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Abstract

Generalized linear modeling (GLM) is becoming a regular tool for insurance ratemaking. Actuaries and underwriters have begun to realize that classes may not simply interact, whether additively or multiplicatively. Some class combinations may synergize, or more than simply interact; others may counteract, or less than simply interact. But lest actuaries be tempted by abundant computer power and affordable GLM software to over-refine rating classes, they must know how to test whether class refinement is statistically significant. This paper provides the theory for this testing, and performs an illustrative test on a small dataset of automobile physical-damage claims.

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One benefit of generalized linear modeling (GLM) is that with it one can test whether the explanatory power of a model with more classes is significantly greater than that of a model with fewer classes. In other words, one can scientifically determine whether a refinement of a classification scheme is worthwhile. This article presents basic statistical theory, and illustrates class-refinement testing with a simple exercise.

The exercise is to estimate frequencies of automobile physical-damage claims by state, sex, and age, a reasonable prerequisite to rating an auto owner's physical-damage insurance coverage. Exhibit 1 shows the summarized data for five states, two sexes, and four age groups. Since every combination is represented, there are 5×2×4 = 40 observations. As per the bottom of Exhibit 2, young insureds are less than twenty-one years old; those in their prime are from twenty-one to forty; middle age ranges from forty-one to sixty-five; and old age is over sixty-five. The age groupings and the data itself are purely illustrative.

A standard actuarial treatment might summarize the exposures and claim counts as in Exhibit 2. It is usual to select certain classes as base, here the base being females in their prime (i.e., from twenty-one to forty years of age). In this dataset the frequency of males is 1.073 times that of females. Frequency decreases by age until old age, the relativities being 1.899, 1.000 (base), 0.825, and 1.142. One can combine these "one-way" relativities to derive that the PD-frequency of old-aged males, for example, is 1.072 × 1.142 = 1.225 times the base frequency. However, one can just as reasonably calculate one table of "two-way" relativities, and conclude that the old-aged-male frequency is 1.288 times greater than base. If premium were proportional to PD-frequency, the "two one-way or one two-way" decision would make a 4.8 percent difference in the premiums of old-aged males.

The two one-way factors require six relativities, or more accurately, four after allowing for one base per factor. The one two-way factor requires eight relativities, or seven without the base. Combining many one-way factors is simpler and easier than using one many-way factor; but it is also less accurate. Only a statistical model can test whether the loss of accuracy is significant. Moreover, in Exhibit 2 sex and age are not controlled for state; but a statistical model can filter out the effect of state.

The standard linear model is $\mathbf{y} = X\beta + \mathbf{e}$, where $\mathbf{y}$ is the $(t\times 1)$ vector of observations, X is the $(t\times k)$ "design" matrix, each of whose columns is called a factor or explanatory variable, β is the $(k\times 1)$ vector of parameters to be estimated, and $\mathbf{e}$ is the $(t\times 1)$ random vector of error terms. The errors are assumed to be of mean zero, of identical variance (i.e., "homoskedastic"), and of zero covariance. In matrix terminology, the error vector has mean 0 $(t\times 1)$ and variance $\sigma^2 I_t$, where I_t is the $(t\times t)$ identity matrix. The columns of X must be linearly independent, i.e., X must be of full column rank. Since the k columns of X explain, or account for, the t observations (with a residue of randomness), it is desirable for t to be much larger than k. With the model so described, the best linear unbiased estimator (BLUE) of β is $\beta = (X'X)^{-1}X'y$. The variance of this estimator, which will be important for hypothesis testing, is $Var[\beta] = \sigma^2(X'X)^{-1}$. Usually σ^2 must be estimated, the formula for the unbiased estimator being:

$$\sigma^2 = \frac{(y - X\beta)'(y - X\beta)}{t - k}$$

Exhibit 3 presents and solves the one-way model. The design matrix consists of zeroes and ones. Therefore, the explanatory variables are categorical. For example, the first column of X tells whether the observation pertains to California (yes = 1, no = 0). The last column tells whether the observation pertains to old-aged insureds. There are no columns for the base classes of female and prime; hence, nine variables account for as much as possible of forty observations, leaving thirty-one degrees of freedom in the estimation of σ^2. One reads down the exhibit through the intermediate calculations of X'y and X'X to $(X'X)^{-1}$ and β. Below that is the estimate of the variance of β. To the right of the design matrix are predicted values and residuals, and below these are sums of squares and cross products. From these we conclude that the model explains 99.6 percent of the actual values, and that the estimate for σ^2 is 0.049.

The logarithm of frequency is here modeled as a linear combination of parameters with an error term, i.e., $ln\boldsymbol{\varphi} = X\beta + \mathbf{e}$. Thus, $\boldsymbol{\varphi} = exp(X\beta + \mathbf{e})$. The exponential function is a monotonic link between frequency and a standard linear model, and it is this link that makes for a generalized linear model (GLM). Sex and age-group relativities, displayed in the "One-Way" table at the bottom of Exhibit 5 are exponentiated β values, e.g., the old-age relativity is $exp(0.261) = 1.298$. This relativity, for which state and age have been controlled, is much larger than the naïve relativity of 1.142.

Similarly, Exhibit 4 works out the two-way model, with its five state parameters and seven combinations of sex and age group (S × AG). Having three more explanatory variables, it explains more than the one-way model (99.7 versus 99.6 percent). However, due to its fewer degrees of freedom its estimated σ^2 exceeds that of the one-way model (0.051 versus 0.049). Exhibit 5 shows that each model outpredicts the other exactly half the time, and neither method prevails on average or squared deviations. At this point most would claim, quite rightly, that the accuracy gain of the two-way model is trifling, and would prefer multiplying the two one-way factors. However, this is an imprecise, unscientific judgment, and it makes a sizeable difference to the relativities of females in their middle age and males in their prime (13.8 and 12.7 percent, according to Exhibit 5).

