 million New required capital is found from

$$\text{One-half of new capital} = 3\ 1\left[(075 \times 740)^2 + (075 \times 740)^2 + (075 \times 750)^2\right]^{1/2}$$

$$- 05(740) - 10(740) - 05(\cancel{790}) = 150\ 8$$

750

$$\text{So} \quad \frac{\Delta\ \text{Profit}}{\Delta\ \text{Input}} = \frac{0\ 5}{1\ 6} = 31.3\%$$

For the 10% line, again suppose Earned Premium is increased by an increment of $10 million Increase in profit would be $1 million

$$\text{One-half of new capital} = 3\ 1\left[(075 \times 740)^2 + (.075 \times 740)^2 + (075 \times 750)^2\right]^{1/2}$$

$$- 05(740)2 - 10(750) = 150\ 3$$

$$\text{So} \quad \frac{\Delta\ \text{Profit}}{\Delta\ \text{Input}} = \frac{1}{0\ 6} = 166\ 7\%$$

This result indicates that more of the 10% profit line should be written, and less of the 5% lines Suppose \$300 million is added to the 10% line, i e , Earned Premium will be \$1,040 million Earned Premium in the other two lines is then solved for in

$$150 = 3\ 1\left[(\ 075 \times 1040)^2 + 2(\ 075LP)^2\right]^{1/2} - \ 05EP(2) - \ 10(1040)$$

EP = 618 or -100 from this equation, and the positive value is chosen

Next step is to again check Δ Profit/Δ Input for each product line to determine which lines should have more Earned Premium and which less When the indicated increments change direction, the absolute size of the increments should be reduced for convergence Iterations should continue until Δ Profit/Δ Input are approximately the same for all product lines and very little improvement in total profit is achieved by successive iterations

Case 6

This case illustrates the use of correlations of results among product lines in the solution for the optimal product mix The initial product mix is found the same way as for Case 3, by solving the following equation

$$3\ 1\left[(\ 075EP)^2 + (\ 075EP)^2 + (\ 075EP)^2 + 0\ 5(\ 075LP)(\ 075EP) + 0\ 5(075LP)(\ 075EP)\right]$$

$$= 3\ (.05EP) + 150$$

Note that the right-hand two terms of the bracketed expression represent the covariance terms that must be included in the expression for the standard deviation of the total result Also note that the covariances follow from the correlations given since the covariance between the two lines is the product of the correlation between the lines and the standard deviations of both lines 4)

The solution to the above equation is \$476 million Earned Premium for each line, with total Earned Premium and profit of \$1,428 million and \$71 4 million respectively

The next step is to determine the Δ Profit/Δ Input for each line The procedure is the same as for Case 3 If \$10 million Earned Premium is added to one of the correlated lines, the profit increment is \$0 5 million, and the new capital requirement is found from the expression:

One-half of new capital

$$= 3\ 1\left[2(\ 075 \times 476)^2 + (\ 075 \times 846)^2 + 2(\ 50)(\ 075 \times 476)(\ 075 \times 486)\right]^{1/2}$$

$$- 2(\ 05)(476) - \ 05(486) = 151\ 19$$

4) Steel & Torrie, Principals and Procedures of Statistics, McGraw-Hill, 1960 pp 183

For the correlated lines, Δ Profit/Δ Input = 21.0%, and a separate calculation showed that Δ Profit/Δ Input for the independent line was 40.9%.

This indicated that Earned Premium should be added to the independent line, and subtracted from the correlated lines. The amount chosen to be added to the independent line was $300 million, and the amount for each of the correlated lines was solved for in the equation

$$3.1\left[(.075 \times 776)^2 + 2(.075EP)^2 + 2(.50)(.075EP)(.075EP)\right]^{1/2}$$
$$= 150 + .05(776) + 2(.05)EP$$

Results are on Exhibit III.

Three general comments concerning the calculations are the following:

1) In determining the new product mix for each successive iteration, increments of Earned Premium may be chosen for all product lines but one (or more if there are identical product lines). The Earned Premium for the remaining product line is solved for to meet the constraint on probability of insolvency/impairment.

