2004 Discussion Paper Program

Applying and Evaluating Generalized Linear Models

Including
Research Papers on the Valuation of P&C Insurance Companies

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2004 CAS Discussion Paper Program Applying and Evaluating Generalized Linear Models

These papers have been prepared in response to a call for papers by the Casualty Actuarial Society to provide discussion material for the 2004 CAS Spring Meeting, May 16-19 at The Broadmoor in Colorado Springs, Colorado.

The CAS Professional Education Policy Committee, formerly the Committee on Continuing Education, coordinated this Discussion Paper Program under the direction of committee member Roger W. Bovard. Committee chairperson Timothy P. Kenefick oversaw the work of the Michelbacher Award Committee, which reviewed the papers.

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Research Papers on the Valuation of P&C Insurance Companies

The 2004 Discussion Paper Program book also contains two additional papers. Research teams from the Czech Republic and the United States authored studies dealing with the fundamental valuation of property/casualty insurers for merger and acquisition purposes. These research projects were funded by the CAS and sponsored by the CAS Committee on Valuation, Finance, and Investments.

For background on the project and to review the original request for proposals, visit www.casact.org/research/vfic_rfp.htm.

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A Practitioner's Guide to Generalized Linear Models

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The *Practitioner's Guide to Generalized Linear Models* is written for the practicing actuary who would like to understand generalized linear models (GLMs) and use them to analyze insurance data The guide is divided into three sections

Section 1 provides a foundation for the statistical theory and gives illustrative examples and intuitive explanations which clarify the theory The intuitive explanations build upon more commonly understood actuarial methods such as linear models and the minimum bias procedures

Section 2 provides practical insights and realistic model output for each stage of a GLM analysis - including data preparation and preliminary analyses, model selection and iteration, model refinement and model interpretation This section is designed to boost the actuary's confidence in interpreting GLMs and applying them to solve business problems

Section 3 discusses other topics of interest relating to GLMs such as retention modeling and scoring algorithms

More technical material in the paper is set out in appendices.

Acknowledgements

The authors would like to thank James Tanser, FIA, for some helpful comments and contributions to some elements of this paper

Contents

1 GLMs - theory and intuition

1.1 Section 1 discusses how GLMs are formularized and solved. The following topics are covered in detail

- background of GLMs - building upon traditional actuarial methods such as minimum bias procedures and linear models
- introduction to the statistical framework of GLMs
- formularization of GLMs - including the linear predictor, the link function, the offset term, the error term, the scale parameter and the prior weights
- typical model forms
- solving GLMs - maximum likelihood estimation and numerical techniques
- aliasing
- model diagnostics - standard errors and deviance tests.

Background

1.2 Traditional ratemaking methods in the United States are not statistically sophisticated Claims experience for many lines of business is often analyzed using simple one-way and two-way analyses. Iterative methods known as minimum bias procedures, developed by actuaries in the 1960s, provide a significant improvement, but are still only part way toward a full statistical framework

1.3 The classical linear model and many of the most common minimum bias procedures are, in fact, special cases of generalized linear models (GLMs) The statistical framework of GLM allows explicit assumptions to be made about the nature of the insurance data and its relationship with predictive variables The method of solving GLMs is more technically efficient than iteratively standardized methods, which is not only elegant in theory but valuable in practice In addition, GLMs provide statistical diagnostics which aid in selecting only significant variables and in validating model assumptions

1.4 Today GLMs are widely recognized as the industry standard method for pricing private passenger auto and other personal lines and small commercial lines insurance in the European Union and many other markets Most British, Irish and French auto insurers use GLMs to analyze their portfolios and to the authors' knowledge GLMs are commonly used in Italy, the Netherlands, Scandinavia, Spain, Portugal, Belgium, Switzerland, South Africa, Israel and Australia The method is gaining popularity in Canada, Japan, Korea, Brazil, Singapore, Malaysia and eastern European countries

1.5 The primary applications of GLMs in insurance analysis are ratemaking and underwriting Circumstances that limit the ability to change rates at will (eg regulation) have increased the use of GLMs for target marketing analysis

The failings of one-way analysis

1.6 In the past, actuaries have relied heavily on one-way analyses for pricing and monitoring performance

1.7 A one-way analysis summarizes insurance statistics, such as frequency or loss ratio, for each value of each explanatory variable, but without taking account of the effect of other variables Explanatory variables can be discrete or continuous Discrete variables are generally referred to as "factors", with values that each factor can take being referred to as a "level", and continuous variables are generally referred to as "variates" The use of variates is generally less common in insurance modeling.

1.8 One-way analyses can be distorted by correlations between rating factors. For example, young drivers may in general drive older cars A one-way analysis of age of car may show high claims experience for older cars, however this may result mainly from the fact that such older cars are in general driven more by higher risk younger drivers. Relativities based on one-way analyses of age of vehicle and age of driver would double-count the effect of age of driver. Traditional actuarial techniques for addressing this problem usually attempt to standardize the data in such a way as to remove the distorting effect of uneven business mix, for example by focusing on loss ratios on a one-way basis, or by standardizing for the effect of one or more factors. These methods are, however, only approximations.

1.9 One-way analyses also do not consider interdependencies between factors in the way they affect claims experience. These interdependencies, or interactions, exist when the effect of one factor varies depending on the levels of another factor For example, the pure premium differential between men and women may differ by levels of age

1.10 Multivariate methods, such as generalized linear models, adjust for correlations and allow investigation into interaction effects.

The failings of minimum bias procedures

1.11 In the 1960s, actuaries developed a ratemaking technique known as minimum bias procedures [1] These procedures impose a set of equations relating the observed data, the rating variables, and a set of parameters to be determined An iterative procedure solves the system of equations by attempting to converge to the optimal solution. The reader seeking more information may reference "The Minimum Bias Procedure· A Practitioner's Guide" by Sholom Feldblum and Dr J Eric Brosius [2]

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[1] Bailey, Robert A. and LeRoy J Simon, "Two studies in automobile insurance ratemaking," Proceedings of the Casualty Actuarial Society, XLVII, 1960.

[2] Feldblum, Sholom and Brosius, J Eric, "The Minimum Bias Procedures A Practitioner's Guide", Casualty Actuarial Society Forum, 2002 Vol. Fall Page(s): 591-684

1.12 Once an optimal solution is calculated, however, the minimum bias procedures give no systematic way of testing whether a particular variable influences the result with statistical significance. There is also no credible range provided for the parameter estimates. The minimum bias procedures lack a statistical framework which would allow actuaries to assess better the quality of their modeling work.

The connection of minimum bias to GLM

1.13 Stephen Mildenhall has written a comprehensive paper showing that many minimum bias procedures do correspond to generalized linear models.[3] The following table summarizes the correspondence for many of the more common minimum bias procedures. The GLM terminology *link function* and *error function* is explained in depth later in this section. In brief, these functions are key components for specifying a generalized linear model.

Minimum Bias Procedures	Generalized Linear Models	
	Link function	Error function
Multiplicative balance principle	Logarithmic	Poisson
Additive balance principle	Identity	Normal
Multiplicative least squares	Logarithmic	Normal
Multiplicative maximum likelihood with exponential density function	Logarithmic	Gamma
Multiplicative maximum likelihood with Normal density function	Logarithmic	Normal
Additive maximum likelihood with Normal density function	Identity	Normal

1.14 Not all minimum bias procedures have a generalized linear model analog and vice versa. For example, the χ^2 additive and multiplicative minimum bias models have no corresponding generalized linear model analog.

Linear models

1.15 A GLM is a generalized form of a linear model. To understand the structure of generalized linear models it is helpful, therefore, to review classic linear models.

1.16 The purpose of both linear models (LMs) and generalized linear models is to express the relationship between an observed response variable, Y, and a number of covariates (also called predictor variables), X. Both models view the observations, Y_i, as being realizations of the random variable Y.

[3] Mildenhall, Stephen, "A systematic relationship between minimum bias and generalized linear models", Proceedings of the Casualty Actuarial Society, LXXXVI, 1999

1.17 Linear models conceptualize Y as the sum of its mean, μ, and a random variable, ε:

$$Y = \mu + \varepsilon$$

1.18 They assume that

a. the expected value of Y, μ, can be written as a linear combination of the covariates, X, and

b. the error term, ε, is Normally distributed with mean zero and variance σ^2

1.19 For example, suppose a simple private passenger auto classification system has two categorical rating variables territory (urban or rural) and gender (male or female). Suppose the observed average claim severities are:

	Urban	Rural
Male	800	500
Female	400	200

1.20 The response variable, Y, is the average claim severity The two factors, territory and gender, each have two levels resulting in the four covariates male (X_1), female (X_2), urban (X_3), and rural (X_4) The variables take the value 1 or 0 For example, the urban indicator variable, (X_3), is equal to 1 if the territory is urban, and 0 otherwise.

1.21 The linear model seeks to express the observed item Y (in this case average claim severity), as a linear combination of a specified selection of the four variables, plus a Normal random variable ε with mean zero and variance σ^2, often written $\varepsilon \sim N(0,\sigma^2)$. One such model might be

$$Y = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \varepsilon$$

1.22 However this model has as many parameters as it does combinations of rating factor levels being considered, and there is a linear dependency between the four covariates X_1, X_2, X_3, X_4. This means that the model in the above form is not uniquely defined - if any arbitrary value k is added to both β_1 and β_2, and the same value k is subtracted from β_3 and β_4, the resulting model is equivalent

1.23 To make the model uniquely defined in the parameters β_i consider instead the model

$$Y = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon$$

1.24 This model is equivalent to assuming that there is an average response for men (β_1) and an average response for women (β_2), with the effect of being an urban policyholder (as opposed to being a rural one) having an additional additive effect (β_3) which is the same regardless of gender

1.25 Alternatively this could be thought of as a model which assumes an average response for the "base case" of women in rural areas (β_2) with additional additive effects for being male (β_2-β_1) and for being in an urban area (β_3).

1.26 Thus the four observations can be expressed as the system of equations:

$$\begin{aligned}
Y_1 &= 800 = \beta_1 + 0 + \beta_3 + \varepsilon_1 \\
Y_2 &= 500 = \beta_1 + 0 + 0 + \varepsilon_2 \\
Y_3 &= 400 = 0 + \beta_2 + \beta_3 + \varepsilon_3 \\
Y_4 &= 200 = 0 + \beta_2 + 0 + \varepsilon_4
\end{aligned}$$

1.27 The parameters β_1 β_2, β_3 which best explain the observed data are then selected For the classical linear model this is done by minimizing the sum of squared errors (SSE).

$$\begin{aligned}
SSE &= \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 \\
&= (800 - \beta_1 - \beta_3)^2 + (500 - \beta_1)^2 + (400 - \beta_2 - \beta_3)^2 + (200 - \beta_2)^2
\end{aligned}$$

1.28 This expression can be minimized by taking derivatives with respect to β_1, β_2 and β_3 and setting each of them to zero The resulting system of three equations in three unknowns is

$$\begin{aligned}
\frac{\partial SSE}{\partial \beta_1} = 0 \quad &\Rightarrow \quad \beta_1 + \beta_3 + \beta_1 = 800 + 500 = 1300 \\
\frac{\partial SSE}{\partial \beta_2} = 0 \quad &\Rightarrow \quad \beta_2 + \beta_3 + \beta_2 = 400 + 200 = 600 \\
\frac{\partial SSE}{\partial \beta_3} = 0 \quad &\Rightarrow \quad \beta_1 + \beta_3 + \beta_2 + \beta_3 = 800 + 400 = 1200
\end{aligned}$$

which can be solved to derive:

$$\beta_1 = 525$$
$$\beta_2 = 175$$
$$\beta_3 = 250$$

Vector and Matrix Notation

1.29 Formulating the system of equations above quickly becomes complex as both the number of observations and the number of covariates increases, consequently, vector notation is used to express these equations in compact form

1.30 Let $\underline{Y}$ be a column vector with components corresponding to the observed values for the response variable

$$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 800 \\ 500 \\ 400 \\ 200 \end{bmatrix}$$

1.31 Let $\underline{X}_1$, $\underline{X}_2$, and $\underline{X}_3$ denote the column vectors with components equal to the observed values for the respective indicator variables (eg the ith element of $\underline{X}_1$ is 1 when the ith observation is male, and 0 if female):

$$\underline{X}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{X}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{X}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

1.32 Let $\underline{\beta}$ denote a column vector of parameters, and for a given set of parameters let $\underline{\varepsilon}$ be the vector of residuals:

$$\underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \qquad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}$$

1.33 Then the system of equations takes the form

$$\underline{Y} = \beta_1 \underline{X}_1 + \beta_2 \underline{X}_2 + \beta_3 \underline{X}_3 + \underline{\varepsilon}$$

1.34 To simplify this further the vectors $\underline{X}_1$, $\underline{X}_2$, and $\underline{X}_3$ can be aggregated into a single matrix X. This matrix is called the design matrix and in the example above would be defined as.

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

1.35 Appendix A shows an example of the form of the design matrix X when explanatory variables include continuous variables, or "variates".

1.36 The system of equations takes the form

$$\underline{Y} = X\underline{\beta} + \underline{\varepsilon}$$

1.37 In the case of the linear model, the goal is to find values of the components of $\underline{\beta}$ which minimize the sum of squares of the components of $\underline{\varepsilon}$. If there are n observations and p parameters in the model, $\underline{\varepsilon}$ will have n components and $\underline{\beta}$ will have p components ($p<n$).

1.38 The basic ingredients for a linear model thus consist of two elements:

a. a set of assumptions about the relationship between $\underline{Y}$ and the predictor variables, and

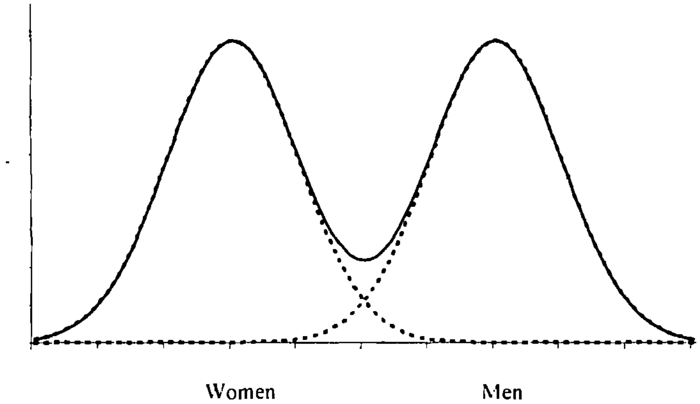
b. an objective function which is to be optimized in order to solve the problem. Standard statistical theory defines the objective function to be the likelihood function. In the case of the classical linear model with an assumed Normal error it can be shown that the parameters which minimize sum of squared error also maximize likelihood.

Classical linear model assumptions

1.39 Linear models assume all observations are independent and each comes from a Normal distribution

1.40 This assumption does not relate to the aggregate of the observed item, but to each observation individually An example may help illustrate this distinction.

Distribution of individual observations



1.41 An examination of average claim amounts by gender may identify that average claim amounts for men are Normally distributed, as are average claim amounts for women, and that the mean of the distribution for men is twice the mean of the distribution for women. The total distribution of average claim amounts across all men and women is not Normally distributed The only distribution of interest is the distribution of the two separate classes (In this case there are only two classes being considered, but in a more complicated model there would be one such class for each combination of the rating factors being considered)

1.42 Linear models assume that the mean is a linear combination of the covariates, and that each component of the random variable is assumed to have a common variance

1.43 The linear model can be written as follows.

$$\underline{Y} = E[\underline{Y}] + \underline{\varepsilon}, \qquad E[\underline{Y}] = \mathbf{X}.\underline{\beta}$$

1.44 McCullagh and Nelder outline the explicit assumptions as follows:[4]

(LM1) *Random component:* Each component of $\underline{Y}$ is independent and is Normally distributed. The mean, μ_i, of each component is allowed to differ, but they all have common variance σ^2

(LM2) *Systematic component:* The p covariates are combined to give the "linear predictor" $\underline{\eta}$:

$$\underline{\eta} = \mathbf{X}.\underline{\beta}$$

(LM3) *Link function:* The relationship between the random and systematic components is specified via a link function. In the linear model the link function is equal to the identity function so that

$$E[\underline{Y}] \equiv \underline{\mu} = \underline{\eta}$$

1.45 The identity link function assumption in **(LM3)** may appear to be superfluous at this point, but it will become more meaningful when discussing the generalization to GLMs

Limitations of Linear Models

1.46 Linear models pose quite tractable problems that can be easily solved with well-known linear algebra approaches However it is easy to see that the required assumptions are not easy to guarantee in applications:

- It is difficult to assert Normality and constant variance for response variables Classical linear regression attempts to transform data so that these conditions hold. For example, Y may not satisfy the hypotheses but $ln(Y)$ may. However there is no reason why such a transformation should exist.
- The values for the response variable may be restricted to be positive. The assumption of Normality violates this restriction
- If the response variable is strictly non-negative then intuitively the variance of Y tends to zero as the mean of Y tends to zero. That is, the variance is a function of the mean

[4] McCullagh, P. and J. A. Nelder, *Generalized Linear Models*, 2nd Ed., Chapman & Hall/CRC, 1989

- The additivity of effects encapsulated in the second (LM2) and third (LM3) assumptions is not realistic for a variety of applications. For example, suppose the response variable is equal to the area of the wings of a butterfly and the predictor variables are the width and length of the wings Clearly, these two predictor variables do not enter additively; rather, they enter multiplicatively More relevantly, many insurance risks tend to vary multiplicatively with rating factors (this is discussed in more detail in Section 2).

Generalized linear model assumptions

1.47 GLMs consist of a wide range of models that include linear models as a special case. The LM restriction assumptions of Normality, constant variance and additivity of effects are removed. Instead, the response variable is assumed to be a member of the exponential family of distributions. Also, the variance is permitted to vary with the mean of the distribution. Finally, the effect of the covariates on the response variable is assumed to be additive on a transformed scale Thus the analog to the linear model assumptions (LM1), (LM2), and (LM3) are as follows

(GLM1) *Random component:* Each component of $\underline{Y}$ is independent and is from one of the exponential family of distributions

(GLM2) *Systematic component*: The p covariates are combined to give the linear predictor $\underline{\eta}$:

$$\underline{\eta} = \mathbf{X}.\underline{\beta}$$

(GLM3) *Link function.* The relationship between the random and systematic components is specified via a link function, g, that is differentiable and monotonic such that:

$$E[\underline{Y}] \equiv \underline{\mu} = g^{-1}(\underline{\eta})$$

1.48 Most statistical texts denote the first expression in (GLM3) with $g(x)$ written on the left side of the equation, therefore, the systematic element is generally expressed on the right side as the inverse function, g^{-1}

Exponential Family of Distributions

1.49 Formally, the exponential family of distributions is a 2-parameter family defined as

$$f_i(y_i, \theta_i, \phi) = \exp\left\{ \frac{y_i \theta_i - b(\theta_i)}{a_i(\phi)} + c(y_i, \phi) \right\}$$

where $a_i(\phi)$, $b(\theta_i)$, and $c(y_i, \phi)$ are functions specified in advance, θ_i is a parameter related to the mean, and ϕ is a scale parameter related to the variance. This formal definition is further explored in Appendix B For practical purposes it is useful to know that a member of the exponential family has the following two properties:

a. the distribution is completely specified in terms of its mean and variance,

b. the variance of Y_i is a function of its mean

1.50 This second property is emphasized by expressing the variance as

$$Var(Y_i) = \frac{\phi V(\mu_i)}{\omega_i}$$

where V(x), called the variance function, is a specified function, the parameter ϕ scales the variance; and ω_i is a constant that assigns a weight, or credibility, to observation i

1.51 A number of familiar distributions belong to the exponential family the Normal, Poisson, binomial, gamma, and inverse Gaussian[5] The corresponding value of the variance function is summarized in the table below

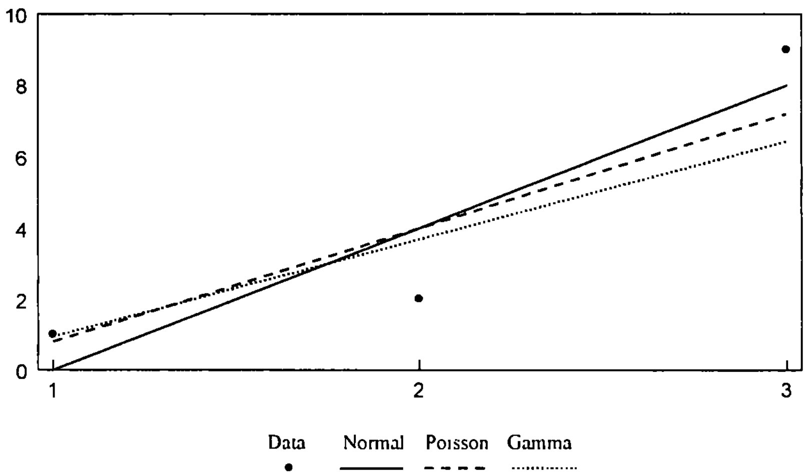
	$V(x)$	
Normal	1	
Poisson	x	
Gamma	x^2	
Binomial	$x(1-x)$	*(where the number of trials = 1)*
Inverse Gaussian	x^3	

1.52 A special member of the exponential family is the Tweedie distribution. The Tweedie distribution has a point mass at zero and a variance function proportional to μ^p (where $p<0$ or $1<p<2$ or $p>2$) This distribution is typically used to model pure premium data directly and is discussed further in Appendix C

[5] A notable exception to this list is the lognormal distribution, which does not belong to the exponential family.

1.53 The choice of the variance function affects the results of the GLM. For example, the graph below considers the result of fitting three different (and very simple) GLMs to three data points In each case the model form selected is a two-parameter model (the intercept and slope of a line), and the three points represent the individual observations (with the observed value Y_i shown on the y-axis for different values of a single continuous explanatory variable shown on the x-axis).

Effect of varying the error term (simple example)



1 54 The three GLMs considered have a Normal, Poisson and gamma variance function respectively It can be seen that the GLM with a Normal variance function (which assumes that each observation has the same fixed variance) has produced fitted values which are attracted to the original data points with equal weight By contrast the GLM with a Poisson error assumes that the variance increases with the expected value of each observation Observations with smaller expected values have a smaller assumed variance, which results in greater credibility when estimating the parameters The model thus has produced fitted values which are more influenced by the observation on the left (with smaller expected value) than the observation on the right (which has a higher expected value and hence a higher assumed variance)

1 55 It can be seen that the GLM with assumed gamma variance function is even more strongly influenced by the point on the left than the point on the right since that model assumes the variance increases with the square of the expected value

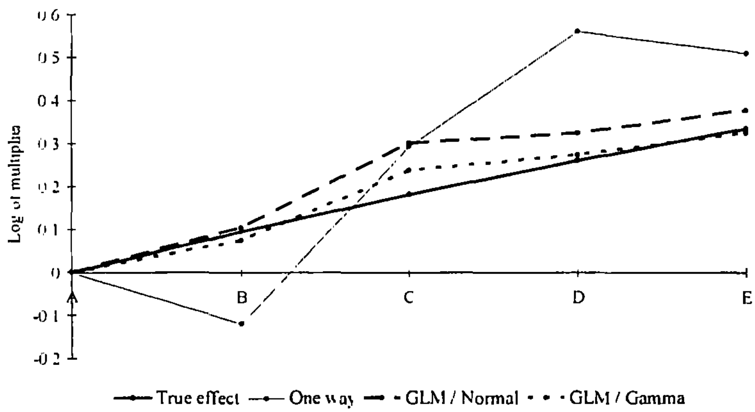
1.56 A further, rather more realistic, example illustrates how selecting an appropriate variance function can improve the accuracy of a model. This example considers an artificially generated dataset which represents an insurance portfolio This dataset contains several rating factors (some of which are correlated), and in each case the true effect of the rating factor is assumed to be known Claims experience (in this case average claim size experience) is then randomly generated for each policy using a gamma distribution, with the mean in each case being that implied by the assumed effect of the rating factors The claims experience is then analyzed using three models to see how closely the results of each model relate to the (in this case known) true factor effect

1.57 The three methods considered are

- a one-way analysis
- a GLM with assumed Normal variance function
- a GLM with assumed gamma variance function

1.58 The results for one of the several rating factors considered are shown on the graph below. It can be seen that owing to the correlations between the rating factors in the data, the one-way analysis is badly distorted The GLM with an assumed Normal distribution is closer to the correct relativities, but it can be seen that it is the GLM with an assumed gamma variance function which yields results which are the closest to the true effect

Effect of varying the error term (insurance rating factor example)



1.59 In addition to the variance function $V(x)$, two other parameters define the variance of each observation, the scale parameter ϕ and the prior weights ω_i

$$Var[Y_i] = \frac{\phi . V(\mu_i)}{\omega_i}$$

Prior weights

1.60 The prior weights allow information about the known credibility of each observation to be incorporated in the model. For example, if modeling claims frequency, one observation might relate to one month's exposure, and another to one year's exposure. There is more information and less variability in the observation relating to the longer exposure period, and this can be incorporated in the model by defining ω_i to be the exposure of each observation. In this way observations with higher exposure are deemed to have lower variance, and the model will consequently be more influenced by these observations

1.61 An example demonstrates the appropriateness of this more clearly. Consider a set of observations for personal auto claims under some classification system Let cell i denote some generic cell defined by this classification system To analyze frequency let

m_{ik} be the number of claims arising from the kth unit of exposure in cell i

ω_i be the number of exposures in cell i

Y_i be the observed claim frequency in cell i:

$$Y_i = \frac{1}{\omega_i} \sum_{k=1}^{\omega_i} m_{ik}$$

1.62 If the random process generating m_{ik} is Poisson with frequency f_i for all exposures k then

$$E[m_{ik}] = f_i = Var[m_{ik}]$$

1 63 Assuming the exposures are independent then

$$\mu_i = E[Y_i] = \frac{1}{\omega_i}\sum_{i=1}^{\omega_i} E[m_{ik}] = \frac{1}{\omega_i}\cdot\omega_i f_i = f_i$$

$$Var[Y_i] = \frac{1}{\omega_i^2}\sum_{i=1}^{\omega_i} Var[m_{ik}] = \frac{1}{\omega_i^2}\cdot\omega_i f_i = \frac{1}{\omega_i} f_i = \mu_i\cdot\frac{1}{\omega_i}$$

1 64 So in this case $V(\mu_i) = \mu_i$, $\phi = 1$, and the prior weights are the exposures in cell i

1 65 An alternative example would be to consider claims severity Let

z_{ik} be the number size of the k[th] claim in cell i

ω_i be the number of claims in cell i

Y_i be the observed mean claim size in cell i·

$$Y_i = \frac{1}{\omega_i}\sum_{k=1}^{\omega_i} z_{ik}$$

1 66 This time assume that the random process generating each individual claim is gamma distributed Denoting

$$E[z_{ik}] = m_i$$

and

$$Var[z_{ik}] = \sigma^2 m_i^2$$

and assuming each claim is independent then

$$\mu_i = E[Y_i] = \frac{1}{\omega_i}\sum_{i=1}^{\omega_i} E[z_{ik}] = \frac{1}{\omega_i}\cdot\omega_i m_i = m_i$$

$$Var[Y_i] = \frac{1}{\omega_i^2}\sum_{i=1}^{\omega_i} Var[z_{ik}] = \frac{1}{\omega_i^2}\cdot\omega_i\sigma^2 m_i^2 = \frac{1}{\omega_i}\sigma^2 m_i^2 = \mu_i^2\cdot\frac{\sigma^2}{\omega_i}$$

1 67 So for severity with a gamma distribution the variance of Y_i follows the general form for all exponential distributions with $V(\mu_i) = \mu_i^2$, $\phi = \sigma^2$, and prior weight equal to the number of claims in cell i

1.68 Prior weights can also be used to attach a lower credibility to a part of the data which is known to be less reliable.

The scale parameter

1.69 In some cases (eg the Poisson distribution) the scale parameter ϕ is identically equal to 1 and falls out of the GLM analysis entirely. However in general and for the other familiar exponential distributions ϕ is not known in advance, and in these cases it must be estimated from the data

1.70 Estimation of the scale parameter is not actually necessary in order to solve for the GLM parameters $\underline{\beta}$, however in order to determine certain statistics (such as standard errors, discussed below) it is necessary to estimate ϕ

1.71 ϕ can be treated as another parameter and estimated by maximum likelihood. The drawback of this approach is that it is not possible to derive an explicit formula for ϕ, and the maximum likelihood estimation process can take considerably longer.

1.72 An alternative is to use an estimate of ϕ, such as

a. the moment estimator (Pearson χ^2 statistic) defined as

$$\hat{\phi} = \sum_i \frac{(Y_i - \mu_i)^2}{V(\mu_i)}$$

b. the total deviance estimator

$$\hat{\phi} = \frac{D}{n-p}$$

where D, the total deviance, is defined later in this paper

Link Functions

1.73 In practice when using classical linear regression practitioners sometimes attempt to transform data to satisfy the requirements of Normality and constant variance of the response variable and additivity of effects. Generalized linear models, on the other hand, merely require that there be a link function that guarantees the last condition of additivity. Whereas **(LM3)** requires that Y be additive in the covariates, the generalization **(GLM3)** instead requires that some transformation of Y, written as $g(Y)$, be additive in the covariates

1.74 It is more helpful to consider μ_i as a function of the linear predictor, so typically it is the inverse of $g(x)$ which is considered:

$$\mu_i = g^{-1}(\eta_i)$$

1.75 In theory a different link function could be used for each observation i, but in practice this is rarely done

1.76 The link function must satisfy the condition that it be differentiable and monotonic (either strictly increasing or strictly decreasing) Some typical choices for a link function include)

	$g(x)$	$g^{-1}(x)$
Identity	x	x
Log	$\ln(x)$	e^x
Logit	$\ln(x/(1-x))$	$e^x/(1+e^x)$
Reciprocal	$1/x$	$1/x$

1.77 Each error structure has associated with it a "canonical" link function which simplifies the mathematics of solving GLMs analytically These are discussed in Appendix D When solving GLMs using modern computer software, however, the use of canonical link functions is not important and any pairing of link function and variance function which is deemed appropriate may be selected

1.78 The log-link function has the appealing property that the effect of the covariates are multiplicative. Indeed, writing $g(x) = ln(x)$ so that $g^{-1}(x) = e^x$ results in

$$\mu_i = g^{-1}(\beta_1 x_{i1} + .. + \beta_p x_{ip}) = \exp(\beta_1 x_{i1}) \cdot \exp(\beta_2 x_{i2}) .. \exp(\beta_p x_{ip})$$

1.79 In other words, when a log link function is used, rather than estimating additive effects, the GLM estimates logs of multiplicative effects

1.80 As mentioned previously, alternative choices of link functions and error structures can yield GLMs which are equivalent to a number of the minimum bias models as well as a simple linear model (see section "The Connection of Minimum Bias to GLM")

The offset term

1.81 There are occasions when the effect of an explanatory variable is known, and rather than estimating parameters $\underline{\beta}$ in respect of this variable it is appropriate to include information about this variable in the model as a known effect. This can be achieved by introducing an "offset term" $\underline{\xi}$ into the definition of the linear predictor $\underline{\eta}$:

$$\underline{\eta} = X.\underline{\beta} + \underline{\xi}$$

which gives

$$E[\underline{Y}] = \underline{\mu} = g^{-1}(\underline{\eta}) = g^{-1}(X.\underline{\beta} + \underline{\xi})$$

1.82 A common example of the use of an offset term is when fitting a multiplicative GLM to the observed number, or count, of claims (as opposed to claim frequency). Each observation may relate to a different period of policy exposure. An observation relating to one month's exposure will obviously have a lower expected number of claims (all other factors being equal) than an observation relating to a year's exposure. To make appropriate allowance for this, the assumption that the expected count of claims increases in proportion to the exposure of an observation (all other factors being equal) can be introduced in a multiplicative GLM by setting the offset term $\underline{\xi}$ to be equal to the log of the exposure of each observation, giving:

$$E[Y_i] = g^{-1}\left(\sum_j X_{ij}\beta_j + \xi_i\right) = \exp\left(\sum_j X_{ij}\beta_j + \log(e_i)\right) = \exp\left(\sum_j X_{ij}\beta_j\right)e_i$$

where e_i = the exposure for observation i.

1.83 In the particular case of a Poisson multiplicative GLM it can be shown that modeling claim counts with an offset term equal to the log of the exposure (and prior weights set to 1) produces identical results to modeling claim frequencies with no offset term but with prior weights set to be equal to the exposure of each observation.

Structure of a generalized linear model

1.84 In summary, the assumed structure of a GLM can be specified as:

$$\mu_i = E[Y_i] = g^{-1}(\sum_j X_{ij}\beta_j + \xi_i)$$
$$Var[Y_i] = \phi V(\mu_i) / \omega_i$$

where

Y_i is the vector of responses

g(x) is the link function, a specified (invertible) function which relates the expected response to the linear combination of observed factors

X_{ij} is a matrix (the "design matrix") produced from the factors

β_j is a vector of model parameters, which is to be estimated

ξ_i is a vector of known effects or "offsets"

ϕ is a parameter to scale the function *V(x)*

V(x) is the variance function

ω_i is the prior weight that assigns a credibility or weight to each observation

1.85 The vector of responses Y_i, the design matrix X_{ij}, the prior weights ω_i, and the offset term ξ_i are based on data in a manner determined by the practitioner. The assumptions which then further define the form of the model are the link function *g(x)*, the variance function *V(x)*, and whether ϕ is known or to be estimated.

Typical GLM model forms

1.86 The typical model form for modeling insurance claim counts or frequencies is a multiplicative Poisson. As well as being a commonly assumed distribution for claim numbers, the Poisson distribution also has a particular feature which makes it intuitively appropriate in that it is invariant to measures of time. In other words, measuring frequencies per month and measuring frequencies per year will yield the same results using a Poisson multiplicative GLM. This is not true of some other distributions such as gamma.

1.87 In the case of claim frequencies the prior weights are typically set to be the exposure of each record. In the case of claim counts the offset term is set to be the log of the exposure.

1.88 A common model form for modeling insurance severities is a multiplicative gamma. As well as often being appropriate because of its general form, the gamma distribution also has an intuitively attractive property for modeling claim amounts since it is invariant to measures of currency In other words measuring severities in dollars and measuring severities in cents will yield the same results using a gamma multiplicative GLM This is not true of some other distributions such as Poisson

1.89 The typical model form for modeling retention and new business conversion is a logit link function and binomial error term (together referred to as a logistic model) The logit link function maps outcomes from the range of (0,1) to (-∞, +∞) and is consequently invariant to measuring successes or failures. If the y-variate being modeled is generally close to zero, and if the results of a model are going to be used qualitatively rather than quantitatively, it may also be possible to use a multiplicative Poisson model form as an approximation given that the model output from a multiplicative GLM can be rather easier to explain to a non-technical audience

1.90 The below table summarizes some typical model forms

$\underline{Y}$	Claim frequencies	Claim numbers or counts	Average claim amounts	Probability (eg of renewing)
Link function $g(x)$	ln(x)	ln(x)	ln(x)	ln(x/(1-x))
Error	Poisson	Poisson	Gamma	Binomial
Scale parameter ϕ	1	1	Estimated	1
Variance function $V(x)$	x	x	x^2	x(1-x)*
Prior weights $\underline{\omega}$	exposure	1	# of claims	1
Offset ξ	0	ln(exposure)	0	0

* where the number of trials=1, or x(t-x)/t where the number of trials = t

GLM maximum likelihood estimators

1.91 Having defined a model form in terms of X, $g(x)$, ξ, $V(x)$, ϕ, and $\underline{\omega}$, and given a set of observations $\underline{Y}$, the components of $\underline{\beta}$ are derived by maximizing the likelihood function (or equivalently, the logarithm of the likelihood function) In essence, this method seeks to find the parameters which, when applied to the assumed model form, produce the observed data with the highest probability.

1.92 The likelihood is defined to be the product of probabilities of observing each value of the y-variate For continuous distributions such as the Normal and gamma distributions the probability density function is used in place of the probability It is usual to consider the log of the likelihood since being a summation across observations rather than a product, this yields more manageable calculations (and any maximum of the likelihood is also a maximum of the log-likelihood). Maximum likelihood estimation in practice, therefore, seeks to find the values of the parameters that maximize this log-likelihood.

1.93 In simple examples the procedure for maximizing likelihood involves finding the solution to a system of equations with linear algebra. In practice, the large number of observations typically being considered means that this is rarely done Instead numerical techniques (and in particular multidimensional Newton-Raphson algorithms) are used Appendix E shows the system of equations for maximizing the likelihood function in the general case of an exponential distribution

1.94 An explicitly solved illustrative example and a discussion of numerical techniques used with large datasets are set out below.

Solving simple examples

1.95 To understand the mechanics involved in solving a GLM, a concrete example is presented Consider the same four observations discussed in a previous section for average claim severity

	Urban	Rural
Male	800	500
Female	400	200

1.96 The general procedure for solving a GLM involves the following steps

a. Specify the design matrix X and the vector of parameters $\underline{\beta}$

b. Choose the error structure and link function

c. Identify the log-likelihood function

d. Take the logarithm to convert the product of many terms into a sum

e. Maximize the logarithm of the likelihood function by taking partial derivatives with respect to each parameter, setting them to zero and solving the resulting system of equations

f. Compute the predicted values

1.97 Recall that the vector of observations, the design matrix, and the vector of parameters are as follows.

$$\underline{Y} = \begin{bmatrix} \text{Male Urban} \\ \text{Male Rural} \\ \text{Female Urban} \\ \text{Female Rural} \end{bmatrix} = \begin{bmatrix} 800 \\ 500 \\ 400 \\ 200 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad \underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

where the first column of X indicates if an observation is male or not, the second column indicates whether the observation is female, and the last column specifies if the observation is in an urban territory or not

1 98 The following three alternative model structures are illustrated.

- Normal error structure with an identity link function
- Poisson error structure with a log link function
- Gamma error structure with an inverse link function.

1 99 These three model forms may not necessarily be appropriate models to use in practice - instead they illustrate the theory involved

1.100 In each case the elements of $\underline{\omega}$ (the "prior weights") will be assumed to be 1, and the offset term $\underline{\xi}$ assumed to be zero, and therefore these terms will, in this example, be ignored.

Normal error structure with an identity link function

1 101 The classical linear model case assumes a Normal error structure and an identity link function. The predicted values in the example take the form

$$E[\underline{Y}] = g^{-1}(X\,\underline{\beta}) = \begin{bmatrix} g^{-1}(\beta_1 + \beta_3) \\ g^{-1}(\beta_1) \\ g^{-1}(\beta_2 + \beta_3) \\ g^{-1}(\beta_2) \end{bmatrix} = \begin{bmatrix} \beta_1 + \beta_3 \\ \beta_1 \\ \beta_2 + \beta_3 \\ \beta_2 \end{bmatrix}$$

1.102 The Normal distribution with mean μ and variance σ^2 has the following density function:

$$f(y,\mu,\sigma^2) = \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)\right\}$$

1.103 Its likelihood function is:

$$L(y,\mu,\sigma^2) = \prod_{i=1}^{n} \exp\{-\frac{(y_i-\mu_i)^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)\}$$

1.104 Maximizing the likelihood function is equivalent to maximizing the log-likelihood function:

$$l(y,\mu,\sigma^2) = \sum_{i=1}^{n} -\frac{(y_i-\mu_i)^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)$$

1.105 With the identity link function, $\mu_i = \Sigma_j X_{ij}\beta_j$ and the log-likelihood function becomes

$$l(y;\mu,\sigma^2) = \sum_{i=1}^{n} -\frac{(y_i - \sum_{j=1}^{p} X_{ij}\beta_j)^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2)$$

1.106 In this example, up to a constant term of $4\ \tfrac{1}{2}\ln(2\pi\sigma^2)$, the log-likelihood is

$$l^*(y;\mu,\sigma^2) = -\frac{(800-(\beta_1+\beta_3))^2}{2\sigma^2} - \frac{(500-\beta_1)^2}{2\sigma^2} - \frac{(400-(\beta_2+\beta_3))^2}{2\sigma^2} - \frac{(200-\beta_2)^2}{2\sigma^2}$$

1.107 To maximize l^* take derivatives with respect to β_1, β_2 and β_3 and set each of them to zero. The resulting system of three equations in three unknowns is

$$\frac{\partial l^*}{\partial \beta_1} = 0 \quad \Rightarrow \quad \beta_1 + \beta_3 + \beta_1 = 800 + 500 = 1300$$

$$\frac{\partial l^*}{\partial \beta_2} = 0 \quad \Rightarrow \quad \beta_2 + \beta_3 + \beta_2 = 400 + 200 = 600$$

$$\frac{\partial l^*}{\partial \beta_3} = 0 \quad \Rightarrow \quad \beta_1 + \beta_3 + \beta_2 + \beta_3 = 800 + 400 = 1200$$

1.108 It can be seen that these equations are identical to those derived when minimizing the sum of squared error for a simple linear model. Again, these can be solved to derive

$$\beta_1 = 525$$
$$\beta_2 = 175$$
$$\beta_3 = 250$$

which produces the following predicted values

	Urban	Rural
Male	775	525
Female	425	175

The Poisson error structure with a logarithm link function

1.109 For the Poisson model with a logarithm link function, the predicted values are given by

$$E[\underline{Y}] = g^{-1}(X\beta) = \begin{bmatrix} g^{-1}(\beta_1 + \beta_3) \\ g^{-1}(\beta_1) \\ g^{-1}(\beta_2 + \beta_3) \\ g^{-1}(\beta_2) \end{bmatrix} = \begin{bmatrix} e^{\beta_1 + \beta_3} \\ e^{\beta_1} \\ e^{\beta_2 + \beta_3} \\ e^{\beta_2} \end{bmatrix}$$

1.110 A Poisson distribution has the following density function

$$f(y;\mu) = e^{-\mu}\mu^{y} / y!$$

1.111 Its log-likelihood function is therefore

$$l(y;\mu) = \sum_{i=1}^{n} \ln f(y_i, \mu_i) = \sum_{i=1}^{n} -\mu_i + y_i \ln \mu_i - \ln(y_i!)$$

1.112 With the logarithm link function, $\mu_i = exp(\Sigma_j X_{ij}\beta_j)$, and the log-likelihood function reduces to

$$l(y;e^{X\beta}) = \sum_{i=1}^{n} -\exp(\sum_{j=1}^{p} X_{ij}\beta_j) + y_i \sum_{j=1}^{p} X_{ij}\beta_j - \ln(y_i!)$$

1.113 In this example, the equation is

$$l(y;\mu) = -e^{(\beta_1 + \beta_3)} + 800 * (\beta_1 + \beta_3) - \ln 800! \quad - e^{\beta_1} + 500 * \beta_1 - \ln 500!$$
$$- e^{(\beta_2 + \beta_3)} + 400 * (\beta_2 + \beta_3) - \ln 400! \quad - e^{\beta_2} + 200 * \beta_2 - \ln 200!$$

1.114 Ignoring the constant of ln800! + ln500! + ln400! + ln200!, the following function is to be maximized:

$$l^*(y,\mu) = -e^{(\beta_1+\beta_3)} + 800(\beta_1+\beta_3) - e^{\beta_1} + 500\,\beta_1 - e^{(\beta_2+\beta_3)} + 400(\beta_2+\beta_3) - e^{\beta_2} + 200.\beta_2$$

1.115 To maximize l^* the derivatives with respect to β_1, β_2 and β_3 are set to zero and the following three equations are derived

$$\frac{\partial l^*}{\partial \beta_1} = 0 \quad => \quad e^{\beta_1} * (e^{\beta_3} + 1) = 1300$$
$$\frac{\partial l^*}{\partial \beta_2} = 0 \quad => \quad e^{\beta_2} * (e^{\beta_3} + 1) = 600$$
$$\frac{\partial l^*}{\partial \beta_3} = 0 \quad => \quad e^{\beta_3} * (e^{\beta_1} + e^{\beta_2}) = 1200$$

1.116 These can be solved to derive the following parameter estimates:

$$\beta_1 = 6.1716$$
$$\beta_2 = 5.3984$$
$$\beta_3 = 0.5390$$

which produces the following predicted values:

	Urban	Rural
Male	821.1	479.0
Female	378.9	221.1

The gamma error structure with an inverse link function

1.117 This example is set out in Appendix F

Solving for large datasets using numerical techniques

1.118 The general case for solving for maximum likelihood in the case of a GLM with an assumed exponential distribution is set out in Appendix E. In insurance modeling there are typically many thousands if not millions of observations being modeled, and it is not practical to find values of $\underline{\beta}$ which maximize likelihood using the explicit techniques illustrated above and in Appendices C and D Instead iterative numerical techniques are used.

1.119 As was the case in the simple examples above, the numerical techniques seek to optimize likelihood by seeking the values of $\underline{\beta}$ which set the first differential of the log-likelihood to zero, as there are a number of standard methods which can be applied to this problem. In practice, this is done using an iterative process, for example Newton-Raphson iteration which uses the formula.

$$\underline{\beta}_{n+1} = \underline{\beta}_n - \boldsymbol{H}^{-1}.\underline{s}$$

where $\underline{\beta}_n$ is the nth iterative estimate of the vector of the parameter estimates $\underline{\beta}$ (with p elements), $\underline{s}$ is the vector of the first derivatives of the log-likelihood and **H** is the (p by p) matrix containing the second derivatives of the log-likelihood. This is simply the generalized form of the one-dimensional Newton-Raphson equation,

$$x_{n+1} = x_n - f'(x_n) / f''(x_n)$$

which seeks to find a solution to $f'(x)=0$

1.120 The iterative process can be started using either values of zero for elements of $\underline{\beta}_0$ or alternatively the estimates implied by a one-way analysis of the data or of another previously fitted GLM.

1.121 Several generic commercial packages are available to fit generalized linear models in this way (such as SAS®, S+, R, etc), and packages specifically built for the insurance industry, which fit models GLMs more quickly and with helpful interpretation of output, are also available (such as Pretium®).

Base levels and the intercept term

1 122 The simple examples discussed above considered a three parameter model, where β_1 corresponded to men, β_2 to women and β_3 to the effect of being in an urban area. In the case of an additive model (with identity link function) this could be thought of as either

- assuming that there is an average response for men, β_1, and an average response for women, β_2, with the effect of being an urban policyholder (as opposed to being a rural one) having an additional additive effect β_3 which is the same regardless of gender

or

- assuming there is an average response for the "base case" of women in rural areas, β_2, with an additional additive effects for being male, β_2-β_1, and for being in an urban area, β_3

1 123 In the case of a multiplicative model this three parameter form could be thought of as

- assuming that there is an average response for men, $exp(\beta_1)$ and an average response for women, $exp(\beta_2)$, with the effect of being an urban policyholder (as opposed to being a rural one) having a multiplicative effect $exp(\beta_3)$ which is the same regardless of gender

or

- assuming there is an average response for the "base case" of women in rural areas $exp(\beta_2)$ with an additional multiplicative effects for being male, $exp(\beta_2$-$\beta_1)$, and for being in an urban area $exp(\beta_3)$

1 124 In the example considered, some measure of the overall average response was incorporated in both the values of β_1 and β_2 The decision to incorporate this in the parameters relating to gender rather than area was arbitrary

1 125 In practice when considering many factors each with many levels it is more helpful to parameterize the GLM by considering, in addition to observed factors, an "intercept term", which is a parameter that applies to all observations

1.126 In the above example, this would have been achieved by defining the design matrix X as

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

that is, by redefining β_1 as the intercept term, and only having one parameter relating to the gender of the policyholder. It would not be appropriate to have an intercept term *and* a parameter for every value of gender since then the GLM would not be uniquely defined - any arbitrary constant k could be added to the intercept term and subtracted from each of the parameters relating to gender and the predicted values would remain the same.

1.127 In practice when considering categorical factors and an intercept term, one level of each factor should have no parameter associated with it, in order that the model remains uniquely defined.

1.128 For example consider a simple rating structure with three factors - age of driver (a factor with 9 levels), territory (a factor with 8 levels) and vehicle class (a factor with 5 levels). An appropriate parameterization might be represented as follows

Age of driver

Factor level	Parameter
17-21	β_2
22-24	β_3
25-29	β_4
30-34	β_5
35-39	β_6
40-49	
50-59	β_7
60-69	β_8
70+	β_9

Territory

Factor level	Parameter
A	β_{10}
B	β_{11}
C	
D	β_{12}
E	β_{13}
F	β_{14}
G	β_{15}
H	β_{16}

Vehicle class

Factor level	Parameter
A	
B	β_{17}
C	β_{18}
D	β_{19}
E	β_{20}

Intercept term	β_1

that is, an intercept term is defined for every policy, and each factor has a parameter associated with each level except one. If a multiplicative GLM were fitted to claims frequency (by selecting a log link function) the exponentials of the parameter estimates β could be set out in tabular form also:

Age of driver

Factor level	Multiplier
17-21	1 6477
22-24	1.5228
25-29	1.5408
30-34	1 2465
35-39	1 2273
40-49	1 0000
50-59	0 8244
60-69	0 9871
70+	0 9466

Territory

Factor level	Multiplier
A	0 9407
B	0 9567
C	1 0000
D	0 9505
E	1 0975
F	1 1295
G	1 1451
H	1 4529

Vehicle class

Factor level	Multiplier
A	1 0000
B	0 9595
C	1.0325
D	0 9764
E	1 1002

Intercept term	0 1412

1.129 In this example the claims frequency predicted by the model can be calculated for a given policy by taking the intercept term 0.1412 and multiplying it by the relevant factor relativities. For the factor levels for which no parameter was estimated (the "base levels"), no multiplier is relevant, and this is shown in the above table by displaying multipliers of 1 The intercept term relates to a policy with all factors at the base level (ie in this example the model predicts a claim frequency of 0 1412 for a 40-49 year old in territory C and a vehicle in class A) This intercept term is not an average rate since its value is entirely dependent upon the arbitrary choice of which level of each factor is selected to be the base level

1.130 If a model were structured with an intercept term but without each factor having a base level, then the GLM solving routine would remove as many parameters as necessary to make the model uniquely defined This process is known as *aliasing*.

Aliasing

1 131 Aliasing occurs when there is a linear dependency among the observed covariates $X_1, \ldots X_p$ That is, one covariate may be identical to some combination of other covariates. For example, it may be observed that

$$\underline{X}_3 = 4 + \underline{X}_1 + 5\underline{X}_2$$

1 132 Equivalently, aliasing can be defined as a linear dependency among the columns of the design matrix $\boldsymbol{X}$

1.133 There are two types of aliasing intrinsic aliasing and extrinsic aliasing.

Intrinsic aliasing

1.134 Intrinsic aliasing occurs because of dependencies inherent in the definition of the covariates. These intrinsic dependencies arise most commonly whenever categorical factors are included in the model.

1.135 For example, suppose a private passenger automobile classification system includes the factor *vehicle age* which has the four levels: 0-3 years (X_1), 4-7 years (X_2), 8-9 years (X_3), and 10+ years (X_4) Clearly if any of X_1, X_2, X_3, is equal to 1 then X_4 is equal to 0, and if all of X_1, X_2, X_3, are equal to 0 then X_4 must be equal to 1 Thus $X_4 = 1 - X_1 - X_2 - X_3$

1.136 The linear predictor

$$\eta = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4$$

(ignoring any other factors) can be uniquely expressed in terms of the first three levels

$$\begin{aligned}\eta &= \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 (1 - X_1 - X_2 - X_3) \\ &= (\beta_1 - \beta_4) X_1 + (\beta_2 - \beta_4) X_2 + (\beta_3 - \beta_4) X_3 + \beta_4\end{aligned}$$

1.137 Upon renaming the coefficients this becomes

$$\eta = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_0$$

1.138 The result is a linear predictor with an intercept term (if one did not already exist) and three covariates.

1.139 GLM software will remove parameters which are aliased Which parameter is selected for exclusion depends on the software The choice of which parameter to alias does not affect the fitted values For example in some cases the last level declared (ie the last alphabetically) is aliased. In other software the level with the maximum exposure is selected as the base level for each factor first, and then other levels are aliased dependent upon the order of declaration (This latter approach is helpful since it minimizes the standard errors associated with other parameter estimates - this subject is discussed later in this paper)

Extrinsic Aliasing

1.140 This type of aliasing again arises from a dependency among the covariates, but when the dependency results from the nature of the data rather than inherent properties of the covariates themselves. This data characteristic arises if one level of a particular factor is perfectly correlated with that of another factor.

1.141 For example, suppose a dataset is enriched with external data and two new factors are added to the dataset: the factors *number of doors* and *color of vehicle*. Suppose further that in a small number of cases the external data could not be linked with the existing data with the result that some records have an unknown color and an unknown number of doors.

Exposures		# Doors				
		2	3	4	5	Unknown
Color	Red	13,234	12,243	15,432	13,432	0
	Green	4,543	4,543	13,243	2,345	0
	Blue	6,544	5,443	15,654	4,565	0
	Black	4,643	1,235	14,565	4,545	0
	Unknown	0	0	0	0	3,242

Selected Base: # Doors = 4, Color = Red

Additional Aliasing: Color = Unknown

1.142 In this case because of the way the new factors were derived, the level *unknown* for the factor *color* happens to be perfectly correlated with the level *unknown* for the factor *# doors*. The covariate associated with *unknown color* is equal to 1 in every case for which the covariate for *unknown # doors* is equal to 1, and vice versa.

1.143 Elimination of the base levels through intrinsic aliasing reduces the linear predictor from 10 covariates to 8, plus the introduction of an intercept term. In addition, in this example, one further covariate needs to be removed as a result of extrinsic aliasing. This could either be the *unknown color* covariate or the *unknown # doors* covariate. Assuming in this case the GLM routine aliases on the basis of order of declaration, and assuming that the *# doors* factor is declared before *color*, the GLM routine would alias *unknown color* reducing the linear predictor to just 7 covariates.

"Near Aliasing"

1.144 When modeling in practice a common problem occurs when two or more factors contain levels that are almost, but not quite, perfectly correlated For example, if the color of vehicle was known for a small number of policies for which the # doors was unknown, the two-way of exposure might appear as follows

Exposures		# Doors			
Color	2	3	4	5	Unknown
Red	13,234	12,343	15,432	13,432	0
Green	4,543	4,543	13,243	2,345	0
Blue	6,544	5,443	15,654	4,565	0
Black	4,643	1,235	14,565	4,545	5
Unknown	0	0	0	0	3,242

Selected Base. # Doors = 4 Color = Red

1 145 In this case the *unknown* level of color factor is not perfectly correlated to the *unknown* level of the # doors factor, and so extrinsic aliasing will not occur

1 146 When levels of two factors are "nearly aliased" in this way, convergence problems can occur. For example, if there were no claims for the 5 exposures indicated in *black color* level and *unknown # doors* level, and if a log link model were fitted to claims frequency, the model would attempt to estimate a very large and negative parameter for *unknown # doors* (for example, -20) and a very large parameter for *unknown color* (for example 20 2) The sum (0 2 in this example) would be an appropriate reflection of the claims frequency for the 3,242 exposures having *unknown # doors* and unknown color, while the value of the *unknown # doors* parameter would be driven by the experience of the 5 rogue exposures having color black with unknown # doors This can either give rise to convergence problems, or to results which can appear very confusing

1 147 In order to understand the problem in such circumstances it is helpful to examine two-way tables of exposure and claim counts for the factors which contain large parameter estimates From these it should be possible to identify those factor combinations which cause the near-aliasing The issue can then be resolved either by deleting or excluding those rogue records, or by reclassifying the rogue records into another factor level.

Model diagnostics

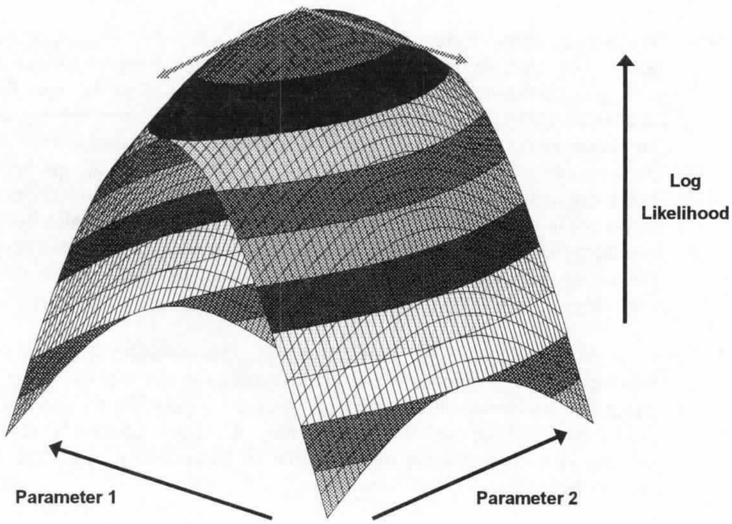
1.148 As well as deriving parameter estimates which maximize likelihood, a GLM can produce important additional information indicating the certainty of those parameter estimates (which themselves are estimates of some true underlying value).

Standard errors

1.149 Statistical theory can be used to give an estimate of the uncertainty. In particular, the multivariate version of the Cramer-Rao lower bound (which states that the variance of a parameter estimate is greater than or equal to minus one over the second derivative of the log likelihood) can define "standard errors" for each parameter estimate. Such standard errors are defined as being the diagonal element of the covariance matrix $-\boldsymbol{H}^{-1}$ where $\boldsymbol{H}$ (the Hessian) is the second derivative matrix of the log likelihood.

1.150 Intuitively the standard errors can be thought of as being indicators of the speed with which log-likelihood falls from the maximum given a change in a parameter. For example consider the below diagram.

Intuitive illustration of standard errors



1 151 This diagram illustrates a simple case with two parameters (β_1 and β_2) and shows how log likelihood varies, for the dataset in question, for different values of the two parameters It can be seen that movements in parameter 1 from the optimal position reduce log likelihood more quickly than similar movements in parameter 2, that is to say the log likelihood curve becomes steeper in the parameter 1 direction than in the parameter 2 direction. This can be thought of as the second partial differential of log likelihood with respect to parameter 1 being large and negative, with the result that the standard error for parameter 1 (being minus one over the second partial differential) is small Conversely the second partial differential of log likelihood with respect to parameter 2 is less large and negative, with the standard error for parameter 2 being larger (indicating greater uncertainty)

1 152 Generally it is assumed that the parameter estimates are asymptotically Normally distributed and consequently it is in theory possible to undertake a simple statistical test on individual parameter estimates, comparing each estimate with zero (ie testing whether the effect of each level of the factor is significantly different from the base level of that factor) This is usually performed using a χ^2 test, with the square of the parameter estimate divided by its variance being compared to a χ^2 distribution This test in fact compares the parameter with the base level of the factor This is not necessarily a fully useful test in isolation as the choice of base level is arbitrary. It is theoretically possible repeatedly to change the base level and so construct a triangle of χ^2 tests comparing every pair of parameter estimates If none of these differences is significant then this is good evidence that the factor is not significant

1 153 In practice graphical interpretation of the parameter estimates and standard errors are often more helpful, and these are discussed in Section 2

Deviance tests

1 154 In addition to the parameter estimate standard errors, measures of deviance can be used to assess the theoretical significance of a particular factor. In broad terms, a deviance is a measure of how much the fitted values differ from the observations

1 155 Consider a deviance function $d(Y_i;\mu_i)$ defined by

$$d(Y_i;\mu_i) = 2\omega_i \int_{\mu_i}^{Y_i} \frac{(Y_i - \zeta)}{V(\zeta)} d\zeta$$

Under the condition that V(x) is strictly positive, $d(Y_i;\mu_i)$ is also strictly positive and satisfies the conditions for being a distance function. Indeed it should be interpreted as such

1.156 Consider an observation Y_i and a GLM that makes a prediction μ_i for that observation. $d(Y_i,\mu_i)$ is a measure of the difference between the fitted and actual observations which gives more weight to the difference between Y_i and μ_i the smaller the variance function $V(x)$ is. That is, if Y_i is known to come from a distribution with small variance then any discrepancy between Y_i and μ_i is given more emphasis

1.157 $d(Y_i,\mu_i)$ can be thought of as a generalized form of the squared error.

1.158 Summing the deviance function across all observations gives an overall measure of deviance referred to as the total deviance D

$$D = \sum_{i=1}^{n} 2\omega_i \int_{\mu_i}^{Y_i} (Y_i - \zeta)/V(\zeta)d\zeta$$

1.159 Dividing this by the scale parameter ϕ gives the scaled deviance D^*, which can be thought of as a generalized form of the sum of squared errors, adjusting for the shape of the distribution

$$D^* = \sum_{i=1}^{n} 2\frac{\omega_i}{\phi} \int_{\mu_i}^{Y_i} (Y_i - \zeta)/V(\zeta)d\zeta$$

1.160 For the class of exponential distributions the scaled deviance can be shown to be equal to twice the difference between the maximum achievable likelihood (ie the likelihood where the fitted value is equal to the observation for every record) and the likelihood of the model

1.161 A range of statistical tests can be undertaken using deviance measures. One of the most useful considers the ratio of the likelihood of two "nested" models, that is to say where one model contains explanatory variables which are a subset of the explanatory variables in a second model. Such tests are often referred to as "type III" tests (as opposed to "type I" tests which consider the significance of factors as they are added sequentially to a model with only an intercept term, referred to as a null model).

1.162 The change in scaled deviance between two nested models (which reflects the ratio of the likelihoods) can be considered to be a sample from a χ^2 distribution with degrees of freedom (defined as number of observations less the number of parameters) equal to the change in degrees of freedom, ie

$$D_1^* - D_2^* \sim \chi^2_{df_1 - df_2}$$

1 163 This allows tests to be undertaken to assess the significance of the parameters that differ between the two models (with the null hypothesis that the extra parameters are not important) Expressed crudely this measures whether the inclusion of an explanatory factor in a model improves the model enough (ie decreases the deviance enough) given the extra parameters which it adds to the model Adding any factor will improve the fit on the data in question - what matters is whether the improvement is significant given the extra parameterization

1 164 The χ^2 tests depend on the scaled deviance. For some distributions (such as the Poisson and the binomial) the scale parameter is assumed to be known, and it is possible to calculate the statistic accurately. For other distributions the scale parameter is not known and has to be estimated, typically as the ratio of the deviance to the degrees of freedom This can decrease the reliability of this test if the estimate of the scale parameter used is not accurate.

1 165 It is possible to show that, after adjusting for the degrees of freedom and the true scale parameter, the estimate of the scale parameter is also distributed with a χ^2 distribution. The F-distribution is the ratio of χ^2 distributions The ratio of the change in deviance and the adjusted estimate of the scale is therefore distributed with an F-distribution

$$\frac{(D_1 - D_2)}{(df_1 - df_2) D_2 / df_2} \sim F_{df_1 - df_2, df_2}$$

1 166 This means that the F-test is suitable for use when the scale parameter is not known (for example when using the gamma distribution) There is no advantage to using this test where the scale is known

2 GLMs in practice

2.1 Section 1 discussed how GLMs are formularized and solved. This section considers practical issues and presents a plan for undertaking a GLM analysis in four general stages:

- pre-modeling analysis - considering data preparation as well as a range of helpful portfolio investigations
- model iteration - typical model forms and the diagnostics used in both factor selection and model validation
- model refinement - investigating interaction variables, the use of smoothing, and the incorporation of artificial constraints
- interpreting the results - how model results can be compared to existing rating structures both on a factor-by-factor basis and overall.

Data required

2.2 GLM claim analyses require a certain volume of experience. Depending on the underlying claim frequencies and the number of factors being analyzed, credible results on personal lines portfolios can generally be achieved with around 100,000 exposures (which could for example be 50,000 in each of two years, etc). Meaningful results can sometimes be achieved with smaller volumes of data (particularly on claim types with adequate claims volume), but it is best to have many 100,000s of exposures. As models fitted to only a single year of data could be distorted by events that occurred during that year, the data should ideally be based on two or three years of experience.

2.3 In addition to combining different years of experience, combining states (or provinces) can also improve stability, assuming inputs are consistent across borders.[6] In the case where one geographic area has sufficient exposure it may be more appropriate to fit a model just to that area's experience. If a countrywide model has been run, the goodness of fit of that model on state data may be investigated, or the state and countrywide model results may be compared side-by-side. Examining the interaction of state with each predictive factor may also identify where state patterns differ from countrywide, interaction variables are discussed later in this paper.

[6] In this sense, inputs refer to explanatory criteria, not necessarily existing rating relativities. Data coding should be reviewed to ensure some level of consistency and care should be taken with recycled coding from one state to another (eg territory 1 in Virginia should not be added to territory 1 in Florida)

2.4 Different types of claim can be affected by rating factors in different ways and so often it is appropriate to analyze different types of claim with separate models. Analyzing different claim elements separately will often identify clearer underlying trends than considering models based on a mixture of claims (for example, liability claims combined with theft claims). Even if a single model is required ultimately, it is generally beneficial to model by individual claim type and later to produce a single model which fits the aggregate of the underlying models by claim type.

2.5 The overall structure of a dataset for GLM claims analysis consists of linked policy and claims information at the individual risk level. Typical data requirements and a more detailed discussion of issues such as dealing with IBNR are set out in Appendix G. In summary, however, the following fields would typically be included in a GLM claims dataset:

- Raw explanatory variables - whether discrete or continuous, internal or external to the company.
- Dummy variables to standardize for time-related effects, geographic effects and certain historical underwriting effects.
- Earned exposure fields - preferably by claim type if certain claim types are only present for some policies. These fields should contain the amount of exposure attributable to the record (eg measured in years).
- Number of incurred claims fields. There should be one field for each claim type, giving the number of claims associated with the exposure period in question.
- Incurred loss amounts fields. There should be one field for each claim type, giving the incurred loss amount of claims associated with the exposure period in question.
- Premium fields. These give the premium earned during the period associated with the record. If it is possible to split this premium between the claim types then this can be used to enhance the analysis. This information is not directly required for modeling claims frequency and severity, however it can be helpful for a range of post-modeling analyses such as measuring the impact of moving to a new rating structure.

2 6 When analyzing policyholder retention or new business conversion, a different form of data is required For example to fit GLMs to policyholder renewal experience, a dataset would contain one record for each invitation to renew and would contain the following fields

- explanatory variables including, for example,
  - rating factors
  - other factors such as distribution channel, method of payment and number of terms with company
  - change in premium on latest renewal[7]
  - change in premium on previous renewal
  - measure of competitiveness on renewal premium
  - details of any mid-term adjustments occurring in the preceding policy period
- number of invitations to renew (typically 1 for each record - this would be the measure of exposure)
- whether or not the policy renewed

2 7 If several risks are written on a single policy, renewal may be defined at the policy level or at the individual risk level (for example, a personal automobile carrier may write all vehicles in a household on a single policy) An understanding of how the model will be used will aid data preparation For example, models that will be part of a detailed model office scenario will benefit from data defined at the individual risk level Models used to gain an overall understanding of which criteria affect policyholder retention (perhaps for marketing purposes) would not require such detail

[7] Separation of premium change into rate change and risk criteria change would be beneficial

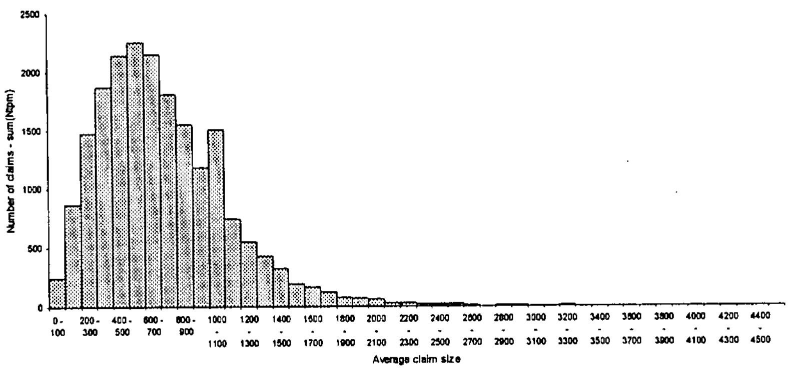
Preliminary analyses

2.8 Before modeling, it is generally helpful to undertake certain preliminary analyses. These analyses include data checks such as identification of records with negative or zero exposures, negative claim counts or losses, and blanks in any of the statistical fields. In addition, certain logical tests may be run against the data - for example, identifying records with incurred losses but with no corresponding claim count.

Analysis of distributions

2.9 One helpful preliminary analysis is to consider the distribution of key data items for the purpose of identifying any unusual features or data problems that should be investigated prior to modeling. Mainly this concerns the distribution of claim amounts (ie number of claim counts by average claim size), which are examined in order to identify features such as unusually large claims and distortions resulting from average reserves placed on newly reported claims. A typical claim distribution graph is shown below.

Distribution of claim amounts



2.10 This distribution, along with a distribution of loss amount by average claim size, will aid in understanding the tail of the distribution for a particular claim type. (When modeling severity, it is often appropriate to apply a large loss threshold to certain claim types, and this helps assess possible thresholds. A tabular representation of the distribution would also help quantify the percent of the claims distribution which would be affected by different large loss thresholds.)

2.11 Distribution analyses can also highlight specific anomalies that might require addressing prior to modeling. For example, if many new claims have a standard average reserve allocated to them, it might be appropriate to adjust the amount of such an average reserve if it was felt that the average level was systematically below or above the ultimate average claims cost.

One and two-way analyses

2.12 Although GLMs are a multivariate method, there is generally benefit in reviewing some one-way and two-way analyses of the raw data prior to modeling.

2.13 Firstly, the one-way distribution of exposure and claims across levels of each raw variable will indicate whether a variable contains enough information to be included in any models (for example, if 99.5% of a variable's exposures are in one level, it may not be suitable for modeling)

2.14 Secondly, assuming there is some viable distribution by levels of the factor, consideration needs to be given to any individual levels containing very low exposure and claim count. If these levels are not ultimately combined with other levels, the GLM maximum likelihood algorithm may not converge (if a factor level has zero claims and a multiplicative model is being fitted, the theoretically correct multiplier for that level will be close to zero, and the parameter estimate corresponding to the log of that multiplier may be so large and negative that the numerical algorithm seeking the maximum likelihood will not converge)

2.15 In addition to investigating exposure and claim distribution, a query of one-way statistics (eg frequency, severity, loss ratio, pure premium) will give a preliminary indication of the effect of each factor.

Factor categorizations

2.16 Before modeling, it is necessary to consider how explanatory variables should be categorized, and whether any variables should be modeled in a continuous fashion as variates (or polynomials in variates). Although variates do not require any artificially imposed categorization, the main disadvantage is that the use of polynomials may smooth over interesting effects in the underlying experience. Often it is better to begin modeling all variables as narrowly defined categorical factors (ensuring sufficient data in each category) and if the categorical factor presents GLM parameter estimates which appear appropriate for modeling with a polynomial, then the polynomial in the variate may be used in place of the categorical factor

2 17 When using categorical factors consideration needs to be given to the way in which the factors are categorized If an example portfolio contained a sufficient amount of claims for each for each age of driver (say from age 16 to 99), the categorization of age of driver may consist of each individual age. This is rarely the case in practice, however, and often it is necessary that levels of certain rating factors are combined.

2.18 In deriving an appropriate categorization, the existing rating structure may provide initial guidance (particularly if the GLMs are to be applied in ratemaking), with factor levels with insufficient exposure then being grouped together and levels with sufficient exposure being considered separately In general such a manual approach tends to be the most appropriate One particular automated approach within the GLM framework is considered in Appendix H This approach, however, would not necessarily produce any more appropriate results than the manual approach

Correlation analyses

2 19 Once categorical factors have been defined, it can also be helpful to consider the degree to which the exposures of explanatory factors are correlated One commonly used correlation statistic for categorical factors is Cramer's V statistic.[8] Further information about this statistic is set out in Appendix I.