The key to the hypothesis is the recognition that the one-way model is a subset of the two-way. Let index i range over female and male, and index j over young, prime, middle, and old. And let β_i be the one-way factor for sex, β_j the one-way factor for age group, and β_{ij} the two-way factor for the combination of sex and age group. The two-way model reduces to the one-way if and only if for all $i \in \{1,2\}$ and for all $j \in \{1,2,3,4\}$, $\beta_{ij} = \beta_i + \beta_j$. Now, because they are bases, $\beta_{i=1}$, $\beta_{j=2}$, and $\beta_{ij=12}$ are zero, and are not paired with explanatory variables. The proper form of the hypothesis must eliminate all β_i and β_j variables, and contain constraints on β_{ij} variables only. The bases allow us to achieve the form. For $i=1$ and for all j, $\beta_{1j} = \beta_{i=1} + \beta_j = 0 + \beta_j = \beta_j$. And for $j=2$ and for all i, $\beta_{i2} = \beta_i + \beta_{j=2} = \beta_i + 0 = \beta_i$. Therefore, for all i and j, $\beta_{ij} = \beta_{i2} + \beta_{1j}$. However, some of these equations are non-binding tautologies. In particular, for $i=1$, $\beta_{1j} = \beta_{12} + \beta_{1j} = 0 + \beta_{1j} \equiv \beta_{1j}$. This leaves constraints of the form $\beta_{2j} = \beta_{22} + \beta_{1j}$. But even here, when $j=2$ we have the tautology $\beta_{22} = \beta_{22} + \beta_{12} = \beta_{22} + 0 \equiv \beta_{22}$. Therefore, the two-way model reduces to the one-way when subjected to the three constraints (corresponding to its three fewer degrees of freedom):

$$\begin{aligned} \beta_{21} &= \beta_{22} + \beta_{11} \\ \beta_{23} &= \beta_{22} + \beta_{13} \\ \beta_{24} &= \beta_{22} + \beta_{14} \end{aligned}$$

Expressed in matrix notation, the hypothesis H_0 that the two-way model is equivalent to a one-way is:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \beta_{11} \\ \beta_{13} \\ \beta_{14} \\ \beta_{21} \\ \beta_{22} \\ \beta_{23} \\ \beta_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_{(3\times 7)}\beta_{(7\times 1)} = r_{(3\times 1)}$$

Matrix R appears in Exhibit 7, but augmented with five columns of zeros to accommodate the state parameters.

Before developing the statistic for testing this hypothesis, it is instructive to see how the two-way model of Exhibit 4 under the constraint $R\beta = r = 0$ reduces to the one-way model of Exhibit 3. In an earlier article[†] the author showed that the restricted BLUE β^* of the model $\mathbf{y} = X\beta + \mathbf{e}$ subject to the constraint $R\beta = r$ falls out of the equation:

$$\begin{bmatrix} X'X & R' \\ R & 0 \end{bmatrix} \begin{bmatrix} \beta^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} X'y \\ r \end{bmatrix}$$

One can discern this form in the three additional rows and columns of the middle section of Exhibit 6. The three lambdas are Lagrange multipliers, byproducts of the constrained least-squares problem that may be ignored. One may check that the seven sex-age parameters are sums of the proper sex and age-group

[†] Leigh J. Halliwell, "Statistical and Financial Aspects of Self-Insurance Funding," Alternative Markets/Self Insurance (CAS 1996 Discussion Paper Program), 1-46. Appendix A and its citations not only solve the restricted least-squares problem, but provide proofs for all the statements herein to follow.

parameters of Exhibit 3. But the equivalence of the two models is apparent from their identical predictions (the $X\beta$ columns) and σ^2 estimates.

The statistical test makes use of the fact the least-squares estimator $\boldsymbol{\beta}$ has mean β and variance $\sigma^2(X'X)^{-1}$. Therefore, $R\boldsymbol{\beta}$ has mean $R\beta$ and variance $\sigma^2R(X'X)^{-1}R'$. Also important is the assumption that the distribution of $\boldsymbol{\beta}$ is multivariate normal. This is true, if $\mathbf{e}$ is multivariate normal; but even under certain robust conditions the distribution of $\boldsymbol{\beta}$ will be asymptotically multivariate normal. Then $R\boldsymbol{\beta}$ will be multivariate normal with mean $R\beta$ and variance $\sigma^2R(X'X)^{-1}R'$, and $R\boldsymbol{\beta} - R\beta$ multivariate normal with mean zero and variance $\sigma^2R(X'X)^{-1}R'$. Wherefore it follows that the expression

$$(R\boldsymbol{\beta} - R\beta)'\left(\sigma^2R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta} - R\beta)$$

$$= \frac{(R\boldsymbol{\beta} - R\beta)'\left(R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta} - R\beta)}{\sigma^2}$$

is chi-square distributed with j degrees of freedom, where j is the number of rows of R. However, normally we do not know σ^2 and have to estimate it. But the sum of the squared residuals divided by σ^2 is chi-square distributed with $t-k$ degrees of freedom, and this sum does not covary with $\boldsymbol{\beta}$. Therefore, under the multivariate normal assumption, the following two expressions are independent chi-square random variables:

$$\frac{(R\boldsymbol{\beta}-R\beta)'\left(R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta}-R\beta)}{\sigma^2}$$

$$\frac{(y-X\beta)'(y-X\beta)}{\sigma^2}=\frac{\sigma^2}{\sigma^2}(t-k)$$

Finally, this implies that the following expression is approximately and asymptotically F-distributed with j and $t-k$ degrees of freedom:

$$\frac{\frac{(R\boldsymbol{\beta}-R\beta)'\left(R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta}-R\beta)}{\sigma^2}\Big/ j}{\frac{(y-X\beta)'(y-X\beta)}{\sigma^2}\Big/ t-k}=\frac{(R\boldsymbol{\beta}-R\beta)'\left(R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta}-R\beta)/j}{\sigma^2}$$