2) The use of the Δ Profit/Δ Input quantities may seem superfluous in the examples in this paper, since there are only two possible directions to move in changing the product mix in each example, and the appropriate direction could be determined simply by observing the change in total profit. The use of these quantities would not be trivial for a greater number of product lines, however, since there would be many options for directions in which to move.

3) A logical question to ask concerning the calculations would be whether and algorithm can be constructed that causes the iterations to converge on an optimal solution in the general case for N product lines. This will not be proven here, but intuitively the answer seems to be yes, since it should always be possible to choose increments for each of the product lines small enough to provide improvement in total profit for each successive iteration.

Exhibit III
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RESULTS OF SUCCESSIVE ITERATIONS IN CALCULATION OF OPTIMAL PRODUCT MIXES
($'s in Millions)

	Line A	Line B	Line C	Total Profit
	5% Profit 7 1/2% S D	5% Profit 7 1/2% S D	10% Profit 7 1/2% S D	
Case 3				
1st Iteration EP	$740	$740	$740	$148 0
$\Delta P/\Delta I$	31 3%	31 3%	166 7%	
2nd Iteration EP	$618	$618	$1,040	$165 8
$\Delta P/\Delta I$	42 2%	42 2%	61 8%	
3rd Iteration EP	$516	$516	$1,150	$166 6
$\Delta P/\Delta I$	52 1%	52 1%	48 0%	
4th Iteration EP	$534	$534	$1,135	$166 9
$\Delta P/\Delta I$	47 5%	47 5%	48 1%	Stop

	Line A	Line B	Line C	Total Profit
	5% Profit 7 1/2% S D	5% Profit 7 1/2% S D	5% Profit 15% S D	
Case 4				
1st Iteration EP	$358	$358	$358	$53 7
$\Delta P/\Delta I$	52 6%	52 6%	7 5%	
2nd Iteration EP	$558	$558	$256	$68 6
$\Delta P/\Delta I$	28 3%	28 3	12 2%	
3rd Iteration EP	$608	$608	$202	$70 9
$\Delta P/\Delta I$	26 2%	26 2%	16 9%	
4th Iteration EP	$658	$658	$ 91	$70 4
$\Delta P/\Delta I$	22 3%	22 3%	57 0%	
5th Iteration EP	$633	$633	$163	$71 5
$\Delta P/\Delta I$	23 7	23 7	21 9	Stop

	Line A	Line B	Line C	Total Profit
	5% Profit 7 1/2% S D	5% Profit 7 1/2% S D	10% Profit 15% S D	
Case 5				
1st Iteration EP	$406 52 3%	$406 52 3%	$406 17 7%	$ 81 2
2nd Iteration EP	$606 27 1%	$606 27 1%	$23[illegible] 30 4%	$ 89 0
3rd Iteration EP	$581 29 1%	$581 29 1%	$309 26 6%	$ 89 [illegible]
4th Iteration EP	$594 29 3%	$594 29 3%	$297 29 1%	$ 89 1 Stop
	5% Profit 7 1/2% S D + 0 5 Correl with B	5% Profit 7 1/2% S D + 0 5 Correl with B	5% Profit 7 1/2% S D	
Case 6				
1st Iteration EP	$476 21 0%	$476 21 0%	$476 40 9%	$ 71 4
2nd Iteration EP	$313 36 3%	$313 36 3%	$776 17 5%	$ 70 1
3rd Iteration EP	$421 20 8%	$421 20 8%	$421 21 2%	$ 73 4 Stop

	Line A	Line B	Line C	Total Profit
	5% Profit 7 1/2% S D -0 5 Correl with B	5% Profit 7 1/2% S D -0 5 Correl with A	5% Profit 7 1/2% S D	
Case 7				
1st Iteration EP	$838	$838	$838	$125 7
$\Delta P/\Delta I$	147%	147%	25%	
2nd Iteration EP	$1,138	$1,138	$496	$138 6
$\Delta P/\Delta I$	43 2%	43 2%	55 5%	
3rd Iteration EP	$1,108	$1,108	$562	$138 9
$\Delta P/\Delta I$	49 2%	49 2%	47 9%	Stop