2 20 Although not used directly in the GLM process, an understanding of the correlations within a portfolio is helpful when interpreting the results of a GLM In particular it can explain why the multivariate results for a particular factor differ from the univariate results, and can indicate which factors may be affected by the removal or inclusions of any other factor in the GLM

Data extracts

2 21 In practice it is not necessary to fit every model to the entire dataset For example, modeling severity for a particular claim type only requires records that contain a claim of that type Running models against data subsets, or extracts, can improve model run speed

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[8] Other correlation statistics for categorical factors include Pearson chi-square, Likelihood ratio chi-square, Phi coefficient and Contingency coefficient A thorough discussion of these statistics is beyond the scope of this paper

2 22 The error term assumed for a model can also influence these data extracts. In the case of claim counts, a particular property of Poisson multiplicative model is that the observed data Y_i can be grouped by unique combination of rating factors being modeled (summing exposure and claim counts for each unique combination) and the GLM parameter estimates and the parameter estimate standard errors will remain unchanged. This is helpful in practice since it can decrease model run times This is not the case for some other distributions

2 23 A gamma multiplicative model does not produce identical results if the observations are grouped by unique combinations of factors. Such a grouping would not change parameter estimates, but it would affect the standard errors Depending on the line of business, however, it may be appropriate to group the small number of multiple claims which occur on the same policy in the same exposure period

Model iteration and the role of diagnostics

2 24 Given data relating to the actual observations and the assumptions about the model form, a GLM will yield parameter estimates which best fit the data given that model form The GLM will not automatically provide information indicating the appropriateness of the model fitted - for this it is necessary to examine a range of diagnostics. This section reviews model forms typically used in practice and discusses the range of diagnostics which aid in both the selection of explanatory factors and the validation of statistical assumptions

Factor selection

2 25 One of the key issues to consider is which explanatory factors should be included in the model The GLM will benefit from including factors which systematically affect experience, but excluding factors which have no systematic effect To distinguish whether a factor effect is systematic or random (and therefore unlikely to be repeated in the future) there are a number of criteria which can be considered, including

- parameter estimate standard errors
- deviance tests (type III tests)
- consistency with time
- common sense

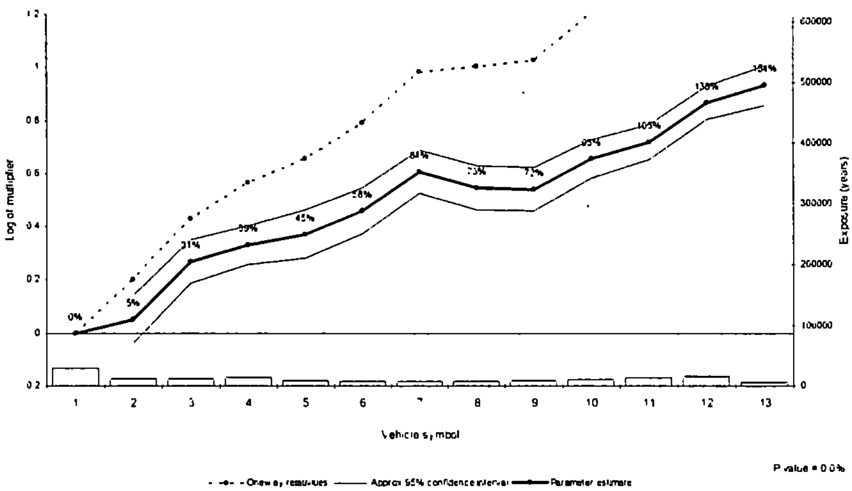
Standard errors

2 26 As discussed in Section 1, as well as deriving parameter estimates which maximize likelihood, a GLM can produce important additional information indicating the certainty of those parameter estimates

2.27 One such helpful diagnostic is the standard errors of the parameter estimates, defined as being the square root of the diagonal element of $-\boldsymbol{H}^{1}$ where $\boldsymbol{H}$ (the Hessian) is the second derivative matrix of the log likelihood.

2.28 Although theoretically tests could be performed on individual parameter estimates using standard errors, in practice it is often more helpful to consider for each factor in the GLM the fitted parameter estimates alongside the associated standard errors (for one base level) in a graphical form thus:

GLM output (example of significant factor)



2.29 One such graph would be shown for each factor in the model. In this case the factor in question is Vehicle Symbol with levels running from 1 to 13.

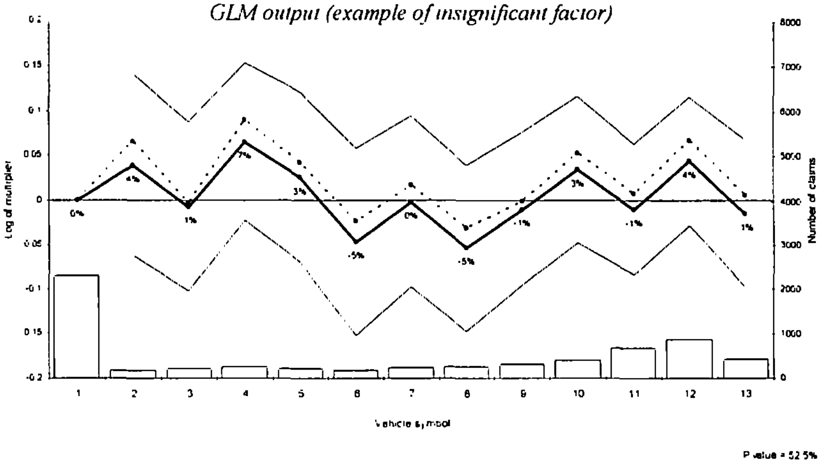
2.30 The thick solid line shows the fitted parameter estimates. In this case the model is a multiplicative model with a log link function and so the parameter estimates represent logs of multipliers. For clarity the implied loadings are shown as labels by each point on the thick solid line. For example - the parameter estimate for Vehicle Symbol 3 has value 0.27. This means that the model estimates that, all other factors being constant, exposures with Vehicle Symbol 3 will have a relativity of $e^{0.27} = 1.31$ times that expected for exposures at the base level (in this example Symbol 1). This multiplier is shown on the graph as a "loading" of 31%.

2.31 The thin solid lines on each graph indicate two standard errors either side of the parameter estimate. Very approximately this means that (assuming the fitted model is appropriate and correct) the data suggests that the true relativity for each level of rating factor will lie between the two thin solid lines with roughly 95% certainty. The two bands will be wide apart, indicating great uncertainty in the parameter estimate where there is low exposure volume, where other correlated factors also explain the risk, or where the underlying experience is very variable

2.32 The dotted lines shows the relativities implied by a simple one-way analysis. These relativities make no allowance for the fact that the difference in experience may be explained in part by other correlated factors These one-way estimates are of interest since they will differ from the multivariate estimates for a given factor when there are significant correlations between that factor and one or more other significant factors. The distribution of exposure for all business considered is also shown as a bar chart at the bottom of each graph. This serves to illustrate which level of each factor may be financially significant.

2.33 Even though the standard errors on the graph only indicate the estimated certainty of the parameter estimates relative to the base level, such graphs generally give a good intuitive feel for the significance of a factor For example in the above case it is clear that the factor is significant since the parameter estimates for Vehicle Symbols 3 to 13 are considerably larger than twice the corresponding standard errors By contrast the graph below (an example of the same factor in a different model for a different claim type) illustrates an example where a factor is not significant - in this case there is no parameter estimate more than two standard errors from zero.

GLM output (example of insignificant factor)



2 34 Sometimes some levels of a categorical factor may be clearly significant, while other levels may be less so Although the factor as a whole may be statistically significant, this may indicate that it is appropriate to re-categorize the factor, grouping together the less significant levels with other levels.

Deviance tests

2.35 As discussed in Section 1, comparing measures of deviance of two nested models allows "type III" tests (χ^2 or F-tests depending on whether or not the scale parameter ϕ is known) to be performed to determine the theoretical significance of individual factors.

2 36 In the Vehicle Symbol examples above (which were based on frequency models of two different claim types, each with a Poisson error structure), the resulting probability values (or P values) from the χ^2 tests are shown as footnotes to the graphs. Each χ^2 test compares a model with Vehicle Symbol to one without. In the first case the χ^2 test shows a probability level close to 0 (displayed to one decimal place as 0 0%) This means that the probability of this factor having such an effect on the deviance by chance is almost zero, ie this factor (according to the χ^2 test) is highly significant Conversely in the second example the probability value is 52 5%, indicating that the factor is considerably less significant and should be excluded from the model Typically factors with χ^2 or F-test probability levels of 5% or less are considered significant.

2 37 These kinds of type III likelihood ratio tests can provide additional information to the graphical interpretation of parameter estimates and standard errors For example if other correlated factors in a model could largely compensate for the exclusion of a factor, this would be indicated in the type III test Also the type III test is not influenced by the choice of the base level in the way that parameter estimate standard errors are

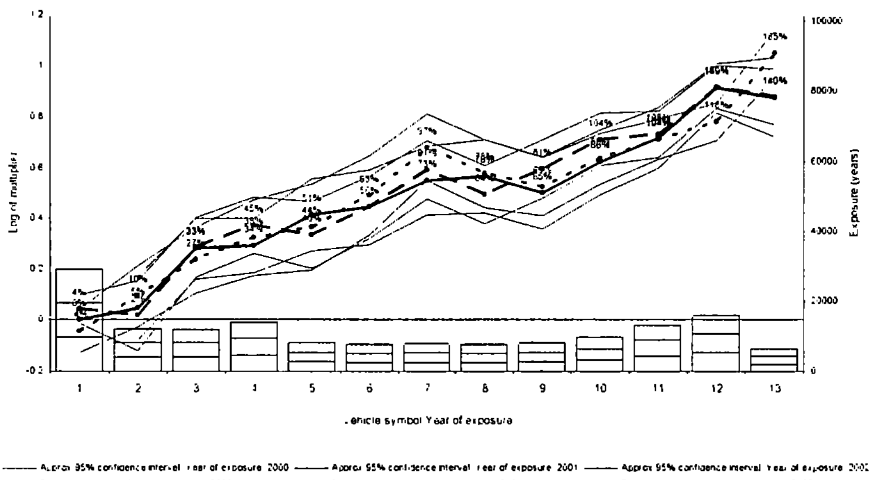
2.38 On the other hand, type III tests can be impractical on occasions - for example if a 20 level factor contained only one level that had any discriminatory effect on experience, a type III test might indicate that the factor was statistically significant, whereas a graphical representation of the model results would show at a glance that the factor contained too many levels and needed to be re-categorized with fewer parameters

Interaction with time

2.39 In addition to classical statistical tests it can often be helpful to consider rather more pragmatic tests such as whether the observed effect of a rating factor is consistent over time For example if more than one year's experience is being considered it is possible to consider the effect of a particular factor in each calendar year of exposure (or alternatively policy year) In theory this could be done by fitting separate models to each year and then comparing the results, however this can be hard to interpret since a movement in one factor in one year may to a large extent be compensated for by a movement in another correlated factor A potentially clearer test, therefore, is to fit a series of models each one of which considers the interaction of a single factor with time (Interactions are discussed in more detail later in this paper)

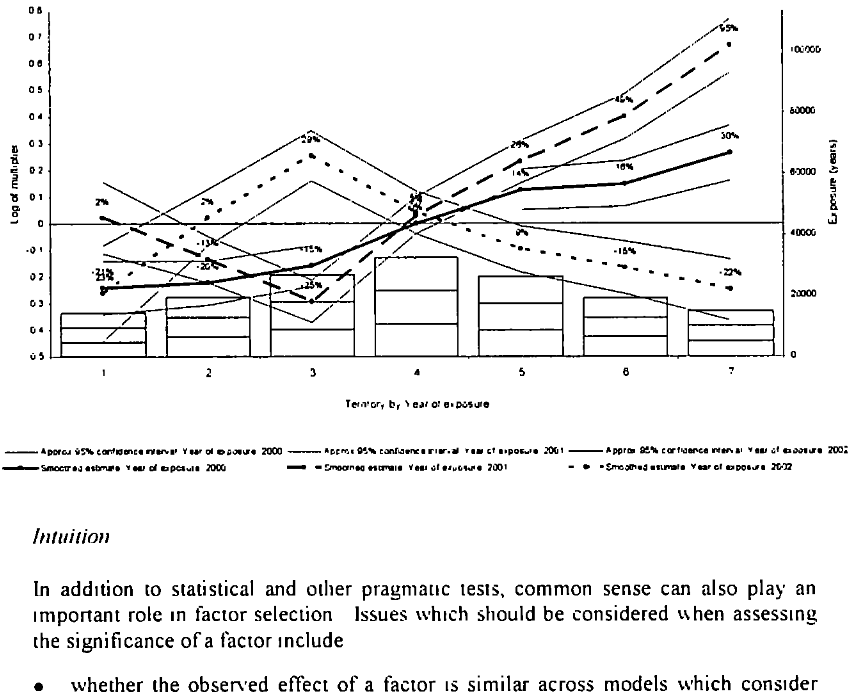
2.40 The below diagram shows one example factor interacted with calendar year of exposure. It is clear from this result showing lines which are largely parallel that the factor effect is mainly consistent from year to year, suggesting that the factor is likely to be a good predictor of future experience.

GLM output - example showing factor consistent over time



2.41 Conversely the graph below shows an example of a factor (in this case territory classification) which, although significant according to classical type III tests, shows a pattern for some levels which differs from year to year In such a case it would be appropriate to investigate whether there was a possible explanation for such variations If the variation can be attributed to some known change (for examples some event in one of the territories during one period) then that can be allowed when interpreting the results If no explanation can be found for variations over time, this may indicate that the factor will be an unreliable predictor of future experience.

GLM output - example showing factor inconsistent over time



Intuition

2.42 In addition to statistical and other pragmatic tests, common sense can also play an important role in factor selection Issues which should be considered when assessing the significance of a factor include

- whether the observed effect of a factor is similar across models which consider related types of claim (eg auto property damage liability and collision)
- whether the observed effect makes logical sense (given the other factors in the model)

- whether the observed effects of a categorical factor which represents a continuous variable (such as the age of a vehicle) show a natural trend - the model has no way of knowing that factor levels have a natural order, therefore if a trend is observed this may suggest that the factor has a more significant effect than the pure statistical tests alone would suggest.

Model iteration : stepwise macros

2 43 It is not generally possible to determine from a single GLM which set of factors are significant since the inclusion or exclusion of one factor will change the observed effects and therefore possibly the significance of other correlated factors in the model. To determine the theoretically optimal set of factors, therefore, it is generally necessary to consider an iterated series of models.

2 44 Often the model iteration starts with a GLM that includes all the main explanatory variables. Insignificant factors can then be excluded, one at a time, refitting the model at each stage

2 45 When a factor is identified as being insignificant it is helpful to compare the GLM parameter estimates for that factor with the equivalent one-way relativities. When the GLM parameter estimates are different from the one-way relativities this indicates that the factor in question is correlated with other factors in the model and that the removal of that factor from the model is likely to affect the parameter estimates for other factors and quite possibly also their significance. Conversely if the one-way relativities are very similar to the GLM relativities for the factor to be excluded, it is likely that there will be no such consequences and that therefore to save time a second insignificant factor could be removed at that iteration also

2 46 If a very large number of factors are to be considered it can be impractical to start the factor iteration process with all possible factors in the model In such cases it is possible to select a model with certain factors which are known to be important, and then to test all other excluded factors by fitting a series of models which, one at a time, tests the consequences of including each of the excluded factors. The most significant of the excluded factors can then be included in the model, and then the other excluded factors can be retested for significance

2.47 Where possible it is generally best to iterate models manually by analyzing the various diagnostics described above for each factor. In practice if many factors are being analyzed this can be impractical In such circumstances automatic "stepwise" model iterating algorithms can be programmed to iterate models on the basis of type III tests alone. Such algorithms start with a specified model, and then:

a. the significance of each factor in the model is tested with a type III test, and the least significant factor is removed from the model if the significance is below a certain specified threshold

b. the significance of each factor not in the model (but in a specified list of potential factors) is tested by (one at a time) creating a new model with each factor included and then measuring the significance of that factor with a type III test. The most significant factor not currently in the model is then included if the significance is above the specified threshold

c. steps **a.** and **b.** are repeated until all factors in the model are deemed significant, and all factors not in the model are deemed insignificant.

2.48 Such algorithms allow no human judgment to be exercised and can take a significant time to complete They are also heavily dependent on the type III test which has some practical shortcomings as described previously Nevertheless they can derive a theoretically optimal model which at the very least could form the starting point for a more considered manual iteration

Model validation

2.49 As well as considering the significance of the modeled rating factors, there are a number of more general diagnostic tests which allow the appropriateness of other model assumptions to be assessed Diagnostics which aid in this investigation include

- residuals which test the appropriateness of the error term
- leverage which identifies observations which have undue influence on a model
- the Box-Cox transformation which examines the appropriateness of the link function

Residuals

2.50 Various measures of residual can be derived to show, for each observation, how the fitted value differs from the actual observation.

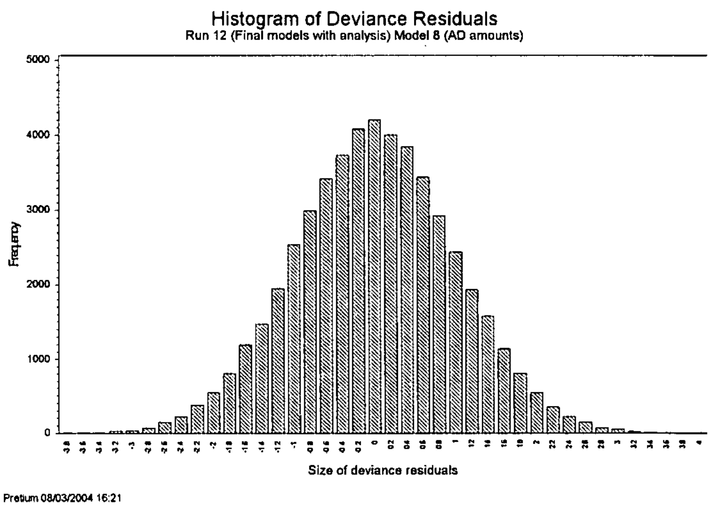
2.51 One measure of residual is the deviance residual

$$r_i^D = sign(Y_i - \mu_i)\sqrt{2\,\omega_i \int_{\mu_i}^{Y_i} (Y_i - \zeta)/V(\zeta)d\zeta}$$

which is the square root of the observation's contribution to the total deviance (ie a measure of the distance between the observation and the fitted value), multiplied by 1 or -1 depending on whether the observation is more than or less than the fitted value.

2.52 The deviance residuals have various helpful properties. In general they will be more closely Normally distributed than the raw residuals (defined simply as the difference between the actual observation and the expected value predicted by the GLM), as the deviance calculation corrects for the skewness of the distributions. For continuous distributions it is possible to test the distribution of the deviance residuals to check that they are Normally distributed. Any large deviation from this distribution is a good indication that the distributional assumptions are being violated.

2.53 The below diagram shows a distribution of deviance residuals from an example model. In this case the residuals appear to be reasonably consistent with a Normal distribution.



2.54 For discrete distributions the deviance residuals based on individual observations tend not to appear Normally distributed. This is because the calculation of the contribution to the deviance can adjust for the shape but not the discreteness of the observations. For example, in the case of fitting a model to claim numbers, a GLM might predict a fitted value for a record of say 0.1 representing an expected claims frequency of 10%. In reality (ignoring multiple claims) either a claim occurs for that record or it does not, with the result that the residual for that record will either correspond to an "actual minus expected" value of (0-0.1) = -0.1, or (with lower probability), the residual will correspond to an "actual minus expected" value of (1-0.1) = 0.9.

2.55 Some practitioners group together the individual residuals into large groups of similar risks. This aggregation can disguise the discreteness allowing some distributional tests to be performed. For example, it is commonly thought that a Poisson with a suitably large mean can be thought of as being almost Normally distributed. At this point the deviance residual calculated on the aggregate data should be smooth enough to test meaningfully.

2.56 The deviance residuals are often standardized before being analyzed. The purpose of this standardization is to transform the residuals so that they have variance 1 if the model assumptions hold. This is achieved by adjusting by the square root of the scale parameter and also by the square root of one minus the "leverage" h_i

$$r_i^{DS} = \frac{sgn(Y_i - \mu_i)}{\sqrt{\phi(1-h_i)}} \sqrt{2\,\omega_i \int_{\mu_i}^{Y_i} (Y_i - \zeta)/V(\zeta)\,d\zeta}$$

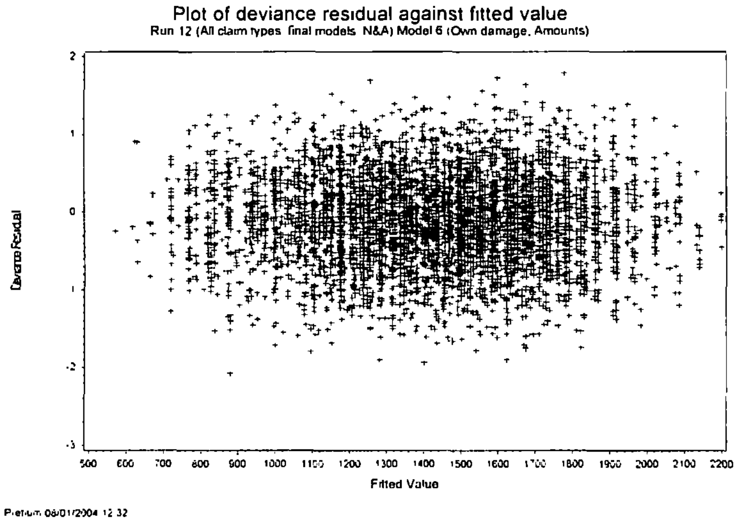
2.57 The leverage h_i is a measure of how much influence an observation has over its own fitted value. Its formal definition is complex but essentially it is a measure of how much a change in an observation affects the fitted value for that observation. Leverage always lies strictly between 0 and 1. A leverage close to 1 means that if the observation was changed by a small amount the fitted value would move by almost the same amount. Where the leverage is close to 1 it is likely that the residual for that observation will be unusually small because of the high influence the observation has on its fitted value. Dividing by the square root of one minus the leverage corrects for this by increasing the residual by an appropriate amount.

2.58 Another commonly used measure of the residual is the Standardized Pearson residual. This is the raw residual adjusted for the expected variance and leverage (as described above)

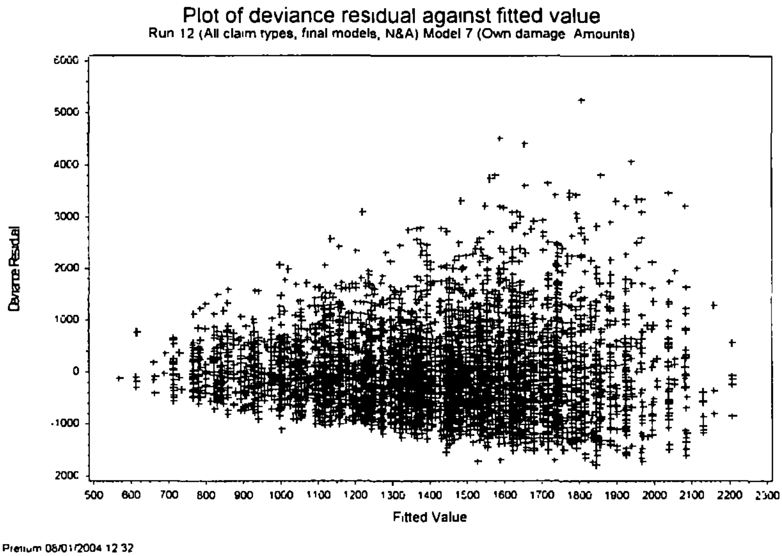
$$r_i^{PS} = \frac{(Y_i - \mu_i)}{\sqrt{\frac{\phi}{\omega_i} V(\mu_i)(1-h_i)}}$$

2 59 This adjustment makes observations with different means comparable, but does not adjust for the shape of the distribution

2 60 Observing scatter plots of residuals against fitted values can give an indication of the appropriateness of the error function which has been assumed For example, if the model form is appropriate then the standardized deviance residuals should be distributed Normal (0,1) regardless of the fitted value The example scatter plot below shows the result of fitting a GLM with a gamma variance function to data which has been randomly generated on a hypothetical insurance dataset from a gamma distribution (with a mean based on assumed factor effects) It can be seen that moving from the left to the right of the graph the general mean and variability of the deviance residuals is reasonably constant, suggesting (as is known to be the case in this artificial example) that the assumed variance function is appropriate.

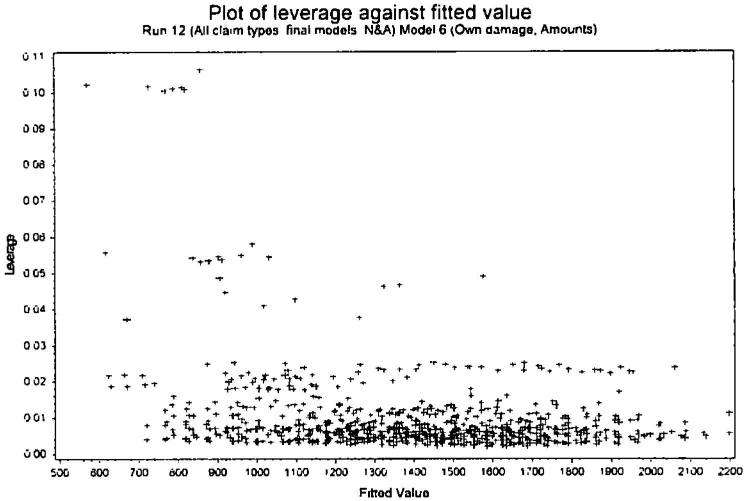


2.61 Conversely the graph below shows the scaled deviance residuals obtained from fitting a GLM with an assumed Normal error to the same gamma data. In this case the variability increases with fitted value, indicating that an inappropriate error function has been selected and that the variance of the observations increases with the fitted values to a greater extent than has been assumed. This could occur, for example, when a Normal model is fitted to Poisson data, when a Poisson model is fitted to gamma data, or (as is the case here) where a Normal model is fitted to gamma data.



Leverage

2 62 As well as being needed to calculate standardized residuals, the leverage statistic is also a helpful diagnostic in its own right, since it can identify particular observations which might have an undue influence on the model For example the graph below shows a scatter plot of leverage against fitted value In this case seven particular observations have clearly higher leverage than other observations (around 0 1) and it is possible that they are having an undue influence on the model An inspection of these observations may indicate whether or not it is appropriate to retain them in the model



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Box Cox transformation and the case for a multiplicative model

2.63 The Box Cox transformation can be used to assess the appropriateness of the assumed link function. The transformation defines the following link function in terms of a scalar parameter λ

$$g(x) = \begin{cases} (x^{\lambda} - 1)/\lambda, & \lambda \neq 0 \\ \ln(x), & \lambda = 0 \end{cases}$$

2.64 If λ=1, g(x) = x-1 This is equivalent to an identity link function (ie an additive model) with a base level shift.

2.65 As λ→0, g(x) → ln(x)[9]

$$\lim_{\lambda \to 0} \frac{x^{\lambda} - 1}{\lambda} = \lim_{\lambda \to 0} \left[\frac{d}{d\lambda}(\exp(\lambda \ln(x)) - 1) \Big/ \frac{d}{d\lambda}\lambda \right] = \lim_{\lambda \to 0} \frac{\ln(x)\, x^{\lambda}}{1} = \ln(x)$$

This is equivalent to a multiplicative model.

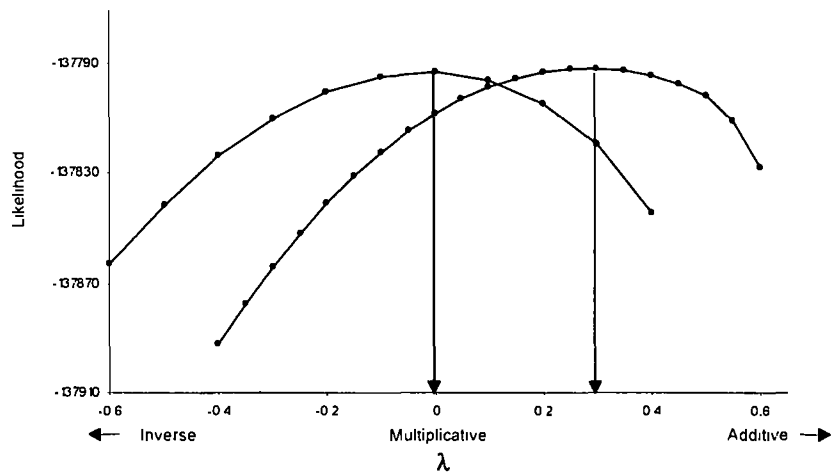
2.66 If λ=-1, g(x) = $1-x^{-1}$ This is equivalent to an inverse link function with a base level shift.

2.67 By fitting a series of GLMs to the data with many different values of λ (including real values between -1 & 0 and 0 & 1), and with all other model features identical in every other respect, it is possible to assess which value of λ is most appropriate for the dataset in question by seeing which value of λ yields the highest likelihood. Optimal values of λ around 0 would suggest that a multiplicative structure with a log link function would be the most appropriate for the data in question, whereas optimal values of λ around 1 would suggest an additive structure would be best, with values around -1 indicating that an inverse link function would be most appropriate

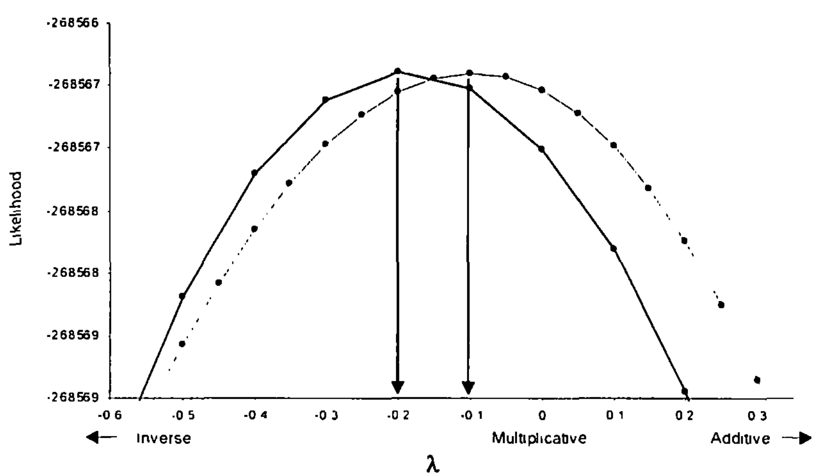
2.68 Examples based on two real datasets are shown below

9 Via L'Hôpital's Rule.

Box Cox transformation results on frequency



Box Cox transformation results on severity

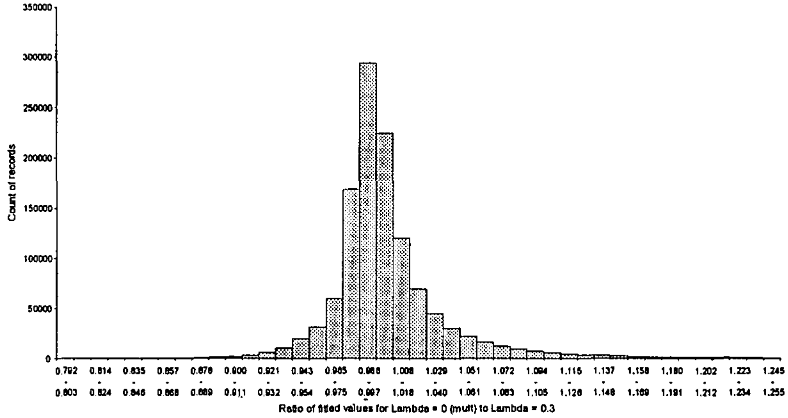


2.69 The first graph shows various values of λ tested on two different datasets containing private passenger automobile property damage liability frequency experience. The optimal λ in one case is very close to zero (suggesting a multiplicative model) but in the other is around 0.3, suggesting that the frequency in that case is largely influenced by explanatory variables in a multiplicative fashion, but to an extent also in an additive fashion.

2.70 The second graph shows the results for claim amounts models for the same data. Here the optimal values of λ are near zero (multiplicative), but this time slightly toward the direction of being partly inverse.

2.71 In order to understand how significant the value of λ is upon the fitted values produced by the model it is helpful to consider the histogram graph below which shows, for one of the two frequency datasets considered above, the distribution of the ratio of fitted values produced by a GLM with $\lambda=0$ to an otherwise identical GLM with $\lambda=0.3$. It can be seen that there is in fact little difference between the fitted values produced by these two models, with the great majority of fitted values being within 2 or 3% of each other.

Distribution of ratio of fitted values between model with $\lambda = 0$ and model with $\lambda = 0.3$



2.72 In practice there are many significant advantages with using a multiplicative structure, not least because it is easy to understand. In the above examples it seems that there is no strong evidence to use a structure other than a multiplicative structure.

2.73 While this should be tested in each case, it is often the case that multiplicative structures and log link functions are the most appropriate practical model for modeling insurance risk, and this may explain the high prevalence of multiplicative rating structures, especially in Europe where GLMs have been in use for many years

Model refinement

Interactions

2.74 Thus far, the discussion has focused on the independent effect of factors in the model. Generalized linear models can also consider the interaction between two or more factors. Interactions occur when the effect of one factor varies according to the level of another factor

2.75 Interactions relate to the effect which factors have upon the risk, and are not related to the correlation in exposure between two factors. This is illustrated with two examples which consider two rating factors in a multiplicative rating structure

Example 1 - correlation but no interaction

Earned exposure

	Town	Countryside	Total
Male	200	100	300
Female	100	200	300
Total	300	300	600

Number of claims

	Town	Countryside	Total
Male	80	20	100
Female	20	20	40
Total	100	40	140

Claims frequency

	Town	Countryside
Male	40%	20%
Female	20%	10%

2.76 In this example the exposure is not distributed evenly amongst the different rating cells - a higher proportion of town dwellers are male than is the case in the countryside. The *effect* of the two factors upon the risk, however, does not in this example depend on each other - men are twice the risk of women (regardless of location) and town dwellers are twice the risk of countryside dwellers (regardless of gender). In this example there is therefore a correlation between the two rating factors, but no interaction

Example 2 - interaction but no correlation

Earned exposure

	Town	Countryside	Total
Male	300	150	450
Female	200	100	300
Total	500	250	750

Number of claims

	Town	Countryside	Total
Male	180	30	210
Female	40	10	50
Total	220	40	260

Claims frequency

	Town	Countryside
Male	60%	20%
Female	20%	10%

2.77 In this example the exposure is distributed evenly amongst the different rating cells - the same proportion of town dwellers are males as are countryside dwellers The effect of the two factors upon the risk, however, in this example depend on each other - it is not possible to represent accurately the effect of being male (compared with being female) in terms of a single multiplier, nor can the effect of location be represented by a single multiplier. To reflect the situation accurately it is necessary in this case to consider multipliers dependent on the combined levels of gender and location.

2.78 An interaction term can be included within a GLM simply by defining an explanatory variable in terms of two or more explanatory variables In the above example, rather than declaring location and gender as two explanatory variables (each with a base level and one parameter), a combined "gender-location" variable could be declared with four levels (a base level and three parameters)

2.79 Interaction terms should only be included where there is statistical justification for the inclusion of the additional parameters In the above example the interaction term only involved the addition of one further parameter to the model, but if an interaction is introduced between two ten level factors (each with a base level and nine parameters), a further 81 parameters would be introduced into the model

"Complete" and "marginal" interactions

2 80 Interactions can be expressed in different ways For example consider the case of two factors each with four levels One way of expressing an interaction is to consider a single factor representing every combination of the two factors (or "complete" interaction) A set of multipliers (in the case of a multiplicative model) could therefore be expressed as follows.

	Factor 1	A	B	C	D
Factor 2·	W	0 72	0 80	0 88	0.96
	X	0.90	1 00	1.10	1 20
	Y	0 97	1 20	1.45	1 66
	Z	1 26	1 40	1.85	2 10

2 81 In this case the base level has been selected to be the level corresponding to level B of factor 1 and level X of factor 2, and the interaction term has 15 parameters

2.82 An alternative representation of this interaction is to consider the single factor effects of factor 1 and factor 2 *and* the additional effect of an interaction term over and above the single factor effects (or "marginal" interaction) A set of multipliers in this form can be set as follows

		Factor 1	A	B	C	D
			0 90	-	1 10	1 20
Factor 2	W	0 80	1	-	1	1
	X	-	-	-	-	-
	Y	1 20	0.9	-	1 1	1·15
	Z	1 40	1	-	1 2	1 25

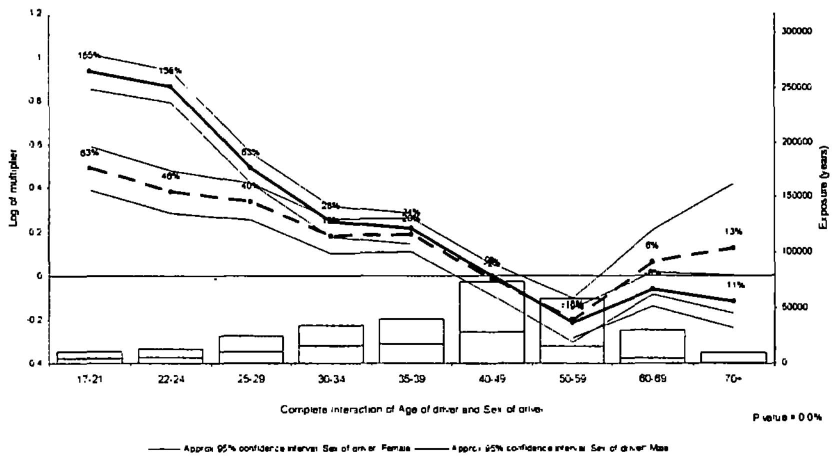
2 83 In this case fewer parameters are present in the additional interaction term because the presence of the single factor effects makes some of the interaction terms redundant When fitted in a GLM (assuming that the single factor effects were declared first) the redundant terms in the additional interaction term would be aliased Overall the three terms combined still have 15 parameters, and result in identical predicted values (for example in the case of factor 1 level D and factor 2 level Z, 1 2 * 1 4 * 1.25 = 2 1)

2 84 In practice sometimes it can be helpful to consider the "complete" interaction (ie just a single factor representation of all combinations of the two factors) and sometimes it can be helpful to consider the additional or "marginal" interaction term over and above the single factor effects While the fitted values from both approaches are identical, what differs is the statistical diagnostics available in the form of parameter estimate standard errors

Example

2 85 For example, the graph below shows the result of a "complete" interaction between the age of driver and the gender of driver for the claims frequency of a certain type of auto claim, with age relativities for men and women superimposed (with solid and dotted lines respectively) on the same x-axis.

Complete interaction

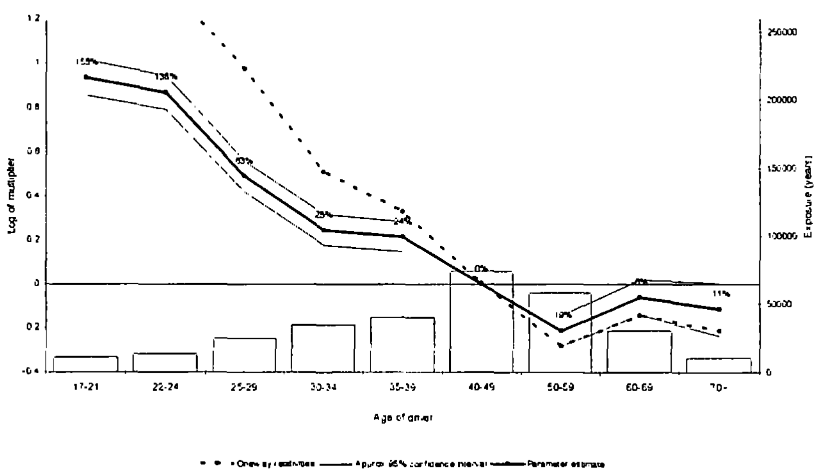


2.86 If there were no significant interaction between these two factors the solid and dotted lines (showing parameters from a log link GLM) would be parallel In this example they are clearly not, showing that while younger drivers have higher frequencies, and while in general male drivers have higher frequencies, in this example (as in many real cases) young men experience a higher frequency than would be predicted by the average independent effects of the two factors.

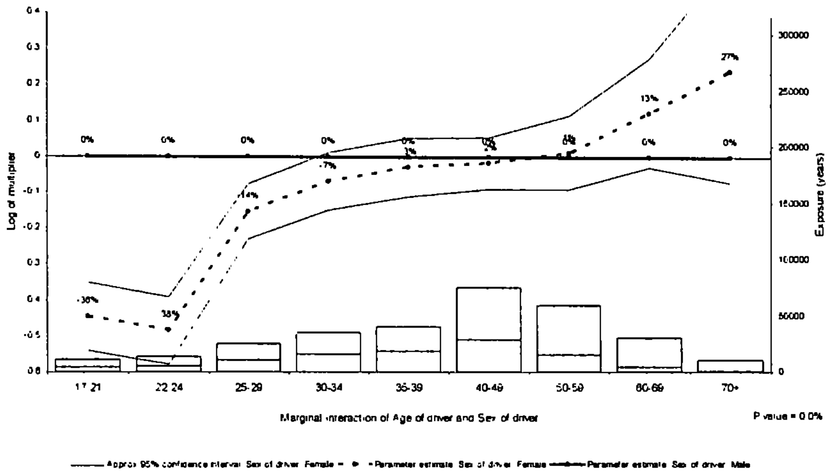
2.87 The narrow standard error bands around the parameter estimate lines suggest the likely significance of the result, however they do not provide any sound theoretical basis for assessing the significance of the factor A more theoretically appropriate test can be applied if a marginal interaction is considered.

2.88 The graphs below show the results of fitting age and then a marginal interaction of age and sex to the same data.

Main effect



Partial marginal interaction



2.89 The first graph shows the single factor effect for age, and the second shows the marginal interaction term over and above this single factor effect (In this case the single factor gender of the driver was not included since it proved not to be significant)

2 90 Since the male level of the gender factor is ordered after the female level, the male levels of the marginal factor have been aliased, with the result that the first graph represents the age effects for males, and the marginal graph shows the additional adjustment which is appropriate for females of different ages.

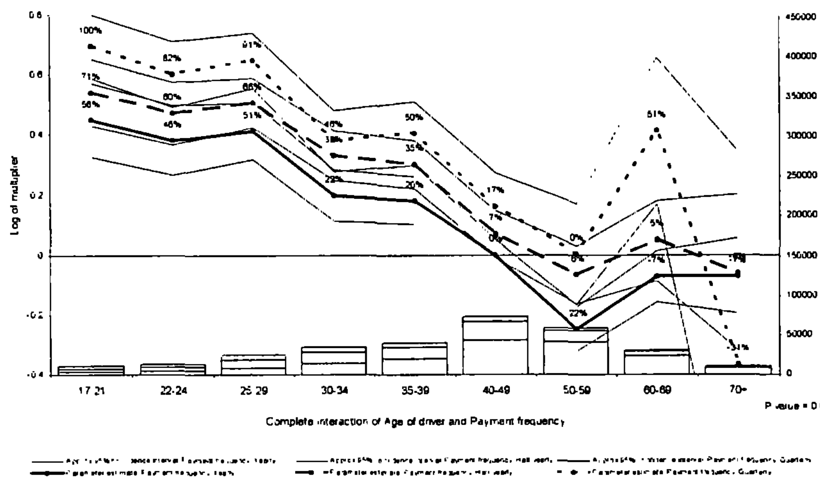
2 91 The implied fitted values from the marginal interaction are the same as the complete interaction - for example:

- Complete interaction effect for age 22-24 female = +46% or multiplier of 1 46
- Marginal approach
  - Single factor age effect for 22-24 = +138% or multiplier of 2 38
  - Marginal effect of women relative to men at age 22-24 = -38% or multiplier of 0 62
  - Combined effect for age 22-24 female = 2.38 x 0 62 = 1 48 (differences due to rounding)

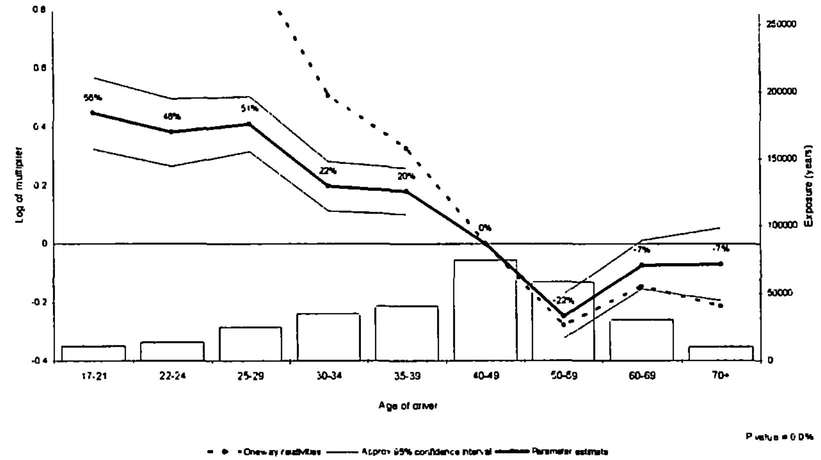
2.92 The marginal approach does however provide more meaningful diagnostics in the form of parameter estimate standard errors and type III tests The standard errors on the graph of the marginal term indicate that the marginal term is indeed significant, and the Type III P-value of 0 00% for this factor confirms that this is the case.

2 93 An example of an interaction term which is not significant is shown below The first graph is the complete interaction (where the parameter estimate lines can be seen to be largely parallel). The second and third graphs show the main effects of age of driver and payment frequency, respectively. The fourth graph shows the marginal interaction (where the marginal interaction term can be seen to be insignificant, both visually and because of the type III p-value of 61.9%)

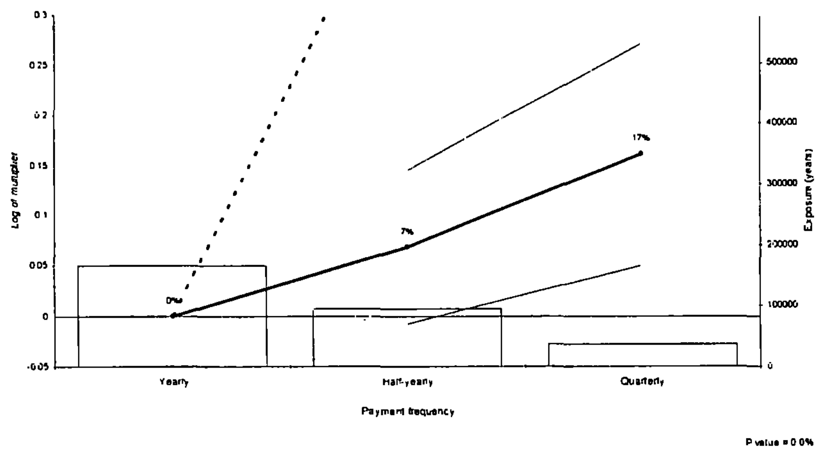
Complete interaction



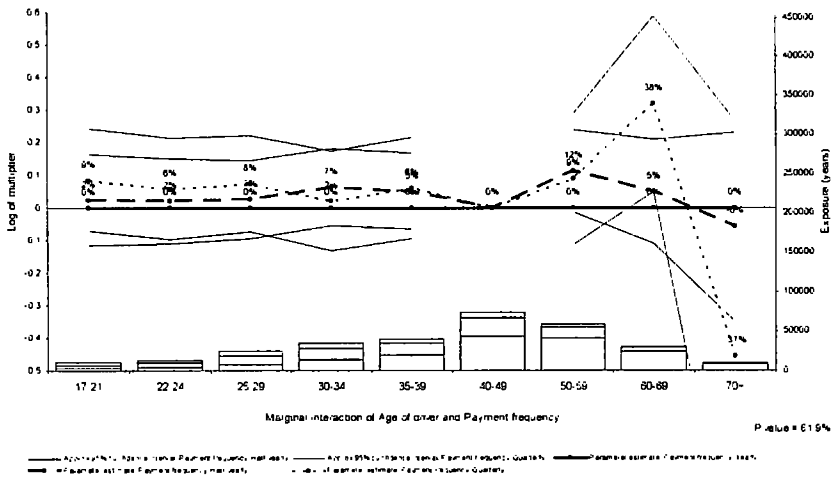
Main effect



Main effect



Marginal interaction



Interpreting marginal interactions

2 94 Although the marginal form of an interaction provides a more sound theoretical basis for assessing the significance of a factor, in practice marginal interactions can be hard to interpret. For example consider the example of two factors each with four levels It might be the case that the true underlying frequency (all other factors being at a certain level) was as follows

	Factor 1:	A	B	C	D
Factor 2:	**W**	7 2%	8 0%	8 8%	9 6%
	X	9 0%	10 0%	11.0%	12 0%
	Y	9 7%	12 0%	14 5%	16 6%
	Z	12 6%	14 0%	18 5%	21 0%

2 95 However in reality the exposure available for this analysis might be low for some combinations of these two factors, for example:

Exposure	Factor 1:	A	B	C	D
Factor 2:	W	1000	1000	1000	1000
	X	1000	1000	1000	1
	Y	1000	1000	1000	1000
	Z	1000	1000	1000	1000

2 96 If in general the claims experience was in line with the underlying frequencies but the one policy with factor 1 level D and factor 2 level X had one claim (resulting in a very high claims frequency of 100%), a marginal interaction would yield results which could be hard to interpret Specifically if a marginal interaction were fitted, the GLM would seek the following parameters

			Factor 1:	A	B	C	D
				β_1	-	β_2	β_3
Factor 2:	W	β_4		β_7	-	β_8	β_9
	X	-		-	-	-	-
	Y	β_5		β_{10}	-	β_{11}	β_{12}
	Z	β_6		β_{13}	-	β_{14}	β_{15}

2 97 Since level X is the base level of factor 2, there is no single term in the marginal interaction which can represent the very high observed frequency for factor 1 level D / factor 2 level X Instead the model will yield parameter β_3 with a very high value, and parameters β_9, β_{12} and β_{15} with low values Although theoretically correct, the parameter estimates and standard errors for parameters β_9, β_{12} and β_{15} would be hard to interpret

Searching for interactions

2 98 In general the significance of an interaction can be assessed by considering

- the standard errors of the parameter estimates of the marginal term
- the type III P-value of the marginal term
- general intuition given the overall "complete" interaction effect
- the consistency of an interaction over time

2.99 In theory all possible combinations of pairs or triplets of factors could be tested as interactions one at a time in a model In practice, the design of the current rating plan, the results of two-way analyses and wider experience will influence the choice of what is tested, as will the ease of interpretation and the ultimate application of the model.

2.100 In some cases rather than considering every combination of two factors with many levels it can be appropriate to consider only the strongest effects For example, a marginal interaction of driver age, car symbol and the interaction of driver age - car symbol (denoted driver age*car symbol) may highlight an interesting effect in one "corner" of the interaction (eg young drivers driving high car symbols). In practice, the interaction may be re-parameterized as a combination of detailed single factors for age of driver and car symbol, and an additional less detailed factor based on the combination of age of driver and car symbol which has the same level for many combinations, and a few levels representing certain combinations of young drivers driving high car symbols

2.101 The inclusion of several meaningful interactions which share factors (eg age*sex, age*multi-car and territory*multi-car) could provide a theoretically correct model but may be very difficult to interpret The practitioner may consider creating separate models for single and multi-car, and continue to investigate other interactions.

Smoothing

2.102 Once models have been iterated to include only significant effects and interactions have been investigated, smoothing of the parameter estimates may be considered in order to improve the predictive power of the model. Much like the offset and prior weight terms in the formularization of GLMs, smoothing is used to incorporate some element of the practitioner's knowledge into the model In this sense, the practitioner may impart knowledge that some factors have a natural order (eg that age of car seven should fall between age of car six and age of car eight). Outliers may also be tempered This tempering is not based on commercial selections at this point (ie tolerance for rate change) but rather an attempt to adjust an anomaly once a proper investigation has been done to ensure that the outlier is truly an anomaly and not something systematic in the experience

2.103 The selection of smoothed parameter estimates can be done in an unscientific fashion (for example - a visual modification to a curve) or in a more scientific fashion (for example - fitting polynomials to the observed parameter estimates, or electing to refit a model using polynomial terms as variates within the GLM). If smoothing is rather severe, the practitioner may consider restricting the values of the smoothed factor and re-running a model to allow other factors to compensate. (The concept of restrictions is discussed later in this paper.) In general, however, this technique may only remove the random element from one factor and move it to another factor, and if often it can be preferable not to refit using restrictions in this way

Risk Premium

2.104 Fitting GLMs separately to frequency and severity experience can provide a better understanding of the way in which factors affect the cost of claims This more easily allows the identification and removal (via smoothing) of certain random effects from one element of the experience Ultimately, however, these underlying models generally need to be combined to give an indication of loss cost, or "risk premium", relativities.[10]

2.105 In the case of multiplicative models for a single claim type, the calculation is straightforward - the frequency multipliers for each factor can simply be multiplied by the severity multipliers for the same factors (which is analogous to adding the parameter estimates when using a log link function) Alternatively, models may be fitted directly to pure premium data using the Tweedie distribution (discussed in Appendix C) The advantages and disadvantages of this alternative approach are discussed in Appendix J.

2.106 Certain market conditions may warrant the development of a single theoretical risk premium model, even if different types of claim have been modeled separately An example is the aggregation of homeowners models by peril into a single rating algorithm at point of sale The derivation of a single model in this situation is not as straightforward since there is no direct way of combining the model results for the underlying claim types into a single overall expected cost of claims model In this situation, however, it is possible to approximate the overall effect of rating factors on the total cost of claims by using a further GLM to calculate a weighted average of the GLMs for each of the underlying frequency and severity models for each of the claim types Specifically this can be done by

- selecting a dataset which most accurately reflects the likely future mix of business
- calculating an expected claim frequency and severity by claim type for each record in the data
- combining these fitted values, for each record, to derive the expected cost of claims (according to the individual GLMs) for each record
- fitting a further generalized linear model to this total expected cost of claims, with this final GLM containing the union of all factors (and interactions) in all of the underlying models.

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[10] The term "risk premium" is used rather than pure premium in order to differentiate between a model fitted directly on pure premium data and a model derived by combining underlying frequency and severity models

2.107 An illustrative example is shown below. The top table represents the intercepts and multipliers from underlying frequency and severity models for claim types 1 and 2. The bottom table shows the calculation of the total risk premium, based on the underlying models, for the first four records in the data. The additional GLM is fitted to this last column in this dataset in order have a single theoretical risk premium model

		Claim type 1		Claim type 2	
		Frequency	**Severity**	**Frequency**	**Severity**
Intercept		0.32	1,000	0.12	4,860
Sex:	**Male**	1.00	1.00	1.00	1.00
	Female	0.75	1.20	0.67	0.90
Area:	**Town**	1.00	1.00	1.00	1.00
	Country	1.25	0.72	0.75	0.83

Policy	Sex	Area	Fitted freq 1	Fitted sev 1	Fitted RP 1	Fitted freq 2	Fitted sev 2	Fitted RP 2	Total RP
82155654	M	T	32.00%	1,000.00	320.00	12.00%	4,860.00	583.20	**903.20**
82168746	F	T	24.00%	1,200.00	288.00	8.04%	4,374.00	351.67	**639.67**
82179481	M	C	40.00%	720.00	288.00	9.00%	4,033.80	363.04	**651.04**
82186845	F	C	30.00%	864.00	259.20	6.03%	3,630.42	218.91	**478.11**
		....	..				.	. ...	

2.108 In addition to combining frequency and severity across multiple claim types, the technique of fitting an overall GLM to fitted values of other GLMs can be used to incorporate non-proportional expense elements into the modeled relativities For example, a constant dollar amount could be added to each observation's expected risk premium and then a GLM re-fitted to this new field The resulting "flattened" risk premium relativities will prevent high risk factor levels from being excessively loaded for expenses

2.109 Alternatively, the amount added to each observation's expected risk premium could be designed to vary according to the results of a separate retention study. This would allow risks with a high propensity to lapse to receive a higher proportion of fixed expense than those risks with a low propensity to lapse As above, a further GLM is fitted to the sum of the expected risk premium and a (lapse-dependent) expense load.

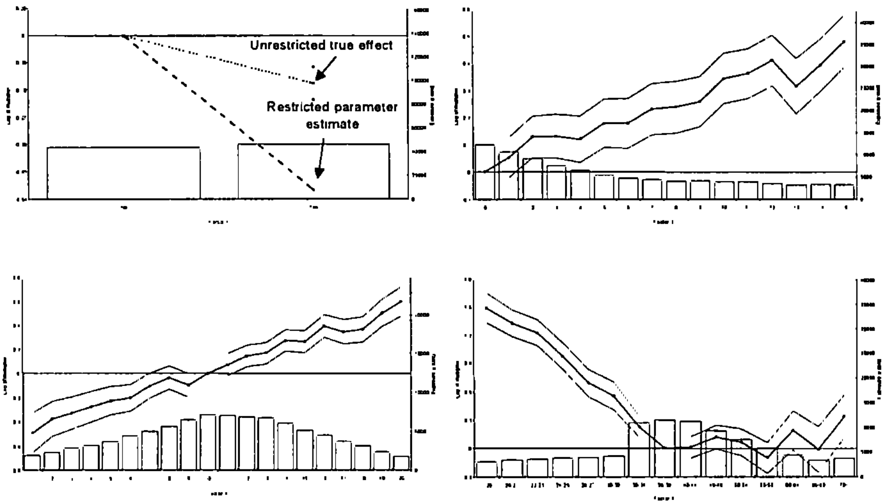
Restrictions

2.110 The theoretical risk premium results from a GLM claims analysis will differ from the rates implemented in practice since consideration needs to be given to price demand elasticity and the competitive situation. There are, however, some situations where legal or commercial considerations may also impose rigid restrictions on the way particular factors are used in practice. Though certain factors need to be restricted, if desired the model may be able to compensate to an extent for this artificial restriction by adjusting the fitted relativities for correlated factors This is achieved using the offset term in the GLM

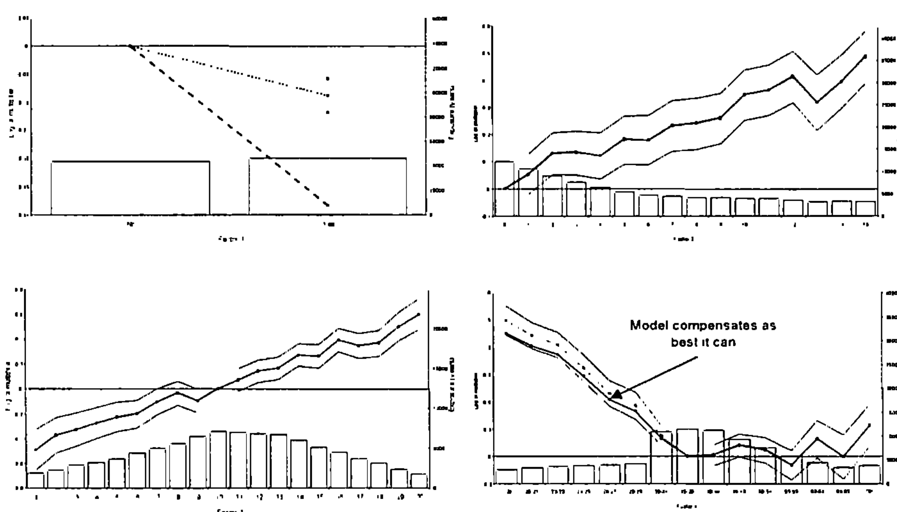
2.111 Specifically, the required parameter estimates (logs of multipliers in the case of a multiplicative model) are calculated for each record and added to the offset term ξ_i. The factor in question is then not included as an explanatory factor in the GLM (This can intuitively be thought of as fixing some selected elements of β_i to be 1 for certain specified columns of the design matrix X_{ij})

2.112 The graphs below illustrate the use of a restriction. In the upper series of graphs, the dotted lines display the theoretically correct parameter estimates indicated by a GLM containing these four rating factors The dashed line in Factor 1 shows the intended restriction In the lower series of graphs, the solid lines show the output of the GLM after the restriction for Factor 1 has been incorporated and Factors 2, 3, 4 have been allowed to compensate It can be seen that the parameter estimates in Factors 2 and 3 have hardly changed, suggesting little correlation between these factors and Factor 1 On the other hand the solid line in Factor 4 has moved away from the theoretically correct dotted line, suggesting a correlation between the restricted levels in Factor 1 and those levels in Factor 4 which moved to compensate for the restriction

Example of restricting a factor



Example of restricting a factor - model compensating for restriction



Although restrictions could be applied either to frequency or amounts models (or in part to both), generally it is more appropriate to impose the restriction on the model at the risk premium stage since this allows a more complete and balanced compensation by the other factors. This can be achieved by calculating the expected cost of claims for each record, according to "unrestricted" GLMs, and then imposing the restriction in the final GLM which is then fitted to the total expected cost of claims. (For restricted risk premium models this approach is necessary even in the case of a single claim type)

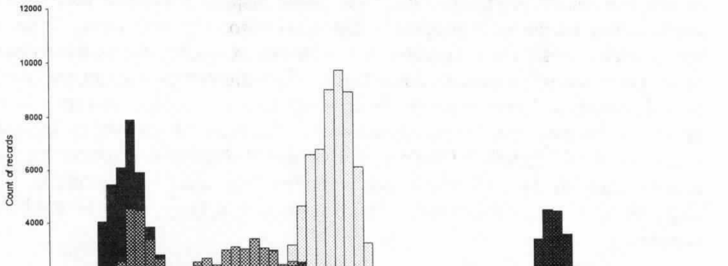
2.113 In the US, many personal lines rating plans contain discounts that were initially implemented for marketing appeal or perhaps mandated by regulation Today's models may indicate that these discounts are not supported by the claims experience - or in many cases may even indicate a surcharge If a company chooses to continue offering such discounts, it is important that these restrictions are incorporated into the modeling process since such restrictions can affect the relativities which become appropriate for other correlated factors. Counterintuitive model results may occur on behavioral factors such as factors which policyholders self-select, for example limits and deductibles. These factors may require restriction if they are to be used directly in ratemaking

2.114 Model restrictions are also used in US ratemaking to mitigate the number of factors which will change in a given rate review. Companies may restrict certain existing rating factors and allow the GLM to measure only the effect of new rating factors. Restrictions may also come into play when applying the results of a countrywide model to a particular state.

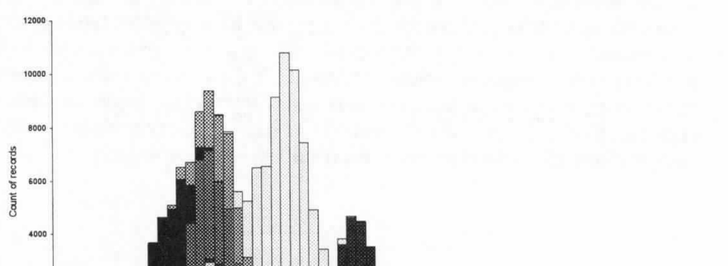
2 115 Prior to incorporating restrictions, it is still important to assess the true effect of all factors upon the risk by initially including them in the analysis as if they were ordinary factors. In addition, a comparison of the fitted values of the theoretical model and the restricted models will demonstrate the degree to which other factors have compensated for the restriction The examples below show two such comparisons. Each graph shows the number of policies (on the y-axis) that have different ratios of restricted to unrestricted fitted values (on the x-axis) The graph is subdivided by levels of the restricted factor (shown in different shading) If the GLM can compensate well for a factor restriction (because there are many other factors in the model correlated with the restricted factor) then this distribution will be narrow Conversely if the GLM cannot compensate well for the restriction, this distribution will be wider

2.116 In this particular example the factors in the upper graph have not compensated well for the restriction. The wide distribution of the restricted to unrestricted ratio implies that the restriction is moving the model away from the theoretical result. The lower graph, on the other hand, shows a model which contains factors that are more correlated with the restricted factor, and which have compensated better for the restriction.

Distribution of ratio of fitted values between restricted and unrestricted models (showing little compensation from other factors)



Distribution of ratio of fitted values between restricted and unrestricted models (showing some compensation from other factors)



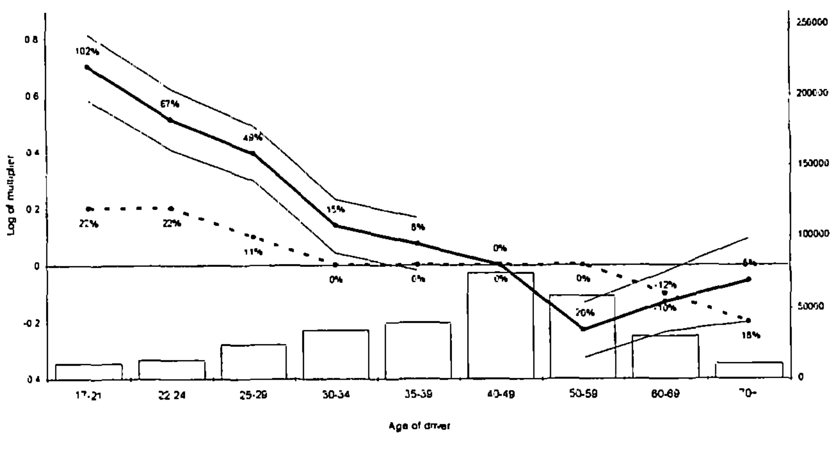
Interpreting the results

2.117 To understand how the results of a GLM claims model differ from the existing rating relativities it is helpful to consider the results both on a factor-by-factor basis and also by measuring the overall effect of all factor differences combined.

Comparing GLM indicated relativities to current relativities

2.118 The final risk premium models can be plotted on graphs similar to those shown in previous sections Another line can be added to display the relativities implicit in the current rating structure This allows easy comparison of the relativities indicated by the model and those which are currently used An example graph is shown below. In this example it can be seen that the current relativities for young drivers (shown as a dotted line) are too low.

Final risk premium model compared to current relativities

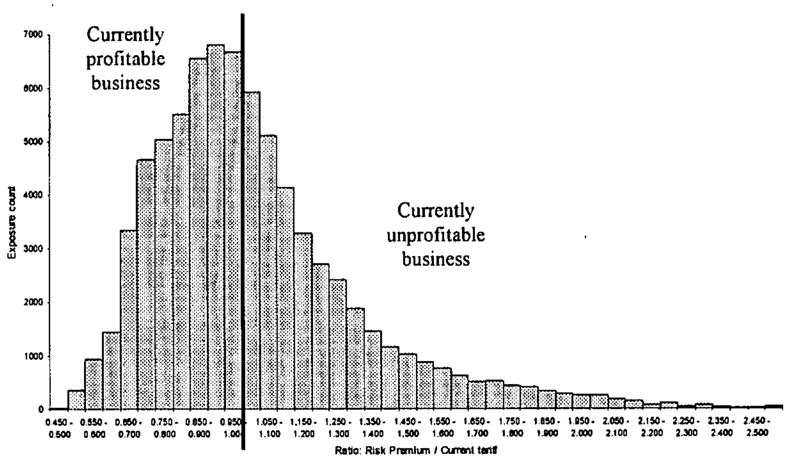


2.119 If the existing rating plan is purely multiplicative, superimposing current relativities on the graph above is very straightforward. Superimposing relativities from a mixed multiplicative/additive rating plan is slightly less straightforward. Some additive components may be re-expressed as an interaction variable (eg {A x B x (C+D)} may be re-expressed to consider the interaction of C and D[11]). Existing rating plans with more complex additive components may be approximated by fitting a multiplicative model to a data field containing existing premium. The appropriateness of this multiplicative proxy to the mixed rating plan can be evaluated by examining the distribution of the ratio of the premium produced by the multiplicative proxy and the actual premium. Proxy models which estimate the rating plan within a narrow distribution (eg +/-5%) may well be appropriate to use.

Impact graphs

2.120 The results of a GLM analysis are interdependent and must be considered together. For example, while a GLM analysis might suggest that young driver relativities are too low, it may also suggest that relativities for inexperienced drivers (eg less than two years licensed) are too high. Although the existing rating structure may be theoretically wrong, it might be the case that to a large extent these errors compensate each other. To understand the true "bottom line" difference between the existing rating structure and the theoretical claims cost, "impact" graphs such as the one below can be considered.

Impact on portfolio of moving to theoretically correct relativities



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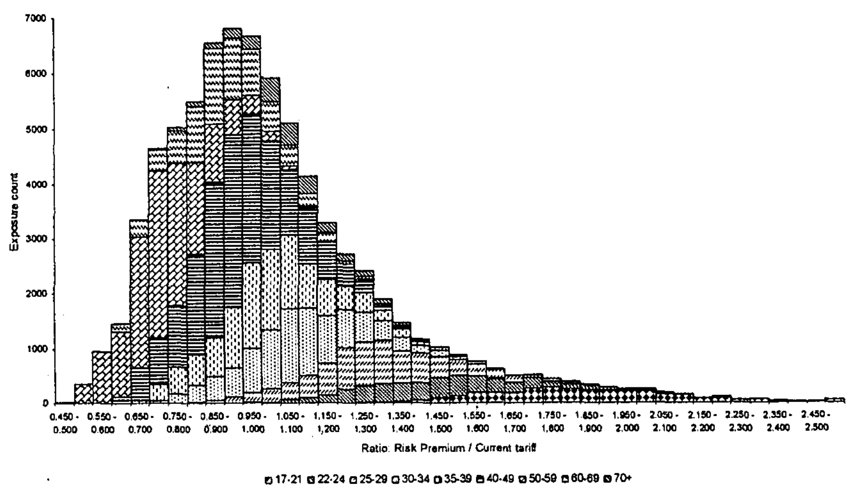
[11] Where A, B, C and D represent factors each of which possibly have a different number of levels.

2.121 This graph above shows the number of exposures in the existing portfolio that would experience different changes in premium if the rating structure were to move from its existing form to the theoretically correct form immediately. It is, of course, exceptionally unlikely that such dramatic change would be implemented in practice. The purpose of this analysis is to understand the magnitude of the existing cross-subsidies by considering the effect of all rating factors at the same time.

2.122 This graph can also be divided by levels of a particular rating factor. (Indeed one such graph can be produced for each rating factor.) This identifies which sectors of the business are currently profitable, and which are currently unprofitable, taking into account the correct theoretical model and considering the effect of all factors at the same time. In the example below, the impact graph is segmented by age of driver (notice the shape does not change, only how the histogram is patterned).

Impact on portfolio of moving to theoretically correct relativities (segmented by age of driver)

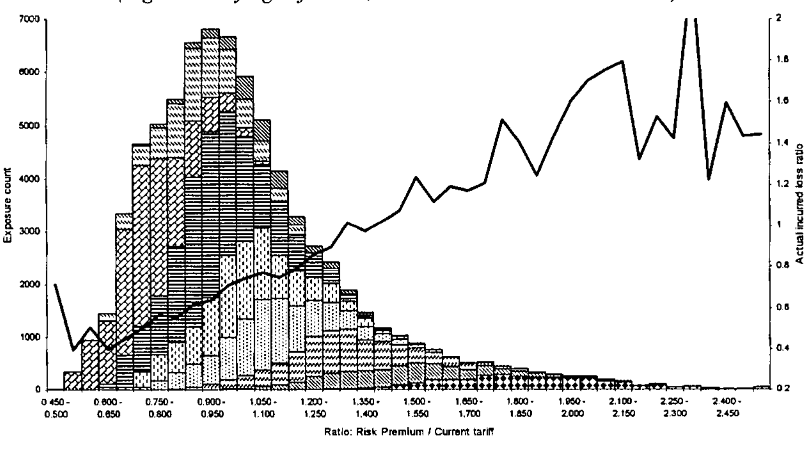


2.123 The histogram shows the impact of all rating factor changes (not just the age of driver factor) by age of driver levels. It can be seen in this example that a large number of exposures which would experience large increases in premium if the rating structure were moved immediately to the theoretically correct structure are young drivers. It had already been seen from the GLM risk premium graphs that young driver relativities were too low. This graph suggests there are no effects from other correlated factors which noticeably mitigate this effect; otherwise, young drivers would not be so strongly on the "unprofitable" side of the impact graph..

2.124 An example may make interpretation of the graph above clearer. Assume the multiplicative claims model uses age, gender, marital status, territory and credit as rating factors. Consider the following young driver profile with indicated rate change for each criterion in parenthesis: age 17-21 (+60%), male (+15%), single (-5%), urban territory (+15%), high credit score (-20%). All factors considered, the total indicated rate change for this risk profile is +61% and so this policy would contribute a count of one to the bar at 1.60-1.65. There are roughly 600 total exposures in this band; roughly one-third of which correspond to drivers age 17-21.

2.125 The graph below adds a second (right hand) y-axis. This y-axis contains the actual loss ratio present in the historical data. This shows very clearly how the GLM has differentiated between segments of differing profitability - each band on the x-axis represents a band of differing expected profitability, and the solid line shows the actual profitability experienced for that band.

Impact on portfolio of moving to theoretically correct relativities (segmented by age of driver, with actual loss ratio also shown)



3 Other applications of GLMs

3.1 This section briefly discusses

- the role of GLMs in the use of credit in personal lines ratemaking
- the use of scoring algorithms in more general terms to consider underwriting and marketing scorecards not necessarily related to credit
- the use of GLMs in retention/conversion analysis.