Hence, under hypothesis H_0: $R\beta = r$, the statistic

$$\frac{(R\boldsymbol{\beta}-R\beta)'\left(R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta}-R\beta)/j}{\sigma^2}$$

is F-distributed with j and $t-k$ degrees of freedom:

Accordingly, Appendix 7 tests whether the β of the two-way model differs significantly from the β of the one-way model, which is the two-way β under hypothesis H_0. R is a 3×12 matrix of zeroes and ones, and r is the 3×1 zero vector. X is the design matrix of Appendix 4, from which β, $(X'X)^{-1}$ and σ^2 also are taken. The F statistic as described above has the value 0.546, which is at the 65.5 percentile of the $F_{3,\,28}$ distribution. In other words, if H_0 be true, in approximately one-third of repeated samples will the F statistic be greater than 0.546. Few statisticians would deem 65.5 percent as significant enough to reject the hypothesis; most would here accept the simpler, one-way model.

Exhibit 1

Automobile Physical-Damage Data

State	Sex	Age Group	Car Years	PD Claims	Freq	Log Freq
CA	F	Young	821.9	94	0.114	-2.168
CA	F	Prime	11,181.7	644	0.058	-2.854
CA	F	Middle	4,792.1	212	0.044	-3.118
CA	F	Old	931.6	39	0.042	-3.173
CA	M	Young	677.6	66	0.097	-2.329
CA	M	Prime	12,418.3	696	0.056	-2.882
CA	M	Middle	5,134.0	222	0.043	-3.141
CA	M	Old	928.5	58	0.062	-2.773
FL	F	Young	515.4	45	0.087	-2.438
FL	F	Prime	5,578.1	229	0.041	-3.193
FL	F	Middle	4,694.1	162	0.035	-3.366
FL	F	Old	1,209.2	54	0.045	-3.109
FL	M	Young	542.0	60	0.111	-2.201
FL	M	Prime	5,796.9	249	0.043	-3.148
FL	M	Middle	4,476.7	171	0.038	-3.265
FL	M	Old	1,166.0	66	0.057	-2.872
MI	F	Young	373.0	6	0.016	-4.130
MI	F	Prime	5,168.0	24	0.005	-5.372
MI	F	Middle	1,545 0	17	0.011	-4.510
MI	F	Old	161.2	3	0.019	-3.984
MI	M	Young	367.8	5	0.014	-4.298
MI	M	Prime	5,966.4	54	0.009	-4.705
MI	M	Middle	1,802.5	13	0.007	-4.932
MI	M	Old	183.8	3	0.016	-4 115
NY	F	Young	447.3	34	0 076	-2.577
NY	F	Prime	5,092.4	220	0.043	-3.142
NY	F	Middle	3,289.3	121	0.037	-3.303
NY	F	Old	611.4	36	0.059	-2.832
NY	M	Young	565.9	59	0.104	-2.261
NY	M	Prime	5,978.7	340	0.057	-2.867
NY	M	Middle	3,528.5	133	0.038	-3.278
NY	M	Old	625.9	30	0.048	-3.038
TX	F	Young	1,009.3	84	0.083	-2.486
TX	F	Prime	8,931.7	513	0.057	-2.857
TX	F	Middle	4,429.9	206	0.047	-3.068
TX	F	Old	705.2	46	0.065	-2.730
TX	M	Young	1,003.4	112	0.112	-2.193
TX	M	Prime	9,067.5	568	0.063	-2.770
TX	M	Middle	4,478.0	225	0.050	-2 991
TX	M	Old	641.7	50	0.078	-2.552

Exhibit 2

Relativity Companson

State	Sex	Age Group	Car Years	PD Claims	Freq
CA			36,885.7	2,031	0.055
FL			23,978.4	1,036	0.043
MI			15,567.8	125	0.008
NY			20,139.4	973	0.048
TX			30,266.5	1,804	0.060
Total			126,837.8	5,969	0.047

Sex	Age Group	Car Years	PD Claims	Freq	Relativity	S × AG Relativity	Diff
F		61,487.7	2,789	0.045	1.000		
M		65,350.1	3,180	0.049	1.073		
	Young	6,323.6	565	0.089	1.899		
	Prime	75,179.7	3,537	0.047	1 000		
	Middle	38,170.0	1,482	0.039	0 825		
	Old	7,164.6	385	0.054	1.142		
F	Young	3,166.9	263	0.083	1.832	1.899	3.7%
F	Prime	35,951.9	1,630	0.045	1.000	1.000	0.0%
F	Middle	18,750.3	718	0.038	0.845	0.825	-2.3%
F	Old	3,618 6	178	0.049	1.085	1.142	5.3%
M	Young	3,156.7	302	0 096	2.110	2.037	-3.4%
M	Prime	39,227 8	1,907	0.049	1.072	1.073	0 1%
M	Middle	19,419.7	764	0.039	0.868	0.885	2.0%
M	Old	3,545.9	207	0.058	1.288	1.225	-4 8%

Age Groups

Young	under 21
Prime	21 to 40
Middle	41 to 65
Old	over 65

Exhibit 3

Linear Model with Two One-Way Factors

State	Sex	Age Group	y	CA	FL	MI	NY	TX	M	Young	Middle	Old	y	Xβ	y - Xβ
CA	F	Young	-2.168	1						1			2.168	-2.382	0.214
CA	F	Prime	-2.854	1									-2.854	-3.053	0.199
CA	F	Middle	-3.118	1							1		-3.118	-3.172	0.053
CA	F	Old	-3.173	1								1	-3.173	-2.792	-0.381
CA	M	Young	-2.329	1					1	1			-2.329	-2.292	-0.036
CA	M	Prime	-2.882	1					1				-2.882	2.963	0.082
CA	M	Middle	-3.141	1					1		1		-3.141	-3.081	-0.059
CA	M	Old	-2.773	1					1			1	-2.773	-2.702	-0.071
FL	F	Young	-2.438		1					1			-2.438	-2.527	0.088
FL	F	Prime	-3.193		1								-3.193	-3.197	0.005
FL	F	Middle	-3.366		1						1		-3.366	-3.316	-0.051
FL	F	Old	-3.109		1							1	-3.109	-2.936	-0.172
FL	M	Young	-2.201		1				1	1			-2.201	-2.438	0.236
FL	M	Prime	-3.148		1				1				-3.148	-3.107	-0.040
FL	M	Middle	-3.265		1				1		1		-3.265	-3.226	-0.039
FL	M	Old	-2.872		1				1			1	-2.872	-2.846	-0.025
MI	F	Young	-4.130			1				1			-4.130	-4.083	-0.046
MI	F	Prime	-5.372			1							-6.372	-4.754	-0.618
MI	F	Middle	-4.510			1					1		-4.510	-4.872	0.363
MI	F	Old	-3.984			1						1	-3.984	-4.493	0.509
MI	M	Young	-4.298			1			1	1			-4.298	-3.993	-0.305
MI	M	Prime	-4.705			1			1				-4.705	-4.664	-0.041
MI	M	Middle	-4.932			1			1		1		-4.932	-4.782	-0.150
MI	M	Old	-4.115			1			1			1	-4.115	-4.403	0.288
NY	F	Young	-2.577				1			1			-2.577	-2.490	-0.087
NY	F	Prime	-3.142				1						-3.142	-3.161	0.019
NY	F	Middle	-3.303				1				1		-3.303	-3.279	-0.024
NY	F	Old	-2.832				1					1	-2.832	-2.900	0.067
NY	M	Young	2.261				1		1	1			-2.261	-2.400	0.139
NY	M	Prime	-2.867				1		1				-2.867	-3.071	0.204
NY	M	Middle	-3.278				1		1		1		-3.278	-3.189	-0.089
NY	M	Old	-3.038				1		1			1	-3.038	-2.810	-0.228
TX	F	Young	-2.486					1		1			-2.486	-2.283	-0.203
TX	F	Prime	-2.857					1					-2.857	-2.954	0.097
TX	F	Middle	-3.068					1			1		-3.068	-3.073	0.004
TX	F	Old	-2.730					1				1	-2.730	-2.693	-0.037
TX	M	Young	-2.193					1	1	1			-2.193	-2.193	0.001
TX	M	Prime	-2.770					1	1				-2.770	-2.664	0.094
TX	M	Middle	-2.991					1	1		1		-2.991	-2.983	-0.008
TX	M	Old	-2.552					1	1			1	-2.552	-2.603	0.051

X'y	X'X								
-22.439	8	0	0	0	0	4	2	2	2
-23.592	0	8	0	0	0	4	2	2	2
-36.046	0	0	8	0	0	4	2	2	2
-23.298	0	0	0	8	0	4	2	2	2
-21.647	0	0	0	0	8	4	2	2	2
-82.610	4	4	4	4	4	20	5	5	5
-27.081	2	2	2	2	2	5	10	0	0
-34.972	2	2	2	2	2	5	0	10	0
-31.179	2	2	2	2	2	5	0	0	10

Cross Products

426.616	425.095	1.521
425.096	425.095	0.000
1.521	0.000	1.521
100.0%	99.6%	0.4%

r	40
k	9
df	31
σ^2	0.049
σ	0.222

	β	$(X'X)^{-1}$								
CA	-3.053	0.225	0.100	0.100	0.100	0.100	-0.050	-0.100	-0.100	-0.100
FL	-3.197	0.100	0.225	0.100	0.100	0.100	-0.050	-0.100	-0.100	-0.100
MI	-4.754	0.100	0.100	0.225	0.100	0.100	-0.050	-0.100	-0.100	-0.100
NY	-3.161	0.100	0.100	0.100	0.225	0.100	-0.050	-0.100	-0.100	-0.100
TX	-2.954	0.100	0.100	0.100	0.100	0.225	-0.050	-0.100	-0.100	-0.100
M	0.090	-0.050	-0.050	-0.050	-0.050	-0.050	0.100	0.000	0.000	0.000
Young	0.671	-0.100	-0.100	-0.100	-0.100	-0.100	0.000	0.200	0.100	0.100
Middle	-0.118	-0.100	-0.100	-0.100	-0.100	-0.100	0.000	0.100	0.200	0.100
Old	0.261	-0.100	-0.100	-0.100	-0.100	-0.100	0.000	0.100	0.100	0.200