The role of GLMs in the use of credit-based insurance scores

3.2 Credit-based insurance scores attempt to measure the predictive power of components of consumer credit report data on the cost of insurance claims. The personal lines insurance industry in the US has been using credit-based insurance scoring for over a decade. A 2001 Conning & Company survey reported that 92% of the 100 largest personal automobile insurance writers in the US use some form of credit scoring.[12]

3.3 The early published actuarial studies on the use of credit information in insurance demonstrated clear differences in univariate loss ratio by different bands of Insurance Bureau credit score. Further studies examined this relationship by components of the Insurance Bureau score and also considered how loss ratio by credit component varied across certain traditional rating variables (ie a two-way approach).[13]

3.4 These studies drew early criticism regarding possible double-counting of effects already present in risk classification schemes.[14] Generalized linear models and other multivariate methods have played a critical role in addressing that criticism. A recent study, conducted by EPIC Actuaries, LLC on behalf of the property-casualty insurance industry's four national trade associations, offered four major findings about credit-based insurance scores:

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[12] "Insurance Scoring in Private Passenger Automobile Insurance – Breaking the Silence", *Conning Report*, Conning, (2001)

[13] The reader seeking more information may reference the summaries of the Tillinghast study and the James Monoghan paper in "Does Credit Score Really Explain Insurance Losses? Multivariate Analysis from a Data Mining Point of View" by Cheng-Sheng Peter Wu and James C Guszcza, Casualty Actuarial Society Forum 2003 Vol Winter Page(s): 120-125.

[14] The use of credit information in insurance underwriting and ratemaking has also drawn serious criticism regarding issues such as social equity, intuitive correlation with loss, disparate impact by race and level of wealth, etc. These issues are beyond the scope of this paper

a. using generalized linear models to adjust for correlations between factors, insurance scores were predictive of propensity for private passenger automobile insurance loss (particularly frequency),

b. insurance scores are correlated with other risk characteristics, but after fully accounting for those correlations, the scores significantly increase the accuracy of risk assessment,

c. insurance scores are among the three most important risk factors for each of the six automobile claim types studied;

d. an analysis of property damage liability frequencies by insurance score group for each of the fifty states suggest consistent results across states [15]

3 5 Model vendors and insurance companies have developed credit-based insurance scoring algorithms which vary in complexity, application and proprietary nature. The 2001 Conning & Company survey concluded that smaller insurers were using credit scoring predominantly in their underwriting processes, whereas larger insurers appeared to be focusing on underwriting, pricing and sophisticated market segmentation.

Insurance scores beyond credit

3 6 Other scoring techniques can be used as a way to share vital information between the actuarial departments and the rest of the insurance organization. For example, scores can be used to predict the profitability of an insurance policy given a certain rating structure This information can be used in underwriting, cession decisions, marketing, and agent compensation schemes.

3 7 The most direct way to manage the profitability of a personal lines product is through effective ratemaking. Sometimes, however, regulatory, practical or commercial conditions restrict the degree to which premiums can be set to reflect the risk. In these circumstances a score based on expected loss ratio can be used by insurers to gauge which customers are likely to be more profitable As various functional areas are familiar with the application of scoring algorithms, this provides a common language for communicating a desired strategy throughout the insurance organization.

[15] "The Relationship of Credit-based Insurance Scores to Private Passenger Automobile Insurance Loss Propensity, an actuarial study by Epic Actuaries LLC"; principal authors Michael J Miller and Richard A Smith

3 8 For example, a scoring algorithm could help target marketing campaigns to those customers who are likely to be more profitable Scores can also be used as part of an incentive scheme for agents, where commission or bonus is linked to the average customer score. Such applications can be particularly useful in highly regulated markets, as the score can include policyholder characteristics that are not permitted in the actual premium.

Producing the score

3 9 One method of deriving a scoring algorithm takes advantage of the "linear" part of generalized linear models (GLMs) The output of a GLM is a series of additive parameters which is then transformed via the link function to give the expected value for an observation. When calculating a score the link function can be omitted, leaving a simple additive structure which orders the risk A straightforward calculation can then transform the additive structure into a scoring algorithm which produces scores between a desired range, for example 0 to 100.

3 10 To derive a profitability score, the starting point would be a standard analysis of claims experience using GLMs This would involve fitting a series of GLMs to historic claims data, considering frequency and severity separately for each claim type. These models would include all standard rating factors, as well as any additional information that will be available at the time the score is to be calculated. Such additional information could include geodemographic data.

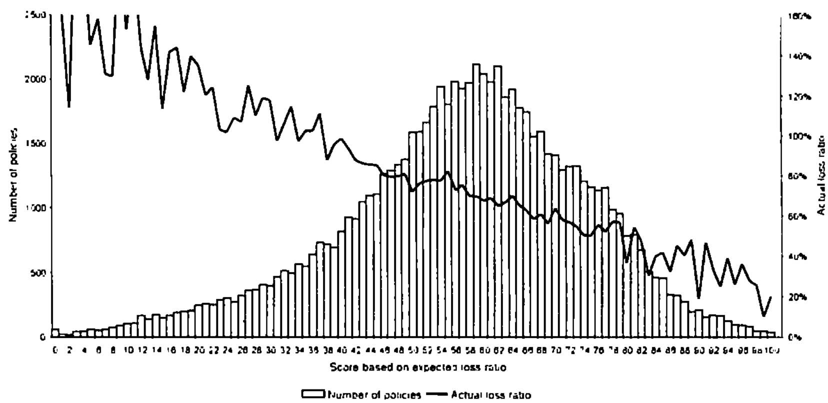
3 11 The expected cost of claims can then be calculated for each record in the data based upon the GLM claims models. For each policy this can then be divided by the premium which will be charged to yield an expected loss ratio, which can then itself be modeled and re-scaled to derive the profitability score

3.12 The model of expected loss ratio should only include those factors that will be considered at the time the score is to be applied For direct mailing campaigns this will usually mean that traditional insurance rating factors used in the premium will have to be excluded at this point (since they are not known at the time of the mailing campaign)

Example results

3 13 The graph below shows how a score can be used to segment very effectively between profitable and unprofitable business The bars on the graph show the number of policies that have been allocated different scores between 0 and 100 The solid line shows the actual loss ratio experienced for business with differing scores. It can be seen that the business towards the left of the graph, with low profitability scores, is experiencing loss ratios of 100% and above, while the business to the right of the graph, with high scores, is returning loss ratios of 50% and below

Distribution of score



3 14 Scores are simple to produce, easy to explain and are increasingly used by insurers Actuaries can play a vital role in the development of scoring models with the aid of generalized linear models

Retention modeling using GLMs

3 15 Traditional ratemaking techniques focus primarily on loss analysis in a static environment Rate changes developed by these techniques, especially when they are large, can actually contribute to a shortfall in projected premium volume and profitability if insufficient consideration is given to the effect of the rate change and other policy characteristics on customer retention and/or new business conversion Modeling retention (or its complement, lapse rate) and new business conversion with GLMs can improve ratemaking decisions and profitability forecasts, as well as improve marketing decisions

3.16 The data for a retention model must include information on individual policies that have been given a renewal offer, and whether or not each policy renewed [16] Similarly, data for a conversion model must contain individual past quotes and whether the quote converted to new business (While most insurers have access to appropriate retention data, many distributing via exclusive agents or independent brokers will not have access to appropriate individual conversion data.) The explanatory variables to include in the data can be divided into three categories customer information, price change data, and information on the competitive position.

3.17 The first category should encompass more than just the standard rating variables (eg age,, territory, claim experience). Other "softer" variables such as number of years with the company, other products held, payment plan and endorsement activity can determine much about a customer's behavior Distribution channel, too, can have a clear effect on the retention rate - and may interact significantly with other factors (eg the effect of age may be different with internet distribution than with agency distribution)

3.18 Prior rate change, whether measured in percent change or dollar change, is often one of the most significant factors in a retention model Though it is intuitive that retention is a function of rate change, the slope of the elasticity curve at different rate changes may not be as obvious In addition, measuring retention using a generalized linear model will adjust for exposure correlations between price elasticity and other explanatory variables (eg a GLM will not show that a particular rating factor level has a low retention rate merely because historically there was aggressive rate activity with that level)

3.19 The third type of variable, information on the competitive position, is often the hardest to gather in practice. An example of a competitive index may be the ratio of the company's renewal quote to third cheapest quote from a specified selection of major competitors at the time of the quote Given the propensity of many small and medium companies to follow the price of the market leaders, ignoring these companies in the model may not have a significant effect since their relationship to the market leaders may be fairly constant over time

3.20 Tracking the myriad of competitor rate changes in a multitude of states can be overwhelming - even with the availability of third party competitive rating software and advances in quote collection procedures Fortunately even the most rudimentary competitive index variables can prove to be predictive in a retention model (and more so in a conversion model)

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[16] Alternatively, retention data may be organized by risk if more than one risk is written on a single policy.

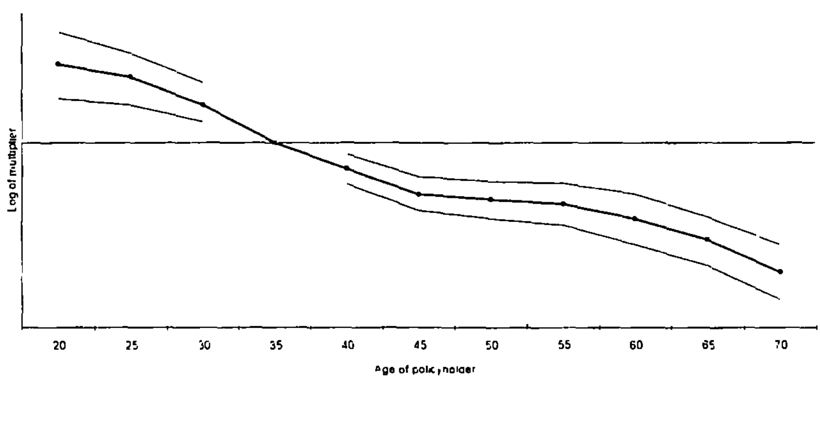
Model form

3 21 As mentioned previously in Section 1, the typical model form for modeling retention (or lapse) and new business conversion is a logit link function and binomial error term (together referred to as a logistic model) The logit link function maps outcomes from the range of (0,1) to (-∞, +∞) and is consequently invariant to measuring successes or failures If the y-variate being modeled is generally close to zero, and if the results of a model are going to be used qualitatively rather than quantitatively, it may also be possible to use a multiplicative Poisson model form as an approximation given that the model output from a multiplicative GLM can be rather easier to explain to a non-technical audience

Example results

3 22 The graph below shows sample GLM output for a lapse model The main line on the graph demonstrates (on a log scale) the measured multiplicative effect of age of policyholder upon lapse rate The effect is measured relative to an arbitrarily selected base level, and the results take into account the effect of all other factors analyzed by the GLM

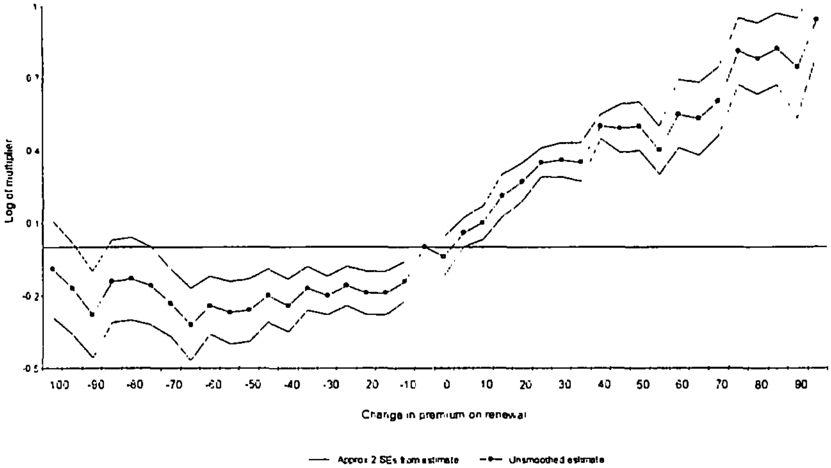
Poisson multiplicative lapse model output



3.23 In this example, which is fairly typical, it can be seen that young policyholders lapse considerably more than older policyholders, perhaps as a result of having more time and enthusiasm in searching for a better quotation, and perhaps also as a result of being generally less wealthy and therefore more interested in finding a competitive price.

3.24 This next graph shows the effect of premium change on lapse rate This GLM output is from a UK Institute of Actuaries General Insurance Research Organisation (GIRO) study[17] based on around 250,000 policies across several major UK insurers in 1996. The premium change is measured in ranges of monetary units (British pounds in this case), but the model could easily be based on the percentage change in premium. As would be expected, increases in premium increase lapses The model, however, quantifies this accurately and enables investigations into potentially optimal rate increases to be undertaken It can be seen in this case (as is generally the case) that decreases in premium beyond a small threshold do not increase retention at all.

Poisson multiplicative lapse model output



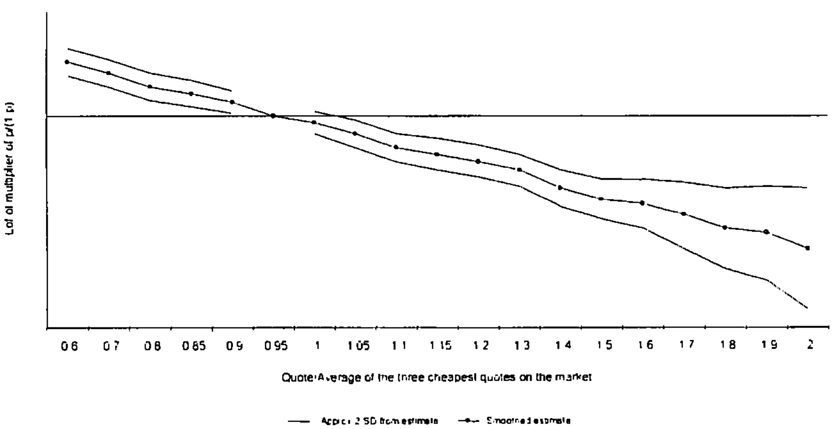
[17] Bland, R H et al, Institute of Actuaries GIRO Customer Selection and Retention Working Party, 1997 - ISBN 0 901066 45 1

3.25 Measures of premium change should ideally consider whether customers have an inherent expectation of premium change. For example, customers with recent claims will anticipate a premium increase and may be prepared to accept their renewal offer rather than face the underwriting guides of a new company Conversely, customers who are rolling off an accident surcharge, hitting a milestone age or a change in marital status may expect a decrease A possible proxy for customer expectation is to adjust the premium change variable to be the ratio of proposed premium (based on new risk criteria and new rates) to adjusted proposed premium (new risk criteria based on last year's rates)

3 26 In addition to including premium change, absolute premium can also be considered as a factor in a model. This approach, though not theoretically incorrect, may make the model difficult to interpret since many other factors in the model will be a component of premium and therefore highly correlated with premium size. Adding absolute premium to the model may significantly alter the observed relativities for other factors which may make the results hard to interpret One alternative to including absolute premium in such a case is to fit separate models for different bands of average premium

3 27 The next graph below shows an example of the effect of competitiveness in a new business conversion model The measure of competitiveness used in this case is the ratio of the proposed premium to the average of the three cheapest alterative premiums from a selection of alternative insurers It can be seen that the less competitive the premium, the lower the conversion rate

Logistic new business conversion model output



3 28 A further analysis which can be undertaken is to superimpose the results of two models on one graph one model that includes the competitiveness measure and one model that does not. The disparity between these two models will show how much of a factor's effects are simply price-related

Applications

3 29 In a fully deregulated market such as the UK, insurance companies can set premium rates according to what the market will bear In the US, insurance companies need to demonstrate that rates are within a reasonable range of loss and expense cost estimates. Companies can, however, measure the sensitivity of various point selections within those ranges (whether the point estimates pertain to overall rate level or classification ratemaking) Future pricing reviews may not only present management with support on actuarial considerations such as trend and loss development, but also a forecast of how various rate change proposals are expected to affect retention, conversion, premium, overall loss ratio (incorporating both overall rate change and portfolio shift of classification changes) and profitability in the short term and/or long term.

3 30 Retention analyses can also lead to operational actions which are unrelated to price. For example, in a highly rate-regulated state, consideration could be given to which segments of the population (given a restricted set of rates) are both profitable and most likely to renew in the future Such a measure could help form new underwriting guides or targeted marketing and cross-sell campaigns

3.31 Insurance expense analysis is another field of study that is often over-shadowed by loss analysis If acquisition expenses are higher than renewal expenses then an understanding of likely retention (and therefore expected life of a policy) can be used to amortize the higher acquisition cost over the expected life of the policy

Conclusion

3.32 A GLM statistically measures the effect that variables have on an observed item In insurance, GLMs are most often used to determine the effect rating variables have on claims experience and the effect that rating variables and other factors have on the probability of a policy renewing or a new business quotation being accepted

3.33 GLMs estimate the true effect of each variable upon the experience, making appropriate allowance for the effect of all other factors being considered Ignoring correlation can produce significant inaccuracies in rates

3.34 GLMs incorporate assumptions about the nature of the random process underlying claims experience Having the flexibility to specify a link function and probability distribution that matches the observed behavior increases the accuracy of the analysis

3.35 A further advantage of using GLMs is that as well as estimating the effect that a given factor has on the experience, a GLM provides information about the certainty of model results

3.36 GLMs are robust, transparent and easy to understand With advances in computer power, GLMs are widely recognized as the industry standard in European personal lines, and fast gaining acceptance from industry professionals in the US and Canada.

3.37 GLMs in insurance are not limited to pricing Alternative applications of GLM claims analyses include underwriting, selective marketing and agency marketing

3.38 GLMs are grounded in statistical theory and offer a practical method for insurance companies to attain satisfactory profitability and a competitive advantage

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A The design matrix when variates are used

Consider the example of a model which is based on two continuous rating variables: age of driver and age of car

Let $\underline{Y}$ be a column vector with components corresponding to the n observed values for the response variable, for example severity:

$$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ . \\ Y_n \end{bmatrix} = \begin{bmatrix} 800 \\ 400 \\ \ldots \\ 200 \end{bmatrix}$$

Let $\underline{X}_1$ and $\underline{X}_2$ denote the column vectors with components equal to the observed values for the continuous variables (eg $\underline{X}_1$ shows the actual age of the driver for each observation):

$$\underline{X}_1 = \begin{bmatrix} 18.1 \\ 32.2 \\ \ldots \\ 44.4 \end{bmatrix} \qquad \underline{X}_2 = \begin{bmatrix} 12.5 \\ 1.6 \\ \ldots \\ 3.8 \end{bmatrix}$$

As before, $\underline{\beta}$ denotes a column vector of parameters, and $\underline{\varepsilon}$ the vector of residuals.

$$\underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \qquad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \ldots \\ \varepsilon_n \end{bmatrix}$$

Then the system of equations takes the form

$$\underline{Y} = \beta_1 \underline{X}_1 + \beta_2 \underline{X}_2 + \underline{\varepsilon}$$

Or, defining the design matrix **X** as

$$X = \begin{bmatrix} 18.1 & 12.5 \\ 32.2 & 1.6 \\ . & \\ 44.4 & 3.8 \end{bmatrix}$$

The system of equations takes the form

$$\underline{Y} = X.\underline{\beta} + \underline{\varepsilon}$$

Polynomials

Rather than assuming that the value of $X.\underline{\beta}$ is linear in the variate, it is also possible to include in the definition of $X.\underline{\beta}$ terms based on polynomials in the variates For example, a model could be based on a third order polynomial in age of driver and a second order polynomial in age of vehicle. In this case the design matrix would be defined as follows

$$X = \begin{bmatrix} 1 & 18.1 & 327.61 & 5929.741 & 12.5 & 156.25 \\ 1 & 32.2 & 1036.84 & 33386.25 & 1.6 & 2.56 \\ . & \cdots & & & . & . \\ 1 & 44.4 & 1971.36 & 87528.38 & 3.8 & 14.44 \end{bmatrix}$$

where

- the first column represents the intercept term $(\text{driver age})^0$
- the second column represents the values of $(\text{driver age})^1$
- the third column represents the values of $(\text{driver age})^2$
- the fourth column represents the values of $(\text{driver age})^3$
- the fifth column represents the values of $(\text{vehicle age})^1$
- the sixth column represents the values of $(\text{vehicle age})^2$

B The exponential family of distributions

Formally the exponential family of distributions is a two-parameter family of functions defined by

$$f(y,\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}$$

where $a(\phi)$, $b(\theta)$, and $c(y, \phi)$ are specified functions Conditions imposed on these functions are that

- **a.** $a(\phi)$ is positive and continuous;
- **b.** $b(\theta)$ is twice differentiable with the second derivative a positive function (in particular $b(\theta)$ is a convex function), and
- **c.** $c(y, \phi)$ is independent of the parameter θ

These three functions are related by the simple fact that f must be a probability density function and so it must integrate to 1 over its domain Different choices for $a(\phi)$, $b(\theta)$, and $c(y, \phi)$ define a different class of distributions and a different solution to the GLM problem The parameter θ is termed the canonical parameter and ϕ the scale parameter. The chart below summarizes some familiar distributions that are members of the exponential family:

	$a(\phi)$	$b(\theta)$	$c(y,\phi)$
Normal	ϕ/ω	$\theta^2/2$	$-\tfrac{1}{2}(\omega y^2/\phi + \ln(2\pi\phi/\omega))$
Poisson	ϕ/ω	e^θ	$-\ln y!$
Gamma	ϕ/ω	$-\ln(-\theta)$	$\tfrac{\omega}{\phi}\ln(\tfrac{\omega y}{\phi}) - \ln(y) - \ln(\Gamma(\tfrac{\omega}{\phi}))$
Binomial (m trials)	ϕ/ω	$m\ln(1+e^\theta)$	$\ln\binom{m}{y}$
Inverse Gaussian	ϕ/ω	$-\sqrt{-2\theta}$	$-\tfrac{1}{2}\{\ln(2\pi\phi y^3\omega) + \omega/(\phi y)\}$

It can be seen that the standard choice for $a(\phi)$ is

$$a(\phi) = \frac{\phi}{\omega}$$

where ω is a *prior weight*, a constant that is specified in advance For insurance applications common choices for the prior weight are equal to 1 (eg when modeling claim counts), the number of exposures (eg when modeling claim frequency), or the total number of claims (eg when modeling claim severity) It is also clear from the chart that for certain distributions, such as the Poisson and binomial distributions, the scale parameter ϕ is equal to 1 and plays no further role in the modeling problem.

A distribution for each observation Y_i needs to be specified It is assumed that

$$f(y_i;\theta_i,\phi) = \exp\left\{\frac{y_i\theta_i - b(\theta_i)}{a(\phi)} + c(y_i,\phi)\right\}$$

Thus each observation has a different canonical parameter θ_i but the scale parameter ϕ is the same across all observations It is further assumed that the functions $a(\phi)$, $b(\theta)$, and $c(y, \phi)$ are the same for all i. So each observation comes from the same class within the exponential family, but allowing θ to vary corresponds to allowing the mean of each observation to vary

The parameters θ_i and ϕ encapsulate the mean and variance information about Y_i It can be shown that for this family of distributions

$$\mu_i = E(Y_i) = b'(\theta_i)$$
$$Var(Y_i) = b''(\theta_i)\,a(\phi)$$

where the prime (') denotes differentiation with respect to θ

The first equation implicitly defines θ_i as a function of μ_i If an explicit expression for the inverse of $b'(\theta_i)$ is known (as is the case for the familiar distributions) then the first equation can be solved to express the canonical parameter θ_i explicitly as a function of the mean of the distribution μ_i

$$\theta_i = (b')^{-1}(\mu_i)$$

Thus the canonical parameter is essentially equivalent to the mean

Section 1 describes how a GLM asserts that μ_i is a function of the linear predictor η_i where the linear predictor is a linear combination of the p covariates $X_{i1}, \ldots X_{ip}$

$$\mu_i = g^{-1}(\beta_1 x_{i1} + \quad + \beta_p x_{ip})$$

Thus θ is ultimately a complicated function of the elements of $\underline{\beta}$

$$\theta_i = (b')^{-1}\left(g^{-1}(\beta_1 x_{i1} + \ . \ + \beta_p x_{ip})\right)$$

This derivation makes explicit the manner in which the distribution of Y_i depends on the GLM parameters $\beta_1, \ldots, \beta_p$.

It can be seen from the table above that the expression for $c(y, \phi)$ can be complicated. Fortunately as long as $c(y, \phi)$ does not depend on θ - and hence not on μ and thus not on the GLM modeling parameters β - then the form of $c(y, \phi)$ is irrelevant to the solution of the maximum likelihood estimator.

Given that θ_i is a function of the mean μ_i the equation

$$Var(Y_i) = b''(\theta_i)\, a(\phi)$$

can be interpreted as establishing the variance of Y_i as a function of the mean of Y_i times some scaling term $a(\phi)$. Thus the scaling parameter ϕ is a function of the mean and variance of the distribution

Thus the exponential family has two desirable properties

- each distribution in the family is completely specified in terms of its mean and variance
- the variance of Y is a function of its mean

This second property is emphasized by writing

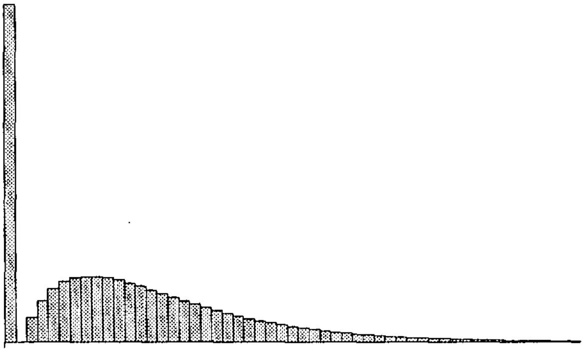
$$Var(Y_i) = \phi \frac{V(\mu_i)}{\omega_i}$$

where the function V is termed the variance function. The chart below summarizes the relationship between the mean and the canonical parameter, expresses f in terms of the standard parameters for the respective distribution, and lists the variance function for the familiar distributions

	Notation	ϕ	$\mu(\theta)$	$V(\mu)$
Normal	$N(\mu,\sigma^2)$	σ^2	θ	1
Poisson	$P(\mu)$	1	e^θ	μ
Gamma	$G(\mu,v)$	v^{-1}	$-1/\theta$	μ^2
Binomial	$B(m,\pi)/m$	$1/m$	$e^\theta/(1+e^\theta)$	$\mu(1-\mu)$
Inverse Gaussian	$IG(\mu,\sigma^2/\omega)$	σ^2	$(-2\theta)^{-1/2}$	μ^3

C The Tweedie distribution

Direct modeling of pure premium or incurred claim data is problematic since a typical pure premium distribution will consist of a large spike at zero (where policies have not had claims) and then a wide range of amounts (where policies have had claims). This is illustrated in the diagram below.



Many of the traditional members of the exponential family of distributions are not appropriate for modeling claims experience from such a distribution since they do not have a point mass at zero combined with an appropriate spread across non-zero amounts.

The Tweedie distribution, which is a special member of the exponential family of distributions, corresponds to the compound distribution of a Poisson claim number process and a Gamma claim size distribution. Consequently its probability density function has a point mass at zero corresponding to the probability of the Poisson number element of the compound distribution being zero.

The Tweedie distribution has three parameters - a mean parameter, a dispersion parameter, and a "shape" parameter α.

Its density function is rather complex and is defined as:

$$f_Y(y;\theta,\lambda,\alpha) = \sum_{n=1}^{\infty} \frac{\left\{(\lambda\omega)^{1-\alpha}\kappa_\alpha(-1/y)\right\}^n}{\Gamma(-n\alpha)n!\,y}.\exp\{\lambda\omega[\theta_0 y - \kappa_\alpha(\theta_0)]\} \quad \text{for } y > 0$$

and

$$p(Y=0) = \exp\{-\lambda\omega\kappa_\alpha(\theta_0)\}$$

where

$$\kappa_\alpha(\theta) = \frac{\alpha - 1}{\alpha}\left(\frac{\theta}{\alpha - 1}\right)^\alpha$$

$$\theta_n = \theta \lambda^{1/(1-\alpha)}$$

ω is the prior weight corresponding to the exposure of the observation in question

It can be shown that the variance function for the above Tweedie distribution is given by

$$V(\mu) = \frac{1}{\lambda}\mu^p$$

where

$$p = \frac{\alpha - 2}{\alpha - 1}$$

Thus the Tweedie distribution can be Poisson-like (as p $\rightarrow$ 1) or Gamma-like (as p $\rightarrow$ 2)

In practice the shape parameter can either be assumed to be a particular value or, more usefully, estimated as part of the maximum likelihood process Typically values of p just under 1 5 seem to be estimated for auto claims experience.

Further information about the Tweedie distribution can be found in the paper "Fitting Tweedie's Compound Poisson Model to Insurance Claims Data" by Jørgenses, B and De Souza, M.C.P, Scand Actuarial J 1994 1·69-93

D Canonical link functions

Each of the exponential distributions has a natural link function called the canonical link. It has the property that $\underline{\theta} = \underline{\eta}$ where $\underline{\theta}$ is the canonical parameter. This property means that the GLM parameters $\beta_1, \ldots \beta_p$ enter the expression for the distribution function in a simple way In general

$$
\begin{aligned}
f_{Y_i}(y_i) &= \exp\left\{\frac{y_i\theta_i - b(\theta_i)}{a(\phi)} + c(y_i, \phi)\right\} \\
&= \exp\left\{\frac{y_i(b')^{-1}(g^{-1}(\eta_i)) - b((b')^{-1}g^{-1}(\eta_i))}{a(\phi)} + c(y_i, \phi)\right\}
\end{aligned}
$$

but if $\underline{\theta} = \underline{\eta}$ this simplifies to

$$
\exp\left\{\frac{y_i\eta_i - b(\eta_i)}{a(\phi)} + c(y_i, \phi)\right\}
$$

and subsequent differentiation with respect to the GLM parameters β_j is thus significantly simplified.

The canonical link functions associated with the familiar distributions are listed below

	Canonical Link
Normal	μ
Poisson	$\ln \mu$
Gamma	$\ln(\mu/(1-\mu))$
Binomial	$1/\mu$
Inverse Gaussian	$1/\mu^2$

Note that the requirement to be a canonical link function:

$$
\theta = (b')^{-1}(g^{-1}(\eta)) = \eta
$$

implies that the inverse of the link function, g^{-1}, is the inverse of b'

In practice, with sophisticated software to solve GLM modeling problems there is no imperative to use the canonical link associated with a particular distribution Instead arbitrary pairings of the link function and the error structure can be made and such non-canonical pairings can in fact yield more predictive models

E Solving for maximum likelihood in the general case of an exponential distribution

In the case of the exponential family of distributions the log likelihood takes the form

$$l = \sum_i \left\{ \frac{y_i\theta_i - b(\theta_i)}{a(\phi)} + c(y_i, \phi) \right\}$$

Log likelihood is maximized by taking, for each j, the first order partial derivative of l with respect to β_i and setting equal to zero

$$\frac{\partial l}{\partial \beta_j} = 0, \qquad j = 1, \ldots, p$$

If there is an explicit expression for θ_i in terms of $\beta_1, \ldots \beta_p$ one can make this substitution into the log likelihood function and then carry out the differentiation However, the calculations become complicated quite quickly It is simpler just to apply the chain rule of calculus three times

$$0 = \frac{\partial l}{\partial \beta_i} = \sum_i \frac{\partial}{\partial \theta_i}\left(\frac{Y_i\theta_i - b(\theta_i)}{a(\phi)} + c(y_i, \phi) \right) \cdot \frac{\partial \theta_i}{\partial \mu_i} \frac{\partial \mu_i}{\partial \eta_i} \cdot \frac{\partial \eta_i}{\partial \beta_i}$$

Recalling the following relationships.

$$\mu_i = b'(\theta_i) \Rightarrow \frac{\partial \mu_i}{\partial \theta_i} = b''(\theta_i) \Rightarrow \frac{\partial \theta_i}{\partial \mu_i} = \frac{1}{b''(\theta_i)}$$

$$\eta_i = g(\mu_i) \Rightarrow \frac{\partial \eta_i}{\partial \mu_i} = g'(\mu_i) \Rightarrow \frac{\partial \mu_i}{\partial \eta_i} = \frac{1}{g'(\mu_i)}$$

$$\eta_i = \beta_1 X_{i1} + \quad + \beta_p X_{ip} \Rightarrow \frac{\partial \eta_i}{\partial \beta_j} = X_{ij}$$

It can be deduced that

$$\frac{\partial l}{\partial \beta_i} = \sum_j \frac{(y_j - \mu_j)}{a(\phi)} \cdot \frac{1}{b''(\theta_j)} \cdot \frac{1}{g'(\mu_j)} \, x_{ij} \, , \qquad i = 1, \ldots, p$$

$$= \sum_j \frac{\omega_j}{V(\mu_j) g'(\beta_1 x_{j1} + \ldots + \beta_p x_{jp})} (y_j - \mu_j) x_{ij} \, , \qquad i = 1, \ldots, p$$

Although the theoretical system of equations which must be satisfied in order to maximize the likelihood can be (relatively) easily written, finding the solution to these equations is more complicated.

F Example of solving for maximum likelihood with a gamma error and inverse link function

For the Gamma error structure with an inverse link function, the predicted values take the form:

$$E[\underline{Y}] = g^{-1}(X\,\underline{\beta}) = \begin{bmatrix} g^{-1}(\beta_1+\beta_3) \\ g^{-1}(\beta_1) \\ g^{-1}(\beta_2+\beta_3) \\ g^{-1}(\beta_2) \end{bmatrix} = \begin{bmatrix} (\beta_1+\beta_3)^{-1} \\ (\beta_1)^{-1} \\ (\beta_2+\beta_3)^{-1} \\ (\beta_2)^{-1} \end{bmatrix}$$

The Gamma error structure has the following density function

$$f(x,\mu,\phi) = \frac{x^{-1}}{\Gamma(1/\phi)}\left(\frac{x}{\mu\phi}\right)^{1/\phi} e^{\left(-\frac{x}{\mu\phi}\right)}$$

Its log-likelihood function is

$$l(x;\mu,\phi) = \sum_{i=1}^{n} \ln f(x_i,\mu_i) = \sum_{i=1}^{n} \frac{1}{\phi}\left(\ln\frac{x_i}{\mu_i} - \frac{x_i}{\mu_i}\right) - \ln x_i - \frac{\ln\phi}{\phi} - \ln\Gamma\left(\frac{1}{\phi}\right)$$

With an inverse link function, $\mu_i = 1/(\Sigma_j X_{ij}\beta_j)$ and the log-likelihood function reduces to

$$l(x,1/X\beta,\phi) = \sum_{i=1}^{n} \frac{1}{\phi}\left(\ln(x_i * \sum_{j=1}^{p} X_{ij}\beta_j) - x_i * \sum_{j=1}^{p} X_{ij}\beta_j\right) - \ln x_i - \frac{\ln\phi}{\phi} - \ln\Gamma\left(\frac{1}{\phi}\right)$$

In this example,

$$\begin{aligned} l(x;\mu) = {} & \frac{1}{\phi}(\ln(800*(\beta_1+\beta_3)) - 800*(\beta_1+\beta_3)) - \ln 800 - \frac{\ln\phi}{\phi} - \ln\Gamma\left(\frac{1}{\phi}\right) \\ & + \frac{1}{\phi}(\ln(500*\beta_1) - 500*\beta_1) - \ln 500 - \frac{\ln\phi}{\phi} - \ln\Gamma\left(\frac{1}{\phi}\right) \\ & + \frac{1}{\phi}(\ln(400*(\beta_2+\beta_3)) - 400*(\beta_2+\beta_3)) - \ln 400 - \frac{\ln\phi}{\phi} - \ln\Gamma\left(\frac{1}{\phi}\right) \\ & + \frac{1}{\phi}(\ln(200*\beta_2) - 200*\beta_2) - \ln 800 - \frac{\ln\phi}{\phi} - \ln\Gamma\left(\frac{1}{\phi}\right) \end{aligned}$$

Ignoring some constant terms and multiplying by ϕ, the following function is to be maximized

$$\begin{aligned} l^*(y,\mu) = {} & \ln(800*(\beta_1+\beta_3)) - 800*(\beta_1+\beta_3) + \ln(500*\beta_1) - 500*\beta_1) \\ & + \ln(400*(\beta_2+\beta_3)) - 400*(\beta_2+\beta_3) + \ln(200*\beta_2) - 200*\beta_2 \end{aligned}$$

Again, to maximize l^* take derivatives with respect to β_1, β_2 and β_3. Set the derivatives to zero and the following three equations are derived:

$$\frac{\partial l^*}{\partial \beta_1} = 0 \quad \Rightarrow \quad \frac{1}{\beta_1 + \beta_3} + \frac{1}{\beta_1} = 1300$$

$$\frac{\partial l^*}{\partial \beta_2} = 0 \quad \Rightarrow \quad \frac{1}{\beta_2 + \beta_3} + \frac{1}{\beta_2} = 600$$

$$\frac{\partial l^*}{\partial \beta_3} = 0 \quad \Rightarrow \quad \frac{1}{\beta_1 + \beta_3} + \frac{1}{\beta_2 + \beta_3} = 1200$$

Solving these simultaneous equation gives the following solutions:

$$\beta_1 = 0.00223804$$
$$\beta_2 = 0.00394964$$
$$\beta_3 = -0.00106601$$

which result in the following predicted values:

	Urban	Rural
Male	853.2	446.8
Female	346.8	253.2

G Data required for a GLM claims analysis

The overall structure of a dataset for GLM claims analysis consists of linked policy and claims information at the individual risk level The definition of individual risk level will vary according to the line of business and the type of model For instance, in a personal automobile claims model, the definition of risk may be a vehicle (In a personal automobile retention model, the definition of risk may be a policy containing several vehicles)

One record should be present for each period of time during which a policy was exposed to the risk of having a claim, and during which all factors remained unchanged Policy amendments should ideally appear as two records, with the previous exposure curtailed at the point of amendment Mid-term policy cancellations should also result in the exposure period being curtailed If this data is not available it is often possible to approximate it from less perfect data - for example the policies in force at one year end could be compared with the policies in force at the previous year end, with matching policies being assumed to be in force for the whole year, and appropriate approximations being made for non-matching policies

The dataset should contain fields defining the earned exposure and the rating factors applicable at the start of the exposure period Additionally, premium information (typically earned premium) can be attached to each record Although premium is not used directly in the development of the claims models, it can provide valuable information for measuring the impact of any new rating or underwriting actions, and for producing summary one-way and two-way analyses including loss ratios

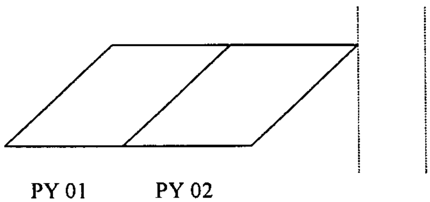
All explanatory variables in the dataset should record the criteria which were applicable at the start of the policy exposure (or, strictly speaking, the point at which the premium was determined for the exposure period in question) In the case of categorical variables such as territory or vehicle class, however, the data recorded should ideally be derived by applying the current method of categorization to the historic situation

Not all explanatory variables will be used to predict future claims experience Dummy variables may be used to absorb certain effects that could bias the parameter estimates For example, if conducting a countrywide study, it may be appropriate to create a dummy variable to standardize for differences in overall loss experience by geography. This dummy variable may be state (province), territory within state (province), or groups of territories within state (province)

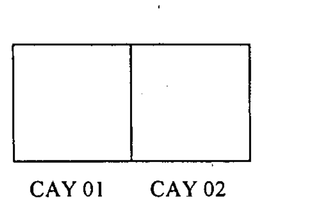
Similarly, if combining data from several companies, a company identifier may be an appropriate dummy variable This dummy variable could absorb differences in underwriting standards and overall quality of business between the companies Dummy variables could also absorb some historical effect which is not expected to continue in the future Though dummy variables can be used in such a way, it is still preferable to have an experience period devoid of strong effects

GLM claims datasets are typically either based on a certain policy year period or a certain calendar-accident year period. An example using the traditional parallelogram and rectangle diagrams illustrates the difference between the two.

Dataset A: policy year



Dataset B: calendar-accident year



Policy year: Annual policies written between 1/1/01 and 12/31/02, earned as of 12/31/03. Claims incurred on these policies before 12/31/03 but losses evaluated as of 6/30/04.

Calendar-accident year: Annual policies earning between 1/1/01 and 12/31/02 in respect of policies written between 1/1/00 and 12/31/02. Claims occurring on policies earning between 1/1/01 and 12/31/02, incurred losses evaluated as of 6/30/03.

There are benefits and disadvantages of each method of organization. The policy year approach has the advantage of relating to a certain period of underwriting and method of selling a product. The earning pattern of any given policy year, however, extends beyond the 12 month period. In order for policies to be fully earned, the cut-off date for exposures needs to extend 12 months (in the case of annual policies) or six months (in the case of semi-annual policies). In addition, the need for some IBNR emergence builds in more delay, resulting in data analyzed being not very recent.

The calendar-accident method of organization requires that each policy be split into its calendar year components (for example, an annual policy written on May 1 will be split into records defined by May 1 through December 31 and January 1 through April 30). Although this adds to system requirements and increases the number of records in the dataset, this allows the creation of an accurate calendar year "dummy" explanatory variable which can be used to absorb trends in claim experience which purely relate to time. If this is not possible, the policy year method of organization can be used, but the effect of any trends can be more difficult to identify.

Claims information

Claim count and loss amount information should be attached to the relevant exposure records, based on the most recent reserve estimates The choice of definition of incurred claim count, specifically whether this pertains to number of claims or number of claimants, is not particularly important if ultimately the claim frequency and claim severity will be combined to the pure premium level. It is generally easier to model loss information net of deductibles, but should ideally not be truncated according to any large loss threshold at this stage since this allows sensitivity testing of several different large loss thresholds when modeling.

It is appropriate to leave some delay between the end of the experience period and the valuation date to allow for some IBNR claims to emerge and to allow for the case estimates to develop If there is a regular (annual or quarterly) review of case estimates, or any other known issue surrounding the reserves, the experience period and valuation date should be selected to take advantage of the most accurate information.

The overall base level adjustment for pure IBNR and development of known claims will be made after models are finalized, but it is necessary to consider whether such time-related influences could bias the model rating factor relativities There is a range of options for investigating the consequences of claims development upon the relativities measured, including:

- ignoring loss development and assuming that parameter estimates are unaffected
- including a dummy variable (eg calendar year or policy year[18]) in the model to absorb time-related influences, once models are finalized, the dummy variable is simply removed and the base levels are adjusted via a separate calculation (this assumes the development of claims is similar for all types of policy)
- before modeling the most recent experience, performing a series of GLM analysis on an older dataset which contains claims statistics as at various periods of development. By comparing GLM relativities based on data as at different development periods it is possible to assess whether claims development differs materially by type of risk - if they do it is possible to use the ratio of two models as at different development periods to derive multivariate development factors which can be applied to analyses based on a more recent dataset

[18] Dummy variables based on quarters or months may contain an element of seasonality

It is also necessary to consider the treatment of claims closed without payment (also known as CWPs) Before modeling, it is generally most appropriate to remove such claims (setting the claim count field to zero in these cases), perhaps also creating a new claim type consisting of only CWPs (if they are to be modeled for expense allocation purposes). If CWPs are not excluded it can become difficult to model average claim amounts since some common GLM forms (eg those with gamma error functions) cannot be fitted to data containing observations equal to zero

Generally, one period of policy exposure will have zero or one claim associated with it. Occasionally, there may be two or more accidents occurring in a given period of exposure. There are a number of alternative ways to deal with this situation.

- Multiple claims could be attached to the single exposure record, with the number of such claims and the total amount of such claims being recorded This is the simplest method A small amount of information is lost as a result of storing information like this, but such a loss is not generally material
- Further records could be created in the database in the case of multiple claims. The exposure end date of the original record could be set to be the date of occurrence of the first accident, with the exposure start date of the second record being the day after Each claim could then be attached to one exposure record (and the "number of claims" fields would always be zero or one) All rating factors recorded in the second record would be identical to the original record.
- Further records could be created in the database as in the second option above, but with the exposure dates in the original record remaining unaltered, and with the exposure start and end dates in the second (and subsequent) copied records being equal to each other, so that the additional records had zero days exposure recorded (When analyzing claim amounts, the exposure information is not required, and when analyzing claim frequencies the experience could be summarized by unique combination of rating factor levels using an appropriate extract of the data, thus compressing this data to derive the correct exposure)

In practice, the easiest way to program the last two of these three methods produces one extra record for every claim, so policies with one claim would produce two records, and policies with two claims would produce three records For example, using the second method, the exposure would be split at every claim date, so that there would always be one record with no claims (the last record).

General

In addition to volume requirements, how the model is to be used should also be considered. If the model were to be used to identify inaccuracies in the current rating plan, a line of business which undergoes significant rate intervention at point of sale would not be appropriate (unless being used to guide underwriters on the acceptable range of their intervention). Similarly, if little is collected or stored in the way of explanatory variables, this too would limit the strength of the GLM.

H Automated approach for factor categorization

One automated approach within the GLM framework is to replace a single factor with many levels with a series of factors each containing just two levels which are then tested for significance For example instead of modeling age of insured with a single factor, a series of binary factors could be created

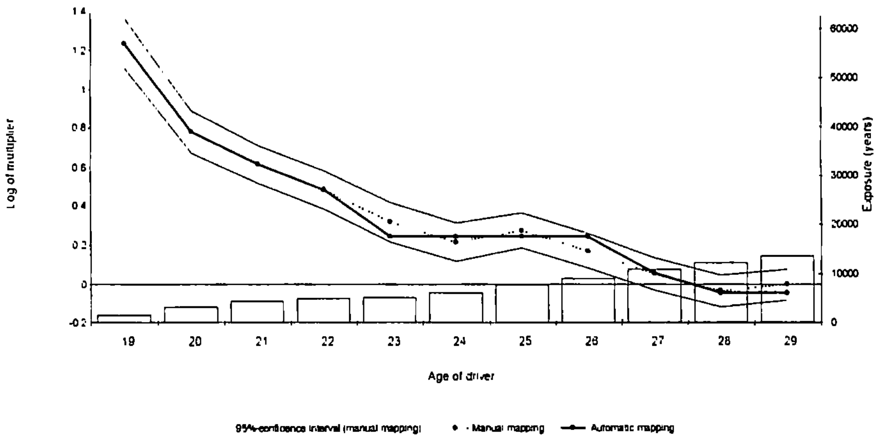
- (binary factor 1) is the age less than 18?
- (binary factor 2) is the age less than 19?
- (binary factor 3) is the age less than 20?
- (binary factor 4) is the age less than 21?
-
- (binary factor 22) is the age less than 39?
- (age 40 is the base level in this example)
- (binary factor 23) is the age less than 41?
- .
- (binary factor 82) is the age less than 100?

These single parameter binary factors could then be tested for significance using an automatic stepwise algorithm as discussed in Section 2

If, for example, ages 23, 24, 25 and 26 did not have a statistically different effect on the risk, the factors "is age less than 24", "is age less than 25" and "is age less than 26" would be deemed insignificant and excluded from the model Those binary factors deemed significant in the model would determine the appropriate age categorization, and implied parameter estimates for each age could then be determined by summing the appropriate binary factors - eg in the above example the implied parameter estimate for "age 20" would be the sum of the parameters for binary factors 4 to 82)

An example result, based on real data, is shown below The dotted line shows the fitted parameter estimates when age is not grouped (and when a parameter is allocated to each individual age rounded to the nearest integer) The solid line shows the parameter estimates implied by the results of the automatic grouping approach described above Only results up to age 29 are shown for reasons of confidentiality.

Example of automatic grouping
(part result only - ages over 29, including base, not shown)



In this case it is not at all clear that the automatic approach produces a better categorization than a manual approach - for example it can be seen from the dotted line that age 23 has a parameter estimate between ages 22 and 24, and intuitively it appears wrong to group this level with ages 25 to 26 as the automatic process suggests. It is often the case that a manual approach to categorization can produce more appropriate results than an automated approach

I Cramer's V

Cramer's V statistic is a measure of correlation between two categorical factors and is defined as

$$\sqrt{\frac{\sum_{i,j} \frac{(n_{ij} - e_{ij})^2}{e_{ij}}}{\min((a-1),(b-1))\, n}}$$

where

a = number of levels of factor one
b = number of levels of factor two
n_{ij} = amount of the exposure measure for the i^{th} level of factor one and j^{th} level of factor two
$n = \Sigma_{ij}(n_{ij})$
$e_{ij} = \Sigma_i(n_{ij})\ \ \Sigma_j(n_{ij}) / n$

The statistic takes values between 0 and 1. A value of 0 means that knowledge of one of the two factors gives no knowledge of the value of the other. A value of 1 means that knowledge of one of the factors allows that value of the other factor to be deduced The two tables below show possible two-way exposure distributions of two categorical factors - each with only two levels, A and B, expressed as either rows or columns. The top table shows a Cramer's V statistic of 0, and the bottom table gives an example of a Cramer's V of 1.

	A	B
A	100	100
B	100	100

	A	B
A	100	0
B	0	100

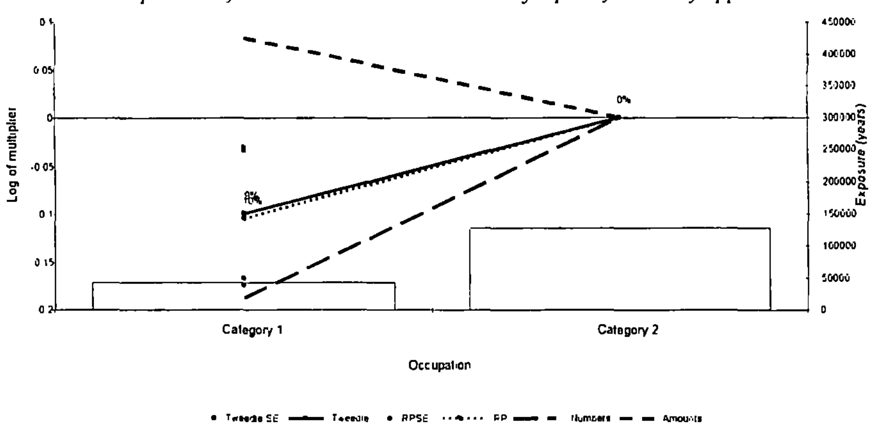
J Benefits of modeling frequency and severity separately rather than using Tweedie GLMs

Tweedie GLMs fitted to pure premium directly can often give very similar results to those derived by the "traditional" approach of combining models fitted to claim frequencies and claim severities separately In these cases using Tweedie GLMs can reduce the amount of iterative modeling work required to produce satisfactory claims models

The traditional approach, however, can provide a better understanding of the way in which factors affect the cost of claims, and can more easily allow the identification and removal of certain random effects from one element of the experience, for example via smoothing or by excluding certain factors from one of the frequency or amounts models

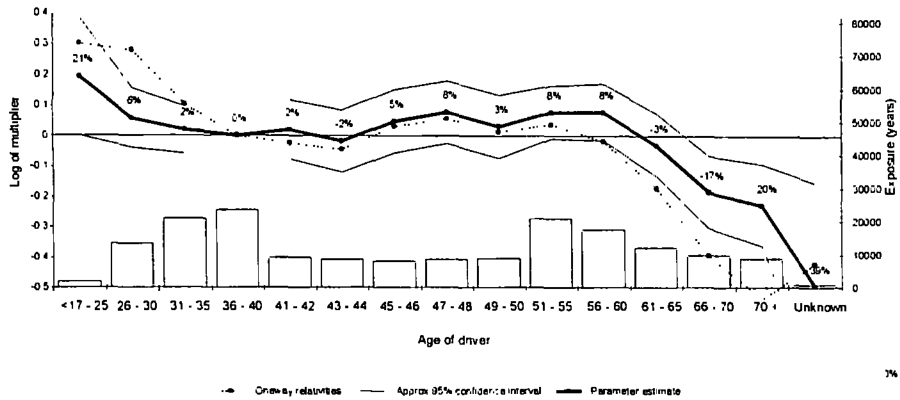
For example, the graph below compares the risk premium results from the Tweedie model to those from the traditional approach for one rating factor Though the results between the two approaches are nearly identical, the traditional approach does provide additional information about the underlying frequency (numbers) and severity (amounts) effects - in this case the factor affects frequencies and severities in completely opposite ways

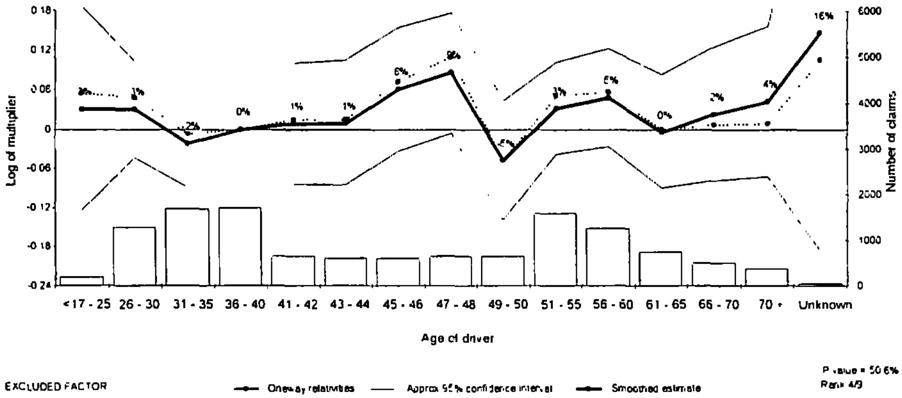
Comparison of Tweedie model with traditional frequency / severity approach



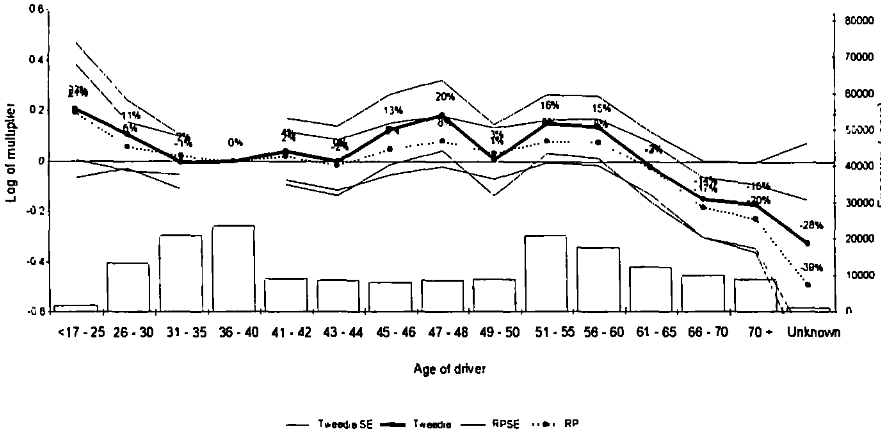
The three graphs below demonstrate a case where the results for a particular factor from a Tweedie GLM differ from those produced by the traditional approach The first two graphs show the underlying frequency and severity model output from the traditional approach. Because of the wide standard errors, meaningless pattern, and insignificant type III test, the factor has been removed from the severity model. Consequently, the traditional risk premium reflects the underlying frequency experience only The Tweedie model is more affected by the volatility from the underlying severity experience, and produces results which may be less appropriate

Comparison of Tweedie model with traditional frequency / severity approach - traditional frequency and severity models





Comparison of Tweedie model with traditional frequency / severity approach



A Primer on the Exponential Family of Distributions

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A PRIMER ON THE EXPONENTIAL FAMILY OF DISTRIBUTIONS

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Abstract

Generalized Linear Model (GLM) theory represents a significant advance beyond linear regression theory, specifically in expanding the choice of probability distributions from the Normal to the Natural Exponential Family. This Primer is intended for GLM users seeking a handy reference on the model's distributional assumptions. The Exponential Family of Distributions is introduced, with an emphasis on variance structures that may be suitable for aggregate loss models in property casualty insurance.

A PRIMER ON THE EXPONENTIAL FAMILY OF DISTRIBUTIONS

INTRODUCTION

Generalized Linear Model (GLM) theory is a significant advance beyond linear regression theory. A major part of this advance comes from allowing a broader family of distributions to be used for the error term, rather than just the Normal (Gaussian) distribution as required in linear regression.

More specifically, GLM allows the user to select a distribution from the *Exponential Family*, which gives much greater flexibility in specifying the variance structure of the variable being forecast (the "response variable"). For insurance applications, this is a big step towards more realistic modeling of loss distributions, while preserving the advantages of regression theory such as the ability to calculate standard errors for estimated parameters. The Exponential family also includes several discrete distributions that are attractive candidates for modeling claim counts and other events, but such models will not be considered here

The purpose of this Primer is to give the practicing actuary a basic introduction to the Exponential Family of distributions, so that GLM models can be designed to best approximate the behavior of the insurance phenomenon.

Insurance Applications

Two major application areas of GLM have emerged in property and casualty insurance. The first is classification ratemaking, which is very clearly illustrated in the papers by Zehnwirth and Mildenhall. The second is in loss reserving, also given an excellent treatment in papers by England & Verrall. In 1991, Mack pointed out a connection between these two applications, so it is not surprising that a common modeling framework works in both contexts.

Both classification ratemaking and reserving seek to find the "best" fitted values μ_i to the observed values y_i. In both cases the <u>response variable</u>, Y_i, of which the observed values y_i are realizations, is measured in units of aggregate loss dollars The response is dependent on predictor variables called <u>covariates</u>. Following Mack, classification ratemaking is performed using at least two covariates, which might include territory and driver age. In the reserving application, the covariates might include accident year and development year.

For our discussions, the choice of covariates used as predictors will not be important, but it will always be assumed that the response variable Y represents aggregate loss dollars. Some of the desirable qualities of the distribution for Y, driven by this assumption, are:

- The distribution is unbiased, or "balanced" with the observed values
- It allows zero values in the response with non-zero probability
- It is positively skewed

Before seeing how specific distributions in the Exponential Family measure up to these desirable qualities, some basic definitions are needed.

DEFINING THE EXPONENTIAL FAMILY

The General and Natural Forms

The general exponential family includes all distributions, whether continuous, discrete or of mixed type, whose probability function or density can be written as follows:

<u>General Form</u> (ignoring parameters other than θ_i):

$$f(y_i;\theta_i) = \exp[d(\theta_i)\ e(y_i) + g(\theta_i) + h(y_i)]$$

where d, e, g, h are all known functions that have the same form for all y_i.

For GLM, we make use of a special subclass called the Natural Exponential Family, for which $d(\theta_i)=\theta_i$ and $e(y_i)=y_i$. Following McCullagh & Nelder, the "natural form" for this family includes an additional dispersion parameter ϕ that is constant for all y_i

Natural Form:

$$f(y_i;\theta_i,\phi)=\exp[\{\theta_i\ \ y_i-b(\theta_i)\}/a(\phi)+c(y_i,\phi)]$$

where a, b, c are all known functions that have the same form for all y_i.

For each form, θ_i is called the canonical parameter for Y_i.

Appendix A shows how the moments are derived for the Natural Exponential Family.

The natural form can also be written in terms of the mean μ_i rather than θ_i by means of a simple transformation: $\mu_i=\tau(\theta_i)=E[y_i;\theta_i]$. This mean value parameterization of the density function, in which μ_i is an explicit parameter, will be the form used in the rest of the paper and the Appendices.

Mean Value Natural Form:

$$f(y_i;\mu_i,\phi)=\exp[\{\tau^{-1}(\mu_i)\ \ y_i-b(\tau^{-1}(\mu_i))\}/a(\phi)+c(y_i,\phi)]$$

To put this in context, a GLM setup based on Y consists of a linear component, which resembles a linear model with several independent variables, and a link function that relates the linear part to a function of the expected value μ_i of Y_i, rather than to μ_i itself. In the GLM, the variables are called covariates, or factors if they refer to qualitative categories. The function $\theta=\tau^{-1}(\mu)$ used in the mean value form is called the canonical link function for a GLM setup based on Y, because it gives the best estimators for the model parameters. Other link functions can be used successfully, so there is no need to set aside practical considerations to use the canonical link function for Y.

For most of this paper, every parameter of the distribution of Y, apart from μ itself, will be considered a known constant. The derogatory-sounding term <u>nuisance parameter</u> is used to identify all parameters that are not of immediate interest.

The Dispersion Function a(ϕ)

The natural form includes a <u>dispersion function</u> $a(\phi)$ rather than a simple constant ϕ. This apparent complication provides an important extra degree of flexibility to model cases in which the Y_i are independent, but not identically distributed. The distributions of the Y_i have the same form, but not necessarily the same mean and variance.

We do not need to assume that every point in the historical sample of n observations has the same mean and variance. The mean μ_i is estimated as a function of a linear combination of predictors (covariates). The variance around this mean can also be a function of external information by making use of the dispersion function $a(\phi)$.

One way in which a model builder might make use of a dispersion function to help improve a model is to set $a(\phi)=\phi/w_i$, where ϕ is constant for all observations and w_i is a weight that may vary by observation. The values w_i are a priori weights based on external information that are selected in order to correct for unequal variances among the observations that would otherwise violate the assumption that ϕ is constant.

Now that we have seen how a non-constant dispersion function can be used to counteract non-constant variance in the response variable, we will assume that the weights are equal to unity, so that each observation is given equal weight.

The Variance Function Var(Y$_i$), and Uniqueness

Before looking at some specific distributions in the Natural Exponential Family, we define a uniqueness property of the variance structure in the natural exponential family. This property, presented concisely on page 51 of Jorgensen, states that the relationship between the variance and the mean (ignoring dispersion parameter ϕ) uniquely identifies the distribution.

In the notation of Appendix A, we write *Var(Y$_i$)* in terms of μ_i as $Var(Y_i) = a(\phi) \cdot V(\mu_i)$, so that the variance is composed of two components: one that depends on ϕ and external factors, and a second that relates the variance to the mean. The function $V(\mu)$, called the <u>unit variance function</u>, is what determines the form of a distribution, given that it is from the natural exponential family with parameters from a particular domain.

The upshot of this result is that, among continuous distributions in this family, $V(\mu) = 1$ implies we have a Normal with mean μ and variance $\phi = \sigma^2$, that $V(\mu) = \mu^2$ arises from a Gamma, and $V(\mu) = \mu^3$ from an Inverse Gaussian. For a discrete response, $V(\mu) = \mu$ means we have a Poisson

<u>Uniqueness Property:</u> *The unit variance function* $V(\mu)$ *uniquely identifies its parent distribution type within the natural exponential family.*

The implications of this Uniqueness Property are important for model design in GLM because it means that once we have defined a variance structure, we have specified the distribution form. Conversely, if a member of the Exponential Family is specified, the variance structure is also determined.

BASIC PROPERTIES OF SPECIFIC DISTRIBUTIONS

Our discussion of the natural exponential family will focus on five specific distributions:

- Normal (Gaussian)
- Poisson
- Gamma
- Inverse Gaussian
- Negative Binomial

The natural exponential family is broader than the specific distributions discussed here. It includes the Binomial, Logarithmic and Compound Poisson/Gamma (sometimes called "Tweedie" – see Appendix C) curves. The interested reader should refer to Jørgensen for details of additional members of the exponential family.

Many other distributions can be written in the general exponential form, if one allows for enough nuisance parameters. For instance, the Lognormal is seen to be a member of the general family by using $e(y) = \ln(y)$ instead of $e(y) = y$, but that excludes it from the natural exponential family. Using a Normal response variable in a GLM with a log link function applied to μ is quite different from applying a log transform to the response itself. The link function relates μ_i to the linear component; it does not apply to Y itself.

In the balance of this discussion, it is assumed that the variable Y is being modeled in currency units. The function $f(y)$ represents the probability or density function over a range of aggregate loss dollar amounts.

Appendix B gives "cheat sheet" summaries of the key characteristics of each distribution.

The Normal (Gaussian) Distribution

The Normal distribution occupies a central role in the historical development of statistics. Its familiar bell shape seems to crop up everywhere. Most linear regression theory depends on Normal approximations to the sampling distribution of estimators. Techniques used in parameter estimation, analysis of residuals, and testing model adequacy are guided largely by intuitions about the Normal curve and its properties.

The Normal has been criticized as a distribution for insurance losses because:

- Its range includes both negative and positive values.
- It is symmetrical, rather than skewed.
- The degree of dispersion supported by the Normal is quite limited.

Besides these criticisms, we should also note that a GLM with an unadjusted Normal response implies that the variance is constant, regardless of the expected loss volume. That is, if a portfolio with a mean of $1,000,000 has a standard deviation of $500,000, a larger portfolio with a $100,000,000 mean would have the same standard deviation.

A weighted dispersion function $a(\phi)=\phi/w_i$ can be used to provide more flexibility in adjusting for non-constant variance. The weights w_i can be set so that the variance for each predicted value μ_i is proportional to some exposure base such as on-level premium or revenue.

For the Normal distribution, this amounts to using weighted least squares. The parameters that minimize the sum of squares are equal to the parameters that maximize the likelihood. The least squares expression then becomes:

Ordinary Least Squares = $\sum (y_i - \mu_i)^2$

Weighted Least Squares = $\sum w_i \cdot (y_i - \mu_i)^2$

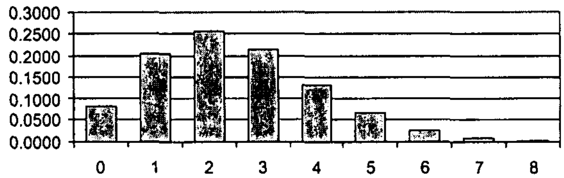
where w_i = 1 / Exposures for category i

Poisson and Over-Dispersed Poisson Distributions

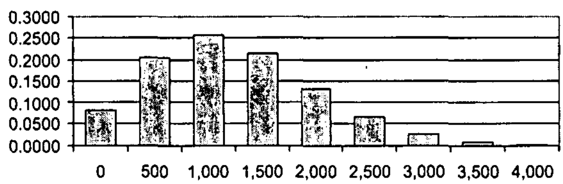
The Poisson distribution is a discrete distribution ranging over the non-negative integers. It has a mean equal to its variance.

The Over-Dispersed Poisson distribution is a generalization of the Poisson, in which the range is a constant ϕ times the positive integers. That is, the variable Y can take on values $\{0, 1\phi, 2\phi, 3\phi, 4\phi, ...\}$. It has a variance equal to ϕ times the mean.

Poisson Distribution



Over-Dispersed Poisson Distribution $\phi = 500$



The first important point to make concerning the Poisson is that, even though it is a discrete distribution, it can still be used as an approximation to a distribution of aggregate losses. There is no need to interpret the probabilities as anything other than a discretized version of an aggregate distribution. In fact, the Poisson immediately shows an advantage over the Normal:

- It is defined only over positive values
- It has positive skewness

An additional advantage of the Poisson is that it allows for a mass point at zero. The assumption that the ratio of the variance to the mean is constant is reasonable for insurance applications. Essentially, this means that when we add together independent random variables, we can add their means and variances. A very convenient property of the Over-Dispersed Poisson (ODP) is that the sum of ODP's that share a common scale parameter ϕ will also be ODP.

Gamma Distribution

The Gamma distribution is defined over positive values and has a positive skew. The probability density function, written in the natural exponential form, is:

$$f(y) = \exp\left[\alpha \cdot \left(\left(\frac{-y}{\mu}\right) - \ln(\mu)\right) + (\alpha - 1) \cdot \ln(\alpha \cdot y) + \ln\left(\frac{\alpha}{\Gamma(\alpha)}\right)\right]$$

From its form, we see that the Gamma belongs to one-parameter natural exponential family, but only if we assume that the shape parameter α is fixed and known By holding α constant, we treat the CV of the response variable as constant regardless of loss volume. As such, portfolios with expected losses of \$1,000,000 and \$100,000,000 would have the same CV. This seems unrealistic for many casualty insurance applications, although the Gamma may work well in high-volume lines of business, where GLM-based classification rating plans and bulk loss reserving models work best.

The Gamma distribution is closed under convolution in certain cases. When the PDF is written in the form below, the sum of two Gamma random variables $X_1 \sim Gamma(\alpha_1, \theta)$ and $X_2 \sim Gamma(\alpha_2, \theta)$ is also Gamma-distributed with $X_{1+2} \sim Gamma(\alpha_1 + \alpha_2, \theta)$, if they have a common θ. Unfortunately, we cannot capitalize on this property in GLM, since we require α to be constant and θ_i to vary.

$$f(y) = \frac{y^{\alpha-1}}{\theta^{\alpha} \cdot \Gamma(\alpha)} \cdot e^{-y/\theta}$$

Inverse Gaussian Distribution

The Inverse Gaussian distribution is occasionally recommended as a model for insurance losses, especially since its shape is very similar to the Lognormal.

The probability density function, written in the natural exponential form is:

$$f(y) = \exp\left[\left\{\left(\frac{-y}{2\mu^2}\right)+\left(\frac{1}{\mu}\right)\right\}\cdot\frac{1}{\phi}-\left(\frac{1}{2\phi y}+\ln\left(\sqrt{2\pi\phi y^3}\right)\right)\right].$$

In this form, the ϕ parameter is again treated as fixed and known. The variance is equal to $\phi\cdot\mu^3$. In other words, the variance is proportional to the mean loss amount cubed. This implies that the CV of a portfolio of losses would <u>increase</u> as the volume of loss increases, which is an unreasonable assumption for insurance phenomena.

The Inverse Gaussian distribution also has a practical difficulty that is worth noting. The difficulty is seen when the cumulative distribution function (CDF) is written:

$$F(y) = \text{NORMSDIST}\left(\frac{(y-\mu)}{\sqrt{y\cdot\mu}}\cdot\frac{1}{CV}\right)+\text{EXP}\left(\frac{2}{CV^2}\right)\cdot\text{NORMSDIST}\left(-\frac{(y+\mu)}{\sqrt{y\cdot\mu}}\cdot\frac{1}{CV}\right)$$

For small values of CV, this expression requires a very accurate evaluation for both EXP(·) and the tails of NORMSDIST(·) function. In practice, this represents a problem since commonly used software often does not provide values in the extreme tails.

The Negative Binomial Distribution

The Negative Binomial distribution, like the Poisson, is a discrete distribution that can be used to approximate aggregate loss dollars. As in the Over-Dispersed Poisson, we can add a scale parameter ϕ to increase the flexibility of the curve.

The Negative Binomial distribution has a variance function equal to:

$$Var(y) \quad = \quad \phi \cdot \mu + \frac{\phi}{k} \cdot \mu^2 \text{ with unit variance } V(\mu) = \mu \cdot \left(1 + \frac{\mu}{k}\right).$$

The variance can be interpreted as the sum of an unsystematic (or "random") component $\phi \cdot \mu$, and a systematic component $\frac{\phi}{k} \cdot \mu^2$. The inclusion of a systematic component implies that some relative variability, as measured by a coefficient of variation, remains even as the mean grows very large. That is,

$$\lim_{\mu \to \infty} CV \quad = \quad \lim_{\mu \to \infty} \frac{\sqrt{Var(y)}}{E[y]} \quad = \quad \lim_{\mu \to \infty} \sqrt{\frac{\phi}{\mu} + \frac{\phi}{k}} \quad = \quad \sqrt{\frac{\phi}{k}}.$$

We would expect the variance of a small portfolio of risks to be driven by random elements represented by the unsystematic component. As the portfolio grows by adding more and more similar risks, the variance would become dominated by the systematic component The parameter k can be interpreted as the expected size of loss μ for which the systematic and unsystematic components are equal.

Stated differently, the k parameter is a selected dollar amount. When the expected loss is below the amount k, the variance is closer to being proportional to the mean and the distribution starts to resemble the Poisson. When the expected loss is above the amount k, the variance is closer to being proportional to the mean squared and the distribution approaches a Gamma shape

This variance structure finds a close parallel to the concept of "mixing", as used in the Heckman-Meyers collective risk model. The unsystematic risk is then typically called the "process variance" and the systematic risk the "parameter variance".

$$Total\ Variance = \underbrace{E[Var(y)]}_{\substack{\text{Process}\\\text{Variance}}} + \underbrace{Var(\mu)}_{\substack{\text{Parameter}\\\text{Variance}}}$$

A practical calculation problem arises if we wish to simultaneously estimate the k and μ parameters. The k parameter is imbedded in a factorial function and is not independent of the scale parameter ϕ, as shown in the probability function below. Because of this complexity, the k will need to be set by the model user separately from the fit of μ. This can be repeated for different values, with a final selection made by the user.

$$\text{Prob}(Y = y) \quad = \quad \exp\left[\left(\ln\left(\frac{\mu}{\mu + k}\right)\cdot y + \ln\left(\frac{k}{\mu + k}\right) k\right)/\phi + \ln\begin{pmatrix}(k + y)/\phi - 1\\ y/\phi\end{pmatrix}\right]$$

The Lognormal Distribution – Not!

Because of its popularity in insurance applications, it is worthwhile to include a brief discussion of the Lognormal distribution.