t-stat	Std(β)	Var(β)								
-29.056	0.105	0.011	0.005	0.005	0.005	0.005	-0.002	-0.005	-0.005	-0.005
-30.428	0.105	0.005	0.011	0.005	0.005	0.005	-0.002	-0.005	-0.005	-0.005
-45.243	0.105	0.005	0.005	0.011	0.005	0.005	-0.002	-0.005	-0.005	-0.005
-30.078	0.105	0.005	0.005	0.005	0.011	0.005	-0.002	-0.005	-0.005	-0.005
-28.115	0.105	0.005	0.005	0.005	0.005	0.011	-0.002	-0.005	-0.005	-0.005
1.285	0.070	-0.002	-0.002	-0.002	-0.002	-0.002	0.005	0.000	0.000	0.000
6.772	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.000	0.010	0.005	0.005
-1.193	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.000	0.005	0.010	0.005
2.636	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.000	0.005	0.005	0.010

Exhibit 4

Linear Model with One Two-Way Factor

State	S = AG	y	CA	FL	MI	NY	TX	FYoung	FMiddle	FOld	MYoung	MPrime	MMiddle	MOld	y	Xβ	y - Xβ
CA	FYoung	2.168	1					1							2.168	-2.389	0.221
CA	FPrime	-2.854	1												-2.854	-3.113	0.259
CA	FMiddle	-3.118	1						1						-3.118	-3.102	-0.016
CA	FOld	-3.173	1							1					-3.173	-2.795	-0.378
CA	MYoung	-2.329	1								1				-2.329	2.286	-0.043
CA	MPrime	-2.882	1									1			-2.882	-2.904	0.022
CA	MMiddle	-3.141	1										1		-3.141	-3.151	0.010
CA	MOld	-2.773	1											1	-2.773	-2.699	-0.074
FL	FYoung	-2.438		1				1							-2.438	-2.533	0.095
FL	FPrime	-3.193		1											-3.193	-3.257	0.064
FL	FMiddle	-3.388		1					1						-3.388	-3.248	-0.120
FL	FOld	-3.109		1						1					-3.108	-2.939	-0.170
FL	MYoung	2.201		1							1				-2.201	-2.430	0.229
FL	MPrime	-3.148		1								1			-3.148	-3.048	-0.100
FL	MMiddle	-3.265		1									1		-3.265	-3.296	0.030
FL	MOld	-2.872		1										1	-2.872	-2.843	-0.028
MI	FYoung	-4.130			1			1							-4.130	-4.090	-0.040
MI	FPrime	-6.372			1										-6.372	-4.814	-0.558
MI	FMiddle	-4.510			1				1						-4.510	-4.803	0.294
MI	FOld	-3.984			1					1					-3.984	-4.496	0.512
MI	MYoung	-4.298			1						1				-4.298	-3.987	-0.312
MI	MPrime	-4.705			1							1			-4.705	-4.605	-0.100
MI	MMiddle	-4.932			1								1		-4.932	-4.652	-0.080
MI	MOld	-4.115			1									1	-4.115	-4.400	0.285
NY	FYoung	-2.577				1		1							-2.577	-2.497	-0.080
NY	FPrime	-3.142				1									-3.142	-3.220	0.078
NY	FMiddle	-3.303				1			1						-3.303	-3.210	-0.093
NY	FOld	-2.832				1				1					-2.832	-2.902	0.070
NY	MYoung	-2.261				1					1				-2.261	-2.393	0.132
NY	MPrime	-2.887				1						1			-2.887	-3.011	0.144
NY	MMiddle	-3.278				1							1		-3.278	-3.258	-0.020
NY	MOld	-3.038				1								1	-3.038	-2.807	-0.231
TX	FYoung	-2.486					1	1							-2.486	-2.290	-0.196
TX	FPrime	-2.857					1								-2.857	-3.014	0.157
TX	FMiddle	-3.068					1		1						-3.068	-3.003	-0.065
TX	FOld	-2.730					1			1					-2.730	-2.698	-0.034
TX	MYoung	-2.193					1				1				2.193	-2.187	-0.006
TX	MPrime	-2.770					1					1			-2.770	-2.805	0.034
TX	MMiddle	-2.991					1						1		-2.991	-3.052	0.061
TX	MOld	-2.552					1							1	-2.552	2.600	0.048

X'y	X'X											
-22.439	8	0	0	0	0	1	1	1	1	1	1	1
-23.582	0	8	0	0	0	1	1	1	1	1	1	1
-36.046	0	0	8	0	0	1	1	1	1	1	1	1
-23.298	0	0	0	8	0	1	1	1	1	1	1	1
-21.647	0	0	0	0	8	1	1	1	1	1	1	1
-13.798	1	1	1	1	1	5	0	0	0	0	0	0
-17.365	1	1	1	1	1	0	5	0	0	0	0	0
-15.628	1	1	1	1	1	0	0	5	0	0	0	0
-13.281	1	1	1	1	1	0	0	0	5	0	0	0
-18.371	1	1	1	1	1	0	0	0	0	5	0	0
17.807	1	1	1	1	1	0	0	0	0	0	5	0
-15.350	1	1	1	1	1	0	0	0	0	0	0	5

Cross Products

426.616	425.179	1.437
425.179	425.179	0.000
1.437	0.000	1.437
100.0%	99.7%	0.3%
I	40	
k	12	
df	28	
σ^2	0.051	
σ	0.227	