The Lognormal distribution is a member of the general exponential family, but its density cannot be written in the natural form:

$$f(y) \quad = \quad \exp\left[(\mu \cdot \ln(y) - \mu^2/2)/\phi - \left(\frac{\ln(y)^2}{2\phi} + \ln\left(\sqrt{2\pi\phi}\right) + \ln(y)\right)\right].$$

To employ a Lognormal model for insurance losses Y, we apply a log transform to the observed values of the response, and fit a Normal distribution assumption to the transformed data. The response variable is therefore $\ln(Y)$ rather than Y.

While it initially seems attractive to be able to use the lognormal along with GLM theory, there are a number of problems with this approach. The first is purely practical. Since we are applying a logarithmic transform $\ln(y)$ to our observed y_i, any zero or negative values make the formula unworkable. One possible workaround is to add a constant amount to each y_i in order to ensure that the logarithms exist.

A second problem is that while the estimate of $\hat{\mu}_i$ (the mean of $\ln(y_j)$) will be unbiased, we cannot simply exponentiate it to estimate the mean of y_i in the original scale of dollars. A bias correction is needed on the GLM results.

A third potential problem arises from the fact that the lognormal model implicitly assumes, as does the Gamma, that all loss portfolios have the same CV. If we believe that the y_i come from distributions with identical CV's, then the GLM model with the Gamma assumption can be used as an alternative to the Lognormal model. This would allow us to steer clear of the first two problems.

HIGHER MOMENT PROPERTIES OF SPECIFIC DISTRIBUTIONS

Now that we have reviewed the basic properties for five specific members of the natural exponential family, including their variance structure, we will examine the overall shape of the curves being used.

Moments

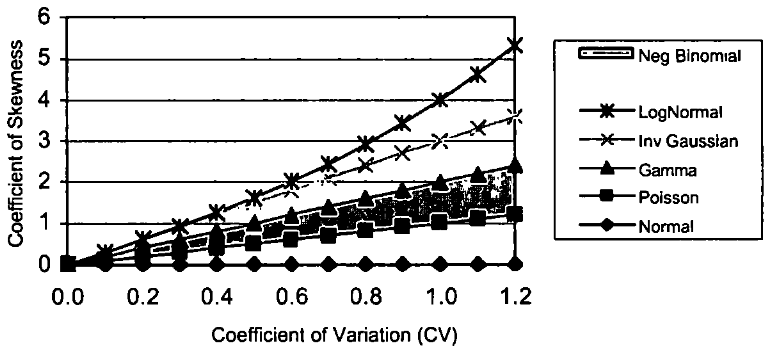
The variances for the natural exponential family members described in the previous section may be summarized as follows

Distribution	Variance
Normal	$Var(y) = \phi$
[Over-Dispersed] Poisson	$Var(y) = \phi \cdot \mu$ (constant V/M)
[Over-Dispersed] Negative Binomial	$Var(y) = \phi \cdot \mu + \frac{\phi}{k} \cdot \mu^2$
Gamma	$Var(y) = \phi \cdot \mu^2$ (constant CV)
Inverse Gaussian	$Var(y) = \phi \cdot \mu^3$

Two higher moments, representing skewness and kurtosis, can be represented in a similar sequence as functions of the CV.

	Skewness	Kurtosis
	$\frac{E\left[(Y-\mu)^3\right]}{Var(Y)^{3/2}}$	$\frac{E\left[(Y-\mu)^4\right]}{Var(Y)^{4/2}}$
Normal	0	3
Poisson	CV	$3 + CV^2$
Negative Binomial	$(2-p) \cdot CV$	$3 + \left(6 \cdot (1-p) + p^2\right) \cdot CV^2$
Gamma	$2 \cdot CV$	$3 + 6 \cdot CV^2$
Inverse Gaussian	$3 \cdot CV$	$3 + 15 \cdot CV^2$

The Negative Binomial distribution can be seen to represent values in the range between the Poisson and Gamma distributions, since $0 < p < 1$. The graph below shows the relationship between the CV and the skewness coefficient.



The Lognormal distribution is shown for comparison sake, and has a coefficient of skewness equal to $(3 + CV^2)\cdot CV$.

Measuring Tail Behavior: The Unit Hazard Function $h_w(y)$

In order to evaluate tail behavior of the curves in the exponential family, we will examine the hazard function $h_w(y)$, the average hazard rate over an interval of fixed width "w".

Unit Hazard Function

$$h_w(y) = \frac{F(y+w) - F(y)}{1 - F(y)} \quad \text{for continuous distributions, } w = \text{layer width}$$

$$h_w(y) = \frac{\Pr(y < Y \le y + w)}{\Pr(Y > y)} \quad \text{for discrete distributions, } w = \text{fixed integer.}$$

The more familiar hazard function $h(y) = f(y)/[1-F(y)]$ presented in Klugman [2003] is sometimes called the "failure rate", because it represents the conditional probability or density of a failure in a given instant of time, given that no failure has yet taken place. The unit hazard function measures the change in $F(y)$ over a small interval of width w, rather than a rate at a given instant in time.

The unit hazard function has a useful interpretation in insurance applications. It is roughly the probability of a partial limit loss in an excess layer. For example, in a layer of $10,000,000 excess $90,000,000, we seek the probability that a loss will not exceed $100,000,000, given that it is in the layer. A high value for $h_w(y)$ would mean that a loss above $90,000,000 would be unlikely to exhaust the full $10,000,000 layer

For most insurance applications, we would expect a decreasing unit hazard function. That is, as we move to higher and higher layers, the chance of a partial loss would decrease. For instance, if we consider a layer such as $10,000,000 xs $990,000,000 we would expect that any loss above $990,000,000 would almost certainly be a full-limit loss. This would imply $h_w(y) \to 0$.

The decreasing hazard function is not what we generally find in the exponential family. For the Normal and Poisson, the hazard function approaches 1, implying that full-limit losses become less likely on higher layers – exactly the opposite of what our understanding of insurance phenomena would suggest. The Negative Binomial, Gamma and Inverse Gaussian distributions asymptotically approach constant amounts, mimicking the behavior of the exponential distribution

The table below shows the asymptotic behavior as we move to higher attachment points for a layer of width w.

Distribution	Limiting Form of $h_w(y)$	Comments
Normal	$\lim_{y \to \infty} h_w(y) = 1$	No loss exhausts the limit
Poisson	$\lim_{y \to \infty} h_w(y) = 1$	
Negative Binomial	$\lim_{y \to \infty} h_w(y) = 1-(1-p)^w$	
Gamma	$\lim_{y \to \infty} h_w(y) = 1-e^{-w\alpha \cdot \mu}$	
Inverse Gaussian	$\lim_{y \to \infty} h_w(y) = 1-e^{-w \cdot (2\phi\mu^2)}$	
Lognormal	$\lim_{y \to \infty} h_w(y) = 0$	Every loss is a full-limit loss

From this table, we see that the members of the natural exponential family have tail behavior that does not fully reflect the potential for extreme events in heavy casualty insurance. It would seem that the natural exponential distributions used with GLM are more appropriate for insurance lines without much potential for extreme events or natural catastrophes.

SMALL SAMPLE ISSUES

The results calculated in Generalized Linear Models generally rely on asymptotic behavior assuming a large number of observations are available. Unfortunately, this is not always the case in Property & Casualty insurance. For instance, in per-risk or per-occurrence excess of loss reinsurance, there may not be a large enough volume of losses to rely upon asymptotic approximations.

While we include here a brief discussion of the uncertainty in our parameter estimates, this is an area in which much more research is needed.

Including Uncertainty in the Mean μ

Most of our discussion of the exponential family has focused on the distribution of future losses around an estimated mean μ. However, the actuary is more often asked to provide a confidence interval around the estimated value of the mean $\hat{\mu}$. The estimate $\hat{\mu}$ is also a random variable, with a mean, variance and higher moments. However, GLM models generally produce an approximation to this distribution by making use of the asymptotic behavior of the coefficients $\hat{\beta}$ in the linear predictor being Normal.

The calculation of the variance in the parameter estimates, which leads to the confidence interval around the estimated mean $\hat{\mu}$, is accomplished using the matrix of second derivatives of the loglikelihood function. A comprehensive discussion of that calculation can be found in McCullagh & Nelder or Klugman [1998].

In general, the distribution of the estimator $\hat{\mu}$ will not be the same exponential family form as that of Y. In other words, the process and parameter variances are variances of different distribution forms. As a practical solution, the actuary will want to select a reasonable curve form (e.g., a gamma or lognormal) with mean and variance that match the estimated $\hat{\mu}$ and $Var(\hat{\mu})$ from the model.

Including Uncertainty in the Dispersion ϕ

In all of the discussion to this point, the dispersion parameter ϕ has been assumed to be fixed and known. It is estimated as a side calculation, separate from the estimate of the parameters $\hat{\beta}$ used to estimate the mean $\hat{\mu}$.

So long as the separate estimate of the dispersion parameter is based on a large number of observations, this approximation is reasonable A problem arises in certain insurance applications where there are relatively few observations, and our estimate of the dispersion is far from certain.

In normal linear regression, the uncertainty in the dispersion parameter (σ^2 instead of ϕ) is modeled by using a Student-t distribution rather than a Normal distribution. The use of a Student-t distribution is equivalent to an assumption that the parameter σ^2 (or ϕ) is distributed as Inverse Gamma with a shape parameter equal to its degrees of freedom ν. That is:

$$g(\phi) = \frac{\lambda^{\frac{\nu}{2}} \cdot e^{-\lambda/\phi}}{\phi^{\frac{\nu}{2}+1} \cdot \Gamma(\nu/2)} \qquad E[\phi^k] = \lambda^k \frac{\Gamma(\nu/2-1)}{\Gamma(\nu/2)}, \text{ for } 2k < \nu,$$

where ν = degrees of freedom.

A similar "mixing" of the dispersion parameter can be made for curves other than the Normal. It is not always easy to explicitly calculate the mixed distribution, but the moments can be found with the formula above.

For calculation purposes, if the distribution is used in a simulation model, the mixing can be accomplished in a two-step process. First we simulate a value for ϕ from an Inverse Gamma distribution. Second we simulate a value from the loss distribution conditional on the simulated ϕ.

The real difficulty with the uncertainty in the dispersion parameter is that it has a significant effect on the higher moments on the distribution, and therefore on the tail – the part of the distribution where the actuary may have the greatest concern. As the formula for the moments of the Inverse Gamma shows, many of the higher moments will not exist.

Another important note on the uncertainty in the dispersion parameter relates to the use of the Lognormal distribution. When the log transform is applied to the observed data in order to use linear regression, we have uncertainty in the dispersion of the logarithms $\ln(y_i)$. When the transformed data $\ln(y_i)$ has a Student-t distribution, the untransformed data y_i follows a Log-T distribution The Log-T has been recommended by Kreps and Murphy for use in estimating confidence intervals in reserving applications.

What neither author noted, however, is that none of the moments of the Log-T distribution exists. We are able to calculate percentiles, but not a "confidence interval" around the mean, because the mean itself does not exist.

CONCLUSIONS

The use of the Natural Exponential Family of distributions in GLM allows for more realistic variance structures to be used in modeling insurance phenomena This is a real advance beyond linear regression models, which are restricted to the Normal distribution.

The Natural Exponential Family also allows the actuary to work directly with their loss data in units of dollars, without the need for logarithmic or other transformations.

However, these advantages do not mean that GLM has resolved all issues for actuarial modeling. The curve forms are generally thin-tailed distributions and should be used with caution in insurance applications with potential for extreme events, or with a small sample of historical data.

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Appendix A: Deriving Moments for the Natural Exponential Family

As stated in this paper, the probability density function *f(y)* for the natural exponential family is given by:

$$f(y;\theta,\phi) = \exp[\,(\theta \cdot y - b(\theta))/a(\phi) + c(y,\phi)]$$

In the natural form, a, b, c are suitable known functions, θ is the <u>canonical parameter</u> for Y, and ϕ is the <u>dispersion parameter</u>. The <u>unit cumulant function</u> $b(\theta)$, which is useful in computing moments of Y, does not depend on y or ϕ. Likewise, the <u>dispersion function</u> $a(\phi)$ does not depend on y or θ. The catch-all function $c(y,\ \phi)$ has no dependence on θ.

The unit cumulant function $b(\theta)$ is so named because it can be used to calculate *cumulants*, which are directly related to the random variable's moments

We recall from Statistics that the Moment Generating Function $MGF(t)$ is defined as:

$$MGF(t) \quad = \quad \int_{-\infty}^{\infty} e^{ty} \cdot f(y)\,dy \qquad\qquad \text{for continuous variables}$$

and that

$$E[Y^r] \quad = \quad \left.\frac{\partial^r MGF(t)}{\partial t^r}\right|_{t=0}.$$

The Cumulant Generating Function $K(t)$ is defined as $\ln[MGF(t)]$, and the cumulants:

$$\kappa_r \quad = \quad \left.\frac{\partial^r K(t)}{\partial t^r}\right|_{t=0}.$$

There is an easy mapping between the first four cumulants and the moments:

$$\kappa_1 \;=\; E[y^1] \;=\; \mu \qquad\qquad \kappa_3 \;=\; E[(y-\mu)^3]$$

$$\kappa_2 \;=\; E[(y-\mu)^2] \;=\; Var(y) \qquad\qquad \kappa_4 \;=\; E[(y-\mu)^4] - 3\cdot Var(y)^2$$

For the Natural Exponential Family, the Cumulant Generating Function can be written in a very convenient form:

$$K(t) \quad = \quad \frac{b(\theta + a(\phi)\cdot t) - b(\theta)}{a(\phi)} \text{, so that}$$

$$\kappa_r \quad = \quad b^{(r)}(\theta)\cdot a(\phi)^{r-1} \qquad \text{where } b^{(r)}(\theta) \quad = \quad \frac{\partial^r b(\theta)}{\partial \theta^r}.$$

In the mean value form, where $\theta = \tau^{-1}(\mu)$, the chain rule is used to find derivatives in terms of μ. The function $b''(\theta)$ is the unit variance function, denoted $V(\mu)$ when expressed in terms of μ.

Mean $E[Y;\theta] = b'(\theta) = \mu$

Variance $Var[Y;\theta] = b''(\theta)\cdot a(\phi) = V(\mu)\cdot a(\phi)$

$$Skewness = \frac{b^{(3)}(\theta)\cdot a(\phi)^2}{Var[Y;\theta]^{3/2}} = \frac{d}{d\mu}[V(\mu)]\cdot\sqrt{\frac{a(\phi)}{V(\mu)}}$$

$$Kurtosis = 3 + \frac{b^{(4)}(\theta)\cdot a(\phi)^3}{Var[Y;\theta]^2} = 3 + \left[\frac{d^2}{d\mu^2}[V(\mu)]\ V(\mu) + \left(\frac{d}{d\mu}V(\mu)\right)^2\right]\cdot\frac{a(\phi)}{V(\mu)}$$

Appendix B1: Normal Distribution

Density Function:

$$f(y) = \frac{1}{\sqrt{2\pi\phi}} \cdot \exp\left(\frac{-(y-\mu)^2}{2\phi}\right)$$

$$y \in (-\infty, \infty)$$

Natural Form:

$$f(y) = \exp\left[\left(\mu\ y - \mu^2/2\right)/\phi - \left(\frac{y^2}{2\phi} + \ln\left(\sqrt{2\pi\phi}\right)\right)\right]$$

Cumulative Distribution Function in Excel® Notation:

$$F(y) = \mathrm{NORMDIST}\left(y, \mu, \sqrt{\phi}, 1\right)$$

Moments:

$$E[Y] = \mu$$

$$Var(Y) = \phi$$

$$Skewness = \frac{E\left[(Y-\mu)^3\right]}{Var(Y)^{3/2}} = 0$$

$$Kurtosis = \frac{E\left[(Y-\mu)^4\right]}{Var(Y)^{4/2}} = 3$$

Convolution of independent Normal random variables:

$$N_x(\mu_x, \phi_x) \otimes N_y(\mu_y, \phi_y) \Rightarrow N_{x+y}(\mu_x + \mu_y, \phi_x + \phi_y)$$

Appendix B2: Over-Dispersed Poisson

Probability Function: $\text{Prob}(Y = y) = \left(\dfrac{\mu}{\phi}\right)^{y/\phi} \dfrac{e^{-\mu/\phi}}{(y/\phi)!}$

$$y \in (0,\ 1\phi,\ 2\phi,\ 3\phi,\ 4\phi,\ \ldots)$$

Natural Form $\text{Prob}(Y = y) = \exp\left[(\ln(\mu)\ y - \mu)/\phi - y \cdot \ln(\phi)/\phi - \ln((y/\phi)!)\right]$

Cumulative Distribution Function in Excel® Notation:

$$\text{Prob}(Y \le y) = 1 - \text{GAMMADIST}\left(\frac{\mu}{\phi}, \frac{y}{\phi} + 1, 1, 1\right)$$

Moments: $E[Y] = \mu$

$$Var(Y) = \phi\ \mu \qquad CV = \sqrt{\frac{\phi}{\mu}}$$

$$Skewness = \frac{E\left[(Y-\mu)^3\right]}{Var(Y)^{3/2}} = \sqrt{\frac{\phi}{\mu}} = CV$$

$$Kurtosis = \frac{E\left[(Y-\mu)^4\right]}{Var(Y)^{4/2}} = 3 + CV^2$$

Convolution of independent Over-Dispersed Poisson random variables:

$$ODP_x(\mu_x, \phi) \otimes ODP_y(\mu_y, \phi) \Rightarrow ODP_{x+y}(\mu_x + \mu_y, \phi)$$

where ϕ is a constant variance/mean ratio

Appendix B3: Gamma

Density Function: $$f(y) = \left(\frac{y\,\alpha}{\mu}\right)^{\alpha}\left(\frac{1}{y}\right)\cdot\frac{e^{-y\alpha/\mu}}{\Gamma(\alpha)}$$

$$y \in (0, \infty)$$

Natural Form: $$f(y) = \exp\left[\alpha\cdot\left(\left(\frac{-y}{\mu}\right)-\ln(\mu)\right)+(\alpha-1)\,\ln(\alpha\cdot y)+\ln\left(\frac{\alpha}{\Gamma(\alpha)}\right)\right]$$

Cumulative Distribution Function in Excel® Notation:

$$F(y) = \text{GAMMADIST}\left(\frac{y\cdot\alpha}{\mu}, \alpha, 1, 1\right)$$

Moments:

$$E[Y] = \mu$$

$$Var(Y) = \frac{\mu^2}{\alpha} \qquad CV = \sqrt{\frac{1}{\alpha}}$$

$$Skewness = \frac{E\left[(Y-\mu)^3\right]}{Var(Y)^{3/2}} = \frac{2}{\sqrt{\alpha}} = 2\cdot CV$$

$$Kurtosis = \frac{E\left[(Y-\mu)^4\right]}{Var(Y)^{4/2}} = 3+6\cdot CV^2$$

Convolution of independent Gamma random variables:

$$G_x(\mu_x, \alpha_x = \mu_x/\beta)\otimes G_y(\mu_y, \alpha_y = \mu_y/\beta) \Rightarrow G_{x+y}(\mu_x+\mu_y, \alpha_x+\alpha_y)$$

where β is a constant variance/mean ratio

Appendix B4: Inverse Gaussian

Density Function. $$f(y) = \frac{1}{\sqrt{2\pi\phi y^3}} \cdot \exp\left(\frac{-(y-\mu)^2}{2\phi\mu^2 y}\right)$$

$$y \in (0, \infty)$$

Natural Form: $$f(y) = \exp\left[\left\{\left(\frac{-y}{2\mu^2}\right) + \left(\frac{1}{\mu}\right)\right\} \cdot \frac{1}{\phi} - \left(\frac{1}{2\phi y} + \ln\left(\sqrt{2\pi\phi y^3}\right)\right)\right]$$

Cumulative Distribution Function in Excel® Notation:

$$F(y) = \text{NORMSDIST}\left(\frac{(y-\mu)}{\mu \cdot \sqrt{\phi \cdot y}}\right) + \text{EXP}\left(\frac{2}{\phi \cdot \mu}\right) \cdot \text{NORMSDIST}\left(-\frac{(y+\mu)}{\mu \cdot \sqrt{\phi \cdot y}}\right)$$

Moments: $$E[Y] = \mu$$

$$Var(Y) = \phi \cdot \mu^3 \qquad CV = \sqrt{\phi \cdot \mu}$$

$$Skewness = \frac{E\left[(Y-\mu)^3\right]}{Var(Y)^{3/2}} = 3 \cdot \sqrt{\phi \cdot \mu} = 3 \cdot CV$$

$$Kurtosis = \frac{E\left[(Y-\mu)^4\right]}{Var(Y)^{4/2}} = 3 + 15 \cdot CV^2$$

Convolution of independent Inverse Gaussian random variables:

$$IG_x\left(\mu_x, \phi_x = \beta / \mu_x^2\right) \otimes IG_y\left(\mu_y, \phi_y = \beta / \mu_y^2\right) \Rightarrow IG_{x+y}\left(\mu_x + \mu_y, \phi_{x+y} = \beta / (\mu_x + \mu_y)^2\right)$$

where β is a constant variance/mean ratio

Appendix B5: [Over-Dispersed] Negative Binomial

Probability Function:

$$\text{Prob}(Y=y) = \binom{(k+y)/\phi - 1}{y/\phi} \cdot p^{k/\phi} \cdot (1-p)^{y/\phi}$$

$$y \in (0,\ 1\phi,\ 2\phi,\ 3\phi,\ 4\phi,\ \ldots)$$

Natural Form:

$$\text{Prob}(Y=y) = \exp\left[\left(\ln\left(\frac{\mu}{\mu+k}\right)\cdot y + \ln\left(\frac{k}{\mu+k}\right)\cdot k\right)/\phi + \ln\binom{(k+y)/\phi - 1}{y/\phi}\right]$$

Cumulative Distribution Function in Excel® Notation:

$$\text{Prob}(Y \le y) = \text{BETADIST}\left(\frac{k}{\mu+k}, \frac{\mu}{\phi}, \frac{y}{\phi}+1\right)$$

Moments.

$$E[Y] = k\cdot\frac{(1-p)}{p} = \mu \qquad \text{so } p = \frac{k}{\mu+k}$$

$$Var(Y) = \phi\cdot k\cdot\frac{(1-p)}{p^2} = \phi\cdot\mu + \frac{\phi}{k}\cdot\mu^2$$

$$CV = \sqrt{\frac{\phi}{\mu}+\frac{\phi}{k}}$$

$$Skewness = \frac{E\left[(Y-\mu)^3\right]}{Var(Y)^{3/2}} = (2-p)\ CV$$

$$Kurtosis = \frac{E\left[(Y-\mu)^4\right]}{Var(Y)^{4/2}} = 3 + \left(6\ (1-p)+p^2\right)\cdot CV^2$$

Convolution of independent Over-Dispersed Negative Binomial random variables:

$$NB_x(\mu_x,\phi,p) \otimes NB_y(\mu_y,\phi,p) \Rightarrow NB_{x+y}(\mu_x+\mu_y,\phi,p)$$

Appendix C: Compound Poisson/Gamma (Tweedie) Distribution

The Tweedie distribution can be interpreted as a collective risk model with a Poisson frequency and a Gamma severity.

Probability Function:

$$f(y \mid \lambda, \theta, \alpha) = \begin{cases} e^{-\lambda} & y = 0 \\ \sum_{k=1}^{\infty} \underbrace{\frac{\lambda^k e^{-\lambda}}{k!}}_{\text{Poisson}} \cdot \underbrace{\frac{y^{k\alpha-1} e^{-y/\theta}}{\theta^{k\alpha} \cdot \Gamma(k\alpha)}}_{\text{Gamma}} & y > 0 \end{cases} \qquad y \in (0, \infty)$$

This form appears complicated, but can be re-parameterized to follow the natural exponential family form.

We set: $\alpha = \dfrac{2-p}{p-1}$ $\lambda = \dfrac{\mu^{2-p}}{\phi \cdot (2-p)}$ $\theta = \phi \cdot (p-1) \cdot \mu^{p-1}$

and $1 < p < 2$, since $p = \dfrac{\alpha+2}{\alpha+1}$ and $\alpha > 0$

$$f(y \mid \mu, \phi, p) = \exp\left[\left(\frac{\mu^{2-p}}{(2-p)} + \frac{y}{(p-1) \cdot \mu^{p-1}}\right) \cdot \frac{-1}{\phi}\right] c(y, \phi)$$

where

$$c(y, \phi) = \begin{cases} 1 & y = 0 \\ \sum_{k=1}^{\infty} \dfrac{y^{k(2-p)/(p-1)-1}}{[\phi \cdot (2-p)]^k \cdot [\phi \cdot (p-1)]^{k(2-p)/(p-1)} \cdot \Gamma(k(2-p)/(p-1)) \cdot k!} & y > 0 \end{cases}$$

The density function $f(y \mid \mu, \phi, p)$ can then be seen to follow the "natural form" for the exponential family.

Moments: $E[Y] \;=\; \lambda \cdot \theta \cdot \alpha \;=\; \mu$

$$Var(Y) \;=\; \lambda \cdot \theta^2 \; \alpha \; (\alpha + 1) \;=\; \phi \; \mu^p$$

$$CV \;=\; \sqrt{\frac{1}{\lambda} + \frac{1}{\lambda\alpha}} \;=\; \sqrt{\frac{\phi}{\mu^{2-p}}}$$

$$Skewness \;=\; \frac{E\left[(Y-\mu)^3\right]}{Var(Y)^{3/2}} \;=\; \frac{\lambda \cdot \theta^3 \cdot \alpha \cdot (\alpha+1) \cdot (\alpha+2)}{\left(\lambda \cdot \theta^2 \cdot \alpha \cdot (\alpha+1)\right)^{3/2}} \;=\; p \cdot CV$$

$$Kurtosis \;=\; \frac{E\left[(Y-\mu)^4\right]}{Var(Y)^{4/2}} \;=\; 3 + p \cdot (2p-1) \; CV^2$$

For GLM, a p value in the (1, 2) range must be selected by the user. The mean μ and dispersion ϕ are then estimated by the model.

The Compound Poisson/Gamma is a continuous distribution, with a mass point at zero. The evaluation of the cumulative distribution function (CDF) is somewhat inconvenient, but can be accomplished using any of the collective risk models available to actuaries.

Finally, we may note that the convolution of independent Tweedie random variables:

$$TW_x(\lambda_x, \theta, \alpha) \otimes TW_y(\lambda_y, \theta, \alpha) \;\Rightarrow\; TW_{x+y}(\lambda_x + \lambda_y, \theta, \alpha)$$

Severity Distributions for GLMs: Gamma or Lognormal? Evidence from Monte Carlo Simulations

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Severity Distributions for GLMs: Gamma or Lognormal? Evidence from Monte Carlo Simulations

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Abstract

Insurance claim costs have been found in numerous studies to be positive and usually positively skewed with variances often proportional to the mean squared. In practice, the gamma and lognormal distributions are the ones with those desired properties most widely used. Most actuarial research in GLMs also report results from normal distributions as a comparison. In this study, we apply Monte Carlo simulation techniques to examine the unbiasedness and stability of the GLM classification relativities assuming gamma, lognormal, and normal distributions. We find that the gamma distribution provides better predictive accuracy and efficiency.

1. Introduction

Generalized Linear Models (GLMs) have been widely used in property-casualty ratemaking recently because they consider all rating factors simultaneously and adjust for interactions and correlations among them. Numerous studies, such as Brown (1988), Holler, Sommer, and Geoff (1999), Mildenhall (1999), and Murphy, Brockman, and Lee (2000) have shown that one-way analysis leads to systematic bias and that GLMs can be used to calculate classification relativities and to reduce estimation errors.

Traditional linear models assume independent and identical normally distributed residuals. Insurance data, such as losses and severities are positive and usually positively skewed. GLMs assume that data is sampled from an exponential family of distributions[1]. Many distributions in this family (e.g., gamma, inverse Gaussian, and negative binomial) are consistent with the nature of insurance data (positive and positively skewed). In practice, GLMs with gamma and normal distributions are usually used for severity relativity calculations. Most empirical analyses of GLMs in actuarial research report results from gamma and normal distributions.

Besides the gamma distribution, the lognormal is the other widely used distribution with the desired characteristics of insurance data. Mildenhall (1999) discussed that the logarithm of the response variable, the variance-stabilizing transformation, is often used

[1] The exponential family of distributions has the probability density function for continuous variables or the probability function for discrete variables in the form of $f(x;\theta,\phi)=\exp\{\frac{x\theta-b(\theta)}{a(\phi)}+c(x,\phi)\}$. The lognormal, Pareto and Weibull distributions are not in the exponential family.

in linear models to improve normality. The log-transformed model also converts multiplicative models to linear ones. The underlying assumption of the log-transformed model is that the response variable follows a lognormal distribution.

McCullagh and Nelder (1989) contend that it is common for data "in the form of continuous measurements" to have variance positively correlated with the mean. So, the constant coefficient of variation (mean / standard deviation) is a more realistic assumption than constant variance. This property of constant coefficient of variation is found to be appropriate for insurance data by Murphy, Brockman, and Lee (2000) and Mildenhall (1999). The former study also shows that the correct selection of the non-constant variance function significantly improves the robustness of parameter estimates.

Both gamma and lognormal distributions have the property of constant coefficient of variation. A gamma distribution with parameters α and θ has the following density function:

$$G(x;\alpha,\theta) = \frac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)\theta^{\alpha}}, \qquad (1)$$

where $\Gamma(\alpha)$ is the gamma function. $G(x,\alpha,\theta)$ has mean $\mu = \alpha\theta$ and variance $\sigma^2 = \alpha\theta^2$. So, the gamma distribution has its variance proportional to its mean squared, i.e., $\sigma^2 = \frac{1}{\alpha}\mu^2$. The lognormal distribution with parameters M and Ω has density:

$$L(x;M,\Omega) = \frac{1}{\Omega x\sqrt{2\pi}} e^{-(\ln x - M)^2/2\Omega^2}. \qquad (2)$$

$L(x,M,\Omega)$ has mean $\mu = e^{M+\Omega^2/2}$ and variance $\sigma^2 = e^{2M+\Omega^2}(e^{\Omega^2}-1)$. The variance of the lognormal distribution is also proportional to its mean squared, i.e.,

$$\sigma^2 = (e^{\Omega^2}-1)\mu^2.$$

In this study, Monte Carlo simulation analysis is applied to investigate the following questions:

1. Under what conditions are the assumed severity distributions important?
2. If the severity distribution is unknown or difficult to test, which of the gamma, lognormal or normal distribution assumptions yield the most robust result (i.e., minimized estimation bias and standard error)?

The individual losses are generated randomly from gamma and lognormal distributions. For each simulated dataset, three models are fit: GLM with a gamma distribution and log link; GLM with a normal distribution and log link; and GLM on the log-transformed severity with a normal distribution and identity link.

This paper assumes the reader is familiar with basic class ratemaking and the fundamentals of GLMs. It is organized as follows. Section 2 discusses the details of our Simulation Methodology, and Section 3 reviews Simulation Results. Section 4 outlines our Conclusions, and Section 5 provides ideas for Future Research.

2. Simulation Methodology

Simulation Assumptions

The simulation is numerically based on the predicted claim severity for private passenger auto collision claims adjusted for severity trend. The data is from Mildenhall (1999) and McCullagh and Nelder (1989)[2], and it includes thirty-two severity observations for two classification variables: eight age groups and four types of vehicle-use (which are based on 8,942 individual claims). In this study, the response variable is the average claim severity and the simulation assumptions include the following:

1). The individual losses have a constant coefficient of variation. Let $L_{i,j}^k$ denote the loss for claim k with age group i and vehicle-use group j. We assume that $L_{i,j}^k$ are independently distributed with mean $E(L_{i,j}^k) = \mu_{i,j}$ and variance $\text{var}(L_{i,j}^k) = \sigma_{i,j}^2 = (\mu_{i,j} * cv)^2$, where cv is the coefficient of variation. Mildenhall (1999) and McCullagh and Nelder (1989) discuss that the assumption of constant variance is unrealistic and the standard deviation of severity is more likely to be positively correlated with the mean severity. Following their research, we assume a constant coefficient of variation (rather than constant variance) in the simulation. The average claim severity $S_{i,j} = \frac{1}{n_{i,j}} \sum_{k=1}^{n_{i,j}} L_{i,j}^k$, where $n_{i,j}$ is the total claims for age group i and vehicle-use group j, while $S_{i,j}$ has mean $\mu_{i,j}$ and variance $\frac{\sigma_{i,j}^2}{n_{i,j}}$.

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[2] We used this dataset, as it was used in these two authoritative studies.

2). The relationship between severities and rating variables is multiplicative. The mean severity equals:

$$\mu_{i,j} = Base * \mathrm{Rel}_i * \mathrm{Rel}_j, \qquad (3)$$

where Rel_i is the severity relativity for age group i and Rel_j is the relativity for vehicle-use group j. This is equivalent to a generalized linear model with log link function:

$$\log(\mu_{i,j}) = Intercept + \sum_i a_i x_i + \sum_j b_j y_j, \qquad (4)$$

where x_i is the dummy variable for age group i ($x_i = 1$ if age is i; 0 otherwise) and y_j is the dummy variable for vehicle-use group j; a_i is the GLM coefficient for x_i and b_j is the coefficient for y_j.

3). The "true" base severity and relativities Rel_i and Rel_j (or GLM coefficients a_i and b_i) distributions are known. Those "true" values are the predicted GLM values in Mildenhall (1999) based on the gamma distribution and log links[3]. The age group "60+" and vehicle-use group "for pleasure" are selected as the base[4], and the base severity is 195. Table 1.1 and Table 1.2 list the "true" relativities for age groups and vehicle-use groups, respectively. Table 1.3 shows the "true" severities and relativities for each combined age and vehicle-use group.

4). The coefficients of variation of the severity distribution are known. We set the coefficients of variation to be 1.0, 2.0 or 3.0[5].

[3] The conclusions in this study are still valid if we use the predicted GLM severities from other distributions in Mildenhall (1999).
[4] The selection of the base does not affect the GLM results numerically.
[5] The data used by Mildenhall (1999) is average claim severity, and the individual claim information is not available. We calculated sixty-five severity coefficients of variation from the major coverages in fifteen states from our company data. The average of variation coefficients was 1.434, the minimum 0.529, the

Distribution Parameters

From the simulation assumptions, the losses $L_{i,j}^k$ follow distributions with means of $\mu_{i,j}$ and variances of $(\mu_{i,j} * cv)^2$. The parameters of the gamma and lognormal distributions can be calculated based on the assumed mean and variance. The gamma distribution $G(x,\alpha,\theta)$ has mean $\alpha\theta$ and variance $\alpha\theta^2$. This implies $L_{i,j}^k$ has gamma parameters of $\alpha = \frac{1}{cv^2}$ and $\theta = \mu_{i,j} * cv^2$. Similarly, the lognormal distribution $L(x,M,\Omega)$ has mean $= e^{M+\Omega^2/2}$ and variance $= e^{2M+\Omega^2}(e^{\Omega^2}-1)$. This implies that $L_{i,j}^k$ has lognormal parameters of $M = \log(\mu_{i,j}) - \log(1+cv^2)/2$ and $\Omega^2 = \log(1+cv^2)$. In other words, $\log(L_{i,j}^k)$ is normally distributed with mean M and variance Ω^2.

Simulation Procedure

1). For each coefficient of variation, the individual losses are generated based on lognormal and gamma distributions. For each combined age group i and vehicle-use group j, $n_{i,j}$ individual losses of $L_{i,j}^k$ are simulated. The average claim severity is calculated as the mean of those individual losses. In total, 8,942 individual losses and thirty-two claim severities $S_{i,j}$ are generated in each round of simulation.

2). For each dataset, three models are fitted: GLM with gamma distribution and log link; GLM with normal distribution and log link; and GLM on the log-transformed

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maximum 3.464, and the standard deviation 0.516. Fifty-seven of the sixty-five coefficients of variation were within the interval [1.0, 3.0].

severities with normal distribution and identity link (the "log-transformed linear model" in Mildenhall 1999). In each regression, we calculate thirteen coefficients (one intercept, eight age groups with 60+ as zero, four vehicle-use with pleasure as zero), seven age relativities, and three vehicle-use relativities. We also calculate thirty-two predicted severities for each combined age and vehicle-use group. Following Mildenhall (1999), we use severity as the response variable and the claim frequencies as the weights in the linear regressions.

$$\log(S_{i,j}) = Intercept + \sum a_i x_i + \sum b_j y_j + \varepsilon_{i,j}, \quad (5)$$

If gamma or normal distributions are assumed, the severity based on a GLM with a log link is:

$$S_{i,j} = e^{\text{intercept}} * e^{a_i + b_j}, \quad (6)$$

For lognormally distributed losses, Klugman, Panjer, and Willmot (1998) and Mildenhall (1999) show that the severity based on log-transformed regression is:

$$S_{i,j} = e^{\text{intercept}} * e^{a_i + b_j + \frac{\Omega_{i,j}^2}{2n_{i,j}}}, \quad (7)$$

where $\Omega_{i,j}^2$ is the variance of the logarithm of the individual loss. We designate $e^{\Omega_{i,j}^2 / 2n_{i,j}}$ as the volatility adjustment factor. In the numerical analysis, $\Omega_{i,j}^2$ is estimated by $\Omega_{i,j}^2 = \log(1 + cv^2)$ for the coefficients of variation 1.0, 2.0, and 3.0, respectively.

3). Steps 1-2 are repeated one thousand times, so the sampling distributions of ten coefficients ($\hat{a}_i$ and $\hat{b}_j$) and thirty-two predicted severities ($\hat{S}_{i,j}$) are generated. The mean and standard error of the coefficients and predicted severities are calculated based on the sampling distributions.

For each combination of the "true" severity distribution and the assumed distribution in the regressions, we evaluate the performance of the models by two criteria. Following Bailey (1963), we use the weighted absolute bias to measure the accuracy of the model. We also evaluate the model from an alternative perspective: the stability of the coefficients and predicted values, which is measured by the weighted standard error.

From the definition of unbiasedness, the mean estimate is equal to the true value, $E(\hat{S}_{i,j}) = \mu_{i,j}$. If the sampling mean of one thousand predicted $\hat{S}_{i,j}$ is equal to the "true" severity and the model is unbiased, then the estimation bias could be measured by $E(\hat{S}_{i,j}) - \mu_{i,j}$. Bailey (1963) suggests using the weighted absolute bias to measure the accuracy of the model:

$$wab = \frac{\sum w_{i,j} \mid E(\hat{S}_{i,j}) - \mu_{i,j} \mid}{\sum w_{i,j}}. \qquad (8)$$

Besides unbiasedness, stability is the other important criteria to measure model performance. The standard error is a commonly used statistic for stability. Similar to (8), we use the weighted standard error to measure the stability of the model:

$$wse = \frac{\sum w_{i,j} \hat{\sigma}_{i,j}}{\sum w_{i,j}}, \qquad (9)$$

where $\hat{\sigma}_{i,j}$ is the sampling standard deviation of one thousand $\hat{S}_{i,j}$.

3. Simulation Results

Data Generated

We repeat the simulation 1,000 times, and 8,942 individual losses are generated in each round of the simulation. As it is inefficient to list all individual losses for all combined groups of age and vehicle-use, we only report the details of simulated average claim severity for two classifications.

Classification I - Age 17-20 and Pleasure Use

Classification II - Age 40-49 and DTW Short (Short Drive to Work, less than 15 miles)

Classification I is used as an example of a small-sample classification (as it only has twenty-one observations. Classification II includes 970 observations and is an example of a large-sample classification. Tables 2.1 and 2.2 show statistical summaries of the 1,000 simulated severities for Classification I with coefficients of variation 1.0, 2.0, and 3.0 for the gamma and lognormal distributions, respectively. Tables 2.3 and 2.4 show statistical summaries for Classification II. Figures 3.1, 3.3, and 3.5 report the scatter plot, density plot, QQ plot, and histogram for gamma distributions with coefficients of variation 1.0, 2.0, and 3.0 for Classification I, respectively. Figures 3.7, 3.9, 3.11 report those plots for lognormal distributions, while Figures 3.2, 3.4, 3.6, 3.8, 3.10, and 3.12 are the corresponding plots for Classification II.

From the simulations, the severity of Classification II is asymptotically normal because of its sample size, even though the individual losses follow gamma or lognormal distributions. The Q-Q plots are close to 45-degree straight lines; and the density function and histogram are close to symmetric. Tables 2.1-2.4 also show that Classification II has much smaller standard deviation and skewness. On the other hand, the severity of Classification I is positively skewed. The Q-Q plots are concave; and the density function and histogram have longer tails on the right side. From Tables 2.1-2.4, the larger the coefficient of variation, the more positively skewed the severity. The severities of lognormal losses have larger skewness because the lognormal distribution has longer right-side tails than the gamma distribution.

Regression Results

For each round of simulation, two datasets are generated based on gamma and lognormal distributions. Six regressions are performed on these two datasets: G-G, G-L, G-N, L-G, L-L, and L-N[6]. For each regression, the GLM coefficients $\hat{a}_i$ for each age group and $\hat{b}_j$ for each vehicle-use group, and predicted severity $\hat{S}_{i,j}$ are calculated. $e^{\hat{a}_i}$ is the relativity for age group i, and $e^{\hat{b}_j}$ is the relativity for vehicle-use group j.

For log-transformed models, a volatility adjustment factor $e^{\Omega^2_{i,j}/2n_{i,j}}$ is applied to reduce the estimation bias. The weighted absolute bias (wab) with and without the adjustment is reported in Table 4.1. Without the adjustment, the overall wab s are 0.38, 1.50, and 2.66 for G-L models with cv = 1.0, 2.0, and 3.0. After the adjustment, $wabs$

[6] G-G implies that the loss follows a Gamma distribution and a Gamma distribution is assumed in the regression; similarly, G-L implies the loss follows a Gamma distribution but a Lognormal is assumed in the regression, and G-N implies the loss follows a Gamma distribution but a Normal is assumed in the regression. The same logic applies for L-G, L-L, and L-N.

are reduced to 0.24, 0.85, and 1.81. On average, the *wab* reduction is 37%. Similarly, the *wab* reduction for L-L models is 35% on average.

Table 4.2 exhibits the *wabs* for the eight age groups with cv = 1.0, 2.0, and 3.0; and Tables 4.3-4.5 show the detailed information for predicted severities and biases for all thirty-two classifications. For the small-sample classifications, the prediction errors without adjustment could be very large. For example, the *wab* for age 17-20 is 39.5 for the G-L model with cv=3.0, and 22.1 for the L-L model if no adjustment is applied. After the adjustment, the *wabs* are reduced to 28.3 and 13.2, respectively. Without the adjustment, the *wabs* of the same models for age 40-49 are 1.25 and 1.83. After the adjustment, the weighted absolute biases are reduced slightly to 1.16 and 1.50, respectively. The volatility adjustment factor could reduce the biases of small-sample classifications significantly. In practice, log-transformed models are often applied. Without adjustment, the predicted severities (or relativities) are underestimated. In the case of G-L with cv=3.0, thirty-one of the thirty-two predicted severities are lower than the "true" relativities. Because the log-transformed model with adjustment is significantly better than the model without adjustment, only the former model is used in the following analysis.

For all the six models (G-G, G-L, G-N, L-G, L-L, and L-N), the weighted absolute biases and weighted standard errors of the predicted severities are used to measure the unbiasedness and stability of the models. The 95% confidence intervals are calculated based on the 2.5% quantile and 97.5% quantile of the sampling distributions. The *wab* and *wse* of the predicted severities for gamma and lognormal losses are reported in Table 5.1 and 5.2, respectively. The detailed information for the mean, bias, and standard deviation of the thirty-two predicted severities is shown in Tables 5.3, 5.5, 5.7, 5.9, 5.11, and 5.13 for G-G, G-L, G-N, L-G, L-L, and L-N, respectively. The corresponding confidence intervals are reported in Tables 5.4, 5.6, 5.8, 5.10, 5.12, and 5.14. Figures 6.1, 6.3, and 6.5 are the scatter and density plots of Classification I predicted severities for gamma losses with coefficient of variations 1.0, 2.0, and 3.0, respectively. Figures 6.2, 6.4, and 6.6 are the same plots for Classification II. Figures 6.7-6.12 show the corresponding plots for lognormal losses.

From Tables 5.3-5.14, the larger size the classification, the smaller bias and standard errors of predicted severities, and the more accurate the classification relativities. For example, the estimation biases of the G-G model with cv=1.0 are 1.65, 1.99, 2.43, and 2.77 for the four age group 17-20 classifications with vehicle-use pleasure, DTW short, DTW long, and business, respectively. The estimation biases for the four age group 40-49 classifications are much smaller (-0.12, 0.05, 0.04, -0.20). This is also true for the standard errors of the models.

When data is less volatile and the sample size of the classification is large enough, the predicted severity is asymptotically normal and the confidence interval is close to symmetric across the mean. On the other hand, when data is volatile and the sample size of the classification is small, the predicted severity is not symmetric across the mean. For example, the confidence interval of the predicted severity with coefficient of variation 1.0

for Classification II is (194.90, 214.72) based on the G-G model. It is symmetric across the "true" mean of 204.54. The confidence interval of the predicted severity with coefficient of variation 3.0 for Classification I is (119.40, 458.67) based on the G-G model. It is far from symmetric with the true "mean" of 254.90. In practice, confidence intervals are usually estimated by adding and subtracting two times the standard error to the mean. Our study shows that this could be very wrong for the small classifications (i.e., asymmetrical confidence intervals might be more appropriate).

Residual Diagnostics

To validate the distribution assumptions in the GLMs, "residual Q-Q" plots and "residual-fitted value" plots are often used to examine the heteroscedasticity within the error structure of the model (e.g., Holler, Sommer, and Geoff 1999 and Murphy, Brockman, and Lee 2000). In contrast to traditional linear models, deviance and Pearson residuals are applied[7]. To make the plots comparable, the residuals are standardized. If the "Q-Q" plots of deviance residuals are nonlinear or the residuals are fanning inwards or outwards (when plotted against the predicted values), the severity distribution assumptions are inappropriate.

We repeat the simulation one thousand times. It is too voluminous to report the residuals plots one thousand times for each model. So, we run an extra simulation independent of the previous ones and show the "residual Q-Q" and "residual-fitted value" plots in Figures 7.1-7.12 for each of the six models with cv=1.0, 2.0, and 3.0. If a gamma

[7] For detailed explanations of deviance and Pearson residuals, please refer to McCullagh and Nelder (1989). If a normal distribution is assumed in GLMs, deviance residuals are equal to Pearson residuals.

distribution is assumed, both Pearson and deviance residual plots are reported. If lognormal or normal distributions are assumed, only deviance residual plots are reported because Pearson and deviance residuals are identical.

From Figures 7.1-7.12, the "residual Q-Q" and "residual-fitted value" plots are similar for the gamma, lognormal, and normal models. It is difficult to examine the assumptions of severity distributions based on average severity data (summarized data). As discussed above, the distribution of average severity may be very different from the distribution of individual losses. When the sample size of the classification is large enough, the average severity is asymptotically normal no matter how the individual losses are distributed. The smoothing effect of summarized data makes the residual plots insensitive to the distributions assumed by GLMs.

If we run GLMs based on the individual losses of the same dataset (with 8,942 observations), residual plots are very sensitive to the distribution assumptions. Figures 7.13 and 7.14 report the "residual Q-Q" and "residual-fitted value" plots for gamma losses with cv=1.0. Figures 7.15 and 7.16 are plots for lognormal losses. It is clear that when the distribution assumptions are consistent with the "true" distribution, the Q-Q plots of deviation residuals are 45-degree straight lines and the residuals are randomly scattered across zero in the "residual-fitted value" plots[8]. Therefore, residual plots work well to examine the distribution assumptions on individual data, but not necessarily on summarized/average data.

[8] For the lognormal models on individual data, the volatility adjustment factor is $e^{\Omega_{i,j}^2/2}$ because each observation represents one claim and has one as the weight.

4. Conclusions

Insurance data with continuous measurement (severities and pure premium) have been found in numerous studies to be non-nonnal: 1) positive and usually positively skewed; and 2) variances are proportionally correlated to the mean squared. In practice, gamma and lognormal are two widely-used distributions with those desired properties. Traditional linear models assume a normal distribution, and don't have those properties, though most GLM actuarial research also report results from normal distributions as a benchmark. In this study, we apply Monte Carlo simulation techniques to examine the gamma, lognormal, and normal distributions, and determine which one provides better estimation in terms of unbiasedness and stability.

The simulation is numerically based on the predicted claim severity for private passenger auto collision claims used by Mildenhall (1999) and McCullagh and Nelder (1989). In each round of 1,000 simulations, six datasets of individual losses are generated based on gamma and lognormal distributions with "true" (known) classification severities and coefficients of variation (1.0, 2.0, and 3.0). For each dataset, three models are fitted on the average severities: GLM with gamma distribution and log link; GLM with normal distribution and log link; and the GLM on the log-transformed severities with normal distribution and identity link.

Based on the simulation results, we find that:

1). When the gamma distribution is "true", the G-G model is dominant in both unbiasedness and stability (except the G-L model is slightly more stable).

2). When the lognormal distribution is "true", the L-L model is dominant in terms of stability.

3). GLMs with a normal distribution never dominate based on any criteria, and they have the worst weighted standard error (wse).

4). GLMs with a gamma distribution are dominant in terms of unbiasedness, no matter whether the "true" distribution is gamma or lognormal.

5). Overall, GLMs with a gamma distribution perform slightly better than the log-transformed model and GLMs with a normal distribution. This result is consistent with the statistical research by Firth (1988).

6). When the data is not volatile, the distribution selection in GLMs may not be as important because all distribution assumptions yield small biases and standard errors.

7). When the log-transformed model is used, the classification relativities should be adjusted by a volatility-adjustment factor. Without the adjustment, the relativities are undervalued.

8). Residual plots may work well to examine the distribution assumptions on individual data, but not necessarily on summarized/average data.

5. Future Research

McCullagh and Nelder (1989) and Mildenhall (1999) discussed that insurance data is more likely to have a constant coefficient of variation rather than constant variance. However, it is possible that the variance increases with the mean but not proportional to the mean squared. Other distributions in the exponential family, such as the negative binomial and inverse Gaussian[9] have those properties. In future research, it might be interesting to generate losses based on negative binomial and inverse Gaussian distributions and also run GLMs assuming those distributions.

We investigated two classifications using private passenger auto severity data. However, simulation research on GLMs could be extended to other response variable (e.g., pure premium), other classifications (e.g., credit, territory), and other lines of business (e.g., homeowners, general liability, workers compensation).

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[9] Negative binomial distribution $f(x;\mu,k)=\frac{\Gamma(x+1/k)}{\Gamma(x+1)\Gamma(1/k)}\frac{(k\mu)^{x}}{(1+k\mu)^{x+1/k}}$ has variance $\mu+k\mu^{2}$.

Inverse Gaussian distribution $f(x;\mu,\sigma)=\frac{1}{\sqrt{2\pi x^{3}}\sigma}\exp\left(-\frac{1}{2x}\left(\frac{x-\mu}{\mu\sigma}\right)^{2}\right)$ has variance $\sigma^{2}\mu^{3}$.

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Appendix 1
"True" Relativities and Severities Assumed from Simulations

Table 1.1: "True" Relativities for Each Age Group

Age	Relativity	GLM Coefficient
17–20	1.307	0 268
21–24	1.301	0.263
25–29	1.206	0.187
30–34	1.156	0.145
35–39	0.931	-0.071
40–49	1.007	0.007
50–59	1.022	0.022
60+	Base	0.000

Table 1.2: "True" Relativities for Each Vehicle-Use Group

Vehicle Use	Relativity	GLM Coefficient
Business	1.644	0.497
DTW Long	1.264	0.234
DTW Short	1.042	0.041
Pleasure	Base	0.000

Table 1.3: "True" Severities and Relativities for Thirty-Two Classifications

Age	Vehicle Use	# of Claims	Severity	Relativity
17–20	Business	5	419.07	2.149
17–20	DTW Long	23	322.17	1.652
17–20	DTW Short	40	265.56	1.362
17–20	Pleasure	21	254.90	1.307
21–24	Business	44	417.10	2.139
21–24	DTW Long	92	320.66	1.644
21–24	DTW Short	171	264 31	1.355
21–24	Pleasure	63	253.70	1.301
25–29	Business	129	386.66	1.983
25–29	DTW Long	318	297.26	1.524
25–29	DTW Short	343	245.02	1.257
25–29	Pleasure	140	235.19	1.206
30–34	Business	169	370.53	1.900
30–34	DTW Long	361	284 85	1.461
30–34	DTW Short	448	234 80	1.204
30–34	Pleasure	123	225.37	1.156
35–39	Business	166	298.35	1.530
35–39	DTW Long	381	229 37	1.176
35–39	DTW Short	479	189 06	0.970
35–39	Pleasure	151	181 47	0.931
40–49	Business	304	322.78	1.655
40–49	DTW Long	719	248.15	1.273
40–49	DTW Short	970	204.54	1.049
40–49	Pleasure	245	196.33	1.007
50–59	Business	162	327.72	1.681
50–59	DTW Long	504	251.95	1 292
50–59	DTW Short	859	207.67	1.065
50–59	Pleasure	266	199.34	1.022
60+	Business	96	320.60	1 644
60+	DTW Long	312	246.47	1.264
60+	DTW Short	578	203.16	1.042
60+	Pleasure	260	195.00	1.000

Appendix 2
Summaries of the Simulated Severities for Selected Classifications

Table 2.1: Statistical Summary of Simulated Gamma Severity for Age 17-20 and Pleasure Use with Variation Coefficients 1.0, 2.0, and 3.0

CV	Min	25% Q	Median	Mean	75% Q	Max	Stdev	Skewness
1.0	86.7	217.2	254.0	258.2	292.5	505.2	57.6	0.605
2.0	51.0	168.1	232.3	251 0	314.7	771.1	111 7	0.846
3.0	4.3	133.4	214.6	255 7	338 4	1,122 0	170.4	1.413

Table 2.2: Statistical Summary of Simulated Lognormal Severity for Age 17-20 and Pleasure Use with Coefficients of Variation 1.0, 2.0, and 3.0

CV	Min	25% Q	Median	Mean	75% Q	Max	Stdev	Skewness
1 0	128.2	215.5	247.5	254.5	284.3	557 6	54.1	0.772
2.0	71.73	181.6	230.6	252.3	291.9	1,097.0	109.6	2.112
3.0	57.95	157.3	220.5	258.1	301.3	4,219.0	200.2	9.162

Table 2.3: Statistical Summary of Simulated Gamma Severity for Age 40-49 and DTW Short Use with Coefficients of Variation 1.0, 2.0, and 3.0

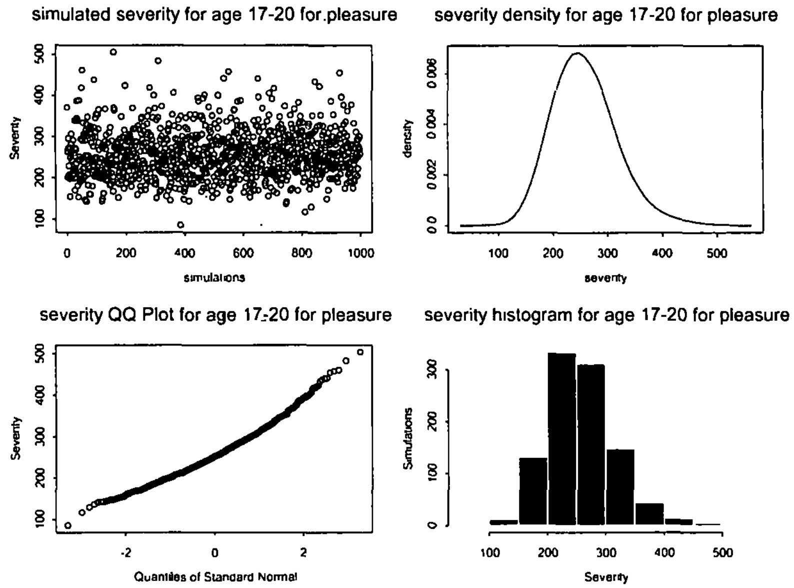
CV	Min	25% Q	Median	Mean	75% Q	Max	Stdev	Skewness
1.0	186.4	200 0	204.5	204.7	209.2	227.9	6.7	0.122
2 0	156.0	193 9	203.0	203.7	213.1	247.5	13.9	0.190
3.0	156.8	190 0	204.0	204.8	217.7	269.3	20.1	0.298

Table 2.4: Statistical Summary of Simulated Lognormal Severity for Age 40-49 and DTW Short Use with Coefficients of Variation 1.0, 2.0, and 3.0

CV	Min	25% Q	Median	Mean	75% Q	Max	Stdev	Skewness
1.0	182.4	200.0	204.4	204.4	208.9	226.5	6.5	0.141
2.0	168.3	195.8	203.3	204.3	212.6	252 0	12 9	0.253
3.0	154.5	191 0	202.2	204.2	215.6	284 8	19 6	0.650

Appendix 3
Plots of Simulated Severities for Selected Classifications

Figure 3.1: Plot Summary of Simulated Gamma Severity for Age 17-20 and Pleasure Use with Coefficient of Variation 1.0



- The density is estimated by the non-parametric method from Silverman (1986).
- A 45-degree straight line in the Q-Q plot implies that the severity follows a normal distribution.

Figure 3.2: Plot Summary of Simulated Gamma Severity for Age 40-49 and DTW Short Use with Coefficient of Variation 1.0

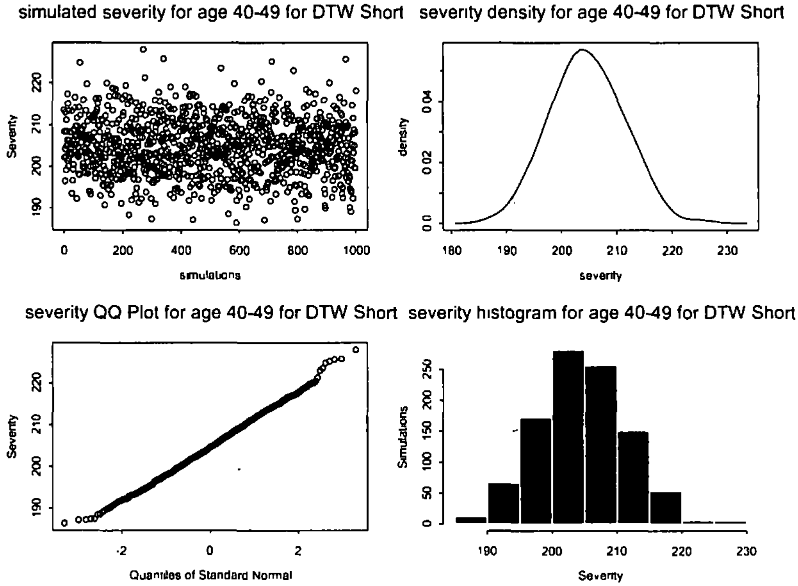


Figure 3.3: Plot Summary of Simulated Gamma Severity for Age 17-20 and Pleasure Use with Coefficient of Variation 2.0

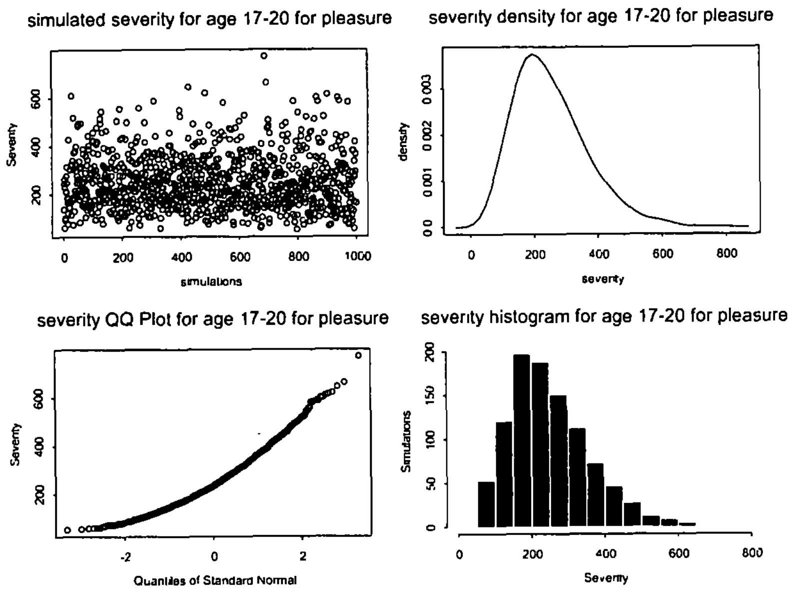


Figure 3.4: Plot Summary of Simulated Gamma Severity for Age 40-49 and DTW Short Use with Coefficient of Variation 2.0

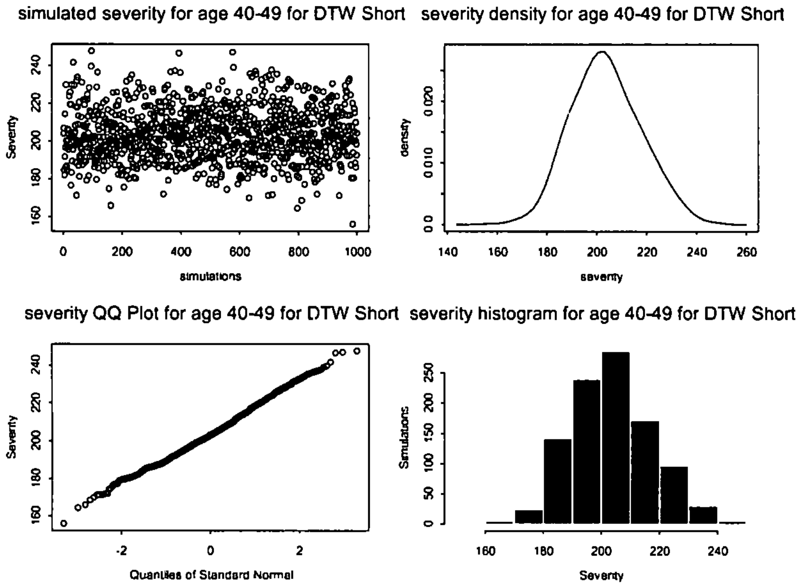


Figure 3.5: Plot Summary of Simulated Gamma Severity for Age 17-20 and Pleasure Use with Coefficient of Variation 3.0

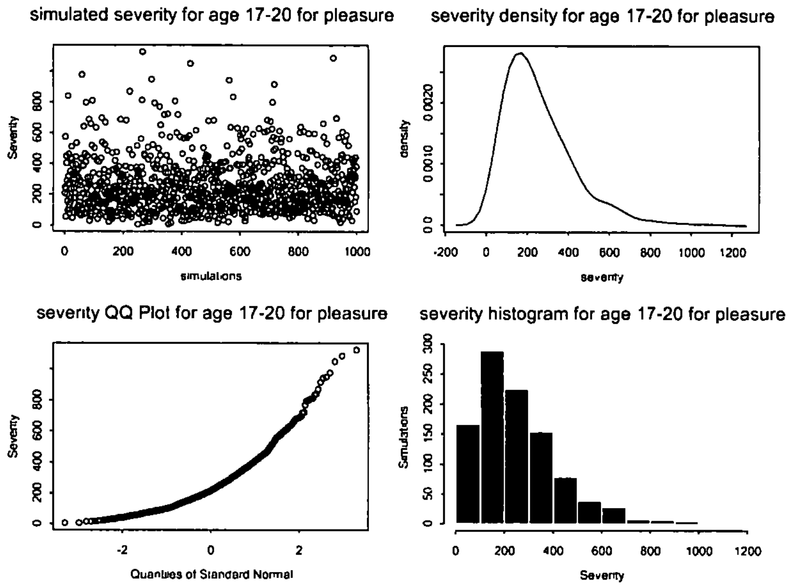


Figure 3.6: Plot Summary of Simulated Gamma Severity for Age 40-49 and DTW Short Use with Coefficient of Variation 3.0

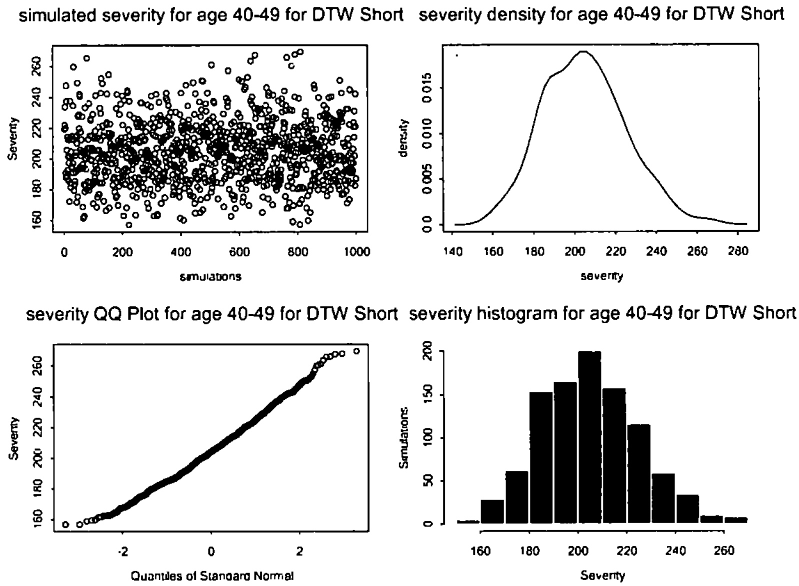


Figure 3.7: Plot Summary of Simulated Lognormal Severity for Age 17-20 and Pleasure Use with Coefficient of Variation 1.0

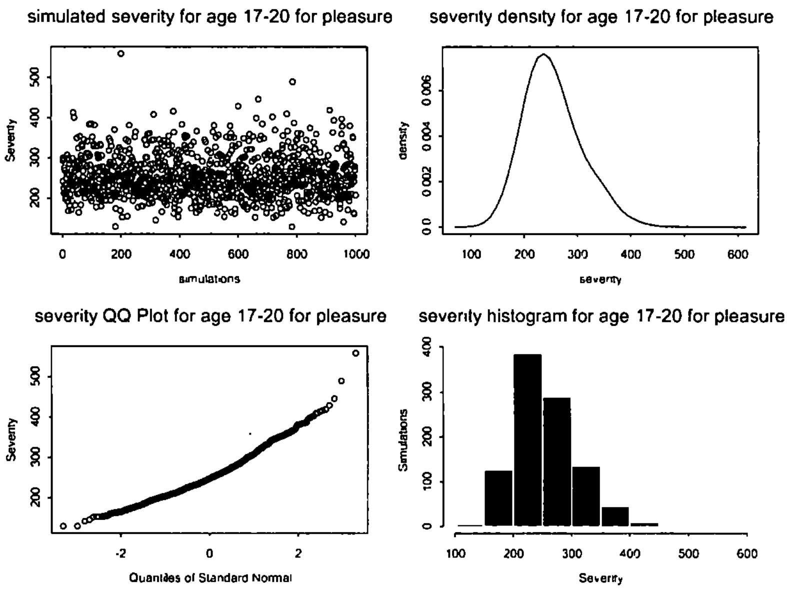


Figure 3.8: Plot Summary of Simulated Lognormal Severity for Age 40-49 and DTW Short Use with Coefficient of Variation 1.0

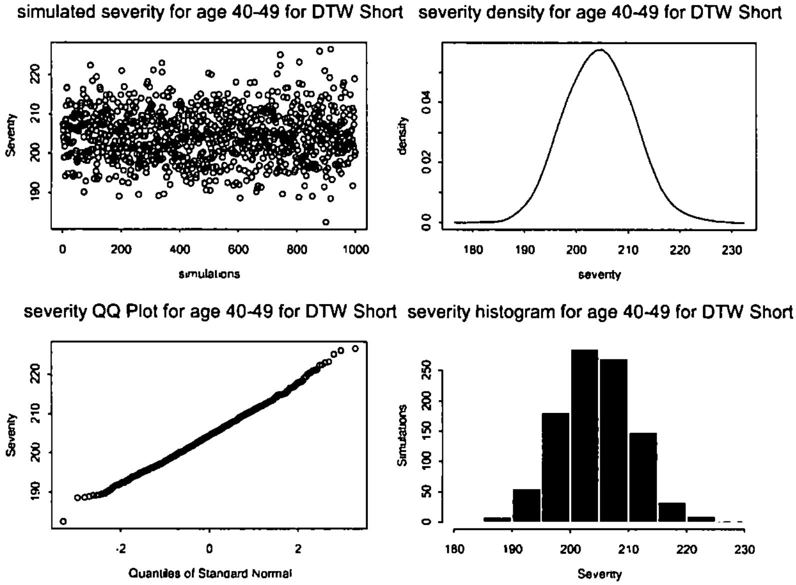


Figure 3.9: Plot Summary of Simulated Lognormal Severity for Age 17-20 and Pleasure Use with Coefficient of Variation 2.0

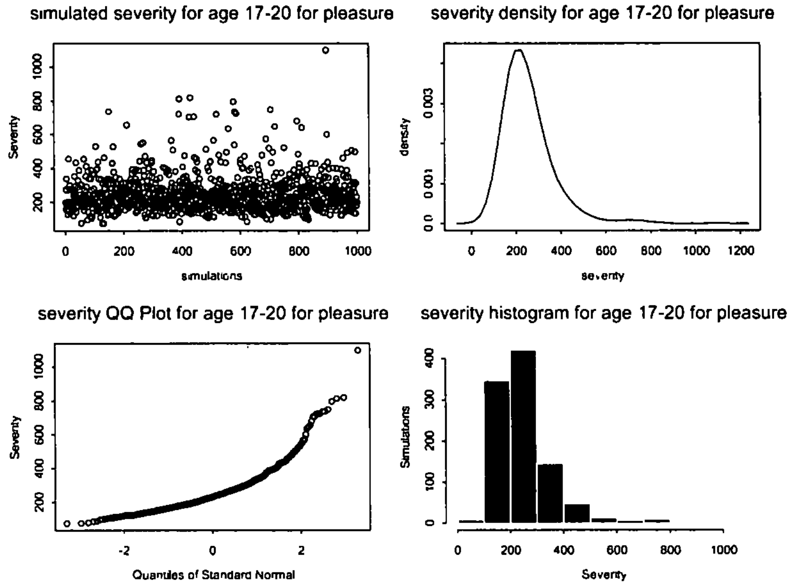


Figure 3.10: Plot Summary of Simulated Lognormal Severity for Age 40-49 and DTW Short Use with Coefficient of Variation 2.0

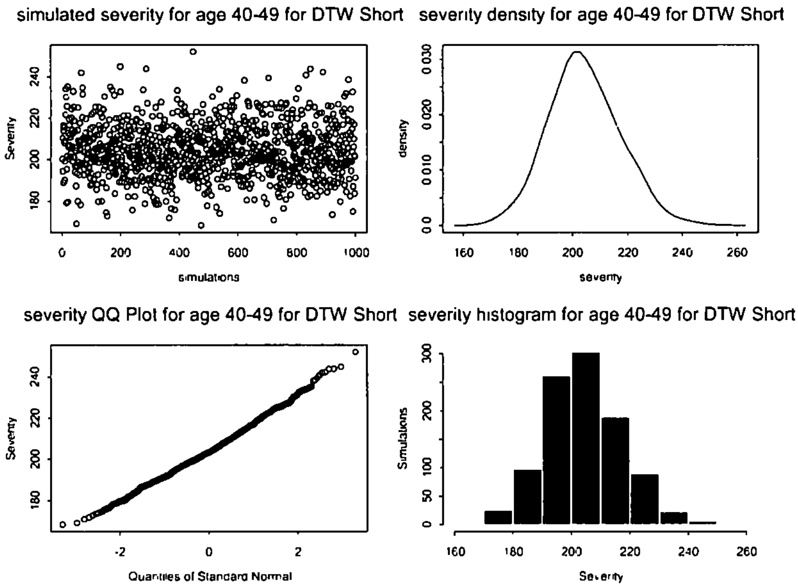


Figure 3.11: Plot Summary of Simulated Lognormal Severity for Age 17-20 and Pleasure Use with Coefficient of Variation 3.0

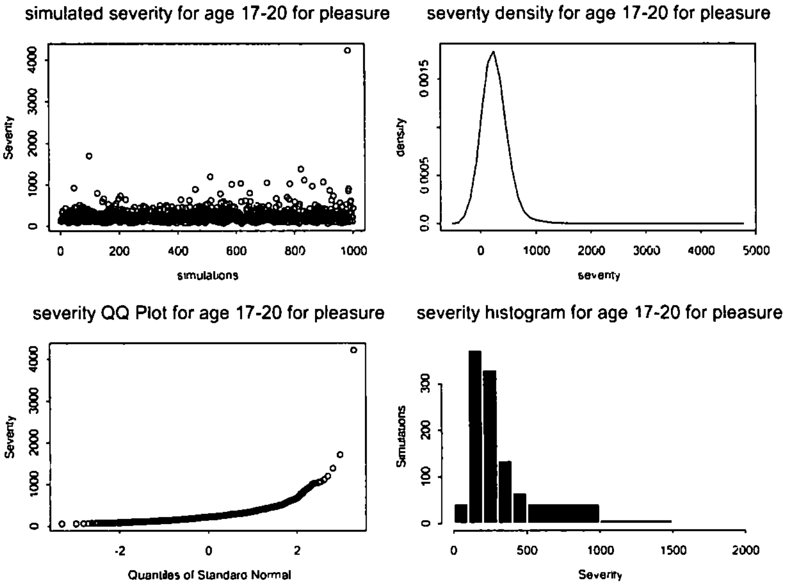
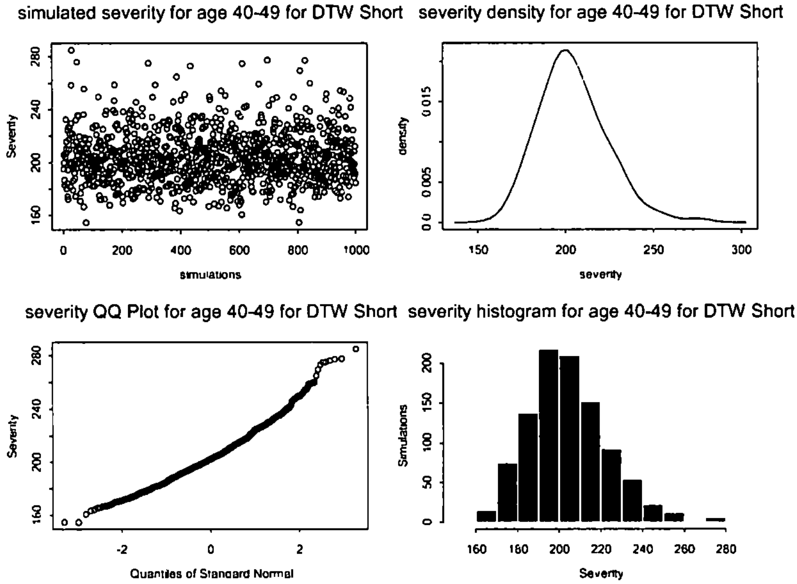


Figure 3.12: Plot Summary of Simulated Lognormal Severity for Age 40-49 and DTW Short Use with Coefficient of Variation 3.0



Appendix 4
Predicted Severities and Biases of Log-transformed Models with and without Adjustment

Table 4.1: Weighted Absolute Biases with and without Adjustment

Coefficient	wab			
Of Variation	G-L1	G-L2	L-L1	L-L2
1.0	0.380	0.240	0.278	0.202
2.0	1.500	0.852	1.539	0.844
3.0	2.661	1.808	2.359	1.589

- G-L1 is the log-transformed model on gamma losses without volatility adjustment;
- G-L2 is the log-transformed model on gamma losses adjusted by the volatility adjustment factors;
- L-L1 is the log-transformed model on lognormal losses without volatility adjustment;
- L-L2 is the log-transformed model on lognormal losses adjusted by the volatility adjustment factors.

Table 4.2: Weighted Absolute Biases with and without Adjustment for Age Groups

	Coefficient	wab			
Age	Of Variation	G-L1	G-L2	L-L1	L-L2
17-20	1 0	2.75	2.20	3.69	2 17
17-20	2.0	21.90	14.56	14.01	7.86
17-20	3.0	39.95	28.30	22.10	13.19
21-24	1.0	1.45	0 35	0.27	1.16
21-24	2.0	6.74	4.06	4.36	1.66
21-24	3.0	8.94	5.13	6.47	2.63
25-29	1.0	0.48	0.13	0.59	0.15
25-29	2.0	1.35	0.48	1 95	0.95
25-29	3.0	2.77	1 34	2 29	1.47
30-34	1.0	0 33	0.37	0.22	0.25
30-34	2.0	1.20	0.51	2.73	1.92
30-34	3.0	1.75	1.10	2.19	1.53
35-39	1.0	0.52	0.26	0 28	0.10
35-39	2 0	1.02	0.41	1.13	0.54
35-39	3.0	2.67	1.80	1 81	1.25
40-49	1.0	0.22	0 19	0.18	0.14
40-49	2.0	1.30	0.95	0.68	0.41
40-49	3.0	1 25	1.16	1.83	1.50
50-59	1.0	0.22	0.21	0.15	0.12
50-59	2 0	0.90	0.45	1.13	0 69
50-59	3 0	1.20	0.73	1.58	1 19
60+	1.0	0.23	0.15	0 21	0.10
60+	2.0	0.55	0.31	0 97	0.35
60+	3.0	3.49	2.62	2 53	1.65

Table 4.3: Predicted Severities and Biases of Log-transformed Models with and without Adjustment for Variation Coefficient 1.0

Age	Vehicle Use	Predicted Severity				Prediction Bias			
		G-L1	G-L2	L-L1	L-L2	G-L1	G-L2	L-L1	L-L2
17-20	Pleasure	252.00	256.20	251.36	255.55	-2.89	1.30	-3.53	0.65
17-20	DTW Short	263.26	265.55	262.24	264.52	-2.30	-0.01	-3.32	-1.04
17-20	DTW Long	319.26	324.11	318.28	323.11	-2.91	1.93	-3.89	0.94
17-20	Business	414.06	443.78	412.76	442.39	-5.00	24.71	-6.31	23.32
21-24	Pleasure	252.06	253.45	253.48	254.88	-1.64	-0.25	-0.22	1.18
21-24	DTW Short	263.30	263.83	264.45	264.98	-1.01	-0.48	0.13	0.67
21-24	DTW Long	319.25	320.46	320.93	322.14	-1.41	-0.20	0.27	1.48
21-24	Business	414.12	417.40	416.20	419.49	-2.97	0.30	-0.90	2.39
25-29	Pleasure	234.43	235.01	234.52	235.10	-0.75	-0.17	-0.67	-0.08
25-29	DTW Short	244.89	245.14	244.65	244.90	-0.14	0.11	-0.37	-0.12
25-29	DTW Long	296.92	297.25	296.88	297.21	-0.33	-0.01	-0.37	-0.05
25-29	Business	385.16	386.20	385.05	386.09	-1.50	-0.46	-1.60	-0.57
30-34	Pleasure	225.05	225.68	225.05	225.69	-0.33	0.31	-0.32	0.32
30-34	DTW Short	235.07	235.25	234.77	234.96	0.27	0.45	-0.03	0.16
30-34	DTW Long	285.03	285.31	284.90	285.17	0.18	0.45	0.04	0.32
30-34	Business	369.71	370.47	369.48	370.24	-0.82	-0.06	-1.04	-0.28
35-39	Pleasure	180.75	181.16	181.10	181.52	-0.72	-0.31	-0.37	0.04
35-39	DTW Short	188.82	188.95	188.93	189.07	-0.25	-0.11	-0.13	0.00
35-39	DTW Long	228.95	229.16	229.27	229.48	-0.41	-0.21	-0.10	0.11
35-39	Business	296.97	297.59	297.34	297.96	-1.38	-0.76	-1.01	-0.39
40-49	Pleasure	195.90	196.18	196.03	196.31	-0.43	-0.15	-0.30	-0.02
40-49	DTW Short	204.63	204.70	204.51	204.58	0.08	0.16	-0.04	0.04
40-49	DTW Long	248.11	248.23	248.16	248.28	-0.03	0.09	0.02	0.14
40-49	Business	321.83	322.20	321.85	322.22	-0.95	-0.58	-0.93	-0.56
50-59	Pleasure	198.95	199.21	199.04	199.30	-0.39	-0.13	-0.29	-0.03
50-59	DTW Short	207.82	207.91	207.65	207.74	0.15	0.23	-0.02	0.06
50-59	DTW Long	252.01	252.18	252.00	252.17	0.06	0.23	0.05	0.23
50-59	Business	326.89	327.59	326.79	327.49	-0.84	-0.14	-0.93	-0.23
60+	Pleasure	194.47	194.73	194.63	194.89	-0.53	-0.27	-0.38	-0.12
60+	DTW Short	203.15	203.27	203.06	203.18	-0.01	0.11	-0.10	0.02
60+	DTW Long	246.34	246.61	246.42	246.70	-0.13	0.14	-0.05	0.23
60+	Business	319.52	320.67	319.59	320.75	-1.08	0.07	-1.00	0.15

Table 4.4: Predicted Severities and Biases of Log-transformed Models with and without Adjustment for Variation Coefficient 2.0

Age	Vehicle Use	Predicted Severity				Prediction Bias			
		G-L1	G-L2	L-L1	L-L2	G-L1	G-L2	L-L1	L-L2
17-20	Pleasure	234.75	243.92	242.18	251.64	-20.15	-10.98	-12.72	-3.26
17-20	DTW Short	245.84	250.83	252.99	258.13	-19.72	-14.73	-12.57	-7.43
17-20	DTW Long	297.79	308.39	306.59	317.51	-24.38	-13.78	-15.58	-4.66
17-20	Business	383.88	450.91	395.37	464.40	-35 19	31.84	-23.70	45.34
21-24	Pleasure	247.25	250.43	249.90	253.11	-6.45	-3.27	-3.80	-0.59
21-24	DTW Short	259.01	260.23	260.96	262.19	-5.30	-4.08	-3.36	-2.13
21-24	DTW Long	313.70	316 46	316.32	319.10	-6.96	-4.20	-4.33	-1 56
21-24	Business	404 78	412.25	407 99	415.52	-12.32	-4.85	-9.11	-1.58
25-29	Pleasure	233.53	234.87	233.61	234.96	-1.66	-0.31	-1.57	-0.23
25-29	DTW Short	244.62	245.20	243.96	244.53	-0 40	0.17	-1.06	-0.49
25-29	DTW Long	296.23	296.98	295.62	296.36	-1.03	-0.28	-1.64	-0.89
25-29	Business	382.31	384.71	381.16	383.55	-4.35	-1.95	-5.50	-3.11
30-34	Pleasure	223.92	225.39	223.19	224 65	-1.45	0.02	-2.18	-0.72
30-34	DTW Short	234.52	234.95	233.05	233.47	-0.28	0.15	-1.75	-1.33
30-34	DTW Long	283.95	284.58	282.43	283.06	-0.91	-0.27	-2.43	-1.80
30-34	Business	366.44	368.19	364 15	365.88	-4.08	-2.33	-6.38	-4.64
35-39	Pleasure	180.24	181.21	180 61	181.57	-1.23	-0.27	-0.87	0.10
35-39	DTW Short	188.75	189.07	188 56	188.88	-0.31	0.01	-0.50	-0.18
35-39	DTW Long	228.59	229.07	228.48	228.96	-0.78	-0.30	-0.89	-0.41
35-39	Business	294.94	296.38	294.63	296.07	-3.41	-1.97	-3.72	-2.29
40-49	Pleasure	194.79	195.43	195.77	196.42	-1.54	-0.90	-0.56	0.09
40-49	DTW Short	204.01	204.18	204.45	204.62	-0.54	-0.37	-0.10	0.07
40-49	DTW Long	247.06	247.34	247.76	248.04	-1.09	-0 81	-0.39	-0 11
40-49	Business	318.75	319.59	319.45	320.30	-4.03	-3.18	-3.32	-2.48
50-59	Pleasure	198.01	198.61	198.25	198.85	-1.33	-0.73	-1.09	-0.49
50-59	DTW Short	207.37	207.56	207.06	207 25	-0.31	-0.11	-0.62	-0.42
50-59	DTW Long	251.16	251.56	250.92	251 32	-0.78	-0.38	-1.03	-0.63
50-59	Business	324.08	325.69	323.51	325 12	-3.64	-2.03	-4 21	-2.60
60+	Pleasure	193.95	194.55	194.01	194 61	-1.06	-0.46	-0 99	-0.39
60+	DTW Short	203.15	203.43	202.64	202.92	-0.01	0.27	-0.52	-0.24
60+	DTW Long	246.10	246.73	245.59	246.23	-0.37	0.26	-0.88	-0.24
60+	Business	317.57	320.24	316.66	319.33	-3.03	-0 36	-3.94	-1.27

Table 4.5: Predicted Severities and Biases of Log-transformed Models with and without Adjustment for Variation Coefficient 3.0

	Vehicle	Predicted Severity				Prediction Bias			
Age	Use	G-L1	G-L2	L-L1	L-L2	G-L1	G-L2	L-L1	L-L2
17-20	Pleasure	218.27	230 58	233.25	246.40	-36.62	-24.32	-21.65	-8.50
17-20	DTW Short	229.70	236 41	246.73	253.93	-35 86	-29.15	-18.84	-11 63
17-20	DTW Long	277.35	291.59	297.28	312.54	-44.82	-30.58	-24.89	-9.63
17-20	Business	354.90	446.80	381.85	480.72	-64.16	27.73	-37.22	61.65
21-24	Pleasure	245.26	249.79	246.14	250.68	-8.44	-3.91	-7.56	-3.02
21-24	DTW Short	257.67	259.41	260.50	262.26	-6.64	-4.90	-3.82	-2.06
21-24	DTW Long	311.91	315.84	313.76	317.71	-8.75	-4.82	-6.90	-2.95
21-24	Business	398.09	408.64	402 80	413.48	-19 01	-8.46	-14.30	-3.62
25-29	Pleasure	232.32	234 24	231.81	233.72	-2.87	-0.95	-3 38	-1.47
25-29	DTW Short	244.17	244 99	245.33	246.15	-0.86	-0.04	0.31	1.13
25-29	DTW Long	295.23	296.30	295.33	296.40	-2.02	-0.95	-1.92	-0.85
25-29	Business	377.08	380.46	379.37	382.77	-9.58	-6.20	-7.29	-3.89
30-34	Pleasure	223.42	225.52	222.07	224.16	-1.95	0.15	-3.30	-1.21
30-34	DTW Short	234.78	235 38	234.81	235.42	-0 02	0.58	0.01	0.62
30-34	DTW Long	283.96	284 86	282.80	283.71	-0.90	0.01	-2 05	-1.15
30-34	Business	362.53	365 01	363.04	365.53	-7 99	-5.51	-7.48	-5 00
35-39	Pleasure	178.68	180.04	178.71	180.08	-2.80	-1.43	-2.77	-1 40
35-39	DTW Short	187.86	188.32	188.98	189.44	-1.20	-0.75	-0.08	0.37
35-39	DTW Long	227.27	227.95	227.62	228.31	-2.10	-1.42	-1.74	-1.06
35-39	Business	290.28	292.30	292.26	294.29	-8.08	-6.06	-6.10	-4.06
40-49	Pleasure	195.13	196.05	193.36	194.27	-1.20	-0.28	-2.97	-2.06
40-49	DTW Short	205.16	205.40	204.50	204.74	0.62	0.86	-0.05	0.20
40-49	DTW Long	248.06	248.46	246.34	246.74	-0.09	0.31	-1.81	-1.41
40-49	Business	316 74	317.94	316 14	317.34	-6.04	-4 83	-6.64	-5 44
50-59	Pleasure	197.46	198.32	196.31	197.17	-1.87	-1.02	-3 02	-2.17
50-59	DTW Short	207 56	207.84	207.65	207.93	-0.11	0.17	-0.02	0.26
50-59	DTW Long	251.10	251.67	250.14	250.71	-0.85	-0.27	-1.81	-1.24
50-59	Business	320.78	323.07	321.03	323.32	-6.94	-4.65	-6.70	-4.41
60+	Pleasure	191.11	191.96	191.14	191.98	-3.89	-3.05	-3.87	-3.02
60+	DTW Short	200 84	201.25	202.26	202.66	-2.32	-1 92	-0.90	-0.50
60+	DTW Long	243 15	244.05	243 67	244.57	-3.32	-2.42	-2 80	-1.90
60+	Business	310 56	314.31	312.80	316.58	-10.04	-6.29	-7.80	-4.02

Appendix 5
Summary Tables of Predicted Severities

Table 5.1: Overall Unbiasedness and Stability of Predicted Severities for Gamma Loss

Coefficient	wab			wse		
Of Variation	G-G	G-L	G-N	G-G	G-L	G-N
1.0	0.180	0.240	0.221	8.170	8.177	8.568
2.0	0.475	0.852	0.509	16.498	16.514	17.239
3.0	0 860	1.808	1.139	25.223	25.097	26.986

G-G implies that the loss follows a Gamma distribution and a Gamma distribution is assumed in the regressions; similarly, G-L implies the loss follows a Gamma distribution but a Lognormal is assumed in the regressions; and G-N implies the loss follows a Gamma distribution but a Normal is assumed in the regressions.

Table 5.2: Overall Unbiasedness and Stability of Predicted Severities for Lognormal Loss

Coefficient	wab			wse		
Of Variation	L-G	L-L	L-N	L-G	L-L	L-N
1.0	0.151	0.202	0 175	8.309	8.284	8.754
2.0	0.498	0.844	0 604	16.426	16 113	17.721
3.0	0.720	1.589	1.006	24 328	23.214	27 608

Table 5.3: Summarized Statistics for Predicted Severities with Gamma Loss and Coefficient of Variation 1.0

	Vehicle	Mean			Bias			Standard Error		
Age	Use	G-G	G-L	G-N	G-G	G-L	G-N	G-G	G-L	G-N
17-20	Pleasure	256.55	256.20	256.42	1.65	1.30	1.53	26.95	27.31	27.80
17-20	DTW Short	267.55	265.55	267.55	1.99	-0.01	1.99	27.64	27.83	28.72
17-20	DTW Long	324.60	324.11	324.69	2.43	1.93	2.52	34.06	34.55	35.56
17-20	Business	421.83	443.78	422.33	2.77	24.71	3.27	44.80	47.88	47.10
21-24	Pleasure	253.27	253.45	253.24	-0.43	-0.25	-0.46	14.48	14.52	15.38
21-24	DTW Short	264.12	263.83	264.18	-0.20	-0.48	-0.13	13.85	13.88	14.77
21-24	DTW Long	320.39	320.46	320.56	-0.27	-0.20	-0.10	16.88	16.93	18.17
21-24	Business	416.43	417.40	417.02	-0.67	0.30	-0.08	24.43	24.56	26.34
25-29	Pleasure	235.00	235.01	234.82	-0.18	-0.17	-0.36	9.61	9.63	9.91
25-29	DTW Short	245.07	245.14	244.98	0.04	0.11	-0.04	8.54	8.54	8.92
25-29	DTW Long	297.28	297.25	297.24	0.02	-0.01	-0.01	10.26	10.26	10.80
25-29	Business	386.39	386.20	386.70	-0.27	-0.46	0.04	16.69	16.66	17.83
30-34	Pleasure	225.54	225.68	225.49	0.16	0.31	0.12	9.06	9.08	9.49
30-34	DTW Short	235.18	235.25	235.22	0.38	0.45	0.42	7.46	7.47	7.88
30-34	DTW Long	285.30	285.31	285.42	0.45	0.45	0.56	9.46	9.46	10.02
30-34	Business	370.80	370.47	371.27	0.27	-0.06	0.75	14.94	14.94	15.71
35-39	Pleasure	181.13	181.16	180.94	-0.34	-0.31	-0.54	6.84	6.84	7.16
35-39	DTW Short	188.90	188.95	188.76	-0.17	-0.11	-0.30	5.98	5.98	6.32
35-39	DTW Long	229.16	229.16	229.04	-0.21	-0.21	-0.32	7.71	7.70	8.00
35-39	Business	297.83	297.59	297.93	-0.52	-0.76	-0.42	12.10	12.09	12.25
40-49	Pleasure	196.21	196.18	196.01	-0.12	-0.15	-0.32	6.80	6.80	7.21
40-49	DTW Short	204.60	204.70	204.47	0.05	0.16	-0.08	5.07	5.07	5.39
40-49	DTW Long	248.19	248.23	248.09	0.04	0.09	-0.06	6.27	6.27	6.51
40-49	Business	322.58	322.20	322.72	-0.20	-0.58	-0.06	11.22	11.22	11.58
50-59	Pleasure	199.30	199.21	199.24	-0.03	-0.13	-0.09	6.73	6.72	7.02
50-59	DTW Short	207.84	207.91	207.85	0.16	0.23	0.18	5.33	5.33	5.58
50-59	DTW Long	252.14	252.18	252.22	0.19	0.23	0.27	7.22	7.21	7.54
50-59	Business	327.71	327.59	328.10	-0.01	-0.14	0.38	12.37	12.37	12.90
60+	Pleasure	194.87	194.73	194.71	-0.13	-0.27	-0.29	7.08	7.08	7.38
60+	DTW Short	203.22	203.27	203.13	0.06	0.11	-0.03	6.03	6.03	6.32
60+	DTW Long	246.54	246.61	246.49	0.07	0.14	0.02	7.99	7.99	8.29
60+	Business	320.42	320.67	320.63	-0.18	0.07	0.03	12.61	12.64	13.08

Table 5.4: 95% Confidence Intervals for Predicted Severities with Gamma Loss and Coefficient of Variation 1.0

Age	Vehicle Use	G-G Lower	G-G Upper	G-L Lower	G-L Upper	G-N Lower	G-N Upper
17-20	Pleasure	207.63	310.60	207.59	311.14	206 28	310.82
17-20	DTW Short	216 34	322.12	214.26	321.27	215.28	322.54
17-20	DTW Long	263.80	392.67	263.01	392.03	261.02	394.83
17-20	Business	340.83	512.07	358.86	541.99	338.00	517 58
21-24	Pleasure	226.25	283 09	226.57	282.19	224.42	284.02
21-24	DTW Short	237.62	290.70	237 59	290.38	236.56	293 08
21-24	DTW Long	289.50	354.99	289.55	356.13	286.43	357.91
21-24	Business	372 10	468.31	372 53	470 09	369.86	473.94
25-29	Pleasure	216.42	253.50	216.61	253.21	216.42	253.86
25-29	DTW Short	229.15	261.89	229.24	262.06	228.08	262.99
25-29	DTW Long	277.55	315.84	277.48	315.65	276.57	317.05
25-29	Business	353.94	418.31	354.08	417.04	354 51	424.11
30-34	Pleasure	208.37	244.26	208.38	244.39	207.70	244.94
30-34	DTW Short	220.31	250.36	220.32	250.59	219.66	251.02
30-34	DTW Long	267.57	304.54	267.29	304.64	266.47	305.14
30-34	Business	343.29	400.92	343.27	400.73	340.73	402.13
35-39	Pleasure	168.49	194.75	168.73	194 88	167.68	194.87
35-39	DTW Short	177.86	201.01	177.93	200.92	176.73	200.90
35-39	DTW Long	214.26	244.34	214.21	244 50	213.75	245.04
35-39	Business	274 05	321.18	273.63	320.86	274.55	322 64
40-49	Pleasure	183.77	209.55	183.81	209.44	182.42	210.65
40-49	DTW Short	194.90	214.72	194.97	214.84	194.43	215.61
40-49	DTW Long	235.56	260.38	235.60	260.46	234.78	260.80
40-49	Business	302.55	344.78	301.93	344.46	301.67	346.83
50-59	Pleasure	186.45	212.74	186.43	212.74	185.40	212.99
50-59	DTW Short	197.56	218.44	197.69	218.57	197.29	219.28
50-59	DTW Long	237.88	265.95	238.07	266.05	237.88	267.72
50-59	Business	304.74	352.71	304.30	352.50	304.37	353.81
60+	Pleasure	180.54	208.23	180.54	208.05	180.35	208.72
60+	DTW Short	190.79	214.47	190.96	214.48	190.91	215.67
60+	DTW Long	230 42	262.11	230.51	262.10	230.67	263.05
60+	Business	296.82	345.96	297.28	346.23	296.32	346.93

Table 5.5: Summarized Statistics of Predicted Severities with Gamma Loss and Coefficient of Variation 2.0

	Vehicle	Mean			Bias			Standard Error		
Age	Use	G-G	G-L	G-N	G-G	G-L	G-N	G-G	G-L	G-N
17-20	Pleasure	252.38	243.92	252.38	-2.52	-10.98	-2.52	55.66	55.58	57.27
17-20	DTW Short	262.56	250.83	262.73	-3.00	-14.73	-2.83	56.54	55.93	58.13
17-20	DTW Long	318.61	308.39	319.25	-3.56	-13.78	-2 92	69.08	69.36	71.04
17-20	Business	414.11	450.91	415.88	-4.95	31.84	-3.19	89.73	101.38	92.78
21-24	Pleasure	252.11	250.43	252.29	-1.59	-3.27	-1.41	29.49	29.59	31 10
21-24	DTW Short	262 37	260.23	262.75	-1.94	-4.08	-1 56	28.87	28.96	30.77
21-24	DTW Long	318.36	316.46	319.33	-2.30	-4.20	-1.33	35.76	35.89	38.73
21-24	Business	414.18	412.25	416 32	-2.92	-4.85	-0.78	50.28	50 63	54.30
25-29	Pleasure	235.66	234.87	235.09	0.48	-0 31	-0.10	19 06	19.18	19.74
25-29	DTW Short	245.25	245.20	244.81	0 23	0.17	-0.21	17.30	17.39	18.03
25-29	DTW Long	297.54	296.98	297.42	0 28	-0.28	0.16	21 24	21.34	22 34
25-29	Business	387.17	384.71	387.86	0.51	-1.95	1.20	33.57	33.60	35.31
30-34	Pleasure	225 81	225.39	225.34	0.44	0.02	-0.03	17.85	17.92	18.71
30-34	DTW Short	234.96	234.95	234.61	0.17	0.15	-0.19	15.33	15 35	16.10
30-34	DTW Long	285.01	284.58	284.95	0.15	-0.27	0.10	18 13	18.07	18.98
30-34	Business	370.86	368.19	371.61	0.33	-2 33	1.09	29.81	29.59	31.50
35-39	Pleasure	181.70	181.21	181.16	0 23	-0.27	-0.31	14 54	14.65	15.20
35-39	DTW Short	189.05	189.07	188.61	-0.02	0.01	-0.45	12.32	12.33	13.20
35-39	DTW Long	229.36	229.07	229.13	0 00	-0.30	-0.24	15.45	15.42	16.25
35-39	Business	298 40	296.38	298.68	0.04	-1.97	0.33	24.05	23.92	24.61
40-49	Pleasure	195.92	195.43	195.58	-0.41	-0.90	-0.75	13.42	13.51	13.95
40-49	DTW Short	203.85	204.18	203.61	-0.69	-0.37	-0.93	10 52	10.54	10.84
40-49	DTW Long	247.32	247.34	247.37	-0.83	-0 81	-0.77	13 36	13.38	13.81
40-49	Business	321.73	319 59	322.46	-1 05	-3.18	-0.32	22.19	22.05	22.61
50-59	Pleasure	199.37	198 61	199.04	0 04	-0.73	-0.29	13.81	13.80	14.37
50-59	DTW Short	207.43	207 56	207.20	-0.25	-0.11	-0.47	10.65	10.67	11.02
50-59	DTW Long	251.70	251.56	251.78	-0.25	-0.38	-0.17	14.13	14.14	14.71
50-59	Business	327 46	325.69	328.25	-0.26	-2.03	0 53	23.59	23.50	24.38
60+	Pleasure	195 53	194.55	195.17	0.53	-0.46	0 17	13.93	13.93	14.57
60+	DTW Short	203.48	203.43	203 22	0.31	0.27	0.06	11.71	11.70	12.33
60+	DTW Long	246.95	246.73	246 99	0.48	0.26	0.52	16.01	16.02	16.92
60+	Business	321.30	320.24	322.03	0.70	-0 36	1.43	25 65	25.61	26.83

Table 5.6: 95% Confidence Intervals of Predicted Severities with Gamma Loss and Coefficient of Variation 2.0

Age	Vehicle Use	G-G Lower	G-G Upper	G-L Lower	G-L Upper	G-N Lower	G-N Upper
17-20	Pleasure	158.35	376.66	149.52	365 56	158.84	376.87
17-20	DTW Short	165.84	381.71	155.59	373.76	163.21	387.07
17-20	DTW Long	200.10	466.14	189.67	458.93	199 68	471.12
17-20	Business	255.66	598.37	277 67	657.76	257.72	616.11
21-24	Pleasure	200.06	312.91	198 77	312.26	195.56	318.09
21-24	DTW Short	209.62	323.74	206 82	320.18	208.76	328.66
21-24	DTW Long	252.26	395.51	250.10	395.99	248.29	400 67
21-24	Business	326 46	518.11	324.64	519 12	323.28	537 11
25-29	Pleasure	197 04	272 30	196.73	271.76	196.67	273.53
25-29	DTW Short	213.02	279 50	212.57	280.36	211 69	281.30
25-29	DTW Long	255.31	338 45	253.74	338.05	254 69	342.84
25-29	Business	324.38	454 40	322.35	452.27	321.71	457.47
30-34	Pleasure	193.78	264.05	193 12	262.77	191.06	265.62
30-34	DTW Short	205.47	264.82	205.22	265.26	203.59	265.85
30-34	DTW Long	251 10	323.21	250.81	322 41	248.71	323 42
30-34	Business	316 42	436.16	314.93	433 69	315.41	439 43
35-39	Pleasure	155 51	212.04	154.64	212.05	154.29	212.11
35-39	DTW Short	166.78	213 24	166.14	213.32	165.21	214.54
35-39	DTW Long	200.79	260 60	200.55	261.02	198 82	263.34
35-39	Business	253.47	345 96	250 98	344.36	254.36	347.61
40-49	Pleasure	171.25	222.39	170.62	222.44	170.23	223.89
40-49	DTW Short	183.16	225.39	183.49	225.63	182.62	225 13
40-49	DTW Long	222.45	274.25	222.16	274.26	223.13	275 54
40-49	Business	277.71	368.54	276.58	364.18	278.55	371.04
50-59	Pleasure	173 47	226.93	172.88	226.73	173.55	228.03
50-59	DTW Short	187.16	228.85	187.30	229 00	186.60	229.03
50-59	DTW Long	225.66	280.26	225.69	280.43	224 79	281.50
50-59	Business	280.90	371.99	279 61	369.27	282.24	375.68
60+	Pleasure	169.76	223.18	168.20	221.76	168.22	223.76
60+	DTW Short	181.97	228 44	181 85	228.06	180.28	228 05
60+	DTW Long	217 10	278.89	217.39	278.64	216.90	280 57
60+	Business	272.16	372.71	272 14	372.18	270.86	376 79

Table 5.7: Summarized Statistics of Predicted Severities with Gamma Loss and Coefficient of Variation 3.0

	Vehicle	Mean			Bias			Standard Error		
Age	Use	G-G	G-L	G-N	G-G	G-L	G-N	G-G	G-L	G-N
17-20	Pleasure	257.52	230.58	255.72	2.62	-24.32	0.83	86.21	82.61	91.05
17-20	DTW Short	267.20	236.41	265.82	1 64	-29.15	0.26	89.76	83.92	96.62
17-20	DTW Long	324.01	291.59	323.56	1.83	-30.58	1.39	107.64	102.76	117.81
17-20	Business	422 28	446.80	425.16	3.22	27.73	6 10	145.96	162.99	166.59
21-24	Pleasure	256 19	249.79	256.05	2.49	-3.91	2.35	46 74	46.19	49.70
21-24	DTW Short	265 13	259.41	265 23	0.82	-4.90	0.92	43 92	43.52	46.77
21-24	DTW Long	322.33	315.84	323.53	1.67	-4.82	2.87	55 62	55.06	60.23
21-24	Business	419.14	408.64	424.14	2.04	-8.46	7.04	77.39	76.22	87.22
25-29	Pleasure	237.39	234.24	236.73	2.20	-0.95	1.54	30.43	30.22	32.09
25-29	DTW Short	245.76	244.99	245.30	0.74	-0.04	0.28	26.57	26.55	28.07
25-29	DTW Long	298.44	296.30	298.80	1.18	-0.95	1.54	32.26	32.16	34.83
25-29	Business	388 38	380.46	391 89	1.72	-6 20	5.23	50.92	50.05	56.39
30-34	Pleasure	227.81	225 52	226.64	2.43	0.15	1.27	27 68	27.37	29.44
30-34	DTW Short	235.78	235.38	234.79	0 98	0.58	-0.01	22.94	22.97	24.44
30-34	DTW Long	286.39	284 86	285.99	1 54	0.01	1.14	28.60	28.59	30.52
30-34	Business	372.51	365.01	374.85	1 99	-5.51	4.33	45.04	44 69	49.22
35-39	Pleasure	181.93	180.04	180 93	0.46	-1.43	-0.54	21.04	20 82	22.97
35-39	DTW Short	188.41	188.32	187 49	-0.66	-0.75	-1.57	18.14	18.17	19.54
35-39	DTW Long	228 91	227.95	228 40	-0.46	-1.42	-0.97	23.28	23.21	24.45
35-39	Business	297.90	292 30	299.37	-0.46	-6 06	1.02	37.73	37.16	39.44
40-49	Pleasure	197.77	196 05	196.66	1.44	-0 28	0.33	19 95	19.84	21.73
40-49	DTW Short	204.82	205.40	203 80	0 27	0 86	-0 74	15.94	15 99	16.94
40-49	DTW Long	248.70	248.46	248.17	0.56	0.31	0 02	19.20	19.25	20.32
40-49	Business	323.57	317.94	325.23	0.79	-4.83	2 45	33.97	33 53	35.81
50-59	Pleasure	200.61	198.32	199.95	1 28	-1.02	0.61	20.93	20 78	22.21
50-59	DTW Short	207.71	207.84	207.21	0.04	0.17	-0.46	16.55	16.58	17.52
50-59	DTW Long	252.35	251.67	252.51	0.40	-0.27	0.56	21.62	21.72	23.34
50-59	Business	328.47	323.07	331.15	0.74	-4 65	3.43	38.02	37.68	41.24
60+	Pleasure	194.72	191.96	193.88	-0 29	-3.05	-1 12	22.21	21.99	23.43
60+	DTW Short	201.57	201.25	200.93	-1.59	-1.92	-2 23	17.96	17.94	19.47
60+	DTW Long	245 07	244.05	244.97	-1 40	-2.42	-1.50	25.02	24 97	26.55
60+	Business	318.89	314.31	321.13	-1.71	-6.29	0 53	40.26	40 16	43.00

Table 5.8: 95% Confidence Intervals of Predicted Severities with Gamma Loss and Coefficient of Variation 3.0

Age	Vehicle Use	G-G Lower	G-G Upper	G-L Lower	G-L Upper	G-N Lower	G-N Upper
17-20	Pleasure	119.40	458.67	100.30	417.64	116.72	464.45
17-20	DTW Short	123.57	463.79	104.96	429.05	121.19	476.23
17-20	DTW Long	152.25	553.60	129.10	522.80	147.50	588.54
17-20	Business	197.98	736.88	192.31	811.30	189 61	791.06
21-24	Pleasure	174 28	354.30	169.13	348.49	168.95	363.52
21-24	DTW Short	187.71	356.20	182.58	347.55	184.32	362.97
21-24	DTW Long	224.39	437.50	216.92	434 11	218.99	450.27
21-24	Business	286.22	582 62	274.97	572.10	282.43	609.41
25-29	Pleasure	182.60	302.19	178.15	297.26	178.27	301.54
25-29	DTW Short	198.08	302 01	198 39	300.56	195.33	301.90
25-29	DTW Long	238.87	369.25	237.79	364.67	235.15	371.60
25-29	Business	295.64	493.23	288 69	486.40	294 77	512.37
30-34	Pleasure	176.23	288.30	175.55	284.70	171.04	290.97
30-34	DTW Short	193.91	284.31	194.30	283.27	189.77	283.53
30-34	DTW Long	230.77	343.30	229.00	341.21	227.96	345.22
30-34	Business	287.59	462.08	281.84	456.24	284.66	482.26
35-39	Pleasure	143.20	227.33	142.39	224.37	138.70	228.38
35-39	DTW Short	153.81	224.66	154.40	225.26	150.07	225.74
35-39	DTW Long	187.28	276.19	186.76	274.11	183.66	277.67
35-39	Business	229.72	377.77	224.79	368.38	227.96	381.40
40-49	Pleasure	160.82	239 99	159.21	238.90	157.29	243.15
40-49	DTW Short	175.46	237 44	176.41	238.21	172.95	238 09
40-49	DTW Long	213.32	287 29	212.09	286.54	210.15	288.20
40-49	Business	261.98	390.65	255.71	386.11	262.30	398.89
50-59	Pleasure	160 21	245.48	159.82	242.52	156.29	245.10
50-59	DTW Short	177.04	241.75	177.40	242.20	174.27	242.14
50-59	DTW Long	213 72	299.75	212.50	299.24	209.77	301.62
50-59	Business	261.28	402.18	258.73	397.92	261.86	418.11
60+	Pleasure	156.33	241.67	154.37	239.29	150.36	243.77
60+	DTW Short	167.59	239.55	167.96	238.12	164.24	241.78
60+	DTW Long	201.44	297.13	200.01	295.99	196.45	298.56
60+	Business	251.86	403.89	247.03	397.12	249 41	419.10

Table 5.9: Summarized Statistics of Predicted Severities with Lognormal Loss and Coefficient of Variation 1.0

	Vehicle	Mean			Bias			Standard Error		
Age	Use	L-G	L-L	L-N	L-G	L-L	L-N	L-G	L-L	L-N
17-20	Pleasure	255.65	255.55	255.69	0.76	0.65	0.79	28.25	27.67	29.69
17-20	DTW Short	266.26	264.52	266 36	0.70	-1.04	0.80	28.82	28.07	30.60
17-20	DTW Long	323.31	323 11	323.50	1.14	0.94	1.32	35.60	34.88	37.76
17-20	Business	420.12	442.39	420.79	1.06	23 32	1.73	47.25	48.78	50.10
21-24	Pleasure	254.78	254.88	254.83	1.08	1.18	1.13	14.83	14.70	15.76
21-24	DTW Short	265.36	264.98	265.44	1.05	0.67	1.12	14.34	14 19	15.29
21-24	DTW Long	322.19	322.14	322.36	1.53	1.48	1.70	18 02	17.82	19 46
21-24	Business	418.66	419.49	419.33	1.56	2 39	2.23	25.10	24.91	27 17
25-29	Pleasure	235.07	235 10	234.98	-0.11	-0.08	-0.21	10.06	10.04	10.37
25-29	DTW Short	244.82	244 90	244.75	-0.21	-0.12	-0.27	8.63	8.61	9.07
25-29	DTW Long	297.22	297.21	297.21	-0.04	-0.05	-0.05	10.69	10.68	11.30
25-29	Business	386.25	386.09	386.65	-0.40	-0.57	-0.01	17.15	17.07	18.17
30-34	Pleasure	225.53	225.69	225.39	0.16	0.32	0.02	9.19	9.18	9.70
30-34	DTW Short	234.88	234.96	234.75	0 08	0.16	-0.04	7.62	7.62	7.97
30-34	DTW Long	285.16	285.17	285.07	0 31	0.32	0.22	9.58	9.56	10.11
30-34	Business	370.56	370.24	370.82	0.03	-0.28	0.29	15.08	15.05	15.67
35-39	Pleasure	181.48	181 52	181.33	0 01	0.04	-0.14	7 01	6.98	7.47
35-39	DTW Short	189.01	189.07	188.87	-0.05	0.00	-0.19	5.91	5.91	6.27
35-39	DTW Long	229.47	229.48	229 35	0.10	0.11	-0.02	7.48	7 47	7.87
35-39	Business	298.19	297 96	298.33	-0.16	-0.39	-0.02	11.90	11.86	12.17
40-49	Pleasure	196.33	196.31	196.26	0.00	-0.02	-0.07	6.63	6.61	6.93
40-49	DTW Short	204.47	204.58	204.43	-0.07	0.04	-0.12	5.03	5.03	5.33
40-49	DTW Long	248.24	248.28	248.23	0.09	0.14	0.09	6.25	6.24	6.73
40-49	Business	322 59	322.22	322.91	-0.19	-0.56	0.13	11.32	11 27	11.64
50-59	Pleasure	199 40	199.30	199.32	0.06	-0.03	-0.02	6.87	6.84	7.14
50-59	DTW Short	207.68	207.74	207.62	0.00	0 06	-0 06	5.47	5.46	5.81
50-59	DTW Long	252.14	252 17	252.12	0.20	0.23	0.18	7.51	7.50	7.83
50-59	Business	327.62	327.49	327.93	-0.10	-0.23	0.21	11.49	11.45	11.90
60+	Pleasure	195.03	194.89	194.92	0.03	-0.12	-0.09	7.17	7.14	7.53
60+	DTW Short	203.14	203.18	203.05	-0.02	0.02	-0.11	6.46	6 45	6.87
60+	DTW Long	246.64	246 70	246.57	0.17	0.23	0.10	8.40	8 39	8.80
60+	Business	320.50	320.75	320 74	-0 10	0 15	0.14	13.31	13.31	13.70

Table 5.10: 95% Confidence Intervals of Predicted Severities with Lognormal Loss and Coefficient of Variation 1.0

Age	Vehicle Use	G-G		G-L		G-N	
		Lower	Upper	Lower	Upper	Lower	Upper
17-20	Pleasure	205.05	316 70	203.40	307.55	203 19	318.52
17-20	DTW Short	214.43	328.01	211.71	320.61	211.73	330.81
17-20	DTW Long	259 17	401.24	256.36	390.40	256.56	402.51
17-20	Business	337.08	522.18	333.09	511.61	330.32	529.09
21-24	Pleasure	226.56	283.43	225.60	281.63	225.85	287.69
21-24	DTW Short	238.46	295.12	237.34	293.23	237.01	298 77
21-24	DTW Long	284 28	358 08	284.01	355.83	284.56	361.52
21-24	Business	370.62	469.04	368.95	464 63	368.65	479.95
25-29	Pleasure	216.26	256.18	216.02	255.22	216.23	256 25
25-29	DTW Short	228 67	262.28	228.51	262.00	228.41	263 20
25-29	DTW Long	276.69	317 48	276.13	317.43	276.64	319.12
25-29	Business	353.85	422.39	353.10	420.98	354 53	426.60
30-34	Pleasure	208 28	243.70	208.13	242.93	207.73	244.50
30-34	DTW Short	220 65	251.47	220.63	250.98	220.31	251 32
30-34	DTW Long	267.84	305.43	267 76	305.27	267.06	306.66
30-34	Business	341.59	400.71	340.58	399.32	341 81	404.39
35-39	Pleasure	168.71	195 98	168.23	195.07	167.11	195 74
35-39	DTW Short	177.91	200 87	177.83	200.70	176.99	200.95
35-39	DTW Long	215.74	245.05	215.66	245 02	214.24	245.06
35-39	Business	276.01	322.57	275.42	321.61	276.00	322.86
40-49	Pleasure	184.24	209.35	183.89	209.06	183.42	210.19
40-49	DTW Short	194.75	214.64	194 72	214 56	194.01	214.94
40-49	DTW Long	236.39	260.36	236 16	260 14	235.00	261.73
40-49	Business	301 33	344.57	300.79	343.99	301.76	345.62
50-59	Pleasure	185 96	212.58	185.85	212.37	185.65	212 70
50-59	DTW Short	197.23	219.06	197 24	218.95	196.83	219.79
50-59	DTW Long	238.18	267.51	238.03	267.33	238 02	268.39
50-59	Business	306.11	351 17	305.46	350.51	306 26	352 32
60+	Pleasure	180.79	209 77	180.52	209.25	179.41	209 82
60+	DTW Short	190.49	217.81	190.44	217.72	189.52	217.20
60+	DTW Long	230.66	263.46	230.34	263.25	229.77	264.51
60+	Business	295 57	348.05	294.77	347.68	295.51	348 73

Table 5.11: Summarized Statistics of Predicted Severities with Lognormal Loss and Coefficient of Variation 2.0

	Vehicle	Mean			Bias			Standard Error		
Age	Use	L-G	L-L	L-N	L-G	L-L	L-N	L-G	L-L	L-N
17-20	Pleasure	255.85	251.64	256.33	0 95	-3 26	1.44	55.75	49.77	59.13
17-20	DTW Short	265.63	258.13	266.16	0.07	-7.43	0.60	57.32	50.41	61.16
17-20	DTW Long	322.42	317.51	323.67	0.25	-4.66	1.50	69.90	62.08	75.29
17-20	Business	418.84	464.40	421.82	-0.22	45.34	2.75	93.53	93.27	101.45
21-24	Pleasure	254.33	253.11	254.09	0.63	-0.59	0.39	28.83	27.88	31.54
21-24	DTW Short	263.96	262.19	263.61	-0.36	-2.13	-0.70	27.46	26 46	29.34
21-24	DTW Long	320.45	319.10	320.57	-0.21	-1.56	-0.09	34 77	33.68	37.51
21-24	Business	416 30	415.52	417.92	-0.80	-1.58	0.82	50 90	49.07	56.94
25-29	Pleasure	235.71	234.96	235.68	0.52	-0.23	0.50	19 58	19.15	20.76
25-29	DTW Short	244.64	244.53	244.60	-0 38	-0 49	-0.42	17.24	17 02	18.44
25-29	DTW Long	296.89	296.36	297.34	-0.37	-0 89	0.08	21.27	20.96	23.18
25-29	Business	385.55	383 55	387.36	-1.11	-3.11	0.70	33.33	32 45	36.85
30-34	Pleasure	225 01	224 65	224.82	-0.36	-0.72	-0.56	17.52	17.25	18.80
30-34	DTW Short	233.52	233.47	233.28	-1.28	-1.33	-1.51	14.69	14.54	15.80
30-34	DTW Long	283 42	283 06	283.59	-1.43	-1.80	-1.26	18.75	18.53	20.24
30-34	Business	368.05	365.88	369.46	-2.48	-4.64	-1.07	29.87	29.08	33.07
35-39	Pleasure	182.07	181.57	181.48	0.60	0 10	0 01	14.76	14.35	16.12
35-39	DTW Short	188.92	188.88	188.26	-0 14	-0 18	-0.80	12.04	11.86	13.02
35-39	DTW Long	229.27	228.96	228.81	-0 10	-0.41	-0.56	14.77	14.53	15 85
35-39	Business	297.76	296.07	298.05	-0.59	-2.29	-0.30	24.34	23.69	25.74
40-49	Pleasure	196.89	196.42	196.61	0.56	0.09	0 28	12 88	12.65	14.00
40-49	DTW Short	204.36	204.62	204.04	-0.19	0 07	-0 51	9.95	9.94	10.79
40-49	DTW Long	248.03	248.04	248.02	-0.12	-0.11	-0.13	12.98	12.92	13.83
40-49	Business	322.09	320 30	323 04	-0.68	-2.48	0.27	22 92	22.28	24.17
50-59	Pleasure	199.53	198.85	198 99	0.20	-0.49	-0.34	13.34	13.09	14 40
50-59	DTW Short	207.12	207.25	206 54	-0.55	-0.42	-1.14	11.03	11.01	11.86
50-59	DTW Long	251 38	251.32	251 07	-0.57	-0.63	-0.88	14.15	14.06	15.27
50-59	Business	326 42	325 12	327.02	-1.30	-2 60	-0 71	23.84	23 34	25.70
60+	Pleasure	195 55	194.61	195.32	0.54	-0.39	0.32	14.17	13.90	15.28
60+	DTW Short	202 99	202.92	202.74	-0.17	-0 24	-0 42	12.32	12.19	13.28
60+	DTW Long	246.40	246.23	246.49	-0 07	-0 24	0 02	16.13	15.91	17.45
60+	Business	319.99	319.33	321 15	-0 61	-1 27	0.55	26.10	25.31	29 18

Table 5.12: 95% Confidence Intervals of Predicted Severities with Lognormal Loss and Coefficient of Variation 2.0

Age	Vehicle Use	L-G Lower	L-G Upper	L-L Lower	L-L Upper	L-N Lower	L-N Upper
17-20	Pleasure	169.68	394.47	168.87	366.93	166.67	399.58
17-20	DTW Short	179 65	405.01	177.92	374.96	175.01	409 57
17-20	DTW Long	210.88	495.47	213.68	463.80	209.70	502 79
17-20	Business	271.22	635.59	304 54	663.21	266.23	670 44
21-24	Pleasure	205.17	312 46	204.53	309.27	201.69	318.60
21-24	DTW Short	216.70	323.67	215 35	321 28	213.56	324.67
21-24	DTW Long	258.44	393.94	258.94	391 20	259.45	398.67
21-24	Business	329.32	530.54	329.47	525.37	327.32	543 69
25-29	Pleasure	200 46	274.84	200.99	273.93	198.76	279.92
25-29	DTW Short	212.92	282 04	213 04	281.60	210.55	283.22
25-29	DTW Long	256.54	343.83	256.74	342.09	256.53	346.75
25-29	Business	328.91	455.32	327.13	451 43	325 82	469.24
30-34	Pleasure	193.74	263.39	193.06	263.06	190.05	264.27
30-34	DTW Short	207.16	264.35	207.40	264.12	205 53	265.69
30-34	DTW Long	250.27	322.50	250.03	321.97	248 98	325.70
30-34	Business	316.23	424 68	314.65	421.11	313.51	437.02
35-39	Pleasure	156.79	212 64	156.19	212.36	152.92	213.55
35-39	DTW Short	168.63	214.29	168.49	214.08	164.60	214.69
35-39	DTW Long	203.37	260.87	203.10	260.29	201.08	260.93
35-39	Business	254.97	347.13	253.51	343.22	253.65	350.60
40-49	Pleasure	174.01	222 20	173 79	221.63	172.78	225.16
40-49	DTW Short	185.78	224.90	185.79	224.75	184.21	226.20
40-49	DTW Long	221.97	274.58	222.26	274 31	220.83	276.09
40-49	Business	280.72	370.43	279.97	367.95	281.36	373.81
50-59	Pleasure	173 89	225.92	173.44	225.01	171.95	229 45
50-59	DTW Short	186.98	229.99	187.14	230.30	184.86	230 96
50-59	DTW Long	224.64	281.15	225 04	280.18	223.36	283.44
50-59	Business	283.77	376.14	282.83	372.19	281.38	382.39
60+	Pleasure	169.52	226.13	169.24	225.40	167.43	227.65
60+	DTW Short	180.88	229.26	181.06	228.47	180 05	229.66
60+	DTW Long	217.11	280.25	216.93	279.46	216 22	281.31
60+	Business	272.80	372.72	273.14	371.94	273 31	382.09

Table 5.13: Summarized Statistics of Predicted Severities with Lognormal Loss and Coefficient of Variation 3.0

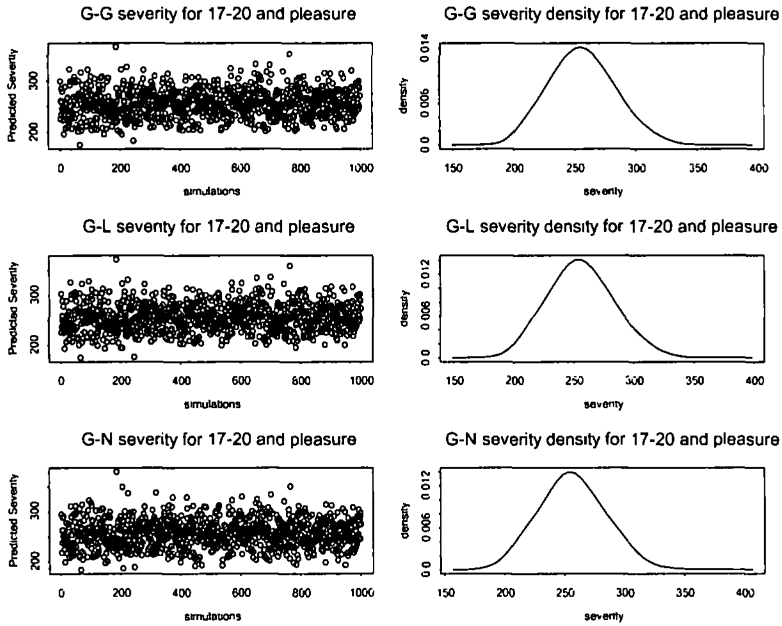
Age	Vehicle Use	Mean			Bias			Standard Error		
		L-G	L-L	L-N	L-G	L-L	L-N	L-G	L-L	L-N
17-20	Pleasure	257.16	246.40	258.06	2.26	-8.50	3.16	85.62	67.59	117.76
17-20	DTW Short	268.54	253.93	267.82	2.98	-11.63	2.26	85.90	67.80	89.92
17-20	DTW Long	324.71	312.54	325.60	2.54	-9.63	3.43	104.78	84.05	115.61
17-20	Business	422.41	480.72	425.05	3.34	61.65	5.98	140.16	134.16	154.98
21-24	Pleasure	254.82	250.68	254.43	1.12	-3.02	0.73	40.94	37.56	45.87
21-24	DTW Short	266.36	262.26	265.81	2.04	-2.06	1.50	40.30	37.18	45.00
21-24	DTW Long	321.82	317.71	322.71	1.16	-2.95	2.05	49.09	45.70	57.36
21-24	Business	418.49	413.48	421.22	1.40	-3.62	4.12	70.58	65.59	83.84
25-29	Pleasure	236.11	233.72	235.41	0.92	-1.47	0.23	27.95	26.32	31.21
25-29	DTW Short	246.79	246.15	245.89	1.77	1.13	0.86	25.80	24.91	28.01
25-29	DTW Long	298.07	296.40	298.13	0.81	-0.85	0.87	30.77	29.33	33.63
25-29	Business	387.89	382.77	389.66	1.23	-3.89	3.00	51.09	47.36	58.73
30-34	Pleasure	226.07	224.16	226.09	0.70	-1.21	0.72	27.30	25.28	33.06
30-34	DTW Short	236.03	235.42	235.68	1.23	0.62	0.88	22.03	21.28	24.80
30-34	DTW Long	285.20	283.71	285.96	0.35	-1.15	1.11	27.71	26.58	31.74
30-34	Business	370.86	365.53	373.41	0.33	-5.00	2.89	44.73	41.68	54.94
35-39	Pleasure	181.61	180.08	180.72	0.13	-1.40	-0.76	20.60	19.56	24.19
35-39	DTW Short	189.67	189.44	188.53	0.61	0.37	-0.53	16.98	16.57	19.27
35-39	DTW Long	229.19	228.31	228.69	-0.18	-1.06	-0.68	21.60	21.00	24.12
35-39	Business	298.06	294.29	298.47	-0.29	-4.06	0.12	35.52	33.60	39.75
40-49	Pleasure	195.69	194.27	195.05	-0.64	-2.06	-1.28	19.43	18.52	23.19
40-49	DTW Short	204.40	204.74	203.48	-0.14	0.20	-1.06	14.83	14.73	16.66
40-49	DTW Long	247.03	246.74	246.92	-1.12	-1.41	-1.23	19.79	19.41	22.24
40-49	Business	321.11	317.34	322.08	-1.67	-5.44	-0.70	33.26	31.46	37.36
50-59	Pleasure	199.01	197.17	198.29	-0.33	-2.17	-1.05	20.50	19.49	23.42
50-59	DTW Short	207.89	207.93	206.95	0.22	0.26	-0.72	16.55	16.31	18.01
50-59	DTW Long	251.24	250.71	251.12	-0.71	-1.24	-0.82	21.46	20.97	23.55
50-59	Business	326.62	323.32	327.89	-1.11	-4.41	0.17	35.76	33.63	44.49
60+	Pleasure	194.21	191.98	193.59	-0.79	-3.02	-1.42	20.49	19.59	23.44
60+	DTW Short	203.00	202.66	202.22	-0.16	-0.50	-0.94	18.29	17.80	20.91
60+	DTW Long	245.37	244.57	245.41	-1.10	-1.90	-1.06	23.93	23.06	27.15
60+	Business	319.04	316.58	320.35	-1.56	-4.02	-0.25	38.26	36.22	45.39

Table 5.14: 95% Confidence Intervals of Predicted Severities with Lognormal Loss and Coefficient of Variation 3.0

Age	Vehicle Use	L-G Lower	L-G Upper	L-L Lower	L-L Upper	L-N Lower	L-N Upper
17-20	Pleasure	150.84	453 79	147.45	408.89	145.28	467 57
17-20	DTW Short	161.08	473.55	153.81	411 66	159.79	483 70
17-20	DTW Long	189.15	565 01	185.34	501.03	183 93	584 01
17-20	Business	239 60	742.03	261.24	752.07	236 26	766 60
21-24	Pleasure	186.60	352 85	185.83	336.33	180 20	365 38
21-24	DTW Short	201 75	357 23	200.30	344.52	196.74	377.53
21-24	DTW Long	242 47	434.58	240.99	419.43	235.19	461.13
21-24	Business	307.98	586.35	306.03	559.09	300.17	627.96
25-29	Pleasure	190.82	297.56	189.29	288.25	185.34	302.97
25-29	DTW Short	205.21	303.08	204.96	300.56	199.10	304.64
25-29	DTW Long	246.31	367.25	245.78	360.33	243.93	376.80
25-29	Business	297.11	503.33	296.69	485.59	297.80	516.34
30-34	Pleasure	184.39	283.27	182.85	279.17	177.12	293.13
30-34	DTW Short	199 74	283.08	199.18	280.41	192.81	288.72
30-34	DTW Long	240 12	348.16	239.32	342.69	232 60	357.79
30-34	Business	293.64	465 80	292.36	452.40	291 62	488.31
35-39	Pleasure	145 54	224.85	145.63	220.97	141 15	231.34
35-39	DTW Short	160.64	225 26	160.97	224.35	156.05	227.58
35-39	DTW Long	193.77	276 10	192.86	273.97	189.01	279.85
35-39	Business	238.76	378.53	235.37	369.80	237.17	394.84
40-49	Pleasure	162.62	239.36	161 18	233.75	157.09	241.60
40-49	DTW Short	178.70	237.68	178 91	238.54	173.96	238.30
40-49	DTW Long	213 76	288.45	213.47	287.55	209.57	292.07
40-49	Business	265.59	391.26	262.73	385.95	262 02	400.44
50-59	Pleasure	163 04	243.77	162.93	238.02	161 17	244.78
50-59	DTW Short	180.15	245 63	180.36	243.67	175 68	244.27
50-59	DTW Long	211 62	297.42	211.79	295 67	207.37	298.99
50-59	Business	265.87	402 81	264.66	396 51	263.59	409 25
60+	Pleasure	158.82	239.61	158.17	234 44	153.71	243.81
60+	DTW Short	171.32	243.17	170 95	240.05	166.97	245 22
60+	DTW Long	206.10	300.94	205.74	296.58	198.58	308 97
60+	Business	257.01	404.61	254 96	394.06	251.04	420.91

Appendix 6
Plots of Predicted Severities for Selected Classifications

Figure 6.1: Scatter and Density Plots of Predicted Severities for Gamma Loss with Coefficient of Variation 1.0 for Age 17-20 and Pleasure Use



- G-G implies that the loss follows a Gamma distribution and a Gamma distribution is assumed in the regressions; similarly, G-L implies the loss follows a Gamma distribution but a Lognormal is assumed in the regressions; and G-N implies the loss follows a Gamma distribution but a Normal is assumed in the regressions.
- The density function is estimated by the non-parametric method from Silverman (1986).

Figure 6.2: Scatter and Density Plots of Predicted Severities for Gamma Loss with Coefficient of Variation 1.0 for Age 40-49 and DTW Short Use

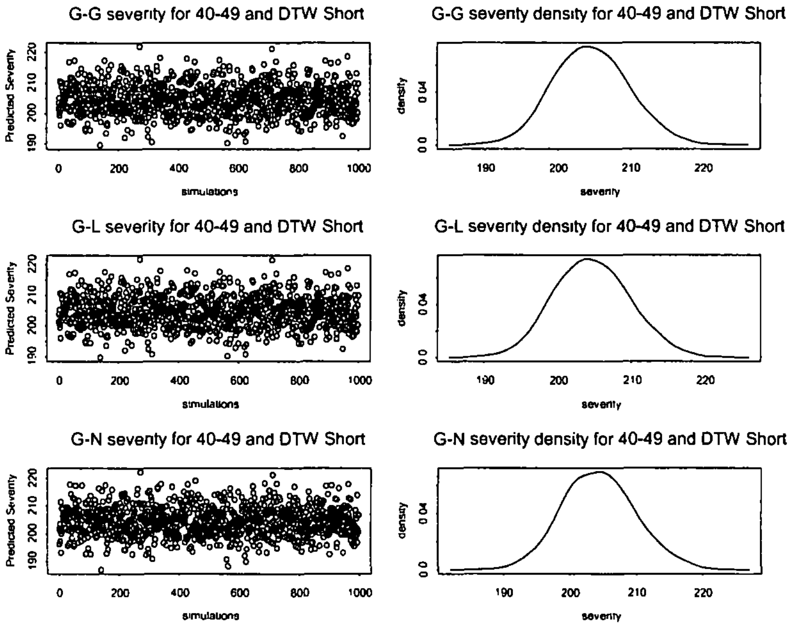


Figure 6.3: Scatter and Density Plots of Predicted Severities for Gamma Loss with Coefficient of Variation 2.0 for Age 17-20 and Pleasure Use

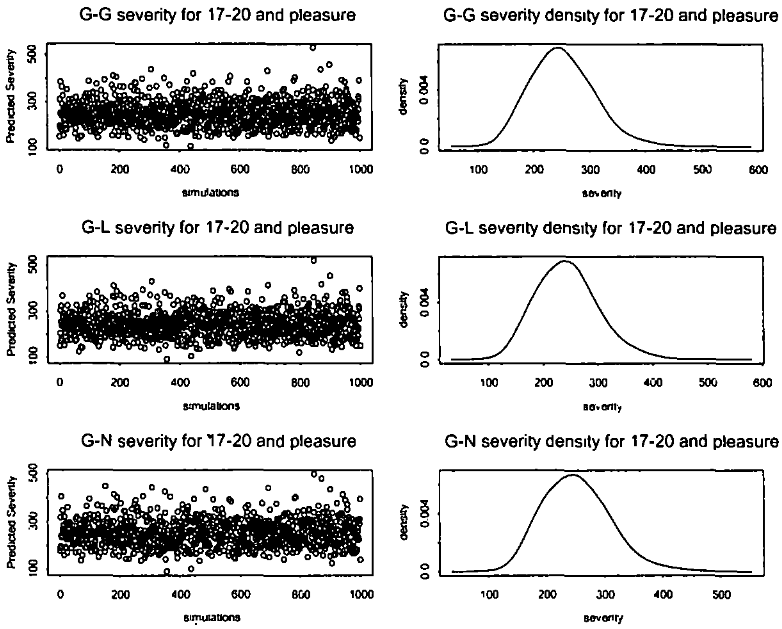


Figure 6.4: Scatter and Density Plots of Predicted Severities for Gamma Loss with Coefficient of Variation 2.0 for Age 40-49 and DTW Short Use

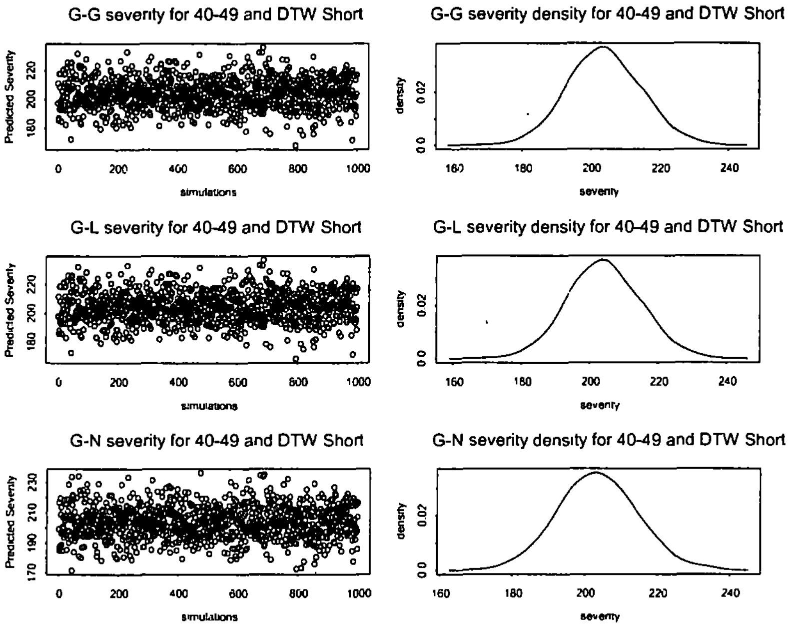


Figure 6.5: Scatter and Density Plots of Predicted Severities for Gamma Loss with Coefficient of Variation 3.0 for Age 17-20 and Pleasure Use

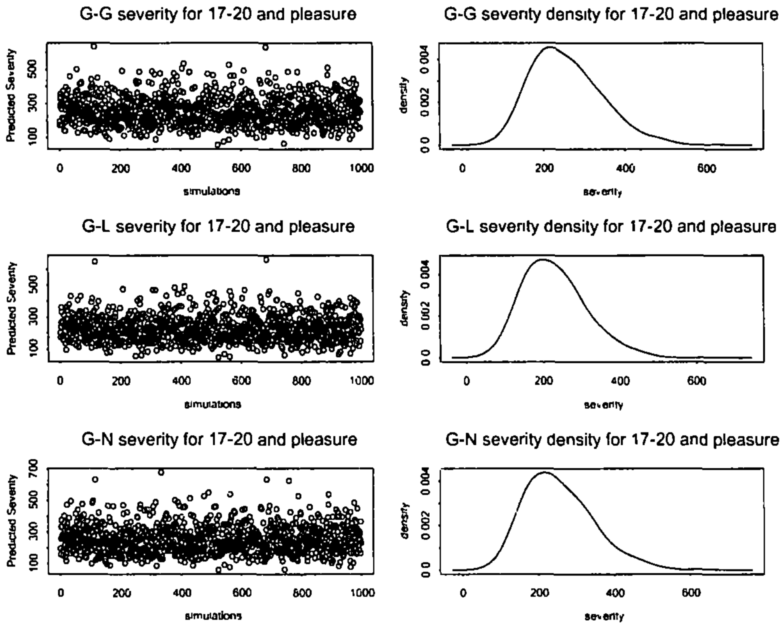


Figure 6.6: Scatter and Density Plots of Predicted Severities for Gamma Loss with Coefficient of Variation 3.0 for Age 40-49 and DTW Short Use

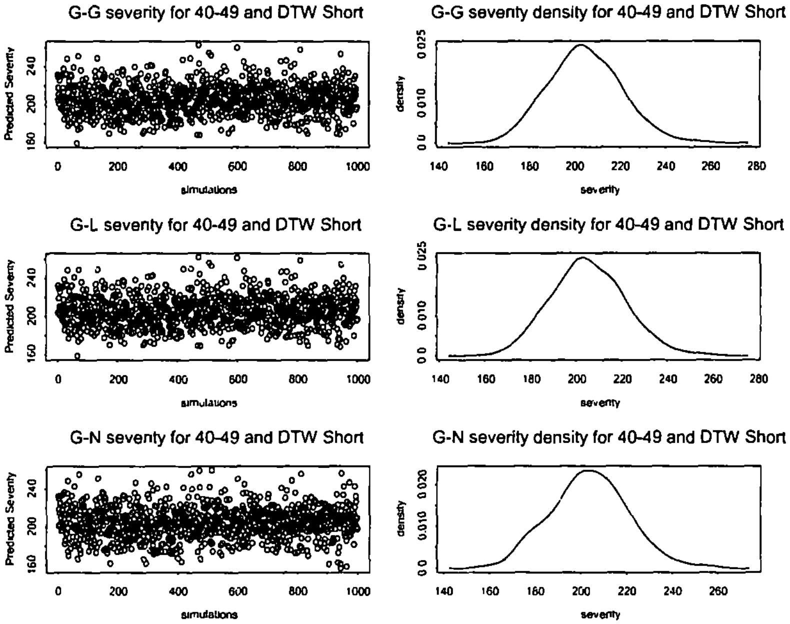


Figure 6.7: Scatter and Density Plots of Predicted Severities for Lognormal Loss with Coefficient of Variation 1.0 for Age 17-20 and Pleasure Use

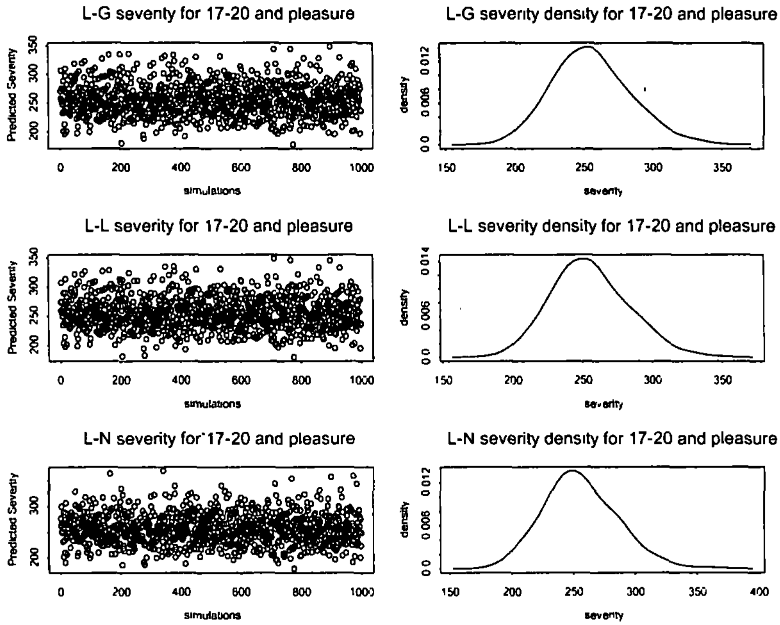


Figure 6.8: Scatter and Density Plots of Predicted Severities for Lognormal Loss with Coefficient of Variation 1.0 for Age 40-49 and DTW Short Use

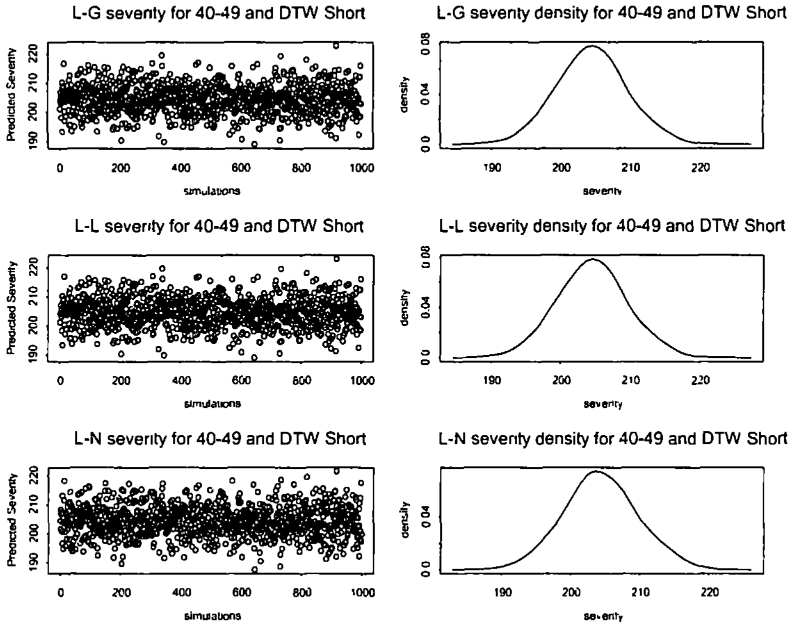


Figure 6.9: Scatter and Density Plots of Predicted Severities for Lognormal Loss with Coefficient of Variation 2.0 for Age 17-20 and Pleasure Use

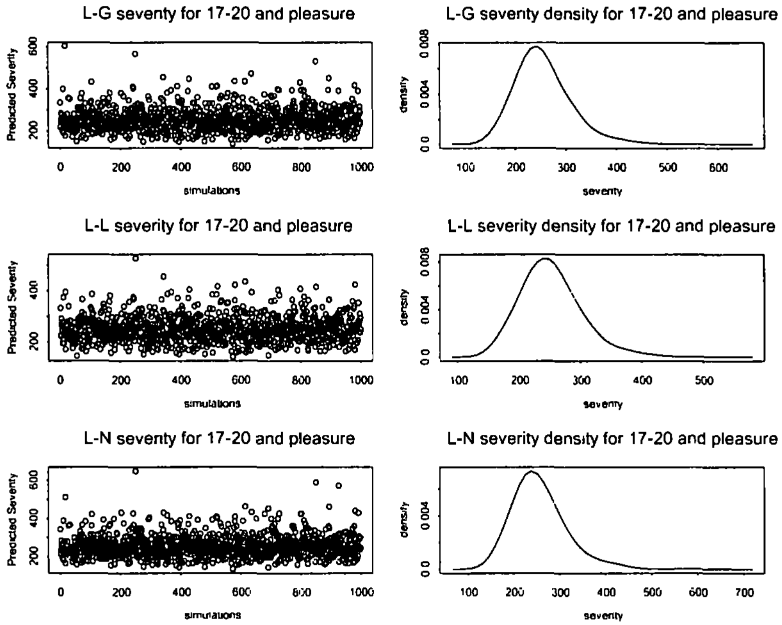


Figure 6.10: Scatter and Density Plots of Predicted Severities for Lognormal Loss with Coefficient of Variation 2.0 for Age 40-49 and DTW Short Use

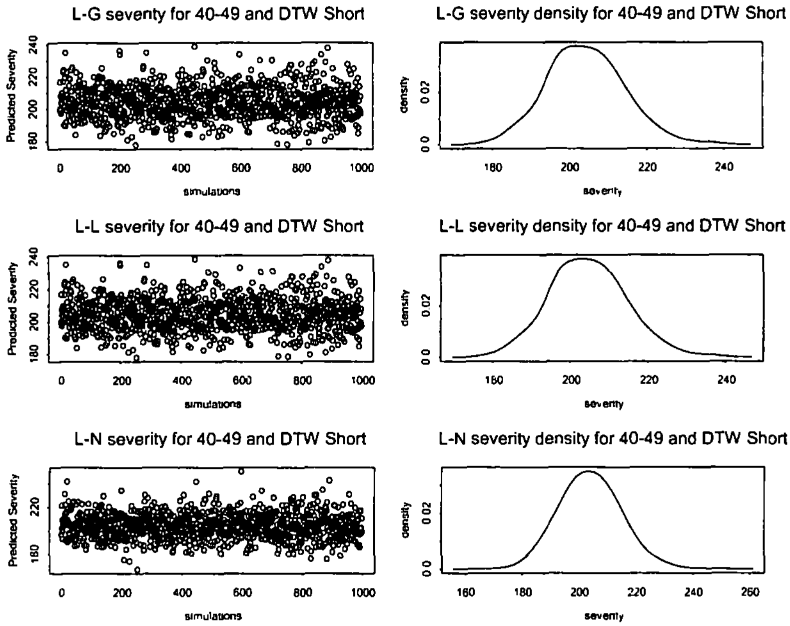


Figure 6.11: Scatter and Density Plots of Predicted Severities for Lognormal Loss with Coefficient of Variation 3.0 for Age 17-20 and Pleasure Use

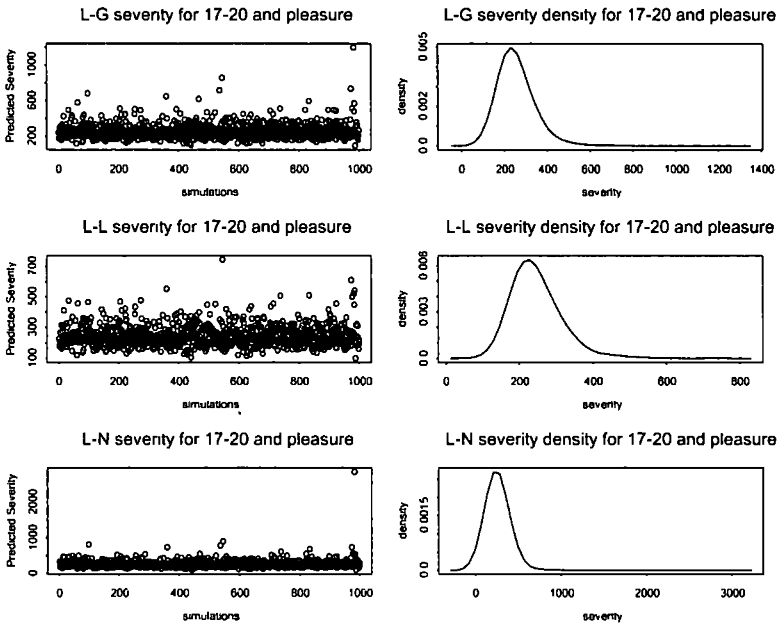
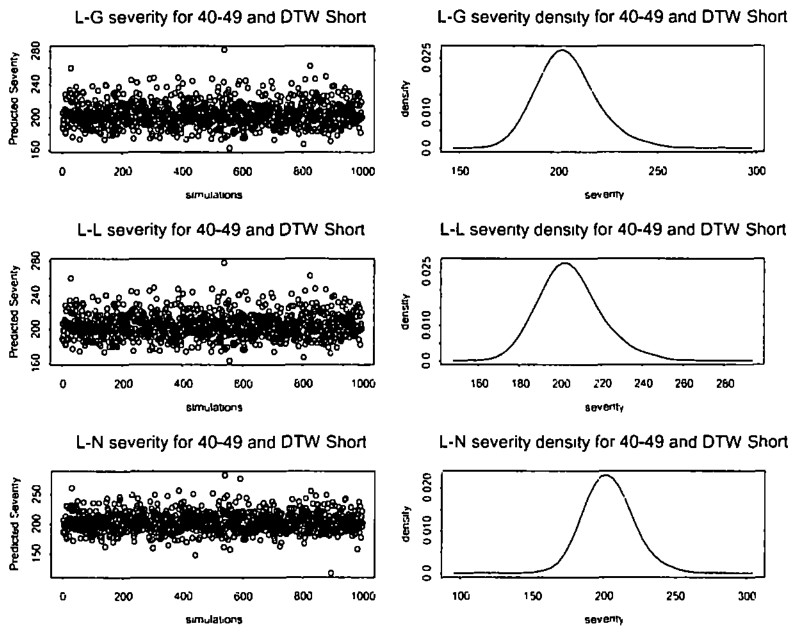
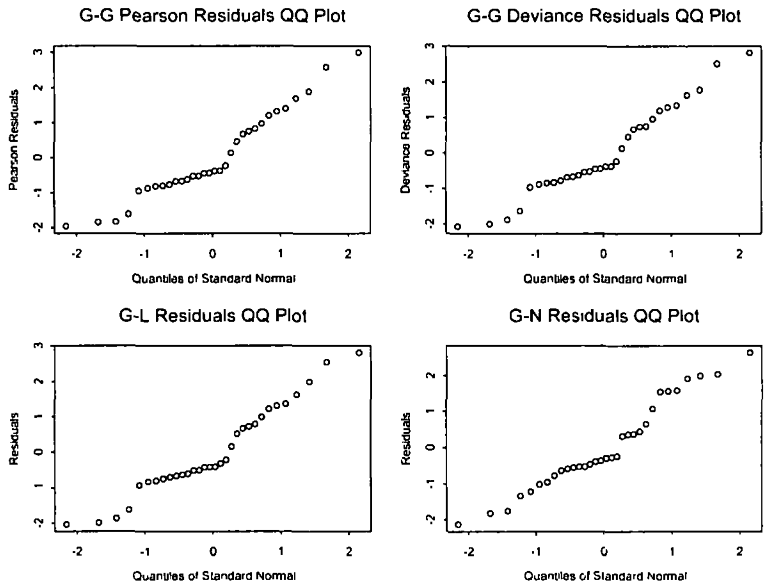


Figure 6.12: Scatter and Density Plots of Predicted Severities for Lognormal Loss with Coefficient of Variation 3.0 for Age 40-49 and DTW Short Use



Appendix 7
Residual Plots for Regression Diagnostics

Figure 7.1: QQ Plots of Standardized Residuals for Gamma Loss with Coefficient of Variation 1.0



G-G implies that the loss follows a Gamma distribution and a Gamma distribution is assumed in the regressions; similarly, G-L implies the loss follows a Gamma distribution but a Lognormal is assumed in the regressions, and G-N implies the loss follows a Gamma distribution but a Normal is assumed in the regressions.