	β	$(X'X)^{-1}$											
CA	-3.113	0.300	0.175	0.175	0.175	0.175	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
FL	-3.257	0.175	0.300	0.175	0.175	0.175	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
MI	-4.814	0.175	0.175	0.300	0.175	0.175	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
NY	-3.220	0.175	0.175	0.175	0.300	0.175	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
TX	3.014	0.175	0.175	0.175	0.175	0.300	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
FYoung	0.724	-0.200	-0.200	-0.200	-0.200	-0.200	0.400	0.200	0.200	0.200	0.200	0.200	0.200
FMiddle	0.011	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.400	0.200	0.200	0.200	0.200	0.200
FOld	0.318	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.200	0.400	0.200	0.200	0.200	0.200
MYoung	0.827	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.200	0.200	0.400	0.200	0.200	0.200
MPrime	0.209	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.200	0.200	0.200	0.400	0.200	0.200
MMiddle	0.038	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.200	0.200	0.200	0.200	0.400	0.200
MOld	0.414	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.400

t-stat	Std[β]	Var[β]											
-25.085	0.124	0.015	0.009	0.009	0.009	0.009	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010
-26.246	0.124	0.009	0.015	0.009	0.009	0.009	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010
38.792	0.124	0.009	0.009	0.015	0.009	0.009	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010
-25.950	0.124	0.009	0.009	0.009	0.015	0.009	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010
-24.288	0.124	0.009	0.009	0.009	0.009	0.015	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010
5.051	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.021	0.010	0.010	0.010	0.010	0.010	0.010
0.074	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.021	0.010	0.010	0.010	0.010	0.010
2.219	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.010	0.021	0.010	0.010	0.010	0.010
5.774	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.010	0.010	0.021	0.010	0.010	0.010
1.461	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.010	0.010	0.010	0.021	0.010	0.010
-0.263	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.010	0.010	0.010	0.010	0.021	0.010
2.886	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.021

Exhibit 5

Comparison of Models

State	Sex	Age Group	Actual	One-Way Predicted	Two-Way Predicted	One-Way Closer	Two-Way Closer
CA	F	Young	-2.168	-2.382	-2.389	1	
CA	F	Prime	-2.854	-3.053	-3.113	1	
CA	F	Middle	-3.118	-3.172	-3.102		1
CA	F	Old	-3.173	-2.792	-2.795		1
CA	M	Young	-2.329	-2.292	-2.286	1	
CA	M	Prime	-2.882	-2.963	-2.904		1
CA	M	Middle	-3.141	-3.081	-3.151		1
CA	M	Old	-2.773	-2.702	-2.699	1	
FL	F	Young	-2.438	-2.527	-2.533	1	
FL	F	Prime	-3.193	-3.197	-3.257	1	
FL	F	Middle	-3.368	-3.316	-3.246	1	
FL	F	Old	-3.109	-2.936	-2.939		1
FL	M	Young	-2.201	-2.436	-2.430		1
FL	M	Prime	-3.148	-3.107	-3.048	1	
FL	M	Middle	-3.265	-3.226	-3.295		1
FL	M	Old	-2.872	-2.846	-2.843	1	
MI	F	Young	-4.130	-4.083	-4.090		1
MI	F	Prime	-5.372	-4.754	-4.814		1
MI	F	Middle	-4.510	-4.872	-4.803		1
MI	F	Old	-3.984	-4.493	-4.496	1	
MI	M	Young	-4.298	-3.993	-3.987	1	
MI	M	Prime	-4.705	-4.664	-4.605	1	
MI	M	Middle	-4.932	-4.782	-4.852		1
MI	M	Old	-4.115	-4.403	-4.400		1
NY	F	Young	-2.577	-2.490	-2.497		1
NY	F	Prime	-3.142	-3.161	-3.220	1	
NY	F	Middle	-3.303	-3.279	-3.210	1	
NY	F	Old	-2.832	-2.900	-2.902	1	
NY	M	Young	-2.261	-2.400	-2.393		1
NY	M	Prime	-2.867	-3.071	-3.011		1
NY	M	Middle	-3.278	-3.189	-3.258		1
NY	M	Old	-3.038	-2.810	-2.807	1	
TX	F	Young	-2.486	-2.283	-2.290		1
TX	F	Prime	-2.857	-2.954	-3.014	1	
TX	F	Middle	-3.068	-3.073	-3.003	1	
TX	F	Old	-2.730	-2.693	-2.696		1
TX	M	Young	-2.193	-2.193	-2.187	1	
TX	M	Prime	-2.770	-2.864	-2.805		1
TX	M	Middle	-2.991	-2.983	-3.052	1	
TX	M	Old	-2.552	-2.603	-2.600		1
Mean Absolute Deviation			0.000	0.136	0.137	20	20
Root Mean Square Deviation			0.000	0.195	0.190		

One-Way Relativies		
F	**1.000**	0.0%
M	1.094	+9.4%
Young	1.956	+95.6%
Prime	**1.000**	0.0%
Middle	0.888	-11.2%
Old	1.298	+29.8%

Two-Way Relativies			One-Ways Multiplied		Diff
F.Young	2.062	+106.2%	1.956	+95.6%	5.4%
F.Prime	**1.000**	0.0%	**1.000**	0.0%	0.0%
F.Middle	1.011	+1.1%	0.888	-11.2%	13.8%
F.Old	1.374	+37.4%	1.298	+29.8%	5.9%
M Young	2.287	+128.7%	2.140	+114.0%	6.9%
M Prime	1.233	+23.3%	1.094	+9.4%	12.7%
M.Middle	0.963	-3.7%	0.972	-2.8%	-1.0%
M Old	1.512	+51.2%	1.421	+42.1%	6.4%

Exhibit 6

Restricted Linear Model with One Two-Way Factor

State	S × AG	y	CA	FL	MI	NY	TX	FYoung	FMiddle	FOld	MYoung	MPrime	MMiddle	MOld
CA	FYoung	-2.168	1					1						
CA	FPrime	2.864	1											
CA	FMiddle	-3.118	1						1					
CA	FOld	-3.173	1							1				
CA	MYoung	-2.329	1								1			
CA	MPrime	-2.882	1									1		
CA	MMiddle	-3.141	1										1	
CA	MOld	2.773	1											1
FL	FYoung	-2.438		1				1						
FL	FPrime	-3.193		1										
FL	FMiddle	-3.366		1					1					
FL	FOld	-3.109		1						1				
FL	MYoung	-2.201		1							1			
FL	MPrime	-3.148		1								1		
FL	MMiddle	-3.265		1									1	
FL	MOld	-2.872		1										1
MI	FYoung	-4.130			1			1						
MI	FPrime	-6.372			1									
MI	FMiddle	-4.610			1				1					
MI	FOld	-3.984			1					1				
MI	MYoung	-4.298			1						1			
MI	MPrime	-4.705			1							1		
MI	MMiddle	-4.932			1								1	
MI	MOld	-4.115			1									1
NY	FYoung	-2.577				1		1						
NY	FPrime	-3.142				1								
NY	FMiddle	-3.303				1			1					
NY	FOld	-2.832				1				1				
NY	MYoung	-2.281				1					1			
NY	MPrime	-2.867				1						1		
NY	MMiddle	-3.278				1							1	
NY	MOld	-3.038				1								1
TX	FYoung	-2.486					1	1						
TX	FPrime	-2.857					1							
TX	FMiddle	-3.068					1		1					
TX	FOld	-2.730					1			1				
TX	MYoung	-2.183					1				1			
TX	MPrime	-2.770					1					1		
TX	MMiddle	-2.991					1						1	
TX	MOld	2.552					1							1

y	Xβ	y−Xβ
-2.168	-2.382	0.214
-2.854	-3.053	0.199
-3.118	-3.172	0.053
-3.173	-2.792	-0.381
-2.329	-2.292	-0.036
-2.882	-2.963	0.082
-3.141	-3.081	-0.059
-2.773	-2.702	-0.071
-2.438	-2.527	0.088
-3.193	-3.197	0.005
-3.366	-3.318	-0.051
-3.109	-2.936	-0.172
-2.201	-2.436	0.236
-3.148	-3.107	-0.040
-3.265	-3.226	-0.039
-2.872	-2.848	-0.025
-4.130	-4.083	-0.046
-6.372	-4.754	-0.818
-4.510	-4.872	0.363
-3.984	-4.493	0.509
-4.298	-3.993	-0.305
-4.705	-4.664	-0.041
-4.932	-4.782	-0.150
-4.116	-4.403	0.288
-2.877	-2.490	-0.087
-3.142	-3.161	0.019
-3.303	-3.279	-0.024
-2.832	-2.900	0.067
-2.261	-2.400	0.139
-2.867	-3.071	0.204
-3.278	-3.189	-0.089
-3.038	-2.810	-0.228
-2.486	-2.283	-0.203
-2.857	-2.954	0.097
-3.068	-3.073	0.004
-2.730	-2.693	-0.037
-2.193	2.193	0.001
-2.770	-2.864	0.094
-2.991	-2.983	-0.008
-2.552	2.603	0.051

Cross Products

426.818	425.096	1.621
425.096	425.096	0.000
1.621	0.000	1.621

100.0%	99.6%	0.4%
n	40	
p	12	
j	3	
df	31	
σ^2	0.049	
σ	0.222	

	X'y	X'X														
	-22.439	8	0	0	0	0	1	1	1	1	1	1	1	0	0	0
	-23.692	0	8	0	0	0	1	1	1	1	1	1	1	0	0	0
	-36.046	0	0	8	0	0	1	1	1	1	1	1	1	0	0	0
	-23.298	0	0	0	8	0	1	1	1	1	1	1	1	0	0	0
	-21.647	0	0	0	0	8	1	1	1	1	1	1	1	0	0	0
	-13.799	1	1	1	1	1	5	0	0	0	0	0	0	1	0	0
	-17.365	1	1	1	1	1	0	5	0	0	0	0	0	0	1	0
	-15.828	1	1	1	1	1	0	0	5	0	0	0	0	0	0	1
	13.281	1	1	1	1	1	0	0	0	5	0	0	0	-1	0	0
	-18.371	1	1	1	1	1	0	0	0	0	5	0	0	1	1	1
	-17.807	1	1	1	1	1	0	0	0	0	0	5	0	0	-1	0
	-16.350	1	1	1	1	1	0	0	0	0	0	0	5	0	0	1
Restriction 1	0.000	0	0	0	0	0	1	0	0	-1	1	0	0	0	0	0
Restriction 2	0.000	0	0	0	0	0	0	1	0	0	1	-1	0	0	0	0
Restriction 3	0.000	0	0	0	0	0	0	0	1	0	1	0	-1	0	0	0