Figure 7.2: Plots of Predicted Severities vs Standardized Residuals for Gamma Loss with Coefficient of Variation 1.0

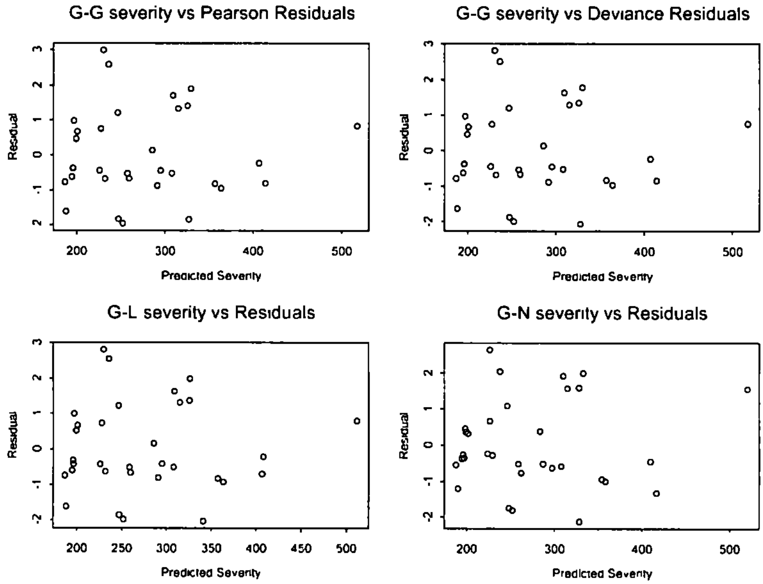


Figure 7.3: QQ Plots of Standardized Residuals for Gamma Loss with Coefficient of Variation 2.0

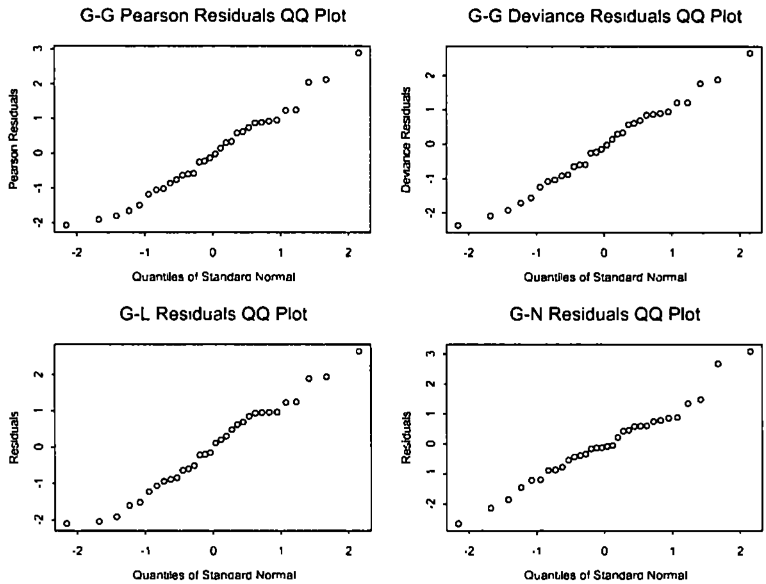


Figure 7.4: Plots of Predicted Severities vs Standardized Residuals for Gamma Loss with Coefficient of Variation 2.0

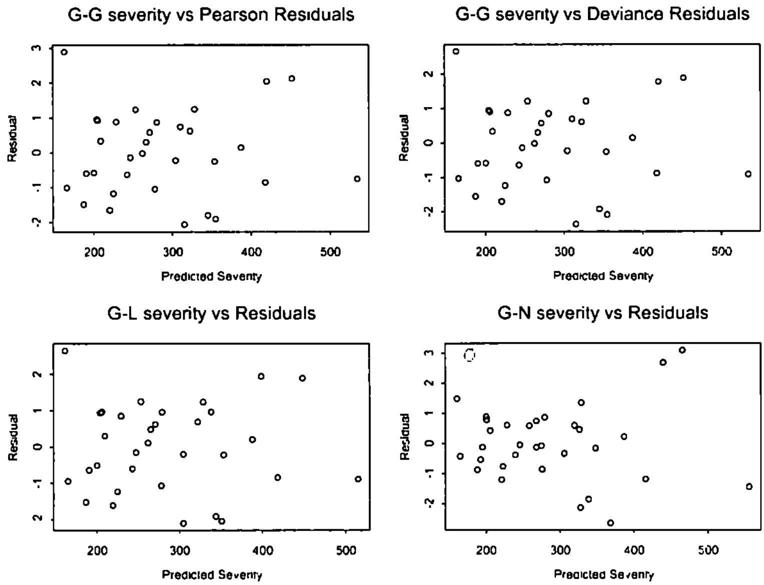


Figure 7.5: QQ Plots of Standardized Residuals for Gamma Loss with Coefficient of Variation 3.0

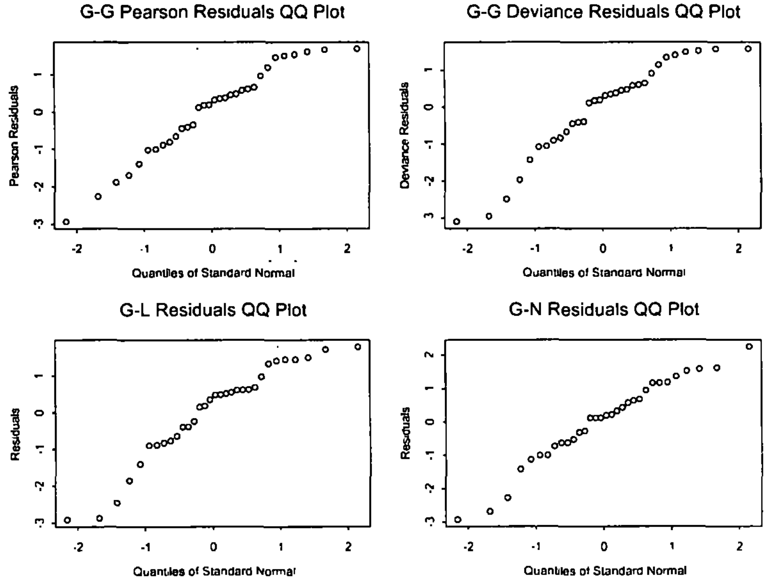


Figure 7.6: Plots of Predicted Severities vs Standardized Residuals for Gamma Loss with Coefficient of Variation 3.0

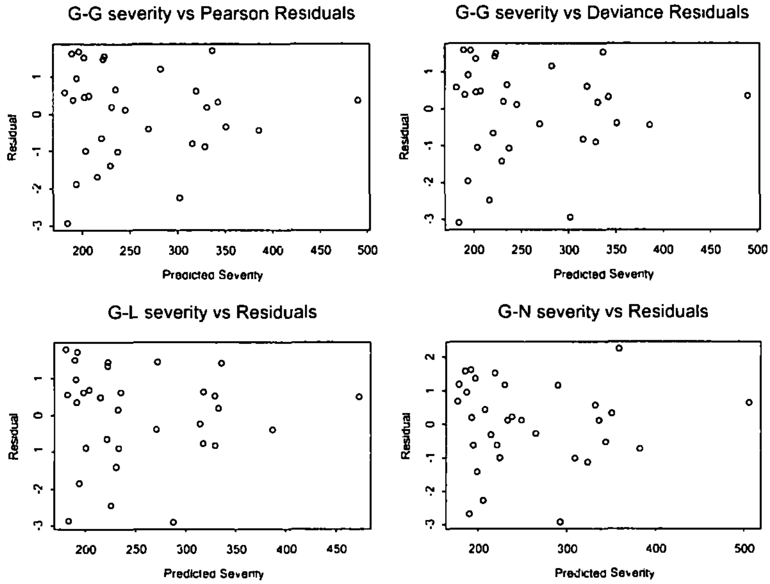


Figure 7.7: QQ Plots of Standardized Residuals for Lognormal Loss with Coefficient of Variation 1.0

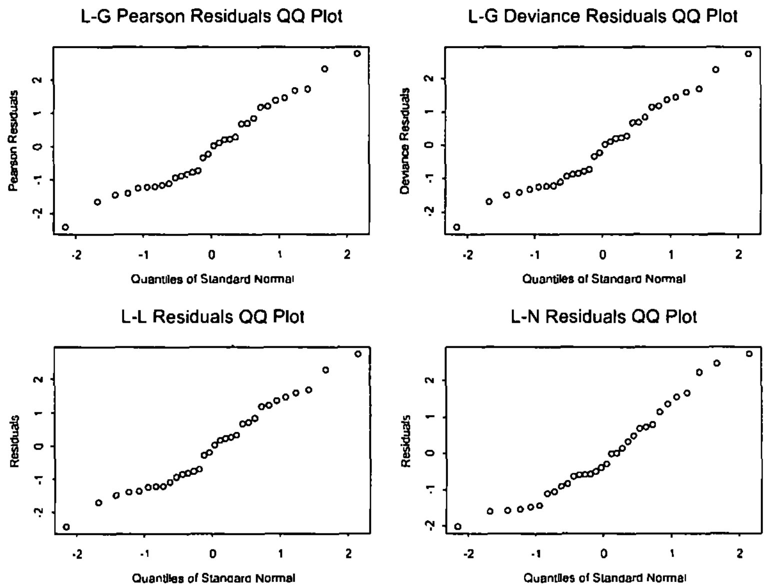


Figure 7.8: Plots of Predicted Severities vs Standardized Residuals for Lognormal Loss with Coefficient of Variation 1.0

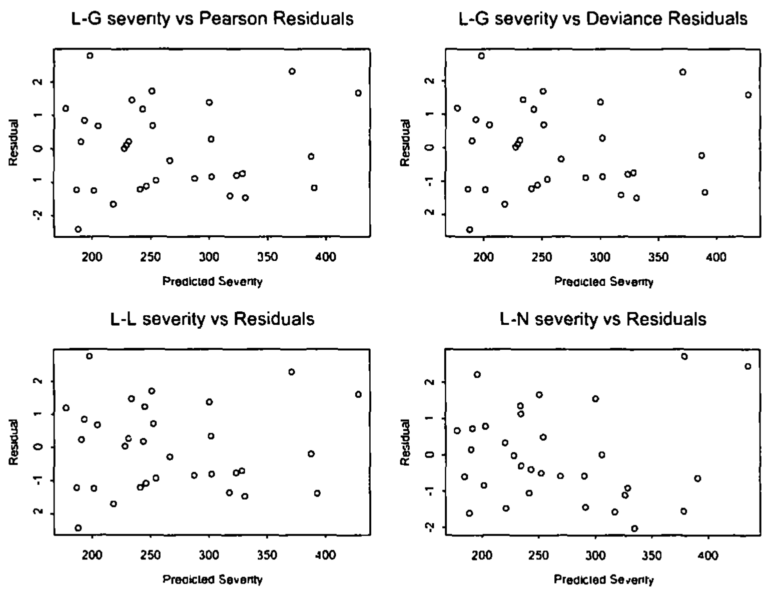


Figure 7.9: QQ Plots of Standardized Residuals for Lognormal Loss with Coëfficient of Variation 2.0

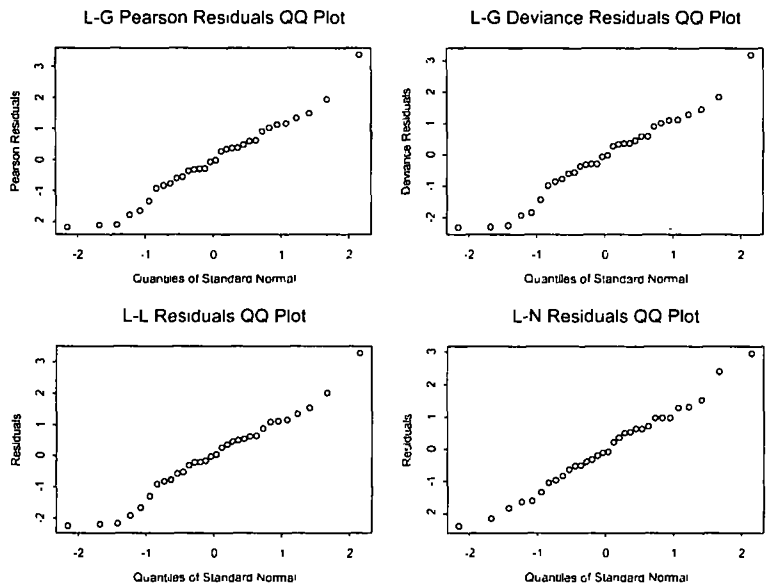


Figure 7.10: Plots of Predicted Severities vs Standardized Residuals for Lognormal Loss with Coefficient of Variation 2.0

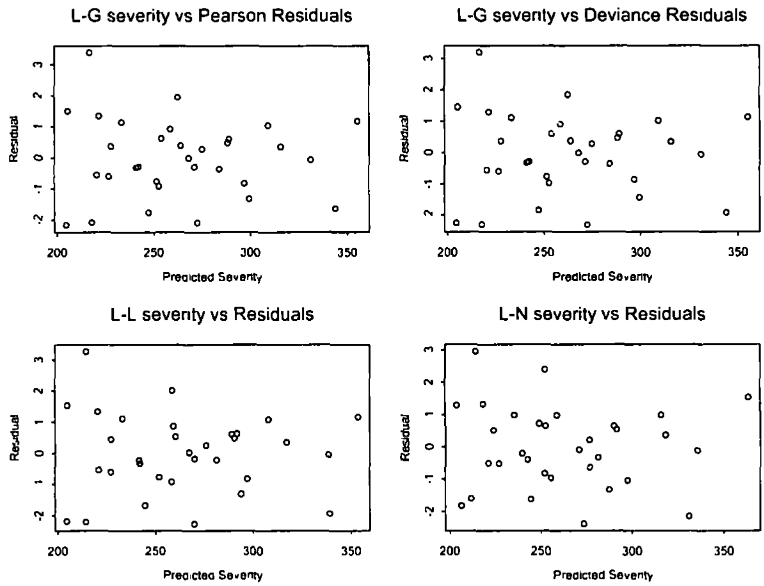


Figure 7.11: QQ Plots of Standardized Residuals for Lognormal Loss with Coefficient of Variation 3.0

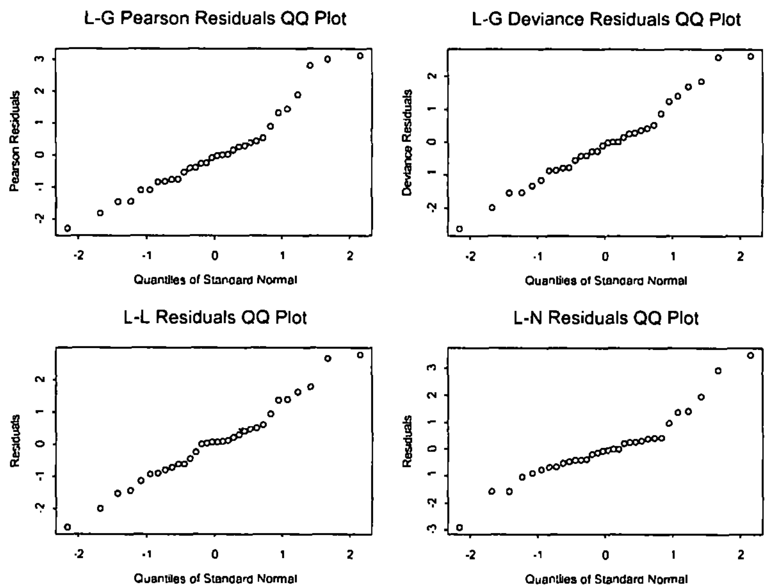


Figure 7.12: Plots of Predicted Severities vs Standardized Residuals for Lognormal Loss with Coefficient of Variation 3.0

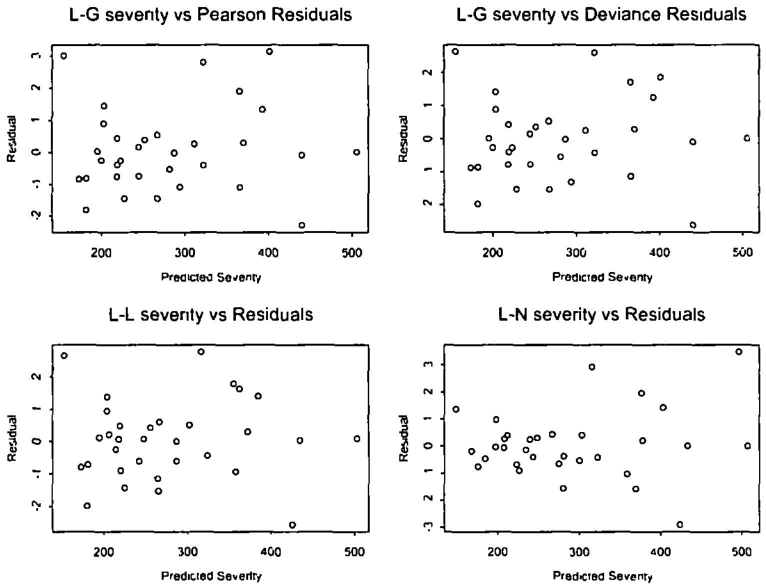


Figure 7.13: QQ Plots of Standardized Residuals for Gamma Loss with Coefficient of Variation 1.0 based on Individual Data

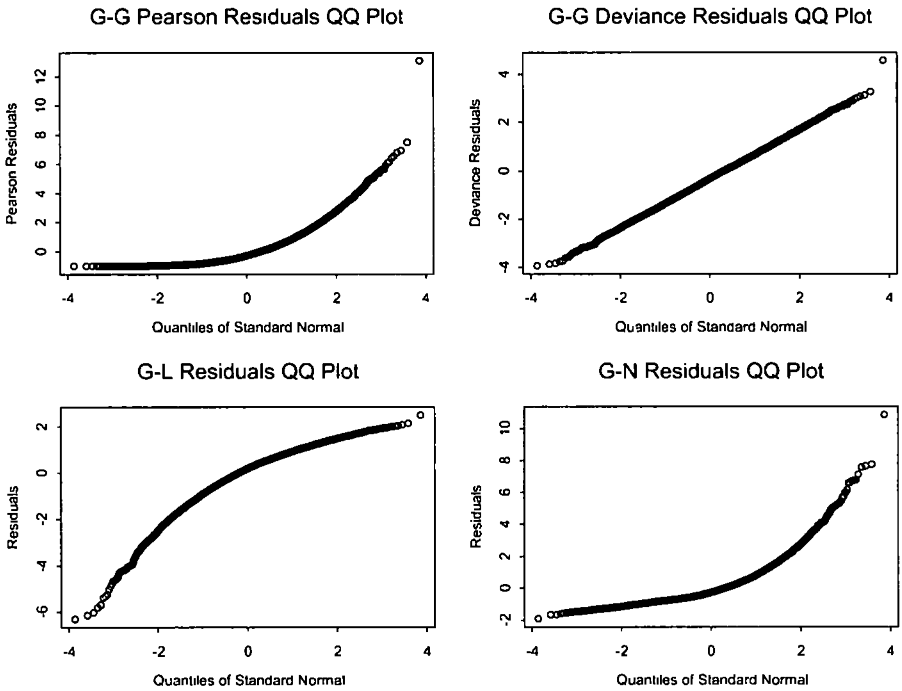


Figure 7.14: Plots of Predicted Severities vs Standardized Residuals for Gamma Loss with Coefficient of Variation 1.0 based on Individual Data

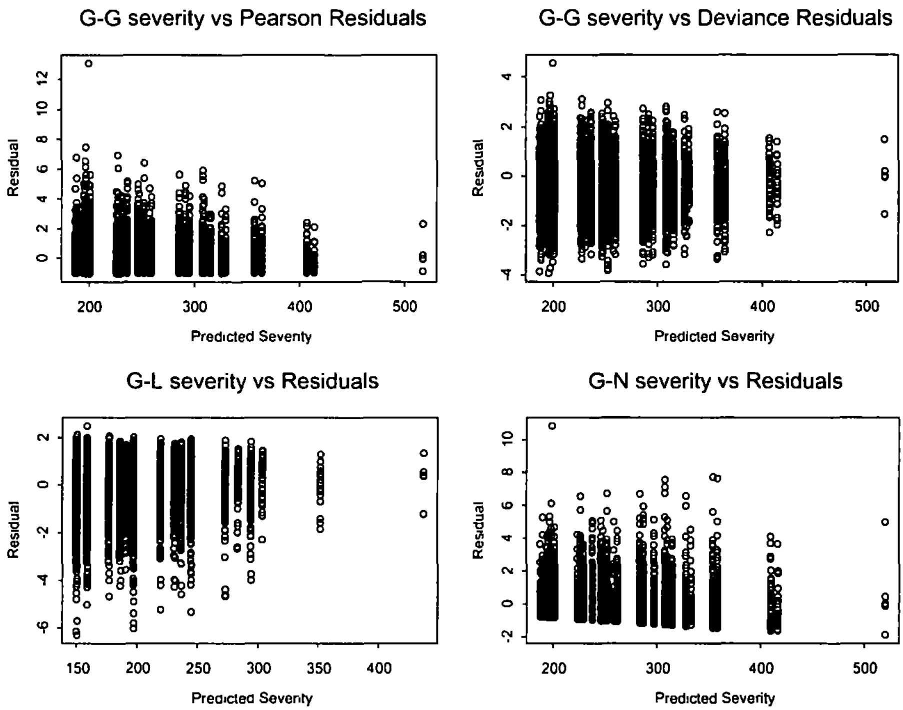


Figure 7.15: QQ Plots of Standardized Residuals for Lognormal Loss with Coefficient of Variation 1.0 based on Individual Data

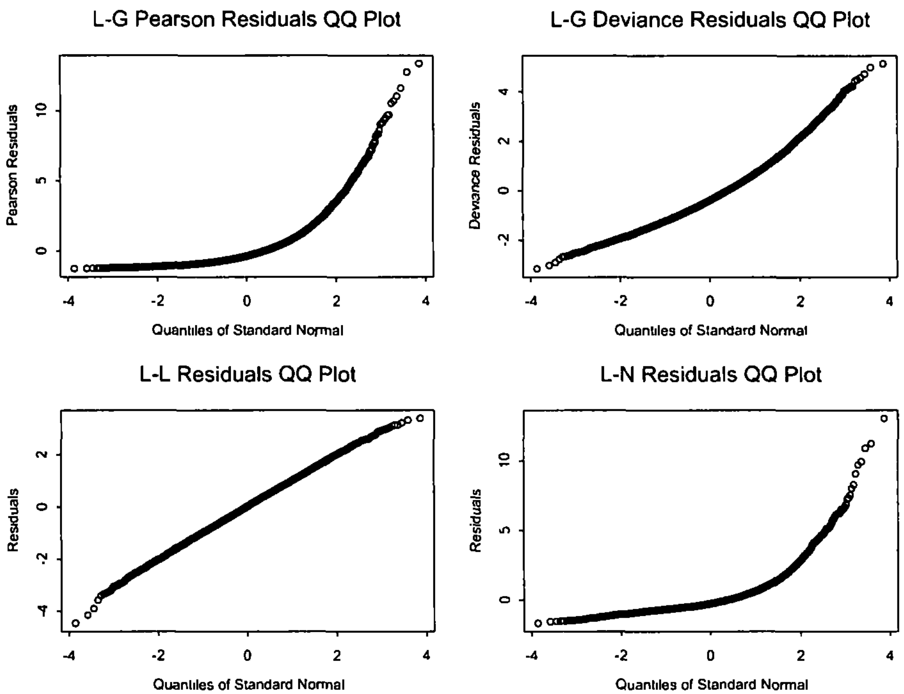
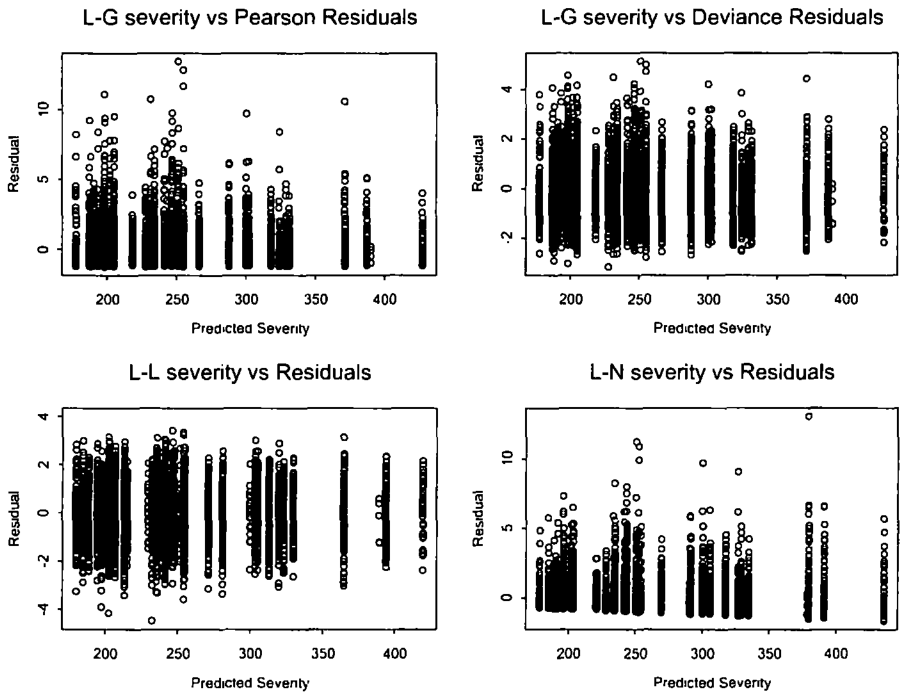


Figure 7.16: Plots of Predicted Severities vs Standardized Residuals for Lognormal Loss with Coefficient of Variation 1.0 based on Individual Data



A Practitioner's Approach to Marine Liability Pricing Using Generalised Linear Models

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A PRACTITIONER'S APPROACH TO MARINE LIABILITY PRICING USING GENERALISED LINEAR MODELS

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ABSTRACT

Marine Liability underwriters – notably those at the Protection and Indemnity (P&I) Clubs – have traditionally used empirical approaches based on individual risk experiences to arrive at their pricing. But P&I is a direct class of insurance and the underwriters have at their disposal significant data volumes. This means that it is more than possible to apply the kind of modelling techniques to P&I (and, for that matter, to other classes in the marine sector) that have become commonplace elsewhere in the General Insurance (Property & Casualty) world. In this paper we note the traditional methods, the data available and indicate how the Generalised Linear Modelling technique can be used to derive rating models that apply to Marine Liability business.

1. INTRODUCTION AND BACKGROUND

1.1.1 As much as 90% of the world's merchant shipping obtains its Marine Liability insurance via the loose network of Protection and Indemnity Associations (the "P&I Clubs") that are members of the International Group of P&I Clubs. Both shipowners and charterers enjoy the benefits of membership of a system that has survived since the middle of the nineteenth century and which itself grew out of the marine hull market based around Lloyd's of London Traditionally, underwriting methods have been empirical in nature and tend to rely heavily on simple loss ratio statements based on simplistic experience models Frequently, they will not even allow for IBNR/IBNER. Risks are usually underwritten as part of a fleet assessment with the historic experience of the vessels being the principal factor taken into account by the underwriter for renewals. Subjective assessments, such as the quality of the fleet's management will often influence the rating decisions.

1.1.2 Larger risks or groups of risks have historically been insured through the P&I Clubs, and the smaller risks, notably smaller vessels requiring limited liability cover or cover on a fixed premium basis, have been insured by the company/Lloyd's market, where special facilities to cater for their needs have developed. However, these facilities have not always proved profitable and few have maintained a consistent place in the market

1.1.3 The total premium for the P&I club market is of the order of $1.8 billion premium (2002). This figure represents the total expected premium receipts, including reinsurance premiums It is based on a total insured gross tonnage of nearly 700 million tons

1.1.4 P&I Clubs (at least, those that are Members of the International Group) are pure mutuals and are owned by their insured members. A typical P&I Club will have two groups of Directors – the first being the Club's main Board who will be elected from amongst the shipowning membership. However, for day-to-day matters, the shipowners are usually content to cede control to the insurance professionals who make up the management companies that run the Clubs. The second group is the management company who will have its own senior management and Directors, who are sometimes, but not always, subject to the formal approval of the Shipowners Board

1 1 5 Shipowners or charterers insure their vessels by "entering" the risks with one or more of the Clubs The shipowner will agree with the underwriter a premium rate per ton entered for each vessel. This is usually known as the "Expected Total Call" (ETC) (the term varies from Club to Club). The Club will expect an agreed proportion of this ETC to be paid up front as an "Advance Call". In recent years, many Clubs have increased the proportion of the ETC called in advance and one Club has recently announced that for 2004/05, the Advance Call will be 100% of the ETC. Where the Advance Call is less than 100% of the ETC, the remainder will be called by the Club at a later stage, possibly a year or eighteen months after the start of the Policy Year A policy year traditionally begins on the 20th February.

1.1.6 Most Clubs review their expected ultimate losses at regular intervals with a view to closing the Policy Year three years after its start Being mutuals, they reserve the right to ask their shipowners for additional premium at any time up to the date of closure Their record of collecting these additional premiums is good, with members' bad debt normally running at less than 5% of total premium Members seeking to leave the Club before the policy year is closed can usually expect to pay a "Release Call", which would normally be set at the Club's highest level of probable future Calls on that policy year.

1.1.7 The Clubs in the International Group operate a Claims Pooling agreement where large claims are shared between them on an equitable basis derived from their entered tonnages, Called Premium and aggregate claims experience over some twenty years. This Pooling agreement has operated since 1993 in two layers, currently between the Pool Retention of $5m and the Upper Pool Limit of $30m.

1.1.8 Beyond $30m, the International Group jointly purchase Excess of Loss Reinsurance in the London Market, using a multi-layered programme. For some years, the Clubs themselves have co-insured up to 25% of the working layer of this programme.

1.1 9 The Group reinsurance currently runs to losses of $2.03bn. Beyond that point, a claim, should it occur, would revert to the Clubs. Funding for such a loss would come from a variety of sources, including overspill reinsurances taken out by some Clubs, calls on Club reserves, and ultimately (as the Clubs are Mutuals) by direct Calls on the members.

1 1.10 Set out in Figure 1 below is a pictorial representation of the 2002 International Group reinsurance programme

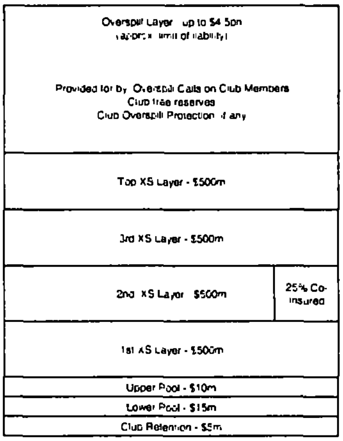

Figure 1 - The 2002 International Group Reinsurance Programme.

1.1.11 Until the late 1990s, standard P&I cover was unlimited, so theoretically a major catastrophe could result in financially crippling calls that could threaten the entire system. Because of this danger, the Group has for some years limited oil pollution risks to first $400m and more recently $500m. For the last few years, the Group has also imposed a liability limit for non-oil pollution claims, using a tonnage based formula derived from the 1976 Athens Convention non-cargo liability limits. This limit effectively establishes a cap on liability claims of around $4.5bn.

1.1.12 Most P&I claims are actually quite small, with only a dozen or so breaching the $5m Pool Retention each year. The largest ever P&I claim was the Exxon Valdez loss in 1989, believed to have cost around $8bn in total. However, as this was oil pollution, the Group loss was limited to the then limit of $400m. The largest non-oil pollution case remains the Betelgeuse loss in 1978 (an explosion off the Irish coast that resulted in several crew deaths), which cost approximately $118m.

1.1.13 It should be noted that there have been incidents in the past which could easily have generated much larger claims. Perhaps the most well known of these was the Texas City explosion in the late 1940s. The cost of that loss at today's prices would run to several billion US dollars - and that loss occurred before US courts started imposing punitive damages on top of other claims.

1.1.14 Today, the P&I risk, while limiting oil pollution losses, still leaves the Clubs exposed to some potential large losses, such as Liquid Petroleum Gas (LPG) tanker explosions and the potentially catastrophic impact of a major passenger cruise liner loss.

2. CURRENT RATING METHODS AND UNDERWRITING MODELS

2.1.1 At some stage during the months leading up to Renewal, every ship owning member or his Broker will have been presented with evidence of his loss record (going back over a period of years). The Club Underwriters will have discussed with the shipowner the Club's overall financial position together with the general level of increase that the Club's Shipowning Board of Directors will have agreed early in the Season should be applied to all Members' rates at the start of negotiations. The Shipowner will argue, perhaps, that they are a special case - they have implemented new stringent levels of ship management and loss prevention; they have replaced ageing elements of their fleet with new state-of-the-art vessels; they no longer carry dangerous cargoes; they no longer sail into potentially litigious US ports and so on. The Shipowner will offer to increase the deductibles operating on their Policy and there will be a healthy discussion as to the effect such an increase might have on the loss ratios.

2.1.2 Clubs use different techniques to aid their arguments. Some will rely on fairly simple gross loss ratio calculations, while others will present rather more sophisticated pricing models to support the discussion. In the end, however, a deal will be done and the business duly renewed.

2.1.3 It is a testimony to the stability of the International Group system that surprisingly little tonnage moves between Clubs at the 20th February Renewal. During any year, mergers and acquisitions between Shipowners result in vessels being moved from Club to Club, but a feature of the renewals process in recent years has been that the vast majority of Shipowners stay with their Clubs. Increasingly, larger (and not so large) Shipowners choose to belong to more than one Club, entering some vessels with one Club, some with another, or occasionally splitting their entry pro-rata between Clubs, so that each Club has, for example, 50% of each vessel in a group of vessels. Such Shipowners may vary their distribution of vessels between their Clubs at renewal, but again, few will make radical changes.

2.1.4 Against this background, rates have fallen in the 1990s. Underwriters always talk of insurance cycles and certainly a soft rate cycle afflicted Lloyd's in those years. It is undoubtedly true that Hull rates fell in the London Market and this generated pressure from Shipowners and Brokers for P&I Underwriters to follow suit. Counter-arguments that Hull and P&I insurance are completely different have tended to fall on deaf ears and the perceived threat from the entry into the market of fixed premium writers reinforced the pressure. The Clubs, it is pointed out, are pure mutuals and their substantial assets are ultimately the property of the Shipowning members. These Shipowning members feel that it is not unreasonable to expect the Clubs to release free reserves in the soft years - reserves that have been built up in the harder years of the cycle, when higher premiums were collected.

2.1.5 Since 2000, a new realism has gripped the market and Rates have substantially increased in the last 2-3 years and are continuing to rise. Most Clubs still believe their rates are too low and that they are continuing to draw down on their reserves. Accordingly, typical general increases sought by the Clubs for the 2004 renewal are still in excess of 15%

2.2 Original Rating Process

2.2.1 Most P&I Club rating procedures currently in force are based on a simple model, with premium rates based on tonnage. Typically, there are several deductions from the gross premium to derive the retained premium, and these are assessed against historic experience on a judgemental basis to reach an acceptable ratio of retained to gross premium. This type of process is typical for many types of risk in the London market.

The deductions may include the following:

- Excess of Loss Reinsurance
- A premium in respect of the upper Pool ($20m - $30m) of losses, possibly based on the reinsurance premium
- A premium in respect of the lower Pool ($5m -$20m). Again possibly based on the reinsurance premium or on some function of the Club's contribution level to lower pool claims.
- An abatement layer below the Pool to smooth out the effect of large losses. Depending on the size of the Club, this might be set at any point between, $100,000 and $2 million and cover the layer from the abatement point to the Pool retention of $5 million.
- Alternatively, some models may not make any allowance for an abatement layer, but may cap claims at the Pool retention point, currently $5 million

2.2.2 The remaining net premium is used to assess the retained loss ratio = premium net of deductions/gross premium. The retained premium for the insured vessel is compared against the corresponding losses If a shortfall arises the rate is adjusted upwards

2.3 Underwriting Models

2.3.1 Larger fleets may be broken down into roughly homogeneous groups of vessels (crude oil tankers, for example, may be assessed together), but it is unusual for the assessment to be any more detailed The simplest underwriting models may do little more than calculate the historic gross loss ratios by underwriting year, with no adjustment for unexpired risk, IBNR or unallocated expenses. These simple calculations will be used to judge whether the rating group is profitable From this judgement, a loading will be applied in addition to the overall increase previously agreed by the Club's board.

2.3.2 There are more sophisticated models in the market P&I Clubs in the International Group pool their losses above $5 million and collectively purchase Excess of Loss reinsurance above $30 million, one variation on the basic loss ratio model, is to cap claims at the $5 million retention point and apply an overall loading to account for the Club's share of Pool and reinsurance claims. A variation on this theme is to recognise that $5 million is far too high a point to share large claims without seriously distorting the loss ratio model for those fleets with a large claim. Therefore, the abatement layers described above are introduced to smooth out the distortions

2.3.3 Other models do exist, with adjustments for IBNR, IBNER, expenses and so on. Some models attempt to relate premium to the risk by developing a simple burning cost model, based on losses per entered ton.

2.3.4 The common factor in all these models is that they are essentially one-dimensional or at best two-dimensional, and make no real statistical use of the wealth of data held on the underwriting systems, and the interaction between the various factors that drive the claims experience

2.4 A Different Approach

2.4.1 The original approach outlined above is simplistic in that it does not fully reflect all of the factors underlying an insured's experience. Using a multifactor approach should give rise to consistent internal premium rates, with the need to increase rates only to reflect the overall market condition, or individual risks that perform badly as a result of poor risk management. A Generalised Linear Model (GLM) approach gives a more scientific basis for estimating rates.

2.4.2 A GLM creates a multi-dimensional representation of the data that enables the inter-dependent relationships in the data to be visualised in a way quite impossible by inspection alone. Such relationships are obvious when there are only two rating factors and can be identified by simple one and two-way tables Even with three factors, and a fair amount of patience, the various combinations of tabular analyses can be explored. But once the number of variables starts to climb, this quickly becomes impossible GLMs explore the data using powerful statistical software and establish the relativities present, as well as evaluating the statistical errors associated with the models derived. In this way, the actuary or statistician can evaluate the possible solutions indicated by the modelling process and select the models that best explain the variation in the data.

2.4.3 GLMs also give an equitable approach to rating between the various fleets or Club members. The rating could be readily extended to the higher layers to allow for the abatements and reinsurance premiums

3. DATA SELECTION

3.1.1 The key to carrying out the GLM modelling process successfully is to obtain as much data from internal underwriting and claims systems as possible. It is important to capture both sides of the data store as valuable descriptive information will often only be reliably held on the underwriting system while the detailed claims cost information will usually only be held on the claims system

3.1.2 The data should be extracted from these systems on an individual risk basis together with measures of exposure period. If a policy has an adjustment mid-term, resulting in a change to information we would wish to use as a rating factor, there should be a single record entry representing each of the rating factors applicable to each portion of the policy. This should not present too much of a problem with P&I business as the incidence of mid-term adjustments to policies is very rare. If an exposure period cannot be calculated within the underwriting system, then enough date information should be extracted for each record produced to allow the accurate calculation of exposure periods

3.1.3 The data extracted for the exercise should be of a recent nature and of sufficient volume to ensure the models fitted accurately reflect the expected claims experience going forward. It is normally expected to cover around four or five policy year's worth of data, although more is acceptable if available Care needs to be taken though as including older data may result in the model no longer reflecting current experience A rough guide is to use a minimum exposure of approximately 15,000 vessel years

3.1.4 The data needed will most likely come as two sources. 1) an underwriting file consisting of single records for each exposure unit – probably a "vessel-year" – representing a unique combination of rating factor details for each period of risk, and 2) a detailed claims file with information relating to every claim incurred by the exposure units on the underwriting file

3.2 Rating Factors

3.2.1 Most existing pricing models analyse actual experience by underwriting year, maybe split according to some approximate vessel classification within a fleet There is a wide range of classifying factors about each vessel routinely captured by the underwriting systems and several of these can be used to analyse the risks. The levels to be modelled for each rating factor are generally easily determined by the nature of a particular rating factor. However, for rating factors with a large number of levels it is more practical to group together levels with similar properties so that more stable parameter estimates are produced within the GLM model.

3.2.2 The common rating factors used in Marine Liability pricing are:

- Type of vessel,
- Age of vessel,
- Classification society (Lloyd's Register, the American Bureau and so on),
- Vessel flag,
- Nationality,
- Tonnage either in terms of gross tonnage or entered tonnage,
- Various types of deductibles.

Other factors can be identified from the existing data such as those vessels with limited liability or those extending their standard P&I Cover to include 4/4ths of vessel collision claims, otherwise known as Running Down Costs (RDC), which by maritime tradition are normally split between P&I and Hull insurances

3.2.3 Up to 150 different types of vessel exist, but for rating purposes these should be aggregated into 10 or so categories at the most. It is practical at this stage to identify vessels that carry dry cargo or tankers carrying clean cargo and rate these as separate factor levels, as different reinsurance arrangements will apply to these vessels at a later stage. The age of vessel factor is most sensibly grouped into bands of 5 years. The classification society factor is usually grouped into 10 or 11 levels representing the major societies plus an 'other' category containing the smaller societies and those vessels where the classification society cannot be identified or is not recorded. The vessel flag factor can usually be split into levels representing 12 or so of the major flag nations plus one level representing the smaller flag nations combined experience

3.2.4 The nationality factor represents the vessel owner's country of origin This is normally best grouped together by geographical region with the major countries such as Greece, USA, Russia, and China identified separately. Deductibles are best grouped into 5 or 6 bands representing the amount of deductible taken. The types of deductible being taken are identifiable from the Rule codes they are attached to and are therefore easily classified into types consistent with the types of claims being analysed. Typical types of deductible are.

- Collision with other vessels,
- Collision with fixed and floating objects,
- Pollution,
- Cargo,
- Personal injury,
- Other

Personal injury deductible types can further be split into crew, passenger and other personal injury if desired. Some vessels can be subject to an all-claims deductible, rather than picking up one or more of these individual deductibles separately.

3.3 Claims Data

3.3.1 Different insurers hold different levels of detail on their claims systems. Some may be able to provide little beyond a total amount paid and a total outstanding estimate for each claim. For modelling purposes however, much more detailed information is required

3.3.2 A P&I Claim can include claims of various types, such as collision damage both in terms of collisions with other vessels and collision with fixed and floating objects, pollution, cargo, personal injury and others. The personal injury element of the claim could also be split further into crew passenger injury, passenger personal injury, stevedore injuries and other injury types if so desired. The type of claim can normally be determined relatively easily, as different aspects of the claims transaction information are normally assigned to Rule codes External fees relating to each claim should be included in the claims amounts to be modelled and are most easily analysed when they are assigned directly to the relevant Rule code for the claim that they apply to

3.3.3 The claims file should be provided on a full transactional basis, allowing full analysis of all claims incurred. In order that the model is fitted to data representing a stable and settled claims position, each incurred claims amount should be increased to take account of any

IBNER The IBNER factors are usually derived from other reserving work carried out upon the same book of business and are applied on a policy year and type of claim basis. The claims transaction file is then summarised on an individual incident per policy per vessel basis.

3.3.4 Each of the individual claims should be capped at an appropriate level to remove the effect of large claims The level at which the claims are capped may be predetermined by the choice of a particular P&I Club's abatement level In other cases this level is too high and an arbitrary figure of $100,000 is chosen. It is also useful to cap individual claims amounts at the abatement level used and at any retention amounts that apply, so that appropriate loadings can be evaluated and applied at a later stage

3.4 Preparation of Modelling Data

3.4.1 The task then is to merge the files, eliminate errors and aggregate claims costs, so as to end up with a manageable data file containing one record for each exposure unit with summarised claims information appended Inevitably at this stage there will be some degree of mismatch when linking the claims data to the underwriting data. Care needs to be taken here to ensure the mismatched claims are investigated. If the mismatched claims are for business to be included in the model, then the mismatch amount needs to be evaluated and a loading for this should be applied at a later stage in the modelling process

3.4.2 Once a single manageable data file has been produced, the data must be examined and any records representing business not required in the model should be removed For instance, a particular Club may want to fit a model to owned-only vessels and not chartered vessels, or they may wish to exclude those vessels insured under consortium arrangements and price this business separately

4. THE GENERALISED LINEAR MODELLING APPROACH

4.1 Modelling

4.1.1 Having generated the database, the most important stage of our work is the modelling process itself For some time now, actuaries and statisticians have been applying a class of mathematical models known as GLMs to mass-volume insurance data to identify relationships between risks and establish relativities between different levels of rating factors.

4.1.2 The underlying assumption in rating Marine Liability business is that the risks are similar in many aspects to those found in personal lines insurance, in particular those found in motor insurance. P&I club risks covered are usually single vessels, each of which is considered to be comparable to a private motor policy Large fleets of vessels on cover are considered to be comparable to a motor fleet policy

4.1.3 There have been a large number of papers written which make use of the GLM techniques to rate motor business. The underlying theory we have used to form the basis of this paper can be found in Brockman & Wright (1992).

4.1.4 We have fitted a basic frequency-severity model to the risk premium per ton calculated using the capped incurred claims cost, using a poisson error structure with a log link, weighted by exposure measured in terms of entered tonnage The log link results in a multiplicative model being fitted, which is preferred as negative fitted values cannot be obtained from the model, unlike in an additive model. Also, a greater level of accuracy can be obtained by fitting multiplicative models as opposed to additive models, where more terms would need to be included in the model to achieve the same outcome. We use a poisson error structure as the incidence of P&I claims being modelled are measured in terms of claims cost per ton over a fixed time period

4.1 5 Another approach is to fit separate models to frequency and severity and examine the results of the two separately. The frequency model would be fitted to the number of observed claims per ton using a poisson error structure with a log link, just like the model used above The severity model is different in that it uses a gamma error structure rather than the poisson error structure. A detailed outline of the theory for severity models can be found in Brockman and Wright (1992).

4.2 Time Dependency

4.2.1 In choosing the data to be included in the model, care must be taken to ensure the exposure periods chosen are suitably recent so that the claims experience being predicted by the model can be expected to be of a similar nature to the historical experience and that the volume of the data being used is large enough to reduce random variation in the parameter estimates. It is common practice to select data that covers the most recent four or five-year period, to ensure that both of these criteria are met. Arguably, very recent claims data should not be used in the model due to its undeveloped nature. This is easily overcome by ensuring that the claims amounts being used are incurred amounts, including both paid amounts and all outstanding estimates, together with an appropriate development for an element of IBNER.

4 2.2 When fitting the models, a time factor should be allowed for as an explanatory variable. This is to ensure the trend in the size of claims due to inflation is identified This way there is no need to remove inflation by making prior adjustments to the claims data. This claims inflation should not be assumed to be the same as the RPI inflation, or the claims inflation experienced in other lines of business. Another reason for fitting a time factor in the model is to remove the effects of any changes in portfolio mix over time as this could result in the parameter estimates being distorted.

4 2.3 To check the stability of parameter estimates over time for a particular rating factor, the selected model should be re-fitted containing an additional interaction term. This interaction term includes both time and the rating factor to be tested. Separate models should be fitted for each of the main rating factors in turn It usual to plot the results of this fit on the same graph as the results of the fit from the selected model. The graph in Figure 2 below shows

the results of fitting an additional term for the interaction of vessel type and time. We can see from this graph that when compared with the fit obtained from a main effects model fit, the same general trend across the vessel types is observed in each of the policy years under analysis.

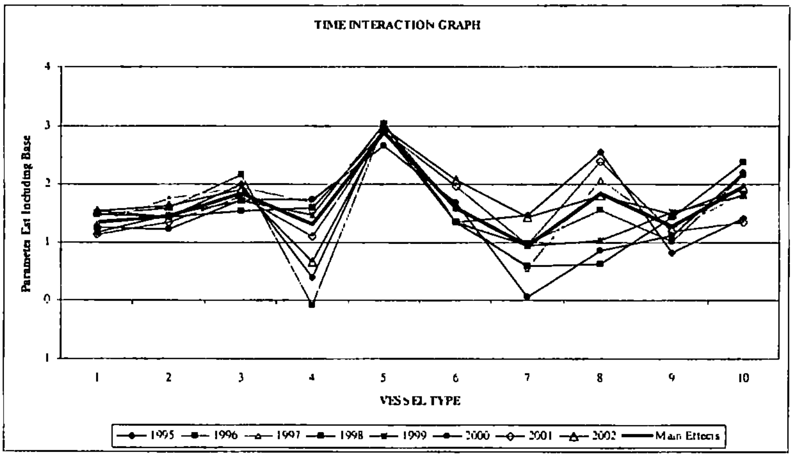

Figure 2 - Time Interaction Graph

4.2.4 If data is available for multiple claim types as mentioned earlier in section 3 3 2, separate models can be fitted to each claim type separately. This will give a much deeper insight into the factors driving the claims experience. Additionally including the time factor in the model provides the ability to estimate the claims inflation for each of the claim types separately, and also to identify trends in the data applicable to individual claim types, without being affected by changes in the portfolio mix over time.

4 2 5 The standard model assumptions of constant variance should be checked by producing a plot of standardised residuals against the fitted values, and also by producing plots of the standardised residuals against the levels of each rating factor in turn The graphs in Figure 3 and 4 below show the plots of standardised residuals against fitted claims costs per entered ton, and standardised residuals by vessel type.

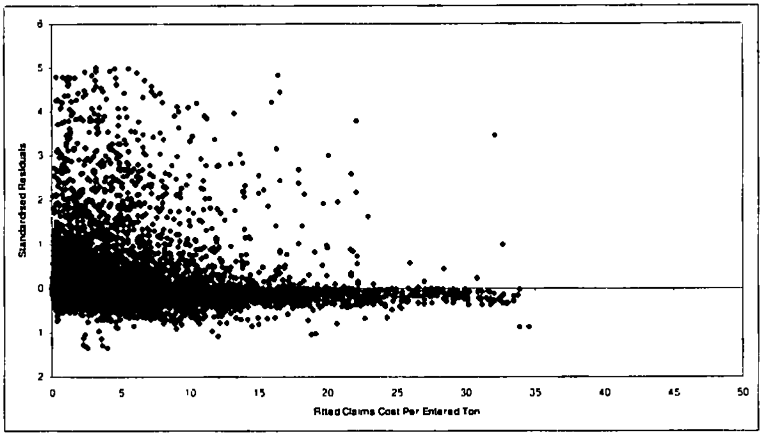

Figure 3 – Plot of Standardised Residuals versus Fitted Values

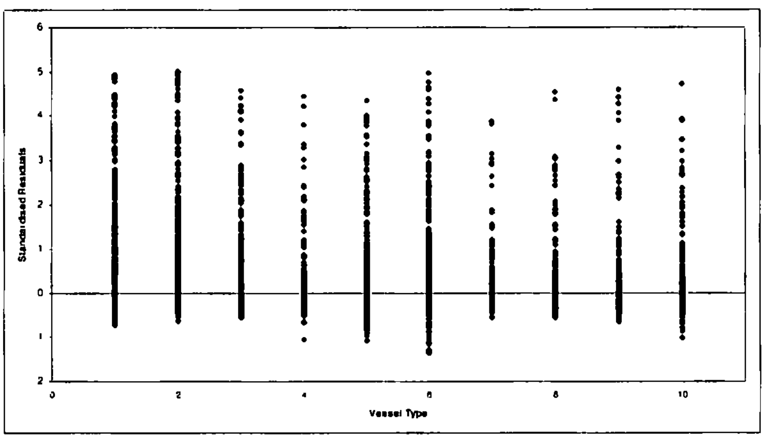

Figure 4 – Plot of Standardised Residuals By Vessel Type

4.3 Validation

4.3.1 The final stage of the modelling process is to turn the GLM output into a set of relativities together with a base rate. This base rate is taken from the model base and will need to be increased to take into account additional costs These will include a loading for any mismatched claims during the data preparation stage, loadings for the capping of claims between the chosen capping level and the abatement level (if the two are different), abated claims, expenses, the club's share of pooled claims, future inflation and reinsurance costs

4 3.2 Having derived the model, it is then applied to the underwriting information to compare the indicated premium for each risk with the actual premium charged The total indicated premium can be examined to ensure that it is sufficient to cover the historic losses.

5. CONCLUSION

5 1.1 In this paper, we have described the background to existing Marine Liability pricing models.

5 1.2 We have gone on to describe the application of powerful new modelling techniques based on Generalised Linear Models to the available data.

5.1 3 The end result is an easy to apply multiplicative rating model that can be used to derive a statistically valid premium for each vessel. Nothing in this work deprives the Underwriter of his or her ability to negotiate a different rate from that indicated by the model. However, with an appropriate modelling technique added to the toolkit of methods, the Underwriter is better placed to conduct a meaningful negotiation with the Shipowner armed with the results of a formal analysis of the past experience.

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Multivariate Spatial Analysis of the Territory Rating Variable

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Multivariate Spatial Analysis of the Territory Rating Variable
by Serhat Guven

ABSTRACT

The average insurer typically utilizes some form of territory ratemaking in its algorithm; thus, in constructing a GLM, one of the major issues revolves around how to reflect location in the statistical solution. The problem arises because there are too many territory categories to directly include in the statistical model. This issue can be resolved by altering the perception of the location dimension from a categorical rating variable to a continuous one.

This paper presents an alternative approach to incorporating the location dimension in the GLM analysis of the rating algorithm. The procedure develops the indicated relativities and boundaries in a statistical multidimensional framework thus removing the distributional effects of other rating variables and measuring the geographic risk alone. Furthermore, the territory procedure is based on the principle of locality, i.e., the expected loss experience at location L is similar to the loss experience around L.

The indicated relativities of each geographic unit are determined by modeling polynomial functions of latitude and longitude in the GLM statistical framework. By expressing the indication in terms of a polynomial the analyst can include location in the statistical model without having to worry about too many additional parameters.

INTRODUCTION

An insurer's rating algorithm consists of a multitude of rating variables to accurately quantify the various insured risks. Insurers that are able to properly segment and appropriately charge the pool of insured risks will not suffer the problems associated with adverse selection. Ideally, the insurer would want to develop a rating scheme that accurately represents the multidimensional framework of the insured population. Generalized Linear Models (GLMs) are an ideal tool to analyze the various dimensions of the rating algorithm in a multivariate framework using distributions common to insurance. Much of the current application of the GLM is to study rating factors such as an insurer's class plan, limit structure, or tier assignments.

Insurers use a wide variety of rating variables, and one of the most common among them is location. The average insurer typically utilizes some form of territory ratemaking in its algorithm. In constructing a GLM, one of the major issues revolves around how to reflect location in the statistical solution. The problem arises because there are too many territory categories to directly include in the statistical model. From a practical point of view, the actuary attempts to identify the best groupings of location that would properly reflect the distributional differences across the rating dimensions without adding an inordinately large number of parameters to the GLM. This approach is somewhat subjective. The problem can be resolved by altering the perception of the location dimension from a categorical rating variable, such as gender, to a continuous one, such as age.

The purpose of this paper is to present an alternative approach to incorporating the location dimension in the GLM analysis of the rating algorithm. The procedure develops the indicated relativities and boundaries in a statistical multidimensional framework thus removing the distributional effects of other rating variables and measuring the geographic risk alone. Furthermore, the territory procedure is based on the principle of locality, i.e., the expected loss experience at location L is similar to the loss experience around L.

Under this approach, a GLM models the dependent random variable as a function of the rating variables including the location dimension. Location is defined as the latitude and longitude coordinates of the geographic unit. Other rating variables are regarded as categorical predictors; however, by using the latitude and longitude coordinates, location can be treated as a continuous predictor. Thus the actuary can measure the geographic risk while upholding the principle of locality. The indicated relativities of each geographic unit are determined by modeling polynomial functions of latitude and longitude in the GLM statistical framework. By expressing the indication in terms of a polynomial the analyst can include location in the statistical model without having to worry about too many additional parameters.

The framework of this paper begins with an overview of traditional territory boundary ratemaking procedures. Traditional methods rely either on loss ratio or one-dimensional adjusted pure premium techniques. GLM or other multivariate methods develop solutions that avoid the problems associated with one-dimensional techniques, however, the emphasis in this section will focus on how traditional methodologies treat the territory variable as a mixture of categorical and continuous concepts.

The next section will introduce some basic ideas for modeling the rating algorithm using GLM. This section will discuss common terminology and strategies used to model a response dependent variable as a function of categorical and continuous independent predictor variables. With this background the problems associated with developing the territory rating variable within the GLM framework will be explained; furthermore, current techniques used to work around these problems will also be presented. The subjective nature and theoretical concerns will be shown for these current methods.

The paper will finally present the proposed solution to the territory issue by modeling the rating algorithm in the GLM framework while treating location as a continuous concept. This approach allows the analyst an alternative solution that avoids the issues and concerns associated with the current techniques. Issues arising from using the proposed methodology and the corresponding practical solutions to these concerns will be presented as well.

SECTION ONE: Traditional Techniques and the Principle of Locality

In 1996 two important papers were presented that discussed the use of geographic information systems in developing the territory rate.

Christopherson and Werland presented "Using a Geographic Information System to Identify Territory Boundaries", which developed rates for a geographic unit reflecting the experience at that unit as well as experience around the unit. The paper eloquently describes the principle of locality as "physical and social conditions around a location impact the risks associated with homes at the location." Once these rates were determined they were aggregated into discrete groupings based on common results.

At the same time, Brubaker presented "Geographic Rating of Individual Risk Transfer Costs without Territorial Boundaries", which also developed rates for a geographic unit; however, he did not recommend aggregation, instead he proposed various interpolation techniques to produce rates for geographic areas that were between the initial geographic units.

Both papers developed a pure premium for each geographic unit. This metric was adjusted to include experience from surrounding locations; however, as the distance from the geographic unit increased, the weight given to the surrounding location decreased. This spatial smoothing approach is used because of the principle of locality.

The principle is based on the concept that the "risk level will vary gradually from one location to another location." From a mathematical perspective this principle allows us to consider the territory rating dimension as a continuous concept. Both papers rely on the concept of distance to develop rates for categorical units, but one can think of distance as a continuous concept (unlike gender or tier dimensions, which are categorical concepts). Furthermore, the Brubaker paper directly utilizes continuity principles in the presentation of interpolation between various categorical geographic units.

Both papers also discuss adjusting the pure premium metric for all other rating dimensions in an attempt to isolate the effect of territory. Without a multivariate approach, this is much easier said than done. Depending on the complexity of the rating algorithm, aggregation assumptions are often made to simplify implementation that results in a greater likelihood of not removing the distributional biases inherent in the rating dimensions of the insurer's data.

The motivation behind these adjustments is to remove the effect of other rating variables and define the "geographic risk as the residual risk after the effects of other rating variables have been controlled." This is a problematic assumption, because from a statistical point of view a model attempts to identify the systematic and unsystematic behavior of the data. The unsystematic behavior is regarded to be the noise that reflects the random nature of the stochastic process. The current procedures imply that the geographic risk should have the qualities associated with systematic as well as unsystematic variation. This is a precarious assumption to make since the unsystematic variation is random noise and the allocation of this randomness to a particular rating dimension is fairly arbitrary.

In summary, traditional ratemaking procedures to develop the territory rating variable rely on the principle of locality, which allows one to consider the location dimension as a continuous concept. Furthermore, the traditional attempts to remove the effects of the distributional biases and the resulting treatment of the territory rating variable to capture the residual risk are problematic.

SECTION TWO: GLM Modeling Techniques

The basic idea behind GLM is to model the dependent response variable as a function of a linear combination of the independent predictor variables. Dependent response variables are defined as the subject that is measured. Examples in the insurance environment are concepts such as frequency and severity. Independent predictor variables are defined as characteristics of the subject that is being measured. Common examples in insurance include concepts such as age and gender. There are three major components of any GLM:

1. The distributional form of the dependent response variable.
2. The structure of the independent predictor variables.
3. The function that links the dependent response variable to the independent predictor variables.

GLM requires the modeler to assume that the dependent response variable be drawn from the exponential family of distributions. In the insurance environment the Poisson and Gamma distributions, which are commonly used to model frequency and severity, are part of the exponential family.

The combination of the independent predictor variables creates the structure of the model. The modeler decides which variables to include or exclude; furthermore, once the variable is included in the model, the analyst must decide on how to include the variable.

If the variable is a rating dimension, should all the levels of the rating dimension be included or should they be grouped into categories? Examples in insurance include grouping of insured ages into categories such as youthful and adult. Should the different variables be modeled so that the effects of one variable depend on the effect of another variable? In homeowners insurance the classic example is the common Protection Construction rating variable.

Can the predictor variable being analyzed be modeled as a categorical or a continuous concept? Categorical concepts allow us to group the individual items into distinct groups; however, the modeler cannot quantify the difference between distinct categories. An example of this type of variable is marital status. The insured can be classified into a particular marital status, but the difference in the levels of marital status cannot be quantified. Continuous variables allow us to quantify and compare the differences in the levels within the variable. The classic example is age. An insured that is forty years old is twenty years older than an insured that is twenty years old. Identifying the predictor variable as continuous allows the modeler to use polynomial functions to describe the behavior of the underlying variable.

Finally, GLM relates the mean of the dependent response variable as a function of the linear combination of independent predictor variables. This function is called the link function. Commonly used link functions are the identity and log functions. The identity function creates an additive model while the log functions are used to build a multiplicative model. Insurers use rating algorithms that have multiplicative as well as additive components. In GLM one can use the structure of the link functions to best reflect the insurers underlying rating algorithm.

As stated earlier, the goal in any GLM is to model the dependent response variable as a function of a linear combination of independent predictor variables. Each predictor, such as gender, has a given number of rating levels, such as male and female. The greater the number of rating levels for a given predictor the more difficult it becomes to interpret the resulting parameters. This becomes quite obvious when we consider the location rating variable. A state or region can be subdivided into countless numbers of geographic units. For example if we decided to use the county zip as the underlying geographic unit, then there are approximately 40,000 unique county zips in the United States. This is a significant challenge to any statistical model. The validity associated with a model that has such a large number of parameters is very questionable.

So one of the challenges associated with application of GLMs in the insurance environment is how to reflect the location rating variable in the statistical solution.

One approach is to define a geographic unit to be large enough so that the total number of location segments is manageable in the GLM. Grouping locations together based on distance and other information, such as population density, usually does this. These techniques are problematic. The first problem with this approach is that the procedure could produce groups that contain heterogeneous data. The second problem is that grouping and clustering procedures can be very subjective.

Another alternative is to derive a GLM using all of the rating variables excluding the geographic dimension. The next step is to examine the residuals of the model and allocate those residuals to the geographic unit. Spatial smoothing techniques are then utilized to insure the principle of locality, and then territory boundaries are derived from the clustering of the geographic units based on the residual of the GLM. Territory relativities are built from the resulting boundaries. The biggest problem with this technique is the residual itself. The residual represents both systematic variation not included in the original GLM (i.e. territory) AND unsystematic variation that is inherent in any stochastic process (i.e. random noise). In this approach both the systematic and unsystematic variation is being allocated to the location rating variable.

One of the commonly relayed themes in this paper has been the principle of locality. As defined earlier, this principle states that experience around location L is similar to experience at location L. This principle can also be thought of a continuous concept. The difference in experience between locations changes gradually. Current methods tend to utilize spatial smoothing techniques thus emphasizing the continuity of this particular dimension. Expanding on this idea, this paper proposes to directly include the location dimension at the lowest geographic unit in the GLM statistical solution; furthermore, by defining the variable using a coordinate system (e.g. latitude/longitude), the analyst can treat the variable as continuous; thus, territory variable can be modeled using polynomial functions which avoids the problems associated with an inordinate number of categorical parameters.

SECTION THREE: Modeling the Geographic Risk

The first step in this process is to identify the geographic unit to be used in the analysis and then assign a coordinate pair to each unit.

One option available is to use the county zip code as the underlying geographic unit, and assign a coordinate latitude-longitude pair for each county zip. County zip codes are easy to use and can be readily extracted from most insurers databases. Furthermore, the coordinate assignments are usually built into most GIS software systems. There are two problems associated with this approach. The first is that county zips are developed by the US Postal Service to allow them to better coordinate and deliver mail. Outside of population density issues, these goals really are not a good representation of the insurer's risk. County zips are constantly changing to meet postal needs; thus, these changes can seriously impair the usefulness of the insurer's data. (See Werner) The other problem is the assignment of the coordinates. Common assignments include the location center OR the population weighted center of the geographic unit.

In the Brubaker paper, the region was segmented into grids with each point being the geographic unit. This is an ideal approach because it avoids all of the problems associated with the county zip. Of course, the problem with this approach is how to define the grids and the technical challenges associated with plotting the data on the grid.

The ideal approach is to have a latitude and longitude coordinate for each record in the insurer's data, but this can be very costly and difficult to implement.

In order to facilitate this discussion, an example will be used to illustrate the underlying equations associated with this approach. Lets assume that we have the following simple rating algorithm:

Premium = BaseRate x TerrRel x Size x Age

Assume that the age rating variable has two rating levels – youthful and adult. Also, the size rating variable has three rating levels – small, medium, and large. Finally, the territory rating variable has fifteen rating levels. Each level represents the geographic unit. Also let the base level for each variable be defined as

Age – Adult
Size – Large
Territory - 4

As stated earlier, the goal of the GLM is to model the dependent response variable as a function of independent predictor variables. The common notation can be expressed as:

$$\mu = h(\eta)$$
$$\eta = \mathbf{XB}$$

The linear combination of parameters is represented by **XB** where **X** is the design matrix and B are the parameters that reflect the charges associated with the risk characteristics.