	β	(X'X)⁻¹														
CA	-3.053	0.225	0.100	0.100	0.100	0.100	-0.100	-0.100	-0.100	-0.150	-0.050	-0.150	-0.150	-0.125	-0.125	-0.125
FL	3.197	0.100	0.225	0.100	0.100	0.100	-0.100	-0.100	-0.100	-0.150	-0.050	-0.150	-0.150	-0.125	-0.125	-0.125
MI	-4.764	0.100	0.100	0.225	0.100	0.100	-0.100	-0.100	-0.100	-0.150	-0.050	-0.150	-0.150	-0.125	-0.125	-0.125
NY	-3.161	0.100	0.100	0.100	0.225	0.100	-0.100	-0.100	-0.100	-0.150	-0.050	-0.150	-0.150	-0.125	-0.125	-0.125
TX	-2.954	0.100	0.100	0.100	0.100	0.225	-0.100	-0.100	-0.100	-0.150	-0.050	-0.150	-0.150	-0.125	-0.125	-0.125
FYoung	0.671	-0.100	-0.100	-0.100	-0.100	-0.100	0.200	0.100	0.100	0.200	0.000	0.100	0.100	0.500	0.000	0.000
FMiddle	-0.118	-0.100	-0.100	-0.100	-0.100	-0.100	0.100	0.200	0.100	0.100	0.000	0.200	0.100	0.000	0.500	0.000
FOld	0.261	-0.100	-0.100	-0.100	-0.100	-0.100	0.100	0.100	0.200	0.100	0.000	0.100	0.200	0.000	0.000	0.500
MYoung	0.781	-0.150	-0.150	-0.150	-0.150	-0.150	0.200	0.100	0.100	0.300	0.100	0.200	0.200	-0.250	0.250	0.250
MPrime	0.090	-0.050	-0.050	-0.050	-0.050	-0.050	0.000	0.000	0.000	0.100	0.100	0.100	0.100	0.250	0.250	0.250
MMiddle	-0.028	-0.150	-0.150	-0.150	-0.150	-0.150	0.100	0.200	0.100	0.200	0.100	0.300	0.200	0.250	-0.250	0.250
MOld	0.351	-0.150	-0.150	-0.150	-0.150	-0.150	0.100	0.100	0.200	0.200	0.100	0.200	0.300	0.250	0.250	-0.750
λ_1	-0.034	-0.125	-0.125	-0.125	-0.125	-0.125	0.500	0.000	0.000	-0.250	0.250	0.250	0.250	1.875	0.625	0.625
λ_2	0.348	-0.125	-0.125	-0.125	-0.125	-0.125	0.000	0.500	0.000	0.250	0.250	-0.250	0.250	0.625	-1.875	0.625
λ_3	-0.014	-0.125	-0.125	-0.125	-0.125	-0.125	0.000	0.000	0.500	0.250	0.250	0.250	-0.250	0.625	0.625	1.875

t-stat	Std(β)	Var(β)										
-29.056	0.105	0.011	0.005	0.005	0.005	0.005	-0.005	-0.005	-0.005	-0.007	-0.002	-0.007	-0.007
-30.428	0.105	0.005	0.011	0.005	0.005	0.005	-0.005	-0.005	-0.005	-0.007	-0.002	-0.007	-0.007
-45.243	0.105	0.005	0.005	0.011	0.005	0.005	-0.005	-0.005	-0.005	-0.007	-0.002	-0.007	-0.007
-30.078	0.105	0.005	0.005	0.005	0.011	0.005	-0.005	-0.005	-0.005	-0.007	-0.002	-0.007	-0.007
-28.115	0.105	0.005	0.005	0.005	0.005	0.011	-0.005	-0.005	-0.005	-0.007	-0.002	-0.007	-0.007
6.772	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.010	0.005	0.005	0.010	0.000	0.005	0.005
-1.193	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.005	0.010	0.005	0.005	0.000	0.010	0.005
2.636	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.005	0.005	0.010	0.005	0.000	0.005	0.010
6.271	0.121	-0.007	-0.007	-0.007	-0.007	-0.007	0.010	0.005	0.005	0.015	0.005	0.010	0.010
1.285	0.070	-0.002	-0.002	-0.002	-0.002	-0.002	0.000	0.000	0.000	0.005	0.005	0.005	0.005
-0.232	0.121	-0.007	-0.007	-0.007	-0.007	-0.007	0.005	0.010	0.005	0.010	0.005	0.015	0.010
2.894	0.121	-0.007	-0.007	-0.007	-0.007	-0.007	0.005	0.005	0.010	0.010	0.005	0.010	0.015

Exhibit 7

Test: Is the two-way model significantly better than the one-way model?

Null Hypothesis H_0: $R\beta = 0$

R

CA	FL	MI	NY	TX	FYoung	FMiddle	FOld	MYoung	MPrime	MMiddle	MOld
0	0	0	0	0	1	0	0	-1	1	0	0
0	0	0	0	0	0	1	0	0	1	-1	0
0	0	0	0	0	0	0	1	0	1	0	-1

Unrestricted Two-Way Estimator

$R\beta$

0.106
0.258
0.114

$R(X'X)^{-1}R'$

0.800	0.400	0.400
0.400	0.800	0.400
0.400	0.400	0.800

$R\beta - 0$

0.106
0.258
0.114

$(R(X'X)^{-1}R')^{-1}$

1.875	-0.625	-0.625
-0.625	1.875	-0.625
-0.625	-0.625	1.875

numerator *df*	3
denominator *df*	28
numerator	0.028
denominator (σ^2)	0.051
F statistic	0.546
Prob[F = 0.546 | H_0]	65.5%