Using the aforementioned rating algorithm we can specifically define η for each combination of rating characteristics:

$\eta_1 = B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_{T1} \times 1 + B_{T2} \times 0 + B_{T3} \times 0 + B_{T5} \times 0 + B_{T6} \times 0 + B_{T7} \times 0 + B_{T8} \times 0 + B_{T9} \times 0 + B_{T10} \times 0 + B_{T11} \times 0 + B_{T12} \times 0 + B_{T13} \times 0 + B_{T14} \times 0 + B_{T15} \times 0$

$\eta_2 = B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_{T1} \times 0 + B_{T2} \times 1 + B_{T3} \times 0 + B_{T5} \times 0 + B_{T6} \times 0 + B_{T7} \times 0 + B_{T8} \times 0 + B_{T9} \times 0 + B_{T10} \times 0 + B_{T11} \times 0 + B_{T12} \times 0 + B_{T13} \times 0 + B_{T14} \times 0 + B_{T15} \times 0$

$\eta_3 = B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_{T1} \times 0 + B_{T2} \times 0 + B_{T3} \times 1 + B_{T5} \times 0 + B_{T6} \times 0 + B_{T7} \times 0 + B_{T8} \times 0 + B_{T9} \times 0 + B_{T10} \times 0 + B_{T11} \times 0 + B_{T12} \times 0 + B_{T13} \times 0 + B_{T14} \times 0 + B_{T15} \times 0$

This pattern of equations continues until all combinations of rating variables have been expressed. For this example there will be ninety (2 x 3 x 15) distinct equations representing the different combinations of Age x Size x Territory.

In matrix notation this system of equations can be expressed as follows

$$\eta = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} B_0 \\ B_{A1} \\ B_{S1} \\ B_{S2} \\ B_{T1} \\ B_{T2} \\ B_{T3} \\ B_{T5} \\ \vdots \\ B_{T15} \end{bmatrix} = \mathbf{XB}$$

Note that the linear combination of the independent predictor variables treats territory as a categorical variable. Thus this model will produce a separate relativity for each geographic unit. The exponential function represents the multiplicative model that best mimics the rating algorithm. Using the earlier example we have the following system of equations:

$h(\eta_1) = \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_{T1} \times 1 + B_{T2} \times 0 + B_{T3} \times 0 + B_{T5} \times 0 + B_{T6} \times 0 + B_{T7} \times 0 + B_{T8} \times 0 + B_{T9} \times 0 + B_{T10} \times 0 + B_{T11} \times 0 + B_{T12} \times 0 + B_{T13} \times 0 + B_{T14} \times 0 + B_{T15} \times 0)$

$h(\eta_2) = \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_{T1} \times 0 + B_{T2} \times 1 + B_{T3} \times 0 + B_{T5} \times 0 + B_{T6} \times 0 + B_{T7} \times 0 + B_{T8} \times 0 + B_{T9} \times 0 + B_{T10} \times 0 + B_{T11} \times 0 + B_{T12} \times 0 + B_{T13} \times 0 + B_{T14} \times 0 + B_{T15} \times 0)$

$h(\eta_3) = \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_{T1} \times 0 + B_{T2} \times 0 + B_{T3} \times 1 + B_{T5} \times 0 + B_{T6} \times 0 + B_{T7} \times 0 + B_{T8} \times 0 + B_{T9} \times 0 + B_{T10} \times 0 + B_{T11} \times 0 + B_{T12} \times 0 + B_{T13} \times 0 + B_{T14} \times 0 + B_{T15} \times 0)$

Note that the exponential function converts the linear combination of parameters into a multiplicative form.

As mentioned earlier, the first step in the territory analysis is the assignment of the coordinates to each geographic unit. For this analysis we will use the coordinates that are displayed in the following manner:

$$
\begin{array}{cc|ccc|}
 & & & y & \\
 & & 1 & 2 & 3 \\
\hline
 & 1 & 1 & 2 & 3 \\
 & 2 & \boxed{4} & 5 & 6 \\
x & 3 & 7 & 8 & 9 \\
 & 4 & 10 & 11 & 12 \\
 & 5 & 13 & 14 & 15 \\
\hline
\end{array}
$$

Thus each territory rating level is assigned a coordinate pair

<u>Territory n: (x, y)</u>
Territory 1: (1, 1)
Territory 2: (1, 2)
Territory 3: (1, 3)
Territory 4: (2, 1) – Base Level
.
.
.
Territory15: (5, 3)

This simple example assumes that the territory rating levels are the smallest geographic unit that describes the risk. In practice a territory rating level covers a much broader area and typically consists of a number of geographic units. The coordinate system assigned to the territory allows the analyst to quantify the difference between separate rating levels. This quantification allows the territory rating variable to be treated as a continuous predictor, which in turn allows the modeler to use polynomial functions to describe the differences in risk experience across the territory rating level.

For the above example we can model the territory variable as a polynomial function of the assigned coordinate system. Assume that a simple one-degree linear relationship for the territory variable is used. Then the system of equations can be described as:

$$
\begin{aligned}
h(\eta_1) &= \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_x \times 1 + B_Y \times 1) \\
h(\eta_2) &= \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_x \times 1 + B_Y \times 2) \\
h(\eta_3) &= \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_x \times 1 + B_Y \times 3) \\
h(\eta_4) &= \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_x \times 2 + B_Y \times 1) \\
h(\eta_5) &= \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_x \times 2 + B_Y \times 2) \\
h(\eta_6) &= \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_x \times 2 + B_Y \times 3) \\
&\;. \\
&\;. \\
&\;. \\
h(\eta_{88}) &= \exp(B_0 \times 1 + B_{A1} \times 0 + B_{S1} \times 0 + B_{S2} \times 0 + B_x \times 5 + B_Y \times 1) \\
h(\eta_{89}) &= \exp(B_0 \times 1 + B_{A1} \times 0 + B_{S1} \times 0 + B_{S2} \times 0 + B_x \times 5 + B_Y \times 2) \\
h(\eta_{90}) &= \exp(B_0 \times 1 + B_{A1} \times 0 + B_{S1} \times 0 + B_{S2} \times 0 + B_x \times 5 + B_Y \times 3)
\end{aligned}
$$

Thus the design and parameter matrices take the following form:

$$\eta = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 & 3 \\ 1 & 1 & 1 & 0 & 2 & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 0 & 0 & 0 & 5 & 1 \\ 1 & 0 & 0 & 0 & 5 & 2 \\ 1 & 0 & 0 & 0 & 5 & 3 \end{bmatrix} \times \begin{bmatrix} B_0 \\ B_{A1} \\ B_{S1} \\ B_{S2} \\ B_X \\ B_Y \end{bmatrix} = \mathbf{XB}$$

Note that if we treated territory as a categorical predictor, then the GLM produces eighteen parameters (Base + Age + Size + Territory = 1 + 1 + 2 + 14). Translating the territory into a continuous predictor, and using the aforementioned structure, the GLM produces only six parameters (Base + Age + Size + Territory = 1 + 1 + 2 + 2). This simple illustration shows that one can use polynomial functions on continuous predictors to reduce the total number of parameters in the statistical solution.

As an alternative to the one-degree polynomials, assume a simple two-degree relationship, then the equations become:

$$h(\eta_1) = \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_X \times 1 + B_{X2} \times 1 + B_Y \times 1 + B_{Y2} \times 1)$$
$$h(\eta_2) = \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_X \times 1 + B_{X2} \times 1 + B_Y \times 2 + B_{Y2} \times 4)$$
$$h(\eta_3) = \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_X \times 1 + B_{X2} \times 1 + B_Y \times 3 + B_{Y2} \times 9)$$
$$h(\eta_4) = \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_X \times 2 + B_{X2} \times 4 + B_Y \times 1 + B_{Y2} \times 1)$$
$$h(\eta_5) = \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_X \times 2 + B_{X2} \times 4 + B_Y \times 2 + B_{Y2} \times 4)$$
$$h(\eta_6) = \exp(B_0 \times 1 + B_{A1} \times 1 + B_{S1} \times 1 + B_{S2} \times 0 + B_X \times 2 + B_{X2} \times 4 + B_Y \times 3 + B_{Y2} \times 9).$$

.

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$$h(\eta_{88}) = \exp(B_0 \times 1 + B_{A1} \times 0 + B_{S1} \times 0 + B_{S2} \times 0 + B_X \times 5 + B_{X2} \times 25 + B_Y \times 1 + B_{Y2} \times 1)$$
$$h(\eta_{89}) = \exp(B_0 \times 1 + B_{A1} \times 0 + B_{S1} \times 0 + B_{S2} \times 0 + B_X \times 5 + B_{X2} \times 25 + B_Y \times 2 + B_{Y2} \times 4)$$
$$h(\eta_{90}) = \exp(B_0 \times 1 + B_{A1} \times 0 + B_{S1} \times 0 + B_{S2} \times 0 + B_X \times 5 + B_{X2} \times 25 + B_Y \times 3 + B_{Y2} \times 9)$$

Again the design matrix takes on the following form:

$$\eta = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 2 & 4 \\ 1 & 1 & 1 & 0 & 1 & 1 & 3 & 9 \\ 1 & 1 & 1 & 0 & 2 & 4 & 1 & 1 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & 0 & 0 & 0 & 5 & 25 & 1 & 1 \\ 1 & 0 & 0 & 0 & 5 & 25 & 2 & 4 \\ 1 & 0 & 0 & 0 & 5 & 25 & 3 & 9 \end{bmatrix} \text{x} \begin{bmatrix} B_0 \\ B_{A1} \\ B_{S1} \\ B_{S2} \\ B_X \\ B_{X^2} \\ B_Y \\ B_{Y^2} \end{bmatrix} = \mathbf{XB}$$

In this case the GLM model produces eight parameters (Base + Age + Size + Territory = 1 + 1 + 2 + 4) to describe the underlying data.

In either case, instead deriving a separate parameter for each territory rating level, the modeler derives a function based on the coordinates of each rating level, which allows the territory relativity to be defined as a function of the latitude and longitude coordinates from each geographic unit.

The immediate benefit of this method is the ability to include the location dimension in the statistical solution without having to rely on clustering routines. Furthermore, the polynomial functions used to describe the territory rating variable are continuous. The continuity of the underlying functions is in line with the principle of locality.

In reality there exist thousands of geographic units and the location dimension would produce an overwhelming number of categorical parameters that significantly affects the validity of the GLM. Instead of treating each unit as a category, this approach allows the analyst to describe the location with a smaller number of manageable parameters.

There are several issues that need to be considered when modeling the territory rating variable in this manner. The first issue is the coordinate system which was described earlier.

The second is the issue of sensitivity. How complex should the polynomial be to describe the geographic risk? The modeler could utilize large degree polynomials in an effort to describe the data as closely as possible. Alternatively, cubic splines could be used to build cubic polynomials for segments across the territory variable. In addition non-linear components (such as xy) can be incorporated in the underlying data to create additional layers of sensitivity. Ultimately the modeler should rely on the principle of parsimony in making this decision.

The next issue is the practical implementation. Most insurers' rating algorithm treat the territory rating variable as a categorical concept; thus implementing continuous curves can create quite a systems cost. To avoid this issue, the analyst would have to aggregate the resulting indications at the geographic unit level to the boundary level. The goal of

GLM is to produce indications that reflect the distributional biases inherent in the underlying rating dimensions. The problem with aggregating the GLM indications can recreate the distributional biases that were trying to be avoided. As with any modeling task, the analyst will have to balance the statistical solution with the practical implementation.

Another issue has to deal with the incorporation of the catastrophe experience. Generally GLM are modeled using non-catastrophe data. It is vital that a catastrophe load be reflected in the results. The problem arises because catastrophe loads are generally not multi dimensional. Catastrophe results tend to be modeled across the location dimension; therefore, the analyst must then allocate an aggregate catastrophe load across the other rating dimensions. Different allocation procedures (such as exposure distributions) imply different homogeneity assumptions.

Finally as with any analysis, it is crucial that the systematic results produced by the statistical model make sense. All actuarial analysis consists of balancing the systematic statistical solution with judgment. In addition to the parameter estimates, models should also produce several statistics that attempt to quantify and describe the validity and variability of the results. It is vital that the analyst leverage this type of information to justify and explain the resulting relativity indications.

CONCLUSION

The purpose of this paper is to present a technique that allows the incorporation of the territory rating variable into the GLM statistical solution. The approach leverages the well known principle of locality whereby the location variable is regarded as a continuous predictor, since the territory dimension can be described via a coordinate system. Specifically the principle of locality allows the modeler the ability to develop closed form polynomial spatial curves that reflect the insurer's geographic risk.

This idea allows one to include directly the location dimension in the analysis. Traditional and current approaches rely on problematic residual assumptions as well as greater subjectivity in the systematic solution. Thus this proposed method would resolve many of these issues surrounding the residual approaches and the loss ratio techniques. The multidimensional GLM allows for a more systematic isolation and quantification of the geographic risk.

As GLM and other multidimensional techniques are becoming more and more common, it is very important that the modeler be able to reflect all of the dimensions that are associated with the insurer's risk into the statistical solution. Historically with GLM, the territory rating variable could not be directly included in the statistical solution. With the approach presented in this paper, the territory rating variable can now be directly included with the other rating variables in the model. The relativities that are derived from a multidimensional model that reflects all of the rating characteristics will produce results that better reflect the distributional dependencies inherent in the rating dimensions. The resulting rating algorithm will better reflect the insured risk thus reducing the insurer's adverse selection

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Testing the Significance of Class Refinement

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Abstract

Generalized linear modeling (GLM) is becoming a regular tool for insurance ratemaking. Actuaries and underwriters have begun to realize that classes may not simply interact, whether additively or multiplicatively. Some class combinations may synergize, or more than simply interact; others may counteract, or less than simply interact. But lest actuaries be tempted by abundant computer power and affordable GLM software to over-refine rating classes, they must know how to test whether class refinement is statistically significant. This paper provides the theory for this testing, and performs an illustrative test on a small dataset of automobile physical-damage claims.

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Testing the Significance of Class Refinement

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One benefit of generalized linear modeling (GLM) is that with it one can test whether the explanatory power of a model with more classes is significantly greater than that of a model with fewer classes. In other words, one can scientifically determine whether a refinement of a classification scheme is worthwhile. This article presents basic statistical theory, and illustrates class-refinement testing with a simple exercise.

The exercise is to estimate frequencies of automobile physical-damage claims by state, sex, and age, a reasonable prerequisite to rating an auto owner's physical-damage insurance coverage. Exhibit 1 shows the summarized data for five states, two sexes, and four age groups. Since every combination is represented, there are 5×2×4 = 40 observations. As per the bottom of Exhibit 2, young insureds are less than twenty-one years old; those in their prime are from twenty-one to forty; middle age ranges from forty-one to sixty-five; and old age is over sixty-five. The age groupings and the data itself are purely illustrative.

A standard actuarial treatment might summarize the exposures and claim counts as in Exhibit 2. It is usual to select certain classes as base, here the base being females in their prime (i.e., from twenty-one to forty years of age). In this dataset the frequency of males is 1.073 times that of females. Frequency decreases by age until old age, the relativities being 1.899, 1.000 (base), 0.825, and 1.142. One can combine these "one-way" relativities to derive that the PD-frequency of old-aged males, for example, is 1.072 × 1.142 = 1.225 times the base frequency. However, one can just as reasonably calculate one table of "two-way" relativities, and conclude that the old-aged-male frequency is 1.288 times greater than base. If premium were proportional to PD-frequency, the "two one-way or one two-way" decision would make a 4.8 percent difference in the premiums of old-aged males.

The two one-way factors require six relativities, or more accurately, four after allowing for one base per factor. The one two-way factor requires eight relativities, or seven without the base. Combining many one-way factors is simpler and easier than using one many-way factor; but it is also less accurate. Only a statistical model can test whether the loss of accuracy is significant. Moreover, in Exhibit 2 sex and age are not controlled for state; but a statistical model can filter out the effect of state.

The standard linear model is $\mathbf{y} = X\beta + \mathbf{e}$, where $\mathbf{y}$ is the $(t\times 1)$ vector of observations, X is the $(t\times k)$ "design" matrix, each of whose columns is called a factor or explanatory variable, β is the $(k\times 1)$ vector of parameters to be estimated, and $\mathbf{e}$ is the $(t\times 1)$ random vector of error terms. The errors are assumed to be of mean zero, of identical variance (i.e., "homoskedastic"), and of zero covariance. In matrix terminology, the error vector has mean 0 $(t\times 1)$ and variance $\sigma^2 I_t$, where I_t is the $(t\times t)$ identity matrix. The columns of X must be linearly independent, i.e., X must be of full column rank. Since the k columns of X explain, or account for, the t observations (with a residue of randomness), it is desirable for t to be much larger than k. With the model so described, the best linear unbiased estimator (BLUE) of β is $\beta = (X'X)^{-1}X'y$. The variance of this estimator, which will be important for hypothesis testing, is $Var[\beta] = \sigma^2(X'X)^{-1}$. Usually σ^2 must be estimated, the formula for the unbiased estimator being:

$$\sigma^2 = \frac{(y - X\beta)'(y - X\beta)}{t - k}$$

Exhibit 3 presents and solves the one-way model. The design matrix consists of zeroes and ones. Therefore, the explanatory variables are categorical. For example, the first column of X tells whether the observation pertains to California (yes = 1, no = 0). The last column tells whether the observation pertains to old-aged insureds. There are no columns for the base classes of female and prime; hence, nine variables account for as much as possible of forty observations, leaving thirty-one degrees of freedom in the estimation of σ^2. One reads down the exhibit through the intermediate calculations of X'y and X'X to $(X'X)^{-1}$ and β. Below that is the estimate of the variance of β. To the right of the design matrix are predicted values and residuals, and below these are sums of squares and cross products. From these we conclude that the model explains 99.6 percent of the actual values, and that the estimate for σ^2 is 0.049.

The logarithm of frequency is here modeled as a linear combination of parameters with an error term, i.e., $ln\boldsymbol{\varphi} = X\beta + \mathbf{e}$. Thus, $\boldsymbol{\varphi} = exp(X\beta + \mathbf{e})$. The exponential function is a monotonic link between frequency and a standard linear model, and it is this link that makes for a generalized linear model (GLM). Sex and age-group relativities, displayed in the "One-Way" table at the bottom of Exhibit 5 are exponentiated β values, e.g., the old-age relativity is $exp(0.261) = 1.298$. This relativity, for which state and age have been controlled, is much larger than the naïve relativity of 1.142.

Similarly, Exhibit 4 works out the two-way model, with its five state parameters and seven combinations of sex and age group (S × AG). Having three more explanatory variables, it explains more than the one-way model (99.7 versus 99.6 percent). However, due to its fewer degrees of freedom its estimated σ^2 exceeds that of the one-way model (0.051 versus 0.049). Exhibit 5 shows that each model outpredicts the other exactly half the time, and neither method prevails on average or squared deviations. At this point most would claim, quite rightly, that the accuracy gain of the two-way model is trifling, and would prefer multiplying the two one-way factors. However, this is an imprecise, unscientific judgment, and it makes a sizeable difference to the relativities of females in their middle age and males in their prime (13.8 and 12.7 percent, according to Exhibit 5).

The key to the hypothesis is the recognition that the one-way model is a subset of the two-way. Let index *i* range over female and male, and index *j* over young, prime, middle, and old. And let β_i be the one-way factor for sex, β_j the one-way factor for age group, and β_{ij} the two-way factor for the combination of sex and age group. The two-way model reduces to the one-way if and only if for all $i \in \{1,2\}$ and for all $j \in \{1,2,3,4\}$, $\beta_{ij} = \beta_i + \beta_j$. Now, because they are bases, $\beta_{i=1}$, $\beta_{j=2}$, and $\beta_{ij=12}$ are zero, and are not paired with explanatory variables. The proper form of the hypothesis must eliminate all β_i and β_j variables, and contain constraints on β_{ij} variables only. The bases allow us to achieve the form. For $i=1$ and for all j, $\beta_{1j} = \beta_{i=1} + \beta_j = 0 + \beta_j = \beta_j$. And for $j=2$ and for all i, $\beta_{i2} = \beta_i + \beta_{j=2} = \beta_i + 0 = \beta_i$. Therefore, for all i and j, $\beta_{ij} = \beta_{i2} + \beta_{1j}$. However, some of these equations are non-binding tautologies. In particular, for $i=1$, $\beta_{1j} = \beta_{12} + \beta_{1j} = 0 + \beta_{1j} \equiv \beta_{1j}$. This leaves constraints of the form $\beta_{2j} = \beta_{22} + \beta_{1j}$. But even here, when $j=2$ we have the tautology $\beta_{22} = \beta_{22} + \beta_{12} = \beta_{22} + 0 \equiv \beta_{22}$. Therefore, the two-way model reduces to the one-way when subjected to the three constraints (corresponding to its three fewer degrees of freedom):

$$\begin{aligned}
\beta_{21} &= \beta_{22} + \beta_{11} \\
\beta_{23} &= \beta_{22} + \beta_{13} \\
\beta_{24} &= \beta_{22} + \beta_{14}
\end{aligned}$$

Expressed in matrix notation, the hypothesis H_0 that the two-way model is equivalent to a one-way is:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \beta_{11} \\ \beta_{13} \\ \beta_{14} \\ \beta_{21} \\ \beta_{22} \\ \beta_{23} \\ \beta_{24} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_{(3\times 7)}\beta_{(7\times 1)} = r_{(3\times 1)}$$

Matrix R appears in Exhibit 7, but augmented with five columns of zeros to accommodate the state parameters.

Before developing the statistic for testing this hypothesis, it is instructive to see how the two-way model of Exhibit 4 under the constraint $R\beta = r = 0$ reduces to the one-way model of Exhibit 3. In an earlier article[†] the author showed that the restricted BLUE β^* of the model $\mathbf{y} = X\beta + \mathbf{e}$ subject to the constraint $R\beta = r$ falls out of the equation:

$$\begin{bmatrix} X'X & R' \\ R & 0 \end{bmatrix} \begin{bmatrix} \beta^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} X'y \\ r \end{bmatrix}$$

One can discern this form in the three additional rows and columns of the middle section of Exhibit 6. The three lambdas are Lagrange multipliers, byproducts of the constrained least-squares problem that may be ignored. One may check that the seven sex-age parameters are sums of the proper sex and age-group

[†] Leigh J. Halliwell, "Statistical and Financial Aspects of Self-Insurance Funding," Alternative Markets/Self Insurance (CAS 1996 Discussion Paper Program), 1-46. Appendix A and its citations not only solve the restricted least-squares problem, but provide proofs for all the statements herein to follow.

parameters of Exhibit 3. But the equivalence of the two models is apparent from their identical predictions (the $X\boldsymbol{\beta}$ columns) and σ^2 estimates.

The statistical test makes use of the fact the least-squares estimator $\boldsymbol{\beta}$ has mean β and variance $\sigma^2(X'X)^{-1}$. Therefore, $R\boldsymbol{\beta}$ has mean $R\beta$ and variance $\sigma^2R(X'X)^{-1}R'$. Also important is the assumption that the distribution of $\boldsymbol{\beta}$ is multivariate normal. This is true, if $\mathbf{e}$ is multivariate normal; but even under certain robust conditions the distribution of $\boldsymbol{\beta}$ will be asymptotically multivariate normal. Then $R\boldsymbol{\beta}$ will be multivariate normal with mean $R\beta$ and variance $\sigma^2R(X'X)^{-1}R'$, and $R\boldsymbol{\beta} - R\beta$ multivariate normal with mean zero and variance $\sigma^2R(X'X)^{-1}R'$. Wherefore it follows that the expression

$$(R\boldsymbol{\beta} - R\beta)'\left(\sigma^2R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta} - R\beta)$$

$$= \frac{(R\boldsymbol{\beta} - R\beta)'\left(R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta} - R\beta)}{\sigma^2}$$

is chi-square distributed with j degrees of freedom, where j is the number of rows of R. However, normally we do not know σ^2 and have to estimate it. But the sum of the squared residuals divided by σ^2 is chi-square distributed with $t - k$ degrees of freedom, and this sum does not covary with $\boldsymbol{\beta}$. Therefore, under the multivariate normal assumption, the following two expressions are independent chi-square random variables:

$$\frac{(R\boldsymbol{\beta} - R\beta)'\left(R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta} - R\beta)}{\sigma^2}$$

$$\frac{(y - X\beta)'(y - X\beta)}{\sigma^2} = \frac{\sigma^2}{\sigma^2}(t - k)$$

Finally, this implies that the following expression is approximately and asymptotically F-distributed with j and $t - k$ degrees of freedom:

$$\frac{\dfrac{(R\boldsymbol{\beta} - R\beta)'\left(R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta} - R\beta)}{\sigma^2} \Big/ j}{\dfrac{(y - X\beta)'(y - X\beta)}{\sigma^2} \Big/ t - k} = \frac{(R\boldsymbol{\beta} - R\beta)'\left(R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta} - R\beta)/j}{\sigma^2}$$

Hence, under hypothesis H_0: $R\beta = r$, the statistic

$$\frac{(R\boldsymbol{\beta} - R\beta)'\left(R(X'X)^{-1}R'\right)^{-1}(R\boldsymbol{\beta} - R\beta)/j}{\sigma^2}$$

is F-distributed with j and $t - k$ degrees of freedom:

Accordingly, Appendix 7 tests whether the β of the two-way model differs significantly from the β of the one-way model, which is the two-way β under hypothesis H_0. R is a 3×12 matrix of zeroes and ones, and r is the 3×1 zero vector. X is the design matrix of Appendix 4, from which β, $(X'X)^{-1}$ and σ^2 also are taken. The F statistic as described above has the value 0.546, which is at the 65.5 percentile of the $F_{3, 28}$ distribution. In other words, if H_0 be true, in approximately one-third of repeated samples will the F statistic be greater than 0.546. Few statisticians would deem 65.5 percent as significant enough to reject the hypothesis; most would here accept the simpler, one-way model.

Exhibit 1

Automobile Physical-Damage Data

State	Sex	Age Group	Car Years	PD Claims	Freq	Log Freq
CA	F	Young	821.9	94	0.114	-2.168
CA	F	Prime	11,181.7	644	0.058	-2.854
CA	F	Middle	4,792.1	212	0.044	-3.118
CA	F	Old	931.6	39	0.042	-3.173
CA	M	Young	677.6	66	0.097	-2.329
CA	M	Prime	12,418.3	696	0.056	-2.882
CA	M	Middle	5,134.0	222	0.043	-3.141
CA	M	Old	928.5	58	0.062	-2.773
FL	F	Young	515.4	45	0.087	-2.438
FL	F	Prime	5,578.1	229	0.041	-3.193
FL	F	Middle	4,694.1	162	0.035	-3.366
FL	F	Old	1,209.2	54	0.045	-3.109
FL	M	Young	542.0	60	0.111	-2.201
FL	M	Prime	5,796.9	249	0.043	-3.148
FL	M	Middle	4,476.7	171	0.038	-3.265
FL	M	Old	1,166.0	66	0.057	-2.872
MI	F	Young	373.0	6	0.016	-4.130
MI	F	Prime	5,168.0	24	0.005	-5.372
MI	F	Middle	1,545 0	17	0.011	-4.510
MI	F	Old	161.2	3	0.019	-3.984
MI	M	Young	367.8	5	0.014	-4.298
MI	M	Prime	5,966.4	54	0.009	-4.705
MI	M	Middle	1,802.5	13	0.007	-4.932
MI	M	Old	183.8	3	0.016	-4 115
NY	F	Young	447.3	34	0 076	-2.577
NY	F	Prime	5,092.4	220	0.043	-3.142
NY	F	Middle	3,289.3	121	0.037	-3.303
NY	F	Old	611.4	36	0.059	-2.832
NY	M	Young	565.9	59	0.104	-2.261
NY	M	Prime	5,978.7	340	0.057	-2.867
NY	M	Middle	3,528.5	133	0.038	-3.278
NY	M	Old	625.9	30	0.048	-3.038
TX	F	Young	1,009.3	84	0.083	-2.486
TX	F	Prime	8,931.7	513	0.057	-2.857
TX	F	Middle	4,429.9	206	0.047	-3.068
TX	F	Old	705.2	46	0.065	-2.730
TX	M	Young	1,003.4	112	0.112	-2.193
TX	M	Prime	9,067.5	568	0.063	-2.770
TX	M	Middle	4,478.0	225	0.050	-2 991
TX	M	Old	641.7	50	0.078	-2.552

Exhibit 2

Relativity Comparison

State	Sex	Age Group	Car Years	PD Claims	Freq
CA			36,885.7	2,031	0.055
FL			23,978.4	1,036	0.043
MI			15,567.8	125	0.008
NY			20,139.4	973	0.048
TX			30,266.5	1,804	0.060
Total			126,837.8	5,969	0.047

Sex	Age Group	Car Years	PD Claims	Freq	Relativity	S × AG Relativity	Diff
F		61,487.7	2,789	0.045	1.000		
M		65,350.1	3,180	0.049	1.073		
	Young	6,323.6	565	0.089	1.899		
	Prime	75,179.7	3,537	0.047	1 000		
	Middle	38,170.0	1,482	0.039	0 825		
	Old	7,164.6	385	0.054	1.142		
F	Young	3,166.9	263	0.083	1.832	1.899	3.7%
F	Prime	35,951.9	1,630	0.045	1.000	1.000	0.0%
F	Middle	18,750.3	718	0.038	0.845	0.825	-2.3%
F	Old	3,618 6	178	0.049	1.085	1.142	5.3%
M	Young	3,156.7	302	0 096	2.110	2.037	-3.4%
M	Prime	39,227 8	1,907	0.049	1.072	1.073	0 1%
M	Middle	19,419.7	764	0.039	0.868	0.885	2.0%
M	Old	3,545.9	207	0.058	1.288	1.225	-4 8%

Age Groups

Young	under 21
Prime	21 to 40
Middle	41 to 65
Old	over 65

Exhibit 3

Linear Model with Two One-Way Factors

State	Sex	Age Group	y	X: CA	FL	MI	NY	TX	M	Young	Middle	Old	y	Xβ	y-Xβ
CA	F	Young	-2.168	1						1			2.168	-2.382	0.214
CA	F	Prime	-2.854	1									-2.854	-3.053	0.199
CA	F	Middle	-3.118	1							1		-3.118	-3.172	0.053
CA	F	Old	-3.173	1								1	-3.173	-2.792	-0.381
CA	M	Young	-2.329	1					1	1			-2.329	-2.292	-0.036
CA	M	Prime	-2.882	1					1				-2.882	2.963	0.082
CA	M	Middle	-3.141	1					1		1		-3.141	-3.081	-0.059
CA	M	Old	-2.773	1					1			1	-2.773	-2.702	-0.071
FL	F	Young	-2.438		1					1			-2.438	-2.527	0.088
FL	F	Prime	-3.193		1								-3.193	-3.197	0.005
FL	F	Middle	-3.366		1						1		-3.366	-3.316	-0.051
FL	F	Old	-3.109		1							1	-3.109	-2.936	-0.172
FL	M	Young	-2.201		1				1	1			-2.201	-2.438	0.236
FL	M	Prime	-3.148		1				1				-3.148	-3.107	-0.040
FL	M	Middle	-3.265		1				1		1		-3.265	-3.226	-0.039
FL	M	Old	-2.872		1				1			1	-2.872	-2.846	-0.025
MI	F	Young	-4.130			1				1			-4.130	-4.083	-0.046
MI	F	Prime	-5.372			1							-5.372	-4.754	-0.618
MI	F	Middle	-4.510			1					1		-4.510	-4.872	0.363
MI	F	Old	-3.984			1						1	-3.984	-4.493	0.509
MI	M	Young	-4.298			1			1	1			-4.298	-3.993	-0.305
MI	M	Prime	-4.705			1			1				-4.705	-4.664	-0.041
MI	M	Middle	-4.932			1			1		1		-4.932	-4.782	-0.150
MI	M	Old	-4.115			1			1			1	-4.115	-4.403	0.288
NY	F	Young	-2.577				1			1			-2.577	-2.490	-0.087
NY	F	Prime	-3.142				1						-3.142	-3.161	0.019
NY	F	Middle	-3.303				1				1		-3.303	-3.279	-0.024
NY	F	Old	-2.832				1					1	-2.832	-2.900	0.067
NY	M	Young	2.261				1		1	1			-2.261	-2.400	0.139
NY	M	Prime	-2.867				1		1				-2.867	-3.071	0.204
NY	M	Middle	-3.278				1		1		1		-3.278	-3.189	-0.089
NY	M	Old	-3.038				1		1			1	-3.038	-2.810	-0.228
TX	F	Young	-2.486					1		1			-2.486	-2.283	-0.203
TX	F	Prime	-2.857					1					-2.857	-2.954	0.097
TX	F	Middle	-3.068					1			1		-3.068	-3.073	0.004
TX	F	Old	-2.730					1				1	-2.730	-2.693	-0.037
TX	M	Young	-2.193					1	1	1			-2.193	-2.193	0.001
TX	M	Prime	-2.770					1	1				-2.770	-2.664	0.094
TX	M	Middle	-2.991					1	1		1		-2.991	-2.983	-0.008
TX	M	Old	-2.552					1	1			1	-2.552	-2.603	0.061

X'y	X'X								
-22.439	8	0	0	0	0	4	2	2	2
-23.592	0	8	0	0	0	4	2	2	2
-36.046	0	0	8	0	0	4	2	2	2
-23.298	0	0	0	8	0	4	2	2	2
-21.847	0	0	0	0	8	4	2	2	2
-62.610	4	4	4	4	4	20	5	5	5
-27.081	2	2	2	2	2	5	10	0	0
-34.972	2	2	2	2	2	5	0	10	0
-31.179	2	2	2	2	2	5	0	0	10

Cross Products

426.616	425.095	1.521
425.095	425.095	0.000
1.521	0.000	1.521
100.0%	99.6%	0.4%

t	40
k	9
df	31
σ^2	0.049
σ	0.222

	β	$(X'X)^{-1}$								
CA	-3.053	0.225	0.100	0.100	0.100	0.100	-0.050	-0.100	-0.100	-0.100
FL	-3.197	0.100	0.225	0.100	0.100	0.100	-0.050	-0.100	-0.100	-0.100
MI	-4.754	0.100	0.100	0.225	0.100	0.100	-0.050	-0.100	-0.100	-0.100
NY	-3.161	0.100	0.100	0.100	0.225	0.100	-0.050	-0.100	-0.100	-0.100
TX	-2.954	0.100	0.100	0.100	0.100	0.225	-0.050	-0.100	-0.100	-0.100
M	0.090	-0.050	-0.050	-0.050	-0.050	-0.050	0.100	0.000	0.000	0.000
Young	0.671	-0.100	-0.100	-0.100	-0.100	-0.100	0.000	0.200	0.100	0.100
Middle	-0.118	-0.100	-0.100	-0.100	-0.100	-0.100	0.000	0.100	0.200	0.100
Old	0.261	-0.100	-0.100	-0.100	-0.100	-0.100	0.000	0.100	0.100	0.200

t-stat	Std(β)	Var(β)								
-29.056	0.105	0.011	0.005	0.005	0.005	0.005	-0.002	-0.005	-0.005	-0.005
-30.428	0.105	0.005	0.011	0.005	0.005	0.005	-0.002	-0.005	-0.005	-0.005
-45.243	0.105	0.005	0.005	0.011	0.005	0.005	-0.002	-0.005	-0.005	-0.005
-30.078	0.105	0.005	0.005	0.005	0.011	0.005	-0.002	-0.005	-0.005	-0.005
-28.115	0.105	0.005	0.005	0.005	0.005	0.011	-0.002	-0.005	-0.005	-0.005
1.285	0.070	-0.002	-0.002	-0.002	-0.002	-0.002	0.005	0.000	0.000	0.000
6.772	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.000	0.010	0.005	0.005
-1.193	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.000	0.005	0.010	0.005
2.636	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.000	0.005	0.005	0.010

Exhibit 4

Linear Model with One Two-Way Factor

State	S = AG	y	CA	FL	MI	NY	TX	FYoung	FMiddle	FOld	MYoung	MPrime	MMiddle	MOld	y	Xβ	y - Xβ
			X														
CA	FYoung	2.168	1					1							2.168	-2.389	0.221
CA	FPrime	-2.854	1												-2.854	-3.113	0.259
CA	FMiddle	-3.118	1						1						-3.118	-3.102	-0.016
CA	FOld	-3.173	1							1					-3.173	-2.795	-0.378
CA	MYoung	-2.329	1								1				-2.329	2.286	-0.043
CA	MPrime	-2.882	1									1			-2.882	-2.904	0.022
CA	MMiddle	-3.141	1										1		-3.141	-3.151	0.010
CA	MOld	-2.773	1											1	-2.773	-2.699	-0.074
FL	FYoung	-2.438		1				1							-2.438	-2.533	0.095
FL	FPrime	-3.193		1											-3.193	-3.257	0.064
FL	FMiddle	-3.366		1					1						-3.366	-3.246	-0.120
FL	FOld	-3.109		1						1					-3.109	-2.939	-0.170
FL	MYoung	2.201		1							1				-2.201	-2.430	0.229
FL	MPrime	-3.148		1								1			-3.148	-3.048	-0.100
FL	MMiddle	-3.265		1									1		-3.265	-3.296	0.030
FL	MOld	-2.872		1										1	-2.872	-2.843	-0.028
MI	FYoung	-4.130			1			1							-4.130	-4.090	-0.040
MI	FPrime	-5.372			1										-5.372	-4.814	-0.558
MI	FMiddle	-4.510			1				1						-4.510	-4.803	0.294
MI	FOld	-3.984			1					1					-3.984	-4.496	0.512
MI	MYoung	-4.298			1						1				-4.298	-3.987	-0.312
MI	MPrime	-4.705			1							1			-4.705	-4.605	-0.100
MI	MMiddle	-4.932			1								1		-4.932	-4.852	-0.080
MI	MOld	-4.115			1									1	-4.115	-4.400	0.285
NY	FYoung	-2.577				1		1							-2.577	-2.497	-0.080
NY	FPrime	-3.142				1									-3.142	-3.220	0.078
NY	FMiddle	-3.303				1			1						-3.303	-3.210	-0.093
NY	FOld	-2.832				1				1					-2.832	-2.902	0.070
NY	MYoung	-2.261				1					1				-2.261	-2.393	0.132
NY	MPrime	-2.867				1						1			-2.867	-3.011	0.144
NY	MMiddle	-3.278				1							1		-3.278	-3.258	-0.020
NY	MOld	-3.038				1								1	-3.038	-2.807	-0.231
TX	FYoung	-2.486					1	1							-2.486	-2.290	-0.196
TX	FPrime	-2.857					1								-2.857	-3.014	0.157
TX	FMiddle	-3.068					1		1						-3.068	-3.003	-0.065
TX	FOld	-2.730					1			1					-2.730	-2.698	-0.034
TX	MYoung	-2.193					1				1				2.193	-2.187	-0.006
TX	MPrime	-2.770					1					1			-2.770	-2.806	0.034
TX	MMiddle	-2.991					1						1		-2.991	-3.052	0.061
TX	MOld	-2.552					1							1	-2.552	2.600	0.048

X'y	X'X											
-22.439	8	0	0	0	0	1	1	1	1	1	1	1
-23.582	0	8	0	0	0	1	1	1	1	1	1	1
-36.046	0	0	8	0	0	1	1	1	1	1	1	1
-23.298	0	0	0	8	0	1	1	1	1	1	1	1
-21.647	0	0	0	0	8	1	1	1	1	1	1	1
-13.799	1	1	1	1	1	5	0	0	0	0	0	0
-17.365	1	1	1	1	1	0	5	0	0	0	0	0
-15.628	1	1	1	1	1	0	0	5	0	0	0	0
-13.281	1	1	1	1	1	0	0	0	5	0	0	0
-18.371	1	1	1	1	1	0	0	0	0	5	0	0
17.807	1	1	1	1	1	0	0	0	0	0	5	0
-15.350	1	1	1	1	1	0	0	0	0	0	0	5

Cross Products

426.616	425.179	1.437
425.179	425.179	0.000
1.437	0.000	1.437

100.0%	99.7%	0.3%
I	40	
k	12	
df	28	
σ^2	0.051	
σ	0.227	

	β	$(X'X)^{-1}$											
CA	-3.113	0.300	0.175	0.175	0.175	0.175	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
FL	-3.257	0.175	0.300	0.175	0.175	0.175	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
MI	-4.814	0.175	0.175	0.300	0.175	0.175	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
NY	-3.220	0.175	0.175	0.175	0.300	0.175	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
TX	3.014	0.175	0.175	0.175	0.175	0.300	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
FYoung	0.724	-0.200	-0.200	-0.200	-0.200	-0.200	0.400	0.200	0.200	0.200	0.200	0.200	0.200
FMiddle	0.011	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.400	0.200	0.200	0.200	0.200	0.200
FOld	0.318	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.200	0.400	0.200	0.200	0.200	0.200
MYoung	0.827	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.200	0.200	0.400	0.200	0.200	0.200
MPrime	0.209	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.200	0.200	0.200	0.400	0.200	0.200
MMiddle	0.038	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.200	0.200	0.200	0.200	0.400	0.200
MOld	0.414	-0.200	-0.200	-0.200	-0.200	-0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.400

t-stat	Std[β]	Var[β]											
-25.085	0.124	0.015	0.009	0.009	0.009	0.009	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010
-26.246	0.124	0.009	0.015	0.009	0.009	0.009	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010
38.792	0.124	0.009	0.009	0.015	0.009	0.009	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010
-25.950	0.124	0.009	0.009	0.009	0.015	0.009	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010
-24.288	0.124	0.009	0.009	0.009	0.009	0.015	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010	-0.010
5.051	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.021	0.010	0.010	0.010	0.010	0.010	0.010
0.074	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.021	0.010	0.010	0.010	0.010	0.010
2.219	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.010	0.021	0.010	0.010	0.010	0.010
5.774	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.010	0.010	0.021	0.010	0.010	0.010
1.461	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.010	0.010	0.010	0.021	0.010	0.010
-0.263	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.010	0.010	0.010	0.010	0.021	0.010
2.886	0.143	-0.010	-0.010	-0.010	-0.010	-0.010	0.010	0.010	0.010	0.010	0.010	0.010	0.021

Exhibit 5

Comparison of Models

State	Sex	Age Group	Actual	One-Way Predicted	Two-Way Predicted	One-Way Closer	Two-Way Closer
CA	F	Young	-2.168	-2 382	-2.389	1	
CA	F	Prime	-2 854	-3.053	-3.113	1	
CA	F	Middle	-3 118	-3 172	-3.102		1
CA	F	Old	-3.173	-2.792	-2 795		1
CA	M	Young	-2 329	-2 292	-2.286	1	
CA	M	Prime	-2 882	-2.963	-2 904		1
CA	M	Middle	-3.141	-3 081	-3 151		1
CA	M	Old	-2 773	-2.702	-2 699	1	
FL	F	Young	-2 438	-2 527	-2.533	1	
FL	F	Prime	-3 193	-3.197	-3 257	1	
FL	F	Middle	-3 368	-3 316	-3 246	1	
FL	F	Old	-3.109	-2 936	-2 939		1
FL	M	Young	-2 201	-2 436	-2.430		1
FL	M	Prime	-3 148	-3 107	-3.048	1	
FL	M	Middle	-3.265	-3.226	-3 295		1
FL	M	Old	-2.872	-2.846	-2.843	1	
MI	F	Young	-4.130	-4 083	-4.090		1
MI	F	Prime	-5.372	-4.754	-4 814		1
MI	F	Middle	-4.510	-4.872	-4 803		1
MI	F	Old	-3.984	-4.493	-4 496	1	
MI	M	Young	-4.298	-3.993	-3 987	1	
MI	M	Prime	-4.705	-4.664	-4 605	1	
MI	M	Middle	-4 932	-4.782	-4 852		1
MI	M	Old	-4 115	-4.403	-4 400		1
NY	F	Young	-2 577	-2 490	-2 497		1
NY	F	Prime	-3 142	-3.161	-3.220	1	
NY	F	Middle	-3 303	-3.279	-3.210	1	
NY	F	Old	-2.832	-2 900	-2.902	1	
NY	M	Young	-2 261	-2 400	-2.393		1
NY	M	Prime	-2.867	-3 071	-3 011		1
NY	M	Middle	-3 278	-3 189	-3.258		1
NY	M	Old	-3 038	-2.810	-2.807	1	
TX	F	Young	-2.486	-2 283	-2 290		1
TX	F	Prime	-2 857	-2.954	-3.014	1	
TX	F	Middle	-3 068	-3 073	-3 003	1	
TX	F	Old	-2.730	-2.693	-2 696		1
TX	M	Young	-2 193	-2.193	-2 187	1	
TX	M	Prime	-2.770	-2.864	-2 805		1
TX	M	Middle	-2 991	-2 983	-3.052	1	
TX	M	Old	-2.552	-2 603	-2 600		1
Mean Absolute Deviation			0 000	0 136	0.137	20	20
Root Mean Square Deviation			0 000	0.195	0 190		

One-Way Relativies		
F	1.000	0.0%
M	1 094	+9 4%
Young	1 956	+95 6%
Prime	1.000	0.0%
Middle	0 888	-11 2%
Old	1 298	+29 8%

Two-Way Relativies			One-Ways Multiplied		Diff
F.Young	2 062	+106 2%	1 956	+95 6%	5 4%
F.Prime	1.000	0 0%	1.000	0 0%	0 0%
F.Middle	1.011	+1.1%	0 888	-11.2%	13.8%
F.Old	1.374	+37 4%	1 298	+29.8%	5 9%
M Young	2.287	+128 7%	2 140	+114 0%	6.9%
M Prime	1.233	+23.3%	1 094	+9 4%	12 7%
M.Middle	0 963	-3 7%	0.972	-2.8%	-1 0%
M Old	1.512	+51 2%	1 421	+42.1%	6 4%

Exhibit 6

Restricted Linear Model with One Two-Way Factor

State	S × AG	y	CA	FL	MI	NY	TX	FYoung	FMiddle	FOld	MYoung	MPrime	MMiddle	MOld
CA	FYoung	-2.168	1					1						
CA	FPrime	2.864	1											
CA	FMiddle	-3.118	1						1					
CA	FOld	-3.173	1							1				
CA	MYoung	-2.329	1								1			
CA	MPrime	-2.882	1									1		
CA	MMiddle	-3.141	1										1	
CA	MOld	2.773	1											1
FL	FYoung	-2.438		1				1						
FL	FPrime	-3.193		1										
FL	FMiddle	-3.386		1					1					
FL	FOld	-3.109		1						1				
FL	MYoung	-2.201		1							1			
FL	MPrime	-3.148		1								1		
FL	MMiddle	-3.285		1									1	
FL	MOld	-2.872		1										1
MI	FYoung	-4.130			1			1						
MI	FPrime	-6.372			1									
MI	FMiddle	-4.610			1				1					
MI	FOld	-3.984			1					1				
MI	MYoung	-4.298			1						1			
MI	MPrime	-4.705			1							1		
MI	MMiddle	-4.932			1								1	
MI	MOld	-4.115			1									1
NY	FYoung	-2.577				1		1						
NY	FPrime	-3.142				1								
NY	FMiddle	-3.303				1			1					
NY	FOld	-2.832				1				1				
NY	MYoung	-2.281				1					1			
NY	MPrime	-2.867				1						1		
NY	MMiddle	-3.278				1							1	
NY	MOld	-3.038				1								1
TX	FYoung	-2.486					1	1						
TX	FPrime	-2.857					1							
TX	FMiddle	-3.068					1		1					
TX	FOld	-2.730					1			1				
TX	MYoung	-2.183					1				1			
TX	MPrime	-2.770					1					1		
TX	MMiddle	-2.991					1						1	
TX	MOld	2.552					1							1

y	Xβ	y − Xβ
-2.168	-2.382	0.214
-2.854	-3.053	0.199
-3.118	-3.172	0.053
-3.173	-2.792	-0.381
-2.329	-2.292	-0.036
-2.882	-2.963	0.082
-3.141	-3.081	-0.059
-2.773	-2.702	-0.071
-2.438	-2.527	0.088
-3.193	-3.197	0.005
-3.368	-3.318	-0.051
-3.109	-2.936	-0.172
-2.201	-2.436	0.236
-3.148	-3.107	-0.040
-3.265	-3.226	-0.039
-2.872	-2.848	-0.025
-4.130	-4.083	-0.048
-6.372	-4.754	-0.818
-4.510	-4.872	0.363
-3.984	-4.493	0.509
-4.298	-3.993	-0.305
-4.705	-4.664	-0.041
-4.932	-4.782	-0.150
-4.116	-4.403	0.288
-2.877	-2.490	-0.087
-3.142	-3.161	0.019
-3.303	-3.279	-0.024
-2.832	-2.900	0.067
-2.261	-2.400	0.139
-2.867	-3.071	0.204
-3.278	-3.189	-0.089
-3.038	-2.810	-0.228
-2.486	-2.283	-0.203
-2.857	-2.954	0.097
-3.068	-3.073	0.004
-2.730	-2.693	-0.037
-2.193	2.193	0.001
-2.770	-2.864	0.094
-2.991	-2.983	-0.008
-2.552	2.603	0.051

	X'y	X'X														
	-22.439	8	0	0	0	0	1	1	1	1	1	1	1	0	0	0
	-23.892	0	8	0	0	0	1	1	1	1	1	1	1	0	0	0
	-36.046	0	0	8	0	0	1	1	1	1	1	1	1	0	0	0
	-23.298	0	0	0	8	0	1	1	1	1	1	1	1	0	0	0
	-21.647	0	0	0	0	8	1	1	1	1	1	1	1	0	0	0
	-13.799	1	1	1	1	1	5	0	0	0	0	0	0	1	0	0
	-17.365	1	1	1	1	1	0	5	0	0	0	0	0	0	1	0
	-15.828	1	1	1	1	1	0	0	5	0	0	0	0	0	0	1
	13.281	1	1	1	1	1	0	0	0	5	0	0	0	-1	0	0
	-18.371	1	1	1	1	1	0	0	0	0	5	0	0	1	1	1
	-17.807	1	1	1	1	1	0	0	0	0	0	5	0	0	-1	0
	-15.350	1	1	1	1	1	0	0	0	0	0	0	5	0	0	1
Restriction 1	0.000	0	0	0	0	0	1	0	0	-1	1	0	0	0	0	0
Restriction 2	0.000	0	0	0	0	0	0	1	0	0	1	-1	0	0	0	0
Restriction 3	0.000	0	0	0	0	0	0	0	1	0	1	0	-1	0	0	0

Cross Products

426.618	425.096	1.521
425.096	425.096	0.000
1.521	0.000	1.521

100.0%	99.6%	0.4%
r	40	
a	12	
j	3	
df	31	
σ^2	0.049	
σ	0.222	

	β	$(X'X)^{-1}$														
CA	-3.053	0.225	0.100	0.100	0.100	0.100	-0.100	-0.100	-0.100	-0.150	-0.050	-0.150	-0.150	-0.125	-0.125	-0.125
FL	3.197	0.100	0.225	0.100	0.100	0.100	-0.100	-0.100	-0.100	-0.150	-0.050	-0.150	-0.150	-0.125	-0.125	-0.125
MI	-4.764	0.100	0.100	0.225	0.100	0.100	-0.100	-0.100	-0.100	-0.150	-0.050	-0.150	-0.150	-0.125	-0.125	-0.125
NY	-3.161	0.100	0.100	0.100	0.225	0.100	-0.100	-0.100	-0.100	-0.150	-0.050	-0.150	-0.150	-0.125	-0.125	-0.125
TX	-2.954	0.100	0.100	0.100	0.100	0.225	-0.100	-0.100	-0.100	-0.150	-0.050	-0.150	-0.150	-0.125	-0.125	-0.125
FYoung	0.671	-0.100	-0.100	-0.100	-0.100	-0.100	0.200	0.100	0.100	0.200	0.000	0.100	0.100	0.500	0.000	0.000
FMiddle	-0.118	-0.100	-0.100	-0.100	-0.100	-0.100	0.100	0.200	0.100	0.100	0.000	0.200	0.100	0.000	0.500	0.000
FOld	0.261	-0.100	-0.100	-0.100	-0.100	-0.100	0.100	0.100	0.200	0.100	0.000	0.100	0.200	0.000	0.000	0.500
MYoung	0.781	-0.150	-0.150	-0.150	-0.150	-0.150	0.200	0.100	0.100	0.300	0.100	0.200	0.200	-0.250	0.250	0.250
MPrime	0.090	-0.050	-0.050	-0.050	-0.050	-0.050	0.000	0.000	0.000	0.100	0.100	0.100	0.100	0.250	0.250	0.250
MMiddle	0.028	-0.150	-0.150	-0.150	-0.150	-0.150	0.100	0.200	0.100	0.200	0.100	0.300	0.200	0.250	-0.250	0.250
MOld	0.351	-0.150	-0.150	-0.150	-0.150	-0.150	0.100	0.100	0.200	0.200	0.100	0.200	0.300	0.250	0.250	-0.750
L	-0.034	-0.125	-0.125	-0.125	-0.125	-0.125	0.500	0.000	0.000	-0.250	0.250	0.250	0.250	1.875	0.625	0.625
λ	0.348	-0.125	-0.125	-0.125	-0.125	-0.125	0.000	0.500	0.000	0.250	0.250	-0.250	0.250	0.625	-1.875	0.625
λ	-0.014	-0.125	-0.125	-0.125	-0.125	-0.125	0.000	0.000	0.500	0.250	0.250	0.250	-0.250	0.625	0.625	1.875

t-stat	Std(β)	Var(β)											
-29.056	0.105	0.011	0.005	0.005	0.005	0.005	-0.005	-0.005	-0.005	-0.007	-0.002	-0.007	-0.007
-30.428	0.105	0.005	0.011	0.005	0.005	0.005	-0.005	-0.005	-0.005	-0.007	-0.002	-0.007	-0.007
-45.243	0.105	0.005	0.005	0.011	0.005	0.005	-0.005	-0.005	-0.005	-0.007	-0.002	-0.007	0.007
-30.078	0.105	0.005	0.005	0.005	0.011	0.005	-0.005	-0.005	-0.005	-0.007	-0.002	-0.007	-0.007
-28.115	0.105	0.005	0.005	0.005	0.005	0.011	-0.005	-0.005	-0.005	-0.007	-0.002	-0.007	-0.007
6.772	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.010	0.005	0.005	0.010	0.000	0.005	0.005
-1.193	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.005	0.010	0.005	0.005	0.000	0.010	0.005
2.636	0.099	-0.005	-0.005	-0.005	-0.005	-0.005	0.005	0.005	0.010	0.005	0.000	0.005	0.010
6.271	0.121	-0.007	-0.007	-0.007	-0.007	-0.007	0.010	0.005	0.005	0.015	0.005	0.010	0.010
1.285	0.070	-0.002	-0.002	-0.002	-0.002	-0.002	0.000	0.000	0.000	0.005	0.005	0.005	0.005
-0.232	0.121	-0.007	-0.007	-0.007	-0.007	-0.007	0.005	0.010	0.005	0.010	0.005	0.015	0.010
2.894	0.121	-0.007	-0.007	-0.007	-0.007	-0.007	0.005	0.005	0.010	0.010	0.005	0.010	0.015

Exhibit 7

Test: Is the two-way model significantly better than the one-way model?

Null Hypothesis H_0: $R\beta = 0$

R

CA	FL	MI	NY	TX	FYoung	FMiddle	FOld	MYoung	MPrime	MMiddle	MOld
0	0	0	0	0	1	0	0	-1	1	0	0
0	0	0	0	0	0	1	0	0	1	-1	0
0	0	0	0	0	0	0	1	0	1	0	-1

Unrestricted Two-Way Estimator

$R\beta$	$R(X'X)^{-1}R'$		
0.106	0.800	0.400	0.400
0.258	0.400	0.800	0.400
0.114	0.400	0.400	0.800

$R\beta - 0$	$(R(X'X)^{-1}R')^{-1}$		
0.106	1.875	-0.625	-0.625
0.258	-0.625	1.875	-0.625
0.114	-0.625	-0.625	1.875

numerator *df*	3
denominator *df*	28
numerator	0.028
denominator (σ^2)	0.051
F statistic	0.546
Prob[$F = 0.546 \mid H_0$]	65.5%

Enhancing Generalised Linear Models with Data Mining

Dr. Inna Kolyshkina, Sylvia Wong, and Steven Lim

Enhancing Generalised Linear Models with Data Mining

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1. Introduction

Generalised linear models (GLM) appear to be a tool that has become very popular and have shown to be effective in the actuarial work over the past decade, see for example Haberman & Renshaw (1998). A detailed description of the GLM methodology is outside of the scope of this paper, and can be found in the other sources such as McCullach and Nelder(1989).

Data mining methodologies are more recent and their popularity in the actuarial community is increasing. They have been used in insurance for risk prediction/assessment, premium setting, fraud detection, health costs prediction, treatment management optimization, investments management optimization, customer retention research and acquisition strategies. Recently a number of publications have examined the use of data mining methods in an insurance and actuarial environment (eg, Francis (2001), Francis (2003). The main reasons for the increasing attractiveness of the data mining approach is that it is very fast computationally and also overcomes some well-known shortcomings of traditional methods.

However the advance of the new methodologies does not mean that the proven, effective techniques such as GLM should be wholly replaced by them. This paper discusses how the advantages and strengths of GLM can be effectively combined with the computational power of data mining methods presenting an example of the combining multivariate adaptive regression splines (MARS®) and GLM approaches by running MARS® model and then building a GLM with MARS® output functions used as predictors. The results of this combined model are compared to the results achievable by hand-fitted GLM. Comparisons are made in terms of time taken, predictive power, selection predictors and their interactions, interpretability of the model, precision and model fit.

2. Enhancing the Linear Modelling Approach by Combining it with Data Mining

GLM being a linear technique shares the usual shortcomings of the linear modelling approach.

Linear models

- operate under the assumption that data is distributed according to a distribution in the exponential family

- are affected by multicollinearity, outliers and missing values in the data

- are often troublesome to use for selecting important predictors and their interactions

- are troublesome to use with categorical predictors that have large numbers of categories (for example, postcode, occupation code etc) as this can lead to unreliable results due to sparsity-related issues

- take longer to build because of the need to address the issues above by transforming both numeric and categorical predictors and choosing predictors and their interactions by hand which can prove to be a lengthy task.

Data mining techniques in contrast

- are typically fast,
- easily select predictors and their interactions,
- are minimally affected with missing values, outliers or collinearity and
- effectively process high-level categorical predictors.

This suggests that combining a linear approach with data mining tools can expedite the modelling process, allowing the modeller to attain equal or better model accuracy in less time with the same level of interpretability. Such models, usually combining decision trees, multivariate adaptive regression splines and GLM have been used by our team in a number of projects (see Kolyshkina and Brookes, 2002).

3. Multivariate Adaptive Regression Splines (MARS®)

Multivariate adaptive regression splines (MARS®) is becoming increasingly popular in the actuarial community; for example, Francis (2003) describes application of MARS® to insurance fraud analysis.

We provide below a brief introduction to the MARS® methodology, a more detailed description can be found in other sources (see for example, Friedman, 1991, Hastie et al. (2001)).

MARS® is an adaptive procedure for regression, and can be viewed as a generalisation of stepwise linear regression or a generalization of the recursive partitioning method to improve the latter's performance in the regression setting (Friedman, 1991; Hastie et al, 2001).

The MARS® procedure builds flexible regression models by fitting separate splines (or basis functions) to distinct intervals of the predictor variables. Both the variables to use and the end points of the intervals for each variable-referred to as "knots" -are found via an exhaustive search procedure, using very fast update algorithms and efficient program coding. Variables, knots and interactions are optimized simultaneously by evaluating a "loss of fit" (LOF) criterion. MARS® chooses the LOF that most improves the model at each step. In addition to searching variables one by one, MARS® also searches for interactions between variables, allowing any degree of interaction to be considered.

The "optimal" MARS® model is selected in a two-phase process. In the first phase, a model is grown by adding basis functions (new main effects, knots, or interactions) until an overly large model is found. In the second phase, basis functions are deleted in order of least contribution to the model until an optimal balance of bias and variance is found. By allowing for any arbitrary shape for the response function as well as for interactions, and by using the two-phase model selection method, MARS® is capable of reliably tracking very complex data structures that often hide in high-dimensional data (Salford Systems, 2002). MARS® is fast, requires less data preparation than some other techniques, can easily handle missing values or noisy data, and the output, for both the model and the basis functions, is easy to interpret. MARS® is implemented in a software package produced by Salford systems. The package is easily available, inexpensive and can work with data in most formats (SAS, SPSS, dbf etc). The output MARS® produces can be combined with any GLM software with minimal effort as it is easy to code in any program language such as SAS which is the main data analysis software package used by many actuaries.

4. How the Use of MARS® Can Expedite GLM Building

Most of the shortcomings of linear models outlined above can be overcome by using MARS® as a way of pre-processing predictors before putting them in a GLM. This will also significantly reduce the time needed for model building. This can be done by feeding MARS® output (in the form of basis functions created by MARS®) as inputs into a GLM.

MARS® is minimally affected by multicollinearity, outliers and missing values in the data, easily handles categorical predictors with large numbers of categories and requires less data preparation than linear methods, it quickly selects important predictors and their interactions and transforms numeric and categorical predictors in such a way that the resulting variables are easy to interpret. The modeller though needs to make sure that the transformed predictors make business sense, and that the MARS® model is stable.

We have seen that although MARS® output functions are not created specifically to be used as the input for a linear model, in practice about 90% of them turn out to be significant predictors in a GLM. Another feature of the MARS® output functions that makes them useful is that they are linearly independent as stated in Friedman (1991) which means that the multicollinearity issues do not arise in the GLM that uses them as explanatory variables.

In the case study below this technique was applied to summarised data, but it would be even more efficient on the individual level data with many predictors, both numeric and categorical.

5. Case Study. Queensland Industry CTP data provided by Motor Accident Insurance Commission (MAIC)

5.1 Background

The methodology described above was applied in order to model the ultimate incurred number of claims based on reported claim data. The data used was industry-wide auto liability data from Queensland (commonly called Compulsory Third Party or "CTP" in Australia).

5.2 Data Description

Individual claim data was aggregated into the number of claims reported for each accident month and development month for input to the GLM. The variables used for the analysis were accident month, accident quarter, number of casualties, development month, development quarter, number of vehicles in the calendar year, and number of vehicles exposed in the month.

5.3 Modelling Methodologies Description

5.3.1 Hand-fitted GLM

An initial GLM was created without using MARS®. This was a Poisson model with the log link, using the number of vehicles exposed in the month as the offset. The transformations and interactions of the input variables were created manually for the purposes of both best model fit and interpretability. The model fit was assessed by usual methods such as ratio of deviance to the degrees of freedom, predictor significance, link test and residual analysis. All the assessments showed adequacy of the model fit.

5.3.2 MARS®- enhanced GLM

A second model was created by preprocessing the variables in MARS® as described above and then including them in a GLM as the inputs. First we built a MARS® model with the ratio of incurred number of claims to number of vehicles exposed in the month as the dependent variable. The model output included explanatory variables pre-processed by MARS®. We then used these variables as the inputs in the Poisson model with log link and the number of vehicles exposed in the month as the offset. The GLM output showed that most of these variables were significant. We then used backward elimination to refine the model by excluding the inputs that were not significant. The resulting model fit was assessed in the same way as for the hand-fitted model above.

5.4 Comparison of Models

5.4.1 Timing

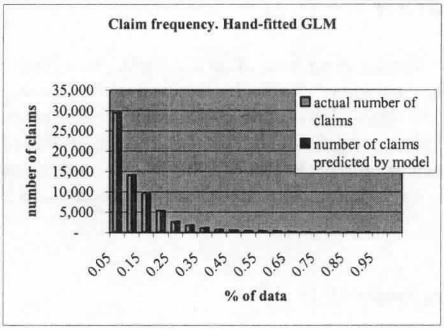
The hand-fitted model of this type would usually take about 5-7 days to build and refine.

The MARS® - enhanced model involved running the model in MARS® with different settings such as finding the optimal level of predictor interaction, then copying and pasting MARS® output into SAS and running and refining the GLM. This took about half a day. The MARS® analysis took less than an hour.

5.4.2 Goodness of fit. Bar charts. Gains chart

The fit of both GLMs was assessed by usual methods such as ratio of deviance to the degrees of freedom, predictor significance, link test and residual analysis. The MARS®- enhanced GLM has shown a similar if not slightly better fit to the hand-fitted GLM.

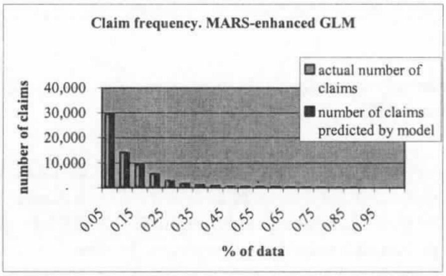
Figure 1. Average actual and predicted values for overall number of claims, hand-fitted GLM



A further diagnostic of model performance is analysis of actual versus expected values of the number of claims.

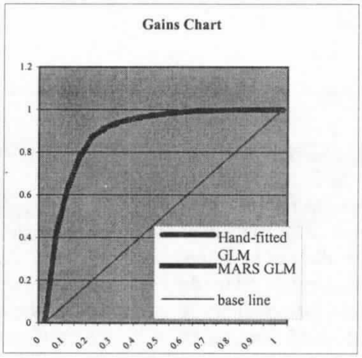
Such analysis can be pictorially represented by a bar chart of averaged actual and predicted values for the number of claims. To create such a chart, the records were ranked from highest to lowest in terms of predicted number of claims for each model, and then segmented into 20 equally sized groups. The average predicted and actual values of the number of claims for each group were then calculated and graphed. The chart for the hand-fitted GLM is shown in Figure 1 and the chart for MARS®-enhanced GLM is shown in Figure 2.

Figure 2 Average actual and predicted values for overall number of claims, MARS-enhanced GLM



Comparison of the charts suggests that the models fit equally well, with the MARS®-enhanced GLM having a marginally better fit. The hand-fitted GLM slightly over-predicts for the fifth group and under-predicts for the third group while the MARS® - enhanced GLM predicts well for the higher expected numbers of claims but slightly overpredicts for the groups with lower numbers of claims. However, the scale of these errors is of little business importance.

Figure 3 Gains chart for number of incurred claims for both models.



Another graphical method used for the comparison of the models was gains charts. Gains charts are described in detail in literature, see for example Berry & Linoff (2000).The gains chart presented in Figure 3 shows that both models are able to predict the segments with high number of claims with a good degree of accuracy. As a rough guide, taking the 15% of records predicted as having the highest number of incurred claims by the model, we end up with 80% of the total number of incurred claims. Taking the 30% of records predicted as having the highest number of incurred claims by the model, we end up with 93% of the total number of incurred claims. The graph above shows that the models perform equally well. Detailed analysis of actual statistical results suggests that the MARS® - enhanced GLM performs marginally better than the hand-fitted model.

It can be seen from the charts above that the MARS® - enhanced GLM fits only slightly better than the hand-fitted model. This effect would be more apparent in raw data modelling than in modelling summarised data as the trends observed are likely to be smoother and easier to identify as random variation cancels out.

5.4.3 MARS® -created vs hand-transformed variables and predictor interactions: similarities and differences

Comparison of the MARS®-created predictor variables with those which were manually created showed a great degree of similarity. For example, if we compare the predictors based on the variable "development month", MARS® placed knots mostly at the same points that were found important by the hand-fitted model. The differences included the fact that the hand-fitted model included the variable "minimum (development month, 10)" which is equal to 10 if development month is greater than 10 and is equal to development month otherwise, while MARS® selected 9 rather than 10 as the "knot point". MARS® also selected interactions of predictors that were not picked up by the hand-fitted model such as the interaction of development month and experience month.

5.4.4 Interpretability of the models

Interpretability of the models was similar. The hand-fitted model was easier to interpret because it included less predictors and less predictor interactions than the MARS®- enhanced model.

5.5 Findings and results

The fit and precision of the models was similar with the MARS®-enhanced GLM showing a slightly better fit than the hand-fitted GLM. The MARS®-enhanced GLM included predictor interactions not picked up by the hand-fitted model. This effect would be more pronounced in raw data modelling than in modelling summarised data, as the trends observed are likely to be smoother and easier to identify as random variation cancels out. The hand-fitted model was easier to interpret because it included less predictors and less predictor interactions than the other model. Building of the MARS®-enhanced GLM was considerably faster and more efficient.
These findings suggest that MARS® is a useful tool to enhance and expedite GLM modelling.

5.6 Future directions

For large data sets, our team has found that combining decision trees (CART®) with MARS® and GLM proves quite effective as described in Kolyshkina & Brookes, 2002.

Also as an additional check of fit of a GLM model, a parallel model built in MARS® can be used. The model will be built as a part of the stage described previously and will not require additional time. The model equation can be copied and pasted from MARS® to SAS or another package directly. Comparison of the predicted values of this model to the GLM model graphically and numerically can suggest some insights and inform the choice of the best of the models.

5.7 Conclusion

The results described above demonstrate that the use of a data mining technique, MARS, to enhance GLM building makes the model-building process considerably faster and more efficient. This approach allows to achieve higher computational speed by expediting the process of the selection of predictors and their interactions and variable transformation. The precision of this model is higher than for the hand-fitted model as shown by traditional GLM assessment methods as well as by using additional goodness-of-fit analyses such as gains chart. The effects described would be even more pronounced in raw data modelling than in for modelling summarised data as described in the case study, especially for large data sets with many potential predictors. The interpretability of the MARS®-enhanced GLM is similar level to that of the hand-fitted model.

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Estimating Claim Settlement Values Using GLM

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Estimating Claim Settlement Values Using GLM

by

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Abstract: The goal of this paper is to demonstrate how generalized linear modeling (GLM) can be applied in non-traditional ways in property and casualty insurance. Specifically, we will use a property and casualty closed claims database to aid in estimating ultimate claim settlement amounts, evaluating claim trends, and assisting in improving claims handling procedures. This specific example will be used to demonstrate the potential of the application of GLM to different areas of an insurance company.

A GLM will be developed with data from the Insurance Research Council (IRC) closed claims study. The model will be populated with characteristics of closed automobile claims along with final settlement amounts. Using this data, the paper will examine how GLM can be used to identify:

1) *Trends in claims severities over time,*
2) *Differences in severities that exist between current ratemaking characteristics (e.g. state, territory), characteristics of the claims and the injured parties, and other factors (e.g. time from reporting to settlement, attorney involvement, use of arbitration), and*
3) *Interactions between these factors.*

Diagnostics will also be discussed which can be used to test the validity and robustness of the GLM models that are developed, and several applications of the results of this type of analysis will be presented.

Over the last several years, Generalized Linear Modeling (GLM) has seen increased usage among actuaries primarily in traditional ratemaking applications. The benefits of GLM are that it allows for a flexible model structure to be fit to insurance ratemaking data, and it also allows for a multivariate model to be generated that simultaneously incorporates a set of independent variables to determine their impact on a dependent variable. This is an improvement over traditional one-way types of analysis (both loss ratio and pure premium) because it adjusts for the impact of distributional biases that are present in all insurance data sets. The result is a set of indications for whatever you are modeling (class plan relativities, tiering relativities, etc.) that reflect the true impact of each variable being analyzed.

GLM has had immediate appeal in the traditional areas of actuarial practice. Most significantly, insurers have used GLM to refine class plan relativities, establish tiering and underwriting plans, and incorporate commercially available insurance scores into rating and underwriting plans, just to name a few applications. These applications have been addressed quickly as insurers move to this type of analysis for a number of reasons: these areas fall within the actuary's normal area of responsibility, the data for these types of analyses is usually readily available, and this type of analysis can provide the most immediate benefit for an insurer.

However, understanding the general statistical nature of GLM, one realizes that a GLM analysis can be applied to other areas within insurance companies, areas that have not necessarily been within the actuaries' traditional realm of responsibility. Specifically, we have used GLM's for a number of non-traditional applications, including developing custom insurance scores, generating vehicle classification systems, evaluating claims and agency personnel and external service providers, and estimating claim settlement value amounts. These types of analyses can provide benefit to many areas of the company, and can display the actuary's skills to a wider audience.

We will demonstrate the concept of applying GLM to non-traditional areas in this paper using the 1994 Insurance Research Council (IRC) Closed Claim Study database. In this example, we use the characteristics of the closed claims as provided in the IRC database to estimate the ultimate settlement value of a claim; however, we will describe this process in general terms such that it might be applied to a variety of different areas. The goal of this paper is not to provide you with a complete analysis of the IRC database, but to use this database as an example of how this general statistical procedure can be applied to other areas.

The Basics of GLM

GLM is a statistical process by which a model is developed in which a specific dependent, or response variable, is predicted by a number of independent, or explanatory variables. For example, as applied to the insurance ratemaking process, the process of setting class premiums for groups of risks can be thought of graphically as shown in Figure 1.

The goal of the classification ratemaking process is to set premiums by class of risk that reflect the risk of each group. This requires estimating the relative loss potential of each insured characteristic in the classification plan to determine how the factor contributes to the overall risk premium. An insured is then charged a premium based on his or her characteristics, and how these characteristics relate to the risk of loss. The traditional approach to analyzing the variables in the class plan was to analyze each of the variables separately, using a one-way loss ratio or pure premium approach. The inherent assumption in the one-way analysis is that, for **each level** of the factor being analyzed, the distribution of all the other factors in the class plan is constant. This means, for example, if one were analyzing auto symbol, model year, and age using a series of one-way analyses, one would be assuming that the same proportion of teenagers drive 10-year old Ford Escorts and brand new Cadillac Escalades. While this is simply one example, there are a number of other violations of this assumption that can be thought of in an auto or homeowners insurance class plan.

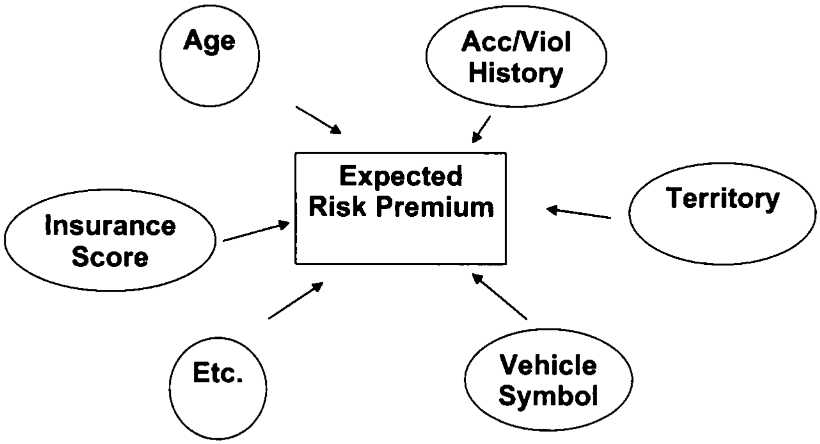

Figure 1: Description of Classification Ratemaking Process

Figure 2 gives an example of how this type of analysis can lead to erroneous results. The first table in Figure 2 gives the results of two separate one-way homeowner's insurance analyses, one for territory and one for protection class. In this particular example, when analyzing the two territories, one assumes that territory A has the same ratio of protection class 1 risks as territory B. The result of the loss ratio analysis shows that the rates for territory A should be increased relative to the territory B rates. Similarly for protection class, the analysis shows that the change in protection class 2 rates should be higher relative to the change in protection class 1 rates. However, when these results are viewed in a two-way table, the true picture becomes clear. The territory loss ratios are identical for both protection classes. The true problem is in the protection class relativities. If one had simply looked at the one-way analysis, the erroneous decision would have been to increase both the territory A rates and the protection class 2 rates, resulting in an over-correction. The reason the one-way loss ratios appear this way is because of the difference in protection class distribution over the two territories. Again, while this is a simple example, one can easily think of the number of different potential scenarios where this can occur in a rating plan.

One Way Analysis

Territory	# of Risks	Loss Ratio
A	10	72%
B	10	48%
Total	20	60%

Protection Class	# of Risks	Loss Ratio
1	10	40%
2	10	80%
Total	20	60%

One Way Analysis

	Protection Class				
	1	1	2	2	
Territory	# Of Risks	Loss Ratio	# Of Risks	Loss Ratio	TOTAL
A	2	40%	8	80%	72%
B	8	40%	2	80%	48%
TOTAL	10	40%	10	80%	60%

Figure 2: Example of one-way loss ratio analysis

GLM corrects for these distributional biases, and also provides a flexible model structure such that it better fits insurance data. One can best think of GLM in terms of one of its simplest forms, classical linear regression. The formula for a simple one-factor linear regression is:

$$y = a + bx + \text{error}$$

This describes the fitting of a line through a series of points, attempting to model a response variable (y) using an explanatory variable (x). The b represents the relationship of the independent variable x to y. There is also an error term which accounts for the fact that the model will not predict the observations perfectly. Under linear regression, the error is assumed to be normally distributed with a mean of zero and a constant variance. A graphical description of this simple regression model can be seen in Figure 3. In this example, the bodily injury severity is being modeled as a function of the time period.

To extend this to GLM, the more general formula for multiple regression is:

$$y = X\beta + \text{error}$$

In this notation, the $X\beta$ represents a matrix, where X represents a series of independent variables and β represents the relationship of these independent variables to the dependent variable. The error term is more general in that it is not restricted to the assumption of normally distributed error terms (as in simple and multiple linear regression). More general error structures, such as Gamma, Poisson, and Negative Binomial can be used which are more representative of insurance data.

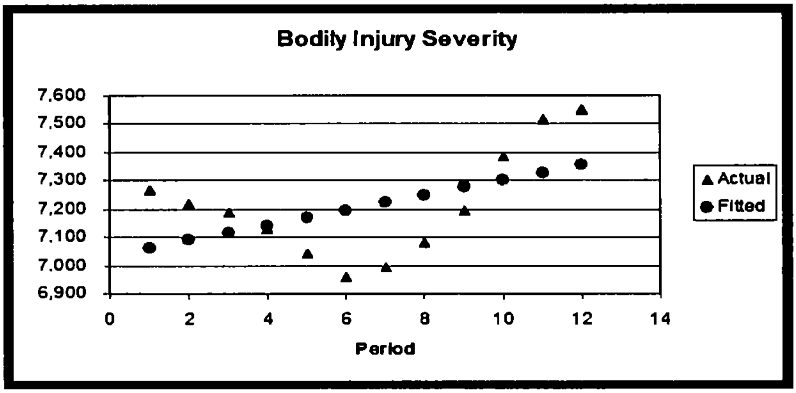

Figure 3: Simple regression example

Non-Traditional Applications

Given the general structure of GLM described above, one can begin to expand the use of GLM beyond the traditional actuarial realm. The general structure of GLM can be described as shown below in Figure 4:

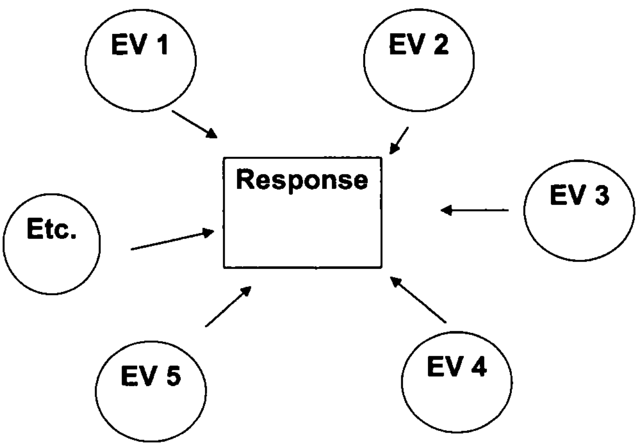

Figure 4: General structure of a GLM model

Because GLM is a general statistical process, it is not limited to estimating class plan relativities. The general structure of the model can be used to describe many different responses by a series of explanatory variables. Depending on what problem GLM is applied to, the explanatory variables and the model error structure will change, but the process of generating and applying the model will remain the same.

CLAIM SETTLEMENT VALUE ESTIMATION

One potential area for the application of GLM in an insurance company is in the estimation of ultimate claim settlement values. The ultimate value of a settled claim can be described as the response variable, and the characteristics of the claim represent the explanatory variables. When a claim is reported to an insurer, the insurer is presented with the facts of the claim. Based on the facts of the claim, an estimate is made of what the final value of that claim will be. This value may be determined based on a claim value estimation software package, guidelines established by the company, the claim persons' expert opinion, or a combination of the three. As the case matures, as payments are made on the case, and as more information regarding the case becomes available, future refinements of that estimate can be made. It is these estimates that are made before the final disposition of a claim that are reflected in an insurers financial results from year to year.

What this GLM example will do is develop a model to estimate the final amount of the claim settlement, which can then be used as part of the overall information that the claims handler uses to determine the expected final value of a claim. The goal of this analysis is not to replace the claims person, no more than the goal of the analysis of traditional class plan relativities by using GLM is to replace the actuary. The goal of this process would be to provide the claims person with additional information on which to base decisions.

This type of model could be used to help estimate the ultimate settlement value of claims based on the information known. It could also be used to assist claims departments in determining the effectiveness of certain claims handling techniques. It can also provide information on areas of focus such that claim handlers might more efficiently handle claims.

Data

To perform this type of analysis, an insurer would need a database of final closed claim settlement amounts, as well as the characteristics of the claims that have been closed. The characteristics available will likely vary between insurers, but examples of the information that could be used are:

- Insured rating and underwriting characteristics
- Type of injuries involved

- Age of injured parties
- Hospitalization involved
- Location of accident
- Types of treatments
- Treatment providers
- Claim reporting lag
- Claim settlement lag

The list of characteristics to be analyzed could continue, and the goal should be to include all the information that is available that might be useful to the analysis. This could be one potential difficulty for an insurer employing this type of analysis technique. For some insurers, this type of closed claim database might simply not exist, or the information might exist in paper form in the claim files.

For this paper, we have analyzed the IRC 1994 Bodily Injury closed claim database. This database was compiled by the IRC as a sample of claims closed during a specific period during 1992 from a number of insurance companies. The database consists of the ultimate settlement value of these claims, a breakdown of these settlement amounts by type of payment (medical, wage loss, etc.), and a number of characteristics of the claim. The variables analyzed from this database reflect many of the items listed above. A complete list of the factors could be obtained from the IRC.

While not a specific issue with the IRC database, an insurer or claims organization that undertakes this type of analysis will need to be aware of claims that are closed without payment. While these claims do not generate any loss dollars, there are at least two other issues that these claims raise. First, they will generate loss adjustment expense dollars because a claim file will be opened on these claims and a claims person will be assigned to handle the claim. Also, because these claims can generate a series of points with no settlement value or a very small settlement value, this can create some difficulty with the determination of a model error structure. One approach to handling this issue would be to use an analysis similar to a claim frequency analysis, but instead analyze the likelihood of a claim closing without payment. This analysis could then be combined with a settlement value analysis to determine the ultimate expected settlement value.

Additionally, a priori there are some factors that we could analyze that would be significant in our analysis of expected claim value but were not present explicitly in the dataset. For example, in the IRC dataset, we knew the date of the accident and the date of the insurance company's initial contact with the claimant, which allowed us to calculate the contact lag. The a priori expectation was that the longer the period between the accident and the initial contact, the larger the ultimate value of the claim. Another example is a difference between the claimant state and the accident state. We assumed a priori that if a claimant has an accident in a state different than their place of residence, it could potentially increase the ultimate settlement value. In an insurer database, there will be variables like these which the modeler will want to derive from information present in the database.

In addition to data from a closed claim database and data from the rating database, there may be information in other parts of the company or external to the company which might be useful to the GLM process. Potential internal information might include marketing information or underwriting information. External available data might include population and vehicle density, medical inflation rates, wage inflation rates, vehicle repair rates, etc. The ultimate goal of the data process is to be confident that you have compiled as complete and correct a dataset as possible with which to generate the model.

Model Considerations

The overall goal of the modeling process is to generate a model that is complex enough to provide a satisfactory degree of predictive accuracy, yet simple enough that it can be explained and understood by users. This delicate balance can be difficult to maintain, but there are some things that can be done to attempt to make this process easier.

In generating a GLM based on the IRC database, we analyzed 150 potential explanatory factors. Needless to say, when analyzing a dataset of this size, we are fully anticipating that the number of explanatory variables included in the final model will be significantly less that 150. Therefore, we need a process by which to determine which variables provide enough predictive value to the modeling process to remain in the final model. There are a number of different approaches that can be undertaken. Three of these approaches are outlined below:

1. <u>Single Inclusion Process</u>: Beginning with the first potential explanatory variable, we add each variable one by one to the model in order of presence in the dataset, keeping the variables that add predictive power to the model and not using the variables that do not provide predictive power. To determine whether or not predictive power was added to the model, we utilize the chi-square test which is based on the deviance of the model, or the difference between the expected claim settlement value as generated by the model and the actual claim settlement value present in the dataset. The disadvantage of this approach is that the order of addition of explanatory variables to the dataset is generally random, and this could result in a less than optimal set of variables being included in the final model.

2. <u>Stepwise Type I Regression</u>: This process begins with a model including no factors, and then generates a one-factor model for all 150 potential explanatory variables. The factor that produces the lowest deviance and proves to be significant by evaluation of the chi-square test results is added to the model (F1). Next, all 149 potential two factors models are generated which include F1 plus all the other explanatory factors, one at a time. The next factor added to the model is the one that produces the lowest deviance and is also significant based on the chi-square test. The process continues until no other additional factors added to the model produce significant results.

While this process is more time consuming than the first process, it helps assure that the factors that provide the most predictive power will, with high likelihood, make it into the final model. Once you have generated a final model, this process will also require a review of the factors in the final model again for significance. There is the potential that a factor that entered the model early in the modeling process might be proved to be insignificant later by the additional variables. To the extent that the model can be simplified by the removal of these redundant factors, this should be done.

3. Stepwise Type III Regression: This is a variation of the Type I regression that starts with a model which includes all 150 factors, then generates a series of models removing the factors one at a time to determine which factor is the least significant. The factor that is not significant as measured by the chi-square test and has the smallest impact on the deviance will be removed from the model. The process continues until there are no more insignificant factors in the final model.

   This approach is the most time consuming of the three, since it requires models with more explanatory factors to be generated.

If we are working with a dataset with a manageable number of explanatory factors (less than 50), we will generally begin with a model that includes all parameters, and investigate each of the independent variables to determine which factors are significant. For analyses that have a larger number of factors, we usually take an automated approach to determining which factors to further investigate. For the purpose of analyzing the IRC database with a larger number of factors, we employed the Type I regression method.

As a general practice for modeling projects, one should consider developing the model based on a portion of the dataset and testing the model that has been developed on the remaining portion of the dataset. There is the potential in generating models that you can "over-fit" the dataset. The process of splitting your dataset, sometimes referred to as "training and testing," can help avoid interpreting a trend when one really is not there. The optimal split will depend on the size of your dataset, but as a general rule of thumb, using 70% of the data to develop the model and 30% to test it works well. In this particular example, we did not divide the dataset due to the size. There were just under 34,000 records in the dataset, and removal of 30% of these records would have significantly impacted our ability to generate the GLM.

In order to generate the model, one must determine an initial model error structure. For claim settlement values, good a priori distribution assumptions are a gamma or a negative binomial distribution. For purposes of this paper, we have chosen the gamma distribution.

Because we are analyzing liability data, the potential always exists for large claims. Large claims present some difficulty in performing a relativity analysis (such as we are performing here or would be performed in a class plan analysis) because one or two large claims can have a significant impact on an indicated relativity or the indicated impact of a claim characteristic on the final settlement value of a claim. However, large claims cannot be ignored because they are covered as part of the insurance contract. Traditionally, insurers have simply generated relativities based on a limited claim severity analysis, and loaded back a fixed amount to each claim for purposes of covering the large claim amount. However, this ignores the fact that the likelihood of large claims is not constant over all claims that are presented. All liability claims have some potential to become large claims, however, there are certain claims that have a higher than average likelihood of becoming large claims. In this analysis of the IRC data, we have analyzed the likelihood of large claims as a basis for generating a large claim load which varies based on the characteristics of the claim.

To generate the large claim analysis, for claims that pierced a $25,000 threshold, we generated a second model, using a logistic error structure, that attempted to determine the likelihood of a claim to pierce the $25,000 threshold based on its particular characteristics. Each total estimated claim amount would then be a combination of the limited claim settlement value estimate and the adjusted large claim load. A description of the models generated is shown in Figure 5.

Expected Limited Loss Portion + **Expected Large Loss Portion** = **Expected Claim Settlement Value**

Figure 5: Claim settlement value model structure

LIMITED CLAIM SETTLEMENT VALUE MODEL

Of the 150 variables analyzed, we selected 35 which were determined to be significant for the limited claim settlement value model. Many of the variables and the effects made intuitive sense, however there were some that may have appeared at first glance to be counterintuitive. We provide a few of the results of the model below, as well as some of the simplifications to the factors in the model.

Presence of an Attorney

One of the factors analyzed in the Bodily Injury dataset was whether or not the claimant was represented by an attorney. Insurers have long alleged that the use of an attorney for an auto insurance claim causes the settlement value of that claim to increase. Attorneys have alleged that the settlement value of claims involving attorneys is higher because they are generally involved in the more serious claims. The results for the involvement of an attorney in the claim settlement process are shown in Figures 6 and 7. Figure 6 shows that, all else being equal, the average final claim settlement value for the base claim characteristics for cases involving attorneys (Code 1) was about $9,500, more than double the cost of claims not involving attorneys (Code 2). Figure 7 simply shows the relative cost of these types of claims due to the impact of attorneys, even after removing the impact of the type of injury. This result is helpful in attempting to determine the final value of a claim and it would also be valuable during the claim handling process in determining which claims should be monitored more closely. The bars at the bottom of graph represent the distribution of claims in each category, and relate to the y-axis on the right side of the graph.

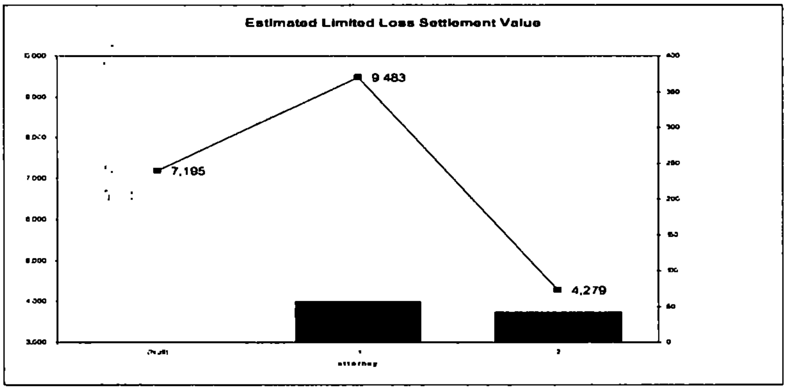

Figure 6: Attorney Involvement

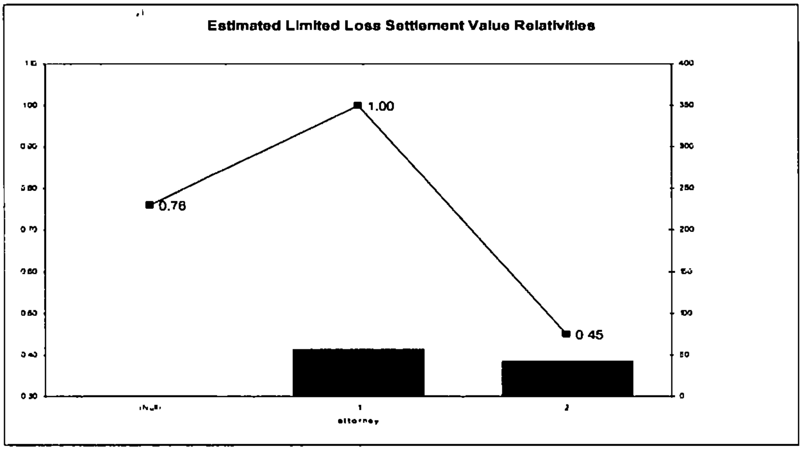

Figure 7: Attorney involvement (Relative to category 1 - yes)

Depending on the type of analysis that you are undertaking, you may have to deal with the issue of unknown explanatory variables. (In the example above the Null category represents an unknown category). Unknown data can come from a couple of different sources. One reason might be that the data collected was just not complete, and therefore there are a number of risks for which you may not have all the desired information. There may also be a systematic reason for unknown variables. For example, in many class plans in the United States, marital status and gender are not used to rate adult risks, so this data is not collected on non-youthful risks. Regardless of the reason for the unknown data, the modeler will need to decide how to handle the unknown values. The best solution would be to try to obtain the missing data fields, however this is not usually feasible. Another option would be to model the unknown variable as a distinct level of a factor, which would make sense if a variable being unknown is a valid occurrence, such as the class plan example given earlier. A third approach would be to group the unknown level with an "average" level, or with the most likely occurrence of the variable. For the purposes of this analysis, since it is likely that there would be information about future claims that is unknown, we chose to model the unknown level as a distinct level.

The graphs shown above represent two different ways of viewing the results of this claim analysis. We will view the results using the relative claim cost method (Figure 7), realizing that we will use the actual claim settlement values when generating the final claim settlement amounts.

Most Significant Injury

Another factor in the analysis dataset is the most significant injury to the claimant. A graph showing the results from this factor is shown in Figure 8. As can be seen from the graph, lower claim amounts were associated with less serious claims, such as minor lacerations (code 3) and various sprains and strains (codes 6-8). The larger claim amounts were associated with more serious claims, such as serious lacerations (code 4), scarring and permanent disfigurement (code 5), Temporomandibular Joint (TMJ) dysfunction (code 16), and loss of senses (code 17).

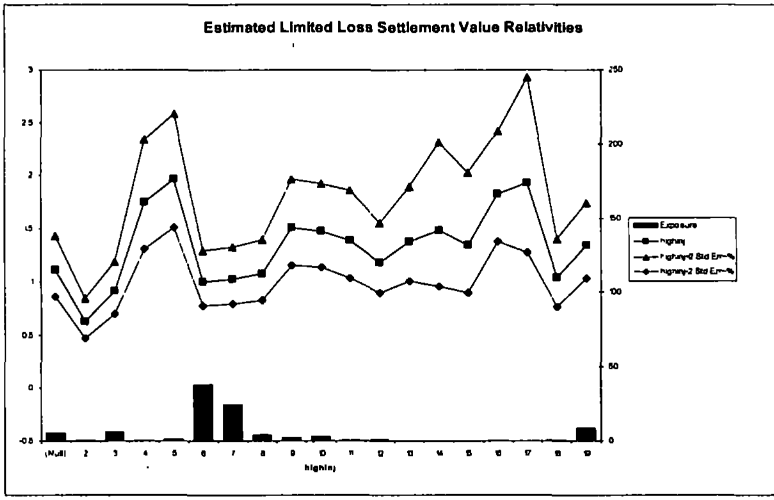

Figure 8: Most significant injury

To assist the modeler in determining the significance of independent variables, the standard error of each parameter estimate is generated. The parameter standard error estimate gives an indication of the reliability of the parameter estimate. For example, the relativity estimate for neck sprains and strains (code 6) is 1.00. Plus and minus two standard errors around this parameter estimate yields 0.77 to 1.29. However, the estimate for the loss of a body part (code 14) is 1.49, two standard errors around this parameter estimate yields a range of 1.13 to 1.93. This is a wider spread, and relecfts the increased uncertainty regarding the serious laceration parameter as compared with the neck sprain/strain parameter. Many times (but not always), increased standard errors for a parameter estimate are caused by a lower number of observations for a particular category. The standard errors will give the modeler information regarding the amount of reliability to place in the estimate.

Year of Accident

The year of the accident occurrence was present in the IRC database, which gives some indication to the length of time the claim had been in the company claim process. Claims were present in this dataset that occurred as far back as 1950. The expectation is that if a claim has been open for a long period of time, it represents a more complex claim, or a claim that may have been contested more fiercely. It is expected that these claims would settle for larger amounts. As can be seen in Figure 9, this trend appears to hold for 1992 back through 1987, but at 1986 the trend appears to break down. This might be a reflection of the trend breaking down, but is more likely a reflection of the data sparseness for years prior to 1988. Due to the lack of data at these points, we decided to combine years 1988 and prior for purposes of this analysis, as shown in Figure 10. Another option would be to potentially extrapolate the trend from 1989 and subsequent onto the 1988 and prior data.

For other variables, levels of the variable that exhibit similar claim settlement values can potentially be combined.

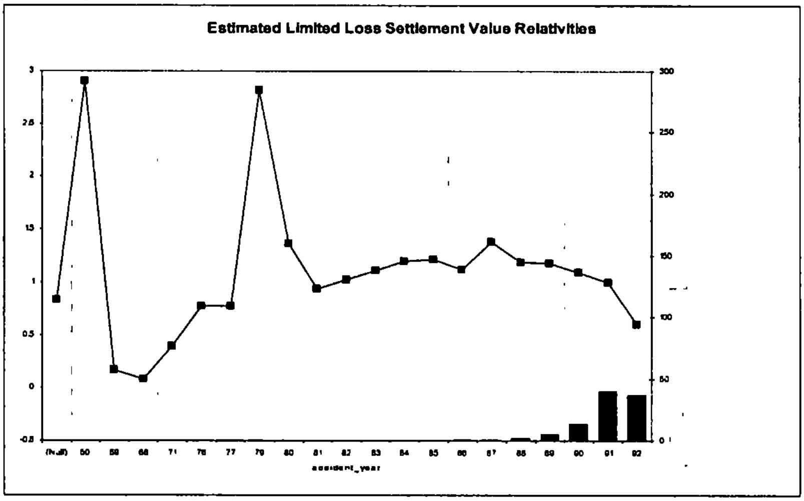

Figure 9: Year of accident

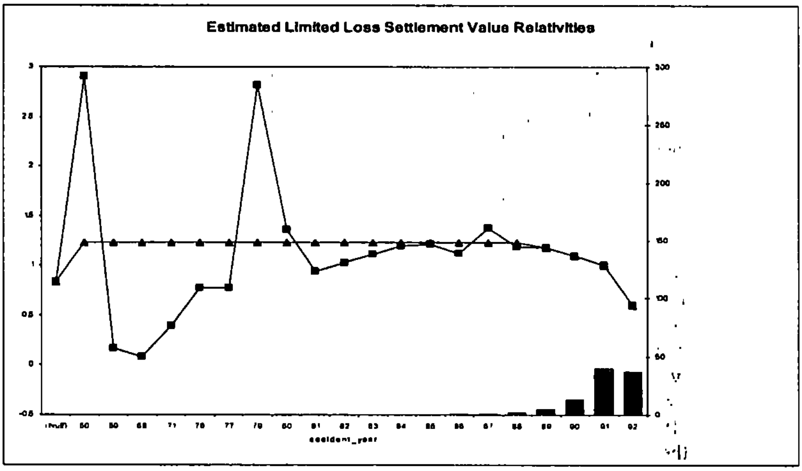

Figure 10: Year of accident grouped

Claimant Age

The age of the claimant was also analyzed as part of the final model. For variables that have a natural scale where successive levels are related, such as age, one can consider fitting a continuous curve representing this factor's impact on the dataset. Figures 11 and 12 represent the initial and final smoothed results of the claimant age factor. In this case, we fit a "mixed" simplification to the claimant age. We fit one curve to ages 0-9, allowed the model to fit separate and distinct factors to ages 10 and 11, and then fit a second curve to ages 12 and over. This demonstrates the flexibility of fitting GLM's. As can be seen, the cost of the ultimate claim tends to increase as the claimant age increases, but then around age 60 begins to decrease again. This may have something to do with the wage earning potential of an injured person. Wages generally tend to increase as a person gets older, and then at older ages the earnings decrease due to retirement.

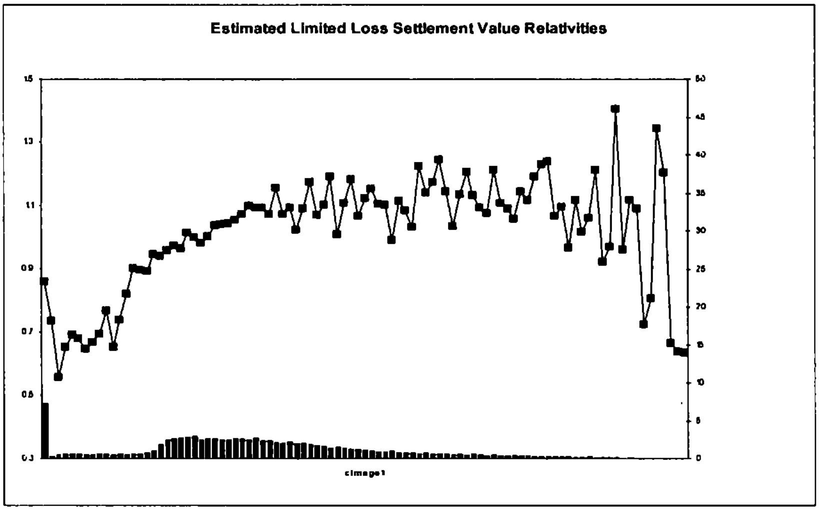

Figure 11: Claimant age

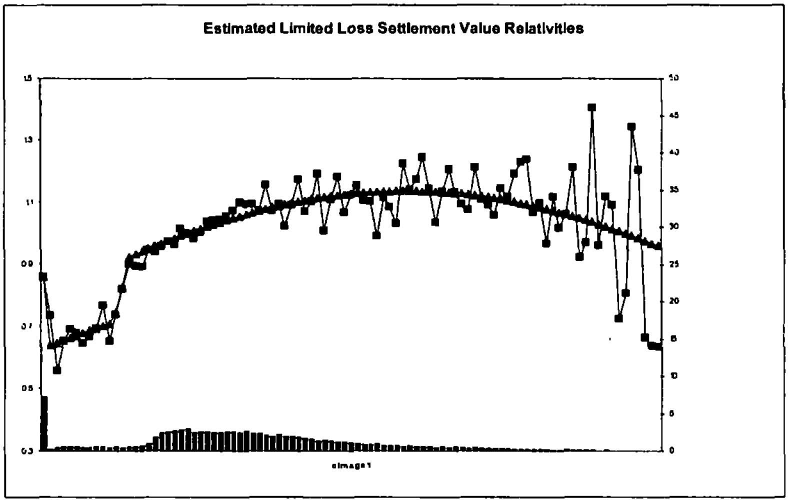

Figure 12: Claimant Age (smoothed)

Injury Type by Attorney Involvement

In addition to the impacts of individual variables on the ultimate settlement value, combinations of factors can have interaction impacts on the final claim settlement amount which can differ from the combined effect of the individual factors. For example, Figures 7 and 8 discussed attorney involvement and injury type, respectively. For a claim that did not involve an attorney, the resulting settlement value was 45% of the value of a claim that did involve an attorney. When considered in combination with injury type, this assumes that all injury types are 45% smaller when an attorney is involved, unless this assumption is specifically relaxed. Figure 13 shows the result of specifically considering this interaction. As can be seen, the presence of an attorney does not have a constant effect when considering different types of injuries. The difference ranges from a 29% increase when dealing with fractures (code 9) to 124% when dealing with other sprains and strains (code 8).

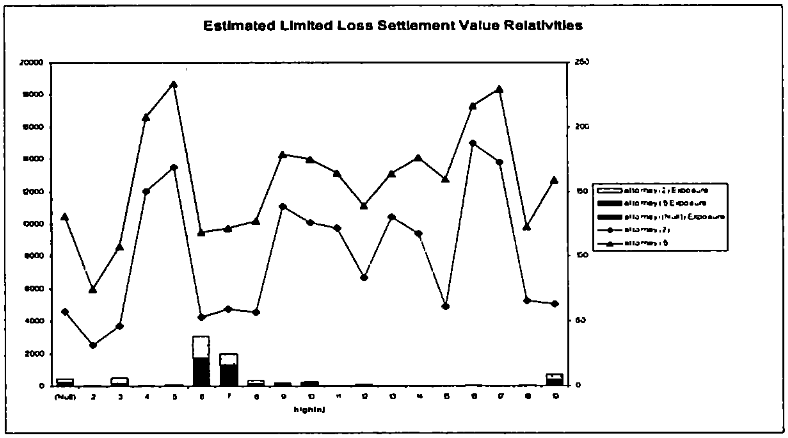

Figure 13: Attorney involvement by injury type

Final Limited Claim Settlement Value Model

An analysis of each of the significant variables was conducted to determine if there were any of the variables that could be simplified, either by grouping of levels of the factor or by fitting of a continuous curve. Also, a series of interactions were tested to determine if they were significant in the final model. After the final limited claim settlement value model is developed, an expected limited claim value is calculated for each record in the dataset. An example of this calculation is shown in Attachment 1. This limited claim settlement value will be combined with the expected excess claim value determined in the next section to reach an overall final expected claim value.

Excess Claim Settlement Value Model

The purpose of developing an excess claim settlement value model is to account for the presence of large claims in the database in a way that recognizes the fact that certain characteristics are more likely to generate large claims than others. We began by generating a model to determine the likelihood of a large claim occurring. This model was developed based on a logistic error structure, with the response variable being whether or not the claim pierced the threshold ($25,000). A logistic model is generally used for the analysis of a yes/no type response variable. We then looked at particular claim characteristics to determine if the presence of certain levels of some characteristics had higher likelihood of large losses than others. The large loss load for each claim was then determined by taking the average excess loss for the base risk and adjusting this excess loss based on the likelihood of a large loss occurring. Below, we show examples of the relative likelihood of large losses for several claim characteristics.

Presence of an Attorney

Again, similar to the limited claim value model, the presence of an attorney significantly increases the likelihood of a large claim. When an attorney is present, the likelihood of a large loss nearly doubles.

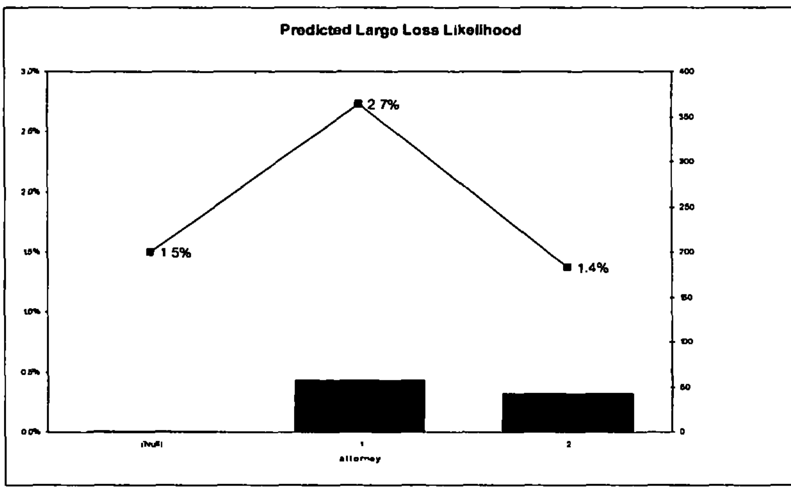

Figure 14: Likelihood of large loss when an attorney is present

Neck Injury

The situation can occur where the results of the large claim frequency analysis might show results that are opposite the results of the limited claim severity. The presence of a neck injury causes a larger limited claim severity. However, the presence of a neck injury is about 15% less likely to produce a large loss (Figure 15).

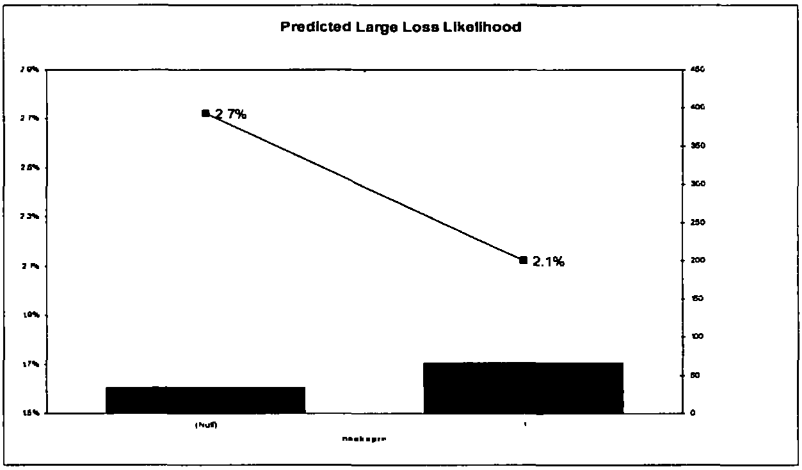

Figure 15: Likelihood of large loss with a neck injury

Accident Year

The accident year was found to be significant in the limited claim settlement analysis, and the older claims had a predicted limited severity of about 25% higher than the base accident year. However, as can be seen in Figure 16, large claims are twice as likely to result from older claims as from less mature claims. This is to be expected, since it is more likely that the more complicated, expensive claims will take longer to settle.

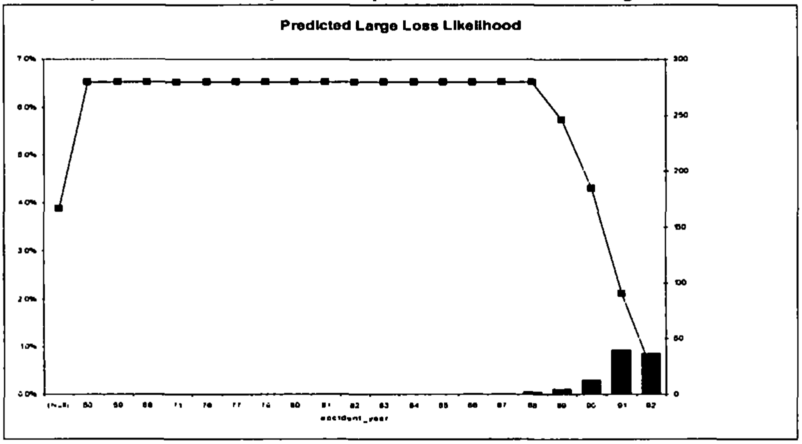

Figure 16: Accident year large claim likelihood

Final Large Claim Load

The final large claim load is calculated by taking the base predicted excess claim amount and adjusting it for the calculated likelihood of a claim turning into an excess claim as determined by the excess claim model. This calculation is different than the limited claim severity example discussed earlier due to the use of the logistic regression model. Taking the product of individual relativities cannot be directly applied here because of the upper limit on the likelihood of 1.0. The formula for the calculation of the likelihood of large loss has natural limits of 0 and 1. For each factor in the excess model, a parameter estimate is developed. The sum of the parameters for risk characteristics of particular claim is then added to the logistic parameter estimate for the base risk, and then the exponent of the negative of this sum is calculated. The final probability is then the inverse of one plus the exponent of the summed parameter. See attachment 2 for the formula and an example of the calculation of the final large claim load. The final expected claim settlement value is simply the sum of the modeled limited claim settlement value and the modeled excess claim settlement value, also shown in Attachment 2.

Evaluating the Overall Model Fit

There are a number of statistical diagnostics that can be applied in order to evaluate the overall fit of the model to the data. These measures include the difference between the observed and fitted values (errors), the standard errors of parameter differences, the chi-square test and the f-test. The last three tests mentioned here are best suited for evaluation of particular factors which may or may not be predictive in the modeling process. There is also the evaluation of the overall model structure which assists in determining if the overall model has been fit with the proper error distribution. In this particular modeling exercise, we modeled the limited claim settlement value data with a Gamma error term. To review the appropriateness of the Gamma model, we looked at residual plots (difference between actual and predicted claim settlement values) to determine whether or not the Gamma assumption makes sense.

Figure 18 shows the resulting residual plot for the limited claim severity model. The residuals have been transformed to adjust for any scale parameter differences in the model so that a better determination can be made regarding the fit of the model. Generally, you would look for a residual plot which is symmetric about 0 on the y-axis and has no obvious asymmetrical tendencies about the x-axis. If you look at the left side of the residual plot in Figure 17, the plot looks reasonable, with a fairly even distribution around 0, and with no obvious distortions, such as a fanning in or fanning out of the plot. However, if you look at the right side of the graph, you will see what appears to be a severe distortion in the residuals. This cut-off along the right side of the graph is due to the fact that we are modeling a capped severity. All of the observed severities have been cut off at $25,000, which causes the residual graph to appear truncated.

Because we are accounting for the excess claim load in a separate model, this residual graph would be acceptable. If there had been other distortions, such as a funnel shaped graph going either way, then these could have been potentially addressed by adjusting the distribution of the error structure. Another potential problem one might see with a residual plot is what appears to be two distinct sets of residuals, aggregating at different places in the residual plot. In this case, there may be a problem with the homogeneity of the underlying data, and segregating the data into more homogenous groups might be the answer. For example, if we attempted to model bodily injury settlement values along with property damage settlement values, we might see a residual plot with two distinct groups of residuals.

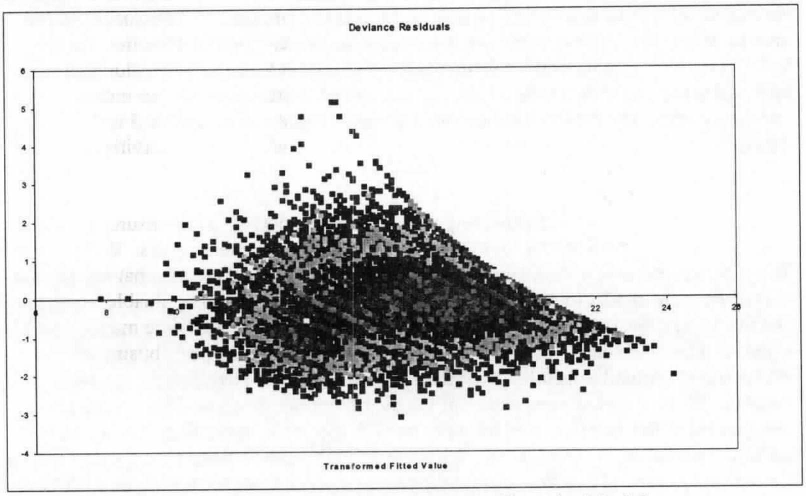

Figure 17: Transformed residual for Limited Claim Severity Model

Applications

There are a number of applications of this type of model to the insurance industry. One potential application would be its use as a tool for claims adjusters in attempting to determine reserve estimates for claims that are made to an insurer or self-insured entity. Once the claim comes in, there are certain characteristics that can be determined. These characteristics could then be used in the claim model to determine an estimated settlement value for the claim. This estimate would not be a replacement for the judgment of the claim adjuster, however, the results of this model would be available as another estimate to assist the claim adjuster in making a final estimate.

Also, there are certain characteristics of the claim that generally lead to larger claim settlement values. As a result of the claim settlement value model, claim persons could be alerted to claims which could potentially become high value claims, and then spend relatively more of their time working on the settlement of these claims. The claims model may simply confirm current common knowledge among claims personnel, such as the presence of an attorney or a fatality would cause the likelihood of a large claim to increase dramatically. It can also provide additional insight into drivers of larger than average claim settlements, especially when considering interactions.

GLM could also allow users to determine trends in claim settlement value estimates. Not only will insurers be able to determine the trend in overall claim settlement values, but it can also be determined if certain factors are increasing in importance over time in estimating the overall claim settlement value. For example, we noticed earlier that the presence of an attorney caused the limited claim settlement value to nearly double. If that relationship between claims with and without attorneys were to be begin to increase from a 2 to 1 ratio with this analysis to 2.25 to 1 with next year's analysis and 2.5 to 1 with the analysis after that, then the company may need to determine why the relativities are trending that way.

Another benefit of this type of claims settlement value model is that the insurer can make use of its own data to determine estimated ultimate claim settlement values. While there may be other models available which have been developed based on data that represents more of the industry, the use of company-specific data can be another valuable estimate that reflects the type of business that the insurance company writes. There may be differences in the claim settlement culture of the company or the type of business the insurer writes which would make a company-specific model valuable.

Conclusion

There are many benefits that the actuary brings to the insurance company. Many of these benefits are thought to be primarily in the area of ratemaking and reserving. However, the ability of the actuary to analyze past statistics and use them to help understand future occurrences has application beyond traditional areas of ratemaking and reserving. Better understanding how to estimate the ultimate claim settlement amount can assist the claims person in better estimating claim reserves. The key here is that the actuary can use his or her unique skills and provide information to claims and other areas. One important tool in providing this assistance is GLM. As actuaries continue to appreciate the potential wide applications of this analysis procedure, innovative solutions can provide value to many areas of the insurance company. Also, this analysis procedure could be applied to many different types of datasets to model different response variables.

Combining Credibility and GLM for Rating of Multi-Level Factors

Esbjörn Ohlsson and Björn Johansson

Combining Credibility and GLM for Rating of Multi-level Factors

Esbjörn Ohlsson [*]   Björn Johansson[†]

March 5, 2004

Abstract

A *multi-level factor* (MLF) is a rating factor with a large number of levels, each of which we want to rate separetely even though many of them do not have a sufficient amount of data. Examples include Car model, Geographic Zone and Company (in experience rating).

Rating of MLFs is a standard situation for employing credibility theory. Traditional credibility theory models MLFs as random effects, but does not treat the situation where there are also ordinary rating factors (like Sex and Age class) alongside with the MLF. The aim of this paper is to show how such a situation can be handled by combining credibility theory and GLM. The method can be seen as an extension of the classical Buhlmann-Straub approach.

The method is presented via an example of experience rating in bus insurance, while the theory and more general results are given in Ohlsson and Johansson (2003).

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1 Multi-level factors and credibility

In non-life insurance rating, it is customary to use Generalized Linear Models (GLMs) to estimate price relativities for a number of rating factors under a multiplicative model. The rating factors are either categorical with a few levels (e.g. Sex) or a grouping of a continous variable (e.g Age group, Mileage class). In case enough data is not available for some group, one can merge groups to get more reliable estimates, *nota bene* at the price of a less detailed tariff However, for rating factors with a large number of levels without an inherent ordering there is no simple way to form groups with sufficient data We introduce the term *multi-level factor* (MLF) for such rating factors and next give some examples.

Example 1.1 (Car model) In private motor car insurance it is well known that the model of the car is an important rating factor, both for third-party liability, hull and theft. In Sweden there is a common basic grouping where car models that are technically very close to each other are put in the same class. Nevertheless, we are left with several thousands of car model classes, some of which represent popular cars with sufficent data available, whereas most classes have moderate or sparse data. Even after taking into account auxiliary variables like weight, effect and brand the car model remains an important rating factor. There is no sensible way to group the models *a priori* and there is not enough data to do a relevant posterior grouping. Hence, car model is a typical MLF. □

Example 1.2 (Experience rating) Using the *customer* as a rating factor is another important example of an MLF. In the commercial lines it is important to base the rating to some extent on the individual claims experience, even though there is often not suffient data for separate rating of each company. This is the classical situation for which North-American actuaries like Whitney and Mowbay introduced credibility estimators in the early 1900's. In the private lines, Lemaire (1995) and others use (European type) credibility estimators for the construction of optimal bonus/malus systems, with the customer as MLF. □

Example 1.3 (Geographic area) In order to get risk homogenous geographic areas one often has to use a very fine subdivision of the country, based on for instance ZIP codes. Neighbouring areas can have quite different risk profiles and hence a prior grouping can be very hard to achieve and we are again left with an MLF. □

As already indicated in example 1.2, a way to solve the rating problem for MLFs is to use *credibility theory*. However, classical Bühlmann-Straub credibility theory does not treat the important situation where we have ordinary rating factors (non-MLFs) besides the MLF This is the problem considered in the present paper, and the proposed solution is to use a combination of GLM and credibility as will now be outlined.

2 Extended Credibility Predictors

For simplicity, we will describe the method in terms of a simple example from bus insurance. For a general treatment we refer to Ohlsson and Johansson (2003).

We consider data for 1993-1998 from the former Swedish insurance company Wasa on 624 bus companies. Here we have just two ordinary rating factors *Age* (with five classes of bus age) and *Zone* (a standard subdivision of Swedish parishes into seven zones). Note that geographic area is not used as an MLF here, as would be the case if we operated on the parishes themselves. Our MLF is the company itself and hence we are looking for a proper experience rating in the presence of the ordinary rating factors Age and Zone

As is standard in insurance applications of GLMs, we perform a separate analysis of *claim frequency* and *average claim cost*. For briefness, we will only consider claim frequency here. Hence let Y_{ijk} be the observed claim frequency for the buses of company k in Age class i operating in Zone j, i.e. Y_{ijk} is the number of claims divided by the exposure weight w_{ijk} measured in policy years. The multiplicative tariff contains a base rate μ, plus factors α_i for Age and β_j for Zone. In credibility theory, the MLF is modelled by a random risk parameter, which suggests that we should introduce a stochastic factor (random effect) U_k for Company in our model in order to get credibility-like results. (This idea was introduced in the actuarial literature by Nelder and Verall, 1997.) The multiplicative model for the expected claim frequency hence becomes

$$E[Y_{ijk}|U_k = u_k] = \mu\alpha_i\beta_j u_k \tag{1}$$

where $\alpha_{i_0} = 1$ and $\beta_{j_0} = 1$ for some base classes i_0 and j_0 and $E(U_k) = 1$, since U_k should be a pure random effect — the systematic part being taken care of by μ. Furthermore, the U_k's are supposed to be independent and identically distributed with common variance $a \doteq \mathrm{Var}[U_k]$. Note that $\mu_{ij} \doteq E[Y_{ijk}] = \mu\alpha_i\beta_j$.

Conditionally on U_k, Y_{ijk} is assumed to follow a (w-weighted) Poisson

GLM with mean given by (1) and hence with variance $\mathrm{Var}[Y_{ijk}|U_k = u_k] = \sigma^2 \mu_{ij} u_k / w_{ijk}$, where σ^2 is the overdispersion parameter (in GLMs usually called ϕ and possibly set to one beforehand).

As in traditional credibility theory, we look for estimators (or rather predictors) $\hat{U}_k$ of U_k which are optimal in the meaning of mean square error, i.e. which minimize $E[(\hat{U}_k - U_k)^2]$. In Section 2.3 of Ohlsson and Johansson (2003) it is shown — as a special case — that the solution to this problem is given, under certain conditions, by the credibility formula

$$\hat{u}_k = z_k \, \overline{u}_k + (1 - z_k) \cdot 1 \tag{2}$$

where

$$\overline{u}_k = \frac{\sum_{i,j} w_{ijk} y_{ijk}}{\sum_{i,j} w_{ijk} \mu_{ij}} \tag{3}$$

and the *credibility factor* z_k is

$$z_k \doteq \frac{\sum_{i,j} w_{ijk} \mu_{ij}}{\sum_{i,j} w_{ijk} \mu_{ij} + \sigma^2 / a} \tag{4}$$

where $\sum_{i,j}$ extends over all tariff cells (i, j) where company k has at least one bus. Note that $E[\overline{U}_k | U_k] = U_k$, and that $\overline{u}_k$ is the ratio between the number of claims by company k and the corresponding expectated value in a tariff with just Age and Zone as rating factors. Hence, $\hat{u}_k$ is a credibility weighted average of our empirical experience of the company, $\overline{u}_k$, and the number 1 — the latter implying rating of the company's buses by their tariff values for Age and Zone only.

Note that we get high credibility if we have large exposure in terms of expected number of claims $w_{ijk} \mu_{ij}$ or if the variance between companies a is large compared to the within company variance σ^2.

Note also that in the case of very high credibility, i.e. $z_k \approx 1$, equation

(2) becomes

$$\sum_{i,j} w_{ijk}\mu_{ij}\hat{u}_k = \sum_{i,j} w_{ijk}y_{ijk} \tag{5}$$

which we recognise as the estimating equations for the ML estimates in a Poisson GLM (equivalent to the method of marginal totals). Hence, the credibility estimator in the case of high credibility is nothing but the ordinary GLM estimator. This is an appealing property since high credibility occurs when there is enough data for the company k.

Remark. If we disregard the ordinary rating factors Age and Zone, so that $\mu_{ij} = \mu$, it is not hard to see that $\mu \cdot \hat{u}_k$ reduces to the ordinary Bühlmann-Straub credibility estimator, see Ohlsson and Johansson (2003), Section 2.3.1. □

The proof of (2)–(4), in a general setting, is given in Ohlsson and Johansson (2003). It is based on an extension of the famous theorem by Jewell (1974) on exact credibility. A fundamental assumption is that some suitable transformation of U_k follows the natural conjugate prior distribution of the GLM distribution (here: of the Poisson distribution). In a forthcoming paper we will show how $\overline{u}_k$ can alternatively be derived without distributional assumptions as the optimal *linear* predictor — in analogy with the original result by Bühlmann and Straub

2.1 Estimation of variance parameters

It remains to estimate the variance parameters σ^2 and a. We use an approach with unbiased estimators based on sums of squares, similar to the one proposed in classical credibility theory (see e.g. Goovaerts and Hoogstad, 1987, p. 48). The derivation is given in Ohlsson and

Johansson (2003) — here we just present the results in our special case. Let

$$\hat{\sigma}_k^2 = \frac{1}{n_k - 1} \sum_{i,j} w_{ijk} \mu_{ij} \left(\frac{Y_{ijk}}{\mu_{ij}} - \overline{u}_k \right)^2$$

where n_k are the number of tariff cells (i, j) where we have $w_{ijk} > 0$.

For each k this gives a separate unbiased estimator of σ^2. As an overall estimator we suggest

$$\hat{\sigma}^2 = \frac{\sum_k (n_k - 1)\hat{\sigma}_k^2}{\sum_k (n_k - 1)} \tag{6}$$

Next we present an unbiased estimator of $a = \mathrm{Var}[U_k]$.

$$\hat{a} = \frac{\sum_k \tilde{w}_k (\overline{u}_k - 1)^2 - K\hat{\sigma}^2}{\sum_k \tilde{w}_k} \tag{7}$$

where $\tilde{w}_k = \sum_{i,j} w_{ijk}\mu_{ij}$ and K is the number of companies Note that this estimator is unbiased only when μ_{ij} is assumed known. In practice we estimate μ_{ij} in a GLM and hence the estimators are not strictly unbiased.

2.2 An algorithm

Nelder and Verall (1997), using a different approach based on hierarchical likelihood, suggested iteration between GLM parameter estimation for ordinary rating factors and prediction of random effects u_k. In our example iteration is also reasonable since there might be confounding between the MLF and the ordinary rating factors (e.g. a good company might operate mainly in one zone and this might lower the factor for that zone). We get the following algorithm for simultaneous rating of ordinary factors and MLFs.

(0) Initially, let $\hat{u}_k = 1$ for all k.

(1) Estimate μ, α_i and β_j in a Poisson GLM, using a log-link and having $\log(\hat{u}_k)$ as *offset*-variable. This yields $\hat{\mu}_{ij} = \hat{\mu}\hat{\alpha}_i\hat{\beta}_j$.

(2) Compute $\hat{\sigma}^2$ and $\hat{a}$, using $\hat{\mu}_{ij}$ from Step 1.

(3) Compute $\hat{u}_k$ for $k = 1, 2, \ldots, K$, using the estimates from Step 1 and 2.

(4) Return to Step 1 with the offset-variable $\log(\hat{u}_k)$ from Step 3.

Repeat Step 1-4 until convergence, which in our experience often takes just a few iterations.

Notice that the computation of ML estimates in ordinary GLMs requires iteration between the estimating equations for the different rating factors. Together with the observations made in connection with equation (5) this means that in case of very high credibility the algorithm will be equivalent to the one for computing the ML estimates when all rating factors are treated as ordinary GLM factors.

3 Numerical results

In Table 1 we show the relativities for the ordinary rating factors Age and Zone, first after running a GLM with these covariates alone, then after 30 iterations of the algorithm with *Company* as MLF. We see that use of the algorithm results in a substantial change in the rating factors for Zone

Next, we list the credibility estimates for a selection of companies in Table 2.

Rating factor	*Level*	*Estimated relativities*	
		GLM only	*Algorithm*
Bus Age	0-2 yrs	2.64	3.05
	3-5 yrs	1.90	1.78
	6-8 yrs	1.77	1.78
	9-11 yrs	1.42	1.37
	12+ yrs	1.00	1.00
Zone	1	1.00	1.00
	2	1.82	1.03
	3	1.43	1.41
	4	1.32	0.94
	5	2.28	1.39
	6	1.44	1.08
	7	0.40	0.95

Table 1: *Estimated values for ordinary rating factors in bus insurance.*

In this rather simple example, the ordinary rating factors explain quite little of the variation in the data and consequently quite high credibility is given even to bus companies with a limited amount of data. The estimated values of the dispersion parameters are $\hat{\sigma}^2 = 1.12$ and $\hat{a} = 1.00$. Note that only the ratio $\hat{\sigma}^2/\hat{a} = 1.12$ enters into the formula for the credibility factor z_k — this value is quite low in our experience. Nevertheless, at the lower end of the table credibility is low and $\hat{u}_k$ is close to one, which means that one has to rely on the ordinary rating factors for these companies.

For an example of the use of the above algorithm in private car insurance, see Section 4 of Ohlsson and Johansson (2003)

k	$w_{\cdot k}$	$\bar{u}_k$	$\hat{u}_k$	z_k
1	219.81	1.81	1.80	0.98
2	269.40	0.90	0.90	0.98
3	181.84	1.36	1.35	0.97
4	102.60	1.16	1.15	0 97
⋮	⋮	⋮	⋮	⋮
201	12.01	0.00	0.45	0.55
202	7.33	0.75	0.86	0.55
203	5.13	0.75	0.86	0.54
204	9.52	2.31	1.70	0.54
⋮	⋮	⋮	⋮	⋮
401	1.89	2.96	1.46	0.23
402	2.43	0.00	0.77	0.23
403	1.90	0.00	0.77	0.23
404	3.22	0.00	0.78	0.22
⋮	⋮	⋮	⋮	⋮
601	0.11	0.00	0.98	0.02
602	0.17	0.00	0.99	0.01
603	0.08	0.00	0.99	0.01
604	0.14	0.00	0.99	0.01
⋮	⋮	⋮	⋮	⋮

Table 2: *Selected bus companies, k, with their number of policy years $w_{\cdot k}$, experience values $\bar{u}_k$, credibility predictors $\hat{u}_k$ and credibility factors z_k. The companies are ordered according to z_k.*

4 Concluding remarks

The method presented in this paper can be seen as either a way to work with random effects in GLMs or as a way to introduce fixed effects into the credibility framework. In any case, combination of GLM and credibility is a very useful and rather simple tool for simultaneous analysis of ordinary and multi-level factors, with many potential applications in different lines of business.

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Loss Reserving with GLMs: A Case Study

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Loss reserving with GLMs: a case study

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Summary

This paper provides a case study in the application of generalised linear models ("GLMs") to loss reserving. The study is motivated by approaching the exercise from the viewpoint of an actuary with a predisposition to the application of the chain ladder ("CL").

The data set under study is seen to violate the conditions for application of the CL in a number of ways. The difficulties of adjusting the CL to allow for these features of the data are noted (Sections 3).

Regression, and particularly GLM regression, is introduced as a structured and rigorous form of data analysis. This enables the investigation and modelling of a number of complex features of the data responsible for the violation of the CL conditions. These include superimposed inflation and changes in the rules governing the payment of claims (Sections 4 to 7).

The development of the analysis is traced in some detail, as is the production of a range of diagnostics and tests used to compare candidate models and validate the final one.

The benefits of this approach are discussed in Section 8.

Keywords: chain ladder, generalised linear model, GLM, loss reserving, regression, superimposed inflation.

1. Introduction

Taylor (2000) surveys many of the methods of loss reserving. Although the **chain ladder** ("CL") (Chapter 3) is, in a number of ways, the most elementary, it is also still the most widely used by practitioners.

This method is based, however, on a very restrictive model whose conditions are likely to be breached quite commonly in practice. When this happens the method is liable to material error in the loss reserve it generates.

If such error is to be corrected, the model itself must be subjected to some form of corrective action. This may be difficult on two scores:

- The CL falls within the category of model labelled **phenomenological** by Taylor, McGuire and Greenfield (2003). This means that it reflects little of the underlying mechanism of claim payment, and consequently the required form of correction may not be readily apparent.
- Even if the required form of correction can be identified, perseverance with the CL may be more tedious and less reliable than its abandonment in favour of a fundamentally different approach.

The present paper is concerned with a data set that manifestly fails to meet the conditions under which application of the CL is valid. It then examines the sorts of corrections required, and how they might be implemented most efficiently.

It should be pointed out that there has been no necessity to trawl through numerous data sets to locate one that breaches CL assumptions. The data set used here relates to the Auto Bodily Injury claims of one of the Australian states. The consultancy with which we associated deals with such claims in four states, and it is fair to say that any one of these could have been used as the example for the present paper.

The viewpoint taken will be that of a reserving actuary with a predisposition to the application of the CL. The validity of its application to the subject data set will be examined (Section 3), as will the materiality of the potential error it introduces. Analysis of the data set will then be directed to the identification of the various breaches of the CL conditions, and their consequences for a loss reserve.

The ultimate purpose of this analysis is not to produce a diatribe against the CL as such, since this may provide a perfectly useful piece of methodology under appropriate conditions. Rather, the purpose is to demonstrate how **Generalised Linear Models ("GLMs")** can provide a structured and rigorous form of data analysis leading to a loss reserving model.

2. The data set

The data set relates to a scheme of Auto Bodily Injury insurance in one state of Australia. This form of insurance is compulsory, and includes no component of property coverage.

The form of coverage, and other conditions under which the scheme operates, are legislated, but it is underwritten by private sector insurers subject to these conditions. Premium rates are partially regulated by the promulgation of acceptable ranges.

Insurers that participate in the underwriting are required to submit their claims data to a centralised data base. The data set used in the present paper is extracted from this data base. It comprises a unit record claim file, containing the following items of information:

- Date of injury;
- Date of notification;
- Histories of:
  - Finalised/unfinalised status (some claims re-open after having been designated finalised), including dates of changes of status
  - Paid losses
  - Case estimates
- Various other claim characteristics (e.g. injury type, injury severity, etc) not used in the present paper.

The scheme of insurance commenced in its present form in September 1994, and the data base contains claims with dates of injury from then. It is current at 30 September 2003.

The purpose of the present paper is to illustrate loss reserving by means of GLMs, rather than to carry out a loss reserving consulting assignment. For this reason, analysis will be limited to **finalised claims**. Some justification for this course will become apparent as the analysis develops, but there will be no attempt to demonstrate beyond doubt that it is the best.

A consequence of this approach is that (for almost all purposes) data are required only in respect of finalised claims. Exceptions are that:

- The ultimate numbers of claims to be notified in each accident quarter have been estimated outside the paper, and will here be taken as given.
- In respect of each accident quarter, the total amount of losses paid to 30 September 2003, whether relating to finalised or unfinalised claims, is used to obtain estimates of outstanding claims in Sections 3.2 and 7.6.

Wherever paid loss amounts are used they have been converted to 30 September 2003 dollar values in accordance with past wage inflation experienced in the state concerned. This is done to eliminate past "normal" inflationary effects on the assumption that wage inflation is the "normal" inflation for this type of claim. Henceforth, any reference to paid losses will carry the tacit implication that they are expressed in these constant dollar values.

Naturally, claims inflation actually experienced differs from wage inflation from time to time, and is the subject of estimation in Sections 7.3.2 and 7.3.3. The excess of claims inflation over wage inflation is referred to as **superimposed inflation** ("SI").

Appendix A.1 provides a triangular summary of the paid loss data in the usual form. In conventional fashion, rows of the triangle represent **accident quarters**, columns **development quarters**, and diagonals **experience quarters** (or quarters of finalisation). Development quarters are labelled 0, 1,..., with development quarter 0 coinciding with the accident quarter.

Let P_{ij} denote claim payments in the (i,j) cell. Let C_{ij} denote their cumulative version:

$$C_{ij} = \sum_{k=0}^{j} P_{ik} \qquad (2.1)$$

Similarly, P^F_{ij} and C^F_{ij} denote the corresponding quantities in respect of just finalised claims. Appendix A.2 provides a triangular summary of these. Each cell of the triangle contains the paid losses, whether paid in that quarter or earlier, **in respect of claims finalised in the cell.**

Let F_{ij} denote number of claims finalised in the (i,j) cell. They are set out in Appendix A.3. Let G_{ij} denote their cumulative version. Define **average sizes** of finalised claims, incremental and cumulative respectively, as follows:

$$S_{ij} = P^F_{ij} / F_{ij} \tag{2.2}$$

$$T_{ij} = C^F_{ij} / G_{ij} . \tag{2.3}$$

Appendices A.4 and A.5 display these average claim sizes.

3. The chain ladder

3.1 Age-to-age factors

Appendix B derives age-to-age factors from the data of Appendix A.

The age-to-age factor linking cells (i,j) and (i,j+1) in the triangle of cumulative paid losses is

$$R^F_{ij} = C^F_{i,j+1} / C^F_{ij}. \tag{3.1}$$

These factors are tabulated in Appendix B.1.

Likewise, the age-to-age factor linking cells (i,j) and (i,j+1) in the triangle of cumulative average claim sizes (Appendix A.4) is

$$Q_{ij} = T_{i,j+1} / T_{ij}. \tag{3.2}$$

These factors are tabulated in Appendix B.2.

Average age-to-age factors are displayed in Appendices B.1 and B.2. Conventionally, these are taken over various past averaging periods, as some sort of test of stability of the factors over time.

Figures 3.1 and 3.2 chart the average age-to-age factors, showing clear indications of **instability**. In development periods 3 to about 10, the factors show a clear tendency toward higher values for more recent experience years (except the latest year, where they are lower).

Figure 3.1

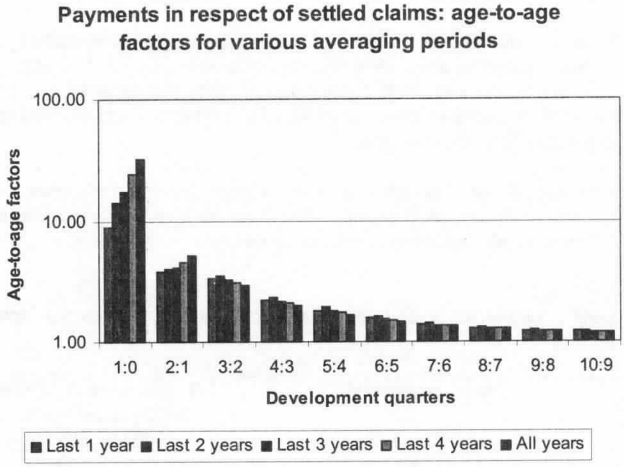
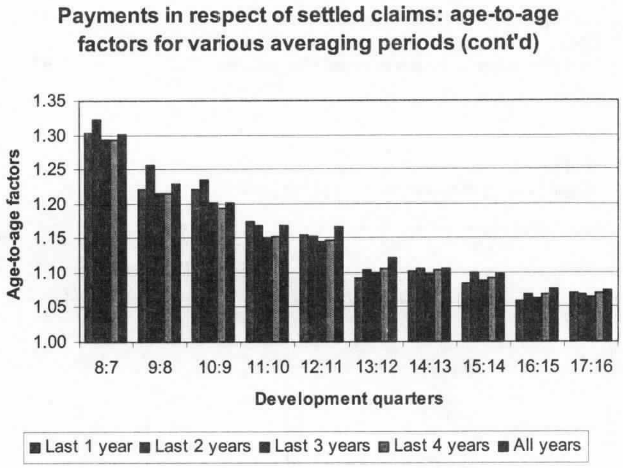


Figure 3.2



3.2 Sensitivity of loss reserve

While Figures 3.1 and 3.2 demonstrate that different averaging periods lead to different age-to-age factors, and therefore to different loss reserves, the materiality of the differences is not apparent. Table 3.1 sets out the loss reserves calculated according to the various averaging periods.

Inspection of Appendix B.1 reveals that, while the age-to-age factors generally showed increasing trends over recent periods, those recorded in the September 2003 experience quarter (the last diagonal, were particularly low. Table 3.1 includes an examination of the effect of including or excluding this quarter's experience from the averaging.

Omission of the September 2003 experience prevents estimation of a loss reserve for that accident period. Therefore, the loss reserves set out in Table 3.1 relate to all accident quarters except that one.

Table 3.1
Loss reserves according to different averaging periods for age-to-age factors

Averaging period	Loss reserve at 30 September 2003 (excluding September 2003 accident quarter)
	$B
All experience quarters	1.61
Last 8 experience quarters	1.68
All experience quarters except September 2003	1.78
Last 8 experience quarters except September 2003	1.92

Table 3.2
Loss reserve dissected by accident period

Accident quarter	Loss reserve at 30 September 2003 (excluding September 2003 accident quarter)
	$M
Sep 00	176
Sep 01	165
Sep 02	171
Dec 02	124
Mar 03	59
Jun 03	58
Total	1,785

The sensitivity of loss reserve to averaging period is considerable. The largest estimate is 19% larger than the smallest. However, a more detailed examination of the loss reserves quickly reveals that the true sensitivity is much greater than this.

Table 3.2 sets out an accident quarter partial dissection of the "All experience quarters except September 2003" reserve from Table 3.1. It is quite evident that the loss reserve is distorted downward in respect of the latest accident quarters.

This is due to the low cumulative paid losses at the end of this quarter, as evidenced by the low age-to-age factors in this quarter, which serve as the baseline for forecasting future paid losses.

The usefulness of the reserves in Table 3.1 is unclear in the presence of this factor. It is natural to correct for it by adjusting any loss reserve at 30 September 2003 (still excluding the September 2003 accident quarter) by forecasting it on the basis of paid losses to 30 June 2003. Specifically, this consists of:

- calculating a standard chain ladder loss reserve at 30 June 2003; and then
- deducting the forecast September 2003 quarter paid losses included in that reserve.

This makes sense only for reserves based on averaging that excludes the September 2003 experience quarter. Table 3.3 augments Table 3.1 to include such corrections.

Table 3.3
Loss reserves corrected and uncorrected for low September 2003 quarter paid loss experience

Averaging period	Loss reserve at 30 September 2003 (excluding September 2003 accident quarter)	
	Uncorrected	Corrected
	$B	$B
All experience quarters	1.61	
Last 8 experience quarters	1.68	
All experience quarters except September 2003	1.78	1.94
Last 8 experience quarters except September 2003	1.92	2.35

Table 3.4, again dealing with the "All experience quarters except September 2003" case, shows that the corrections introduced into the last two rows of Table 3.3 do at least remove the most obvious implausibility in the trends of those loss reserves over recent accident periods.

This comes, however, at the cost of a considerable widening of the gap between the two versions of the chain ladder that respectively use all experience or just the last 8 experience quarters with the exception of the last. The larger of these two estimates is now 21% larger than the other, compared with 8% previously.

Table 3.4
Loss reserve by accident quarter

Accident quarter	Loss reserve at 30 September 2003 (excluding September 2003 accident quarter) – corrected as in Table 3.3
	$M
Sep 00	96
Sep 01	121
Sep 02	137
Dec 02	119
Mar 03	101
Jun 03	114
Total	1,943

It is submitted that the actuary attempting application of the CL to the example data set is now confronted with a bewildering array of models, corrections to models, and corrections to the corrections.

The principal facts are that:

- There are clear time trends in the data;
- One can attempt to deal with this by limiting the data on which the model relies to those of recent period. Here the example of averaging over the last 8 experience quarters is used, but there is no clear guidance to prefer 8 over say 4, or 6, or some other number.
- In any event, the last experience quarter appears fundamentally different from the preceding 7, and the extremely *ad hoc* procedure of dropping it has been adopted.

While the CL can be applied to any choice of data set, there is no apparent criterion for reliable choice of that data set. Moreover, the CL's phenomenological treatment of the trends is deeply unsatisfying. These trends must have a cause that resides somewhere in the detailed mechanics of loss payment. However, the formulaic nature of the CL renders it incurious as to these details.

3.3 The effect of operational time

It is common for the above type of instability to occur when rates of settlement of claims are changing over time. Berquist and Sherman (1972) suggest adjustment to loss reserving methods to take such movements into account.

They refer to "ultimate claims disposed ratio" to denote the proportion of an accident period's claims settled, and suggest that its outstanding claims should be in some way commensurate with the complement of settlement time. Reid (1978) introduced the term **operational time** to take the same meaning, and this terminology will be used below. This quantity is also referred to sometimes as "settlement time".

Let N_i denote the estimated number of claims incurred in accident quarter i, i.e. the number ultimately to be notified in respect of this accident quarter. Then the operational time associated with (the end of) the (i,j) cell, denoted t_{ij} is

$$t_{ij} = G_{ij} / N_i. \tag{3.3}$$

Figure 3.3 plots how the operational times associated with various numbers of development years have changed over past accident quarters. It is seen that the operational time attained after 2 development years (i.e. at the end of development year 1) increased from 33% for the September 1994 accident quarter to the 54% for the December 1998 accident quarter, and then declined somewhat for subsequent accident quarters.

Similar trends affected development years 2 and 3, but not lower or higher development periods.

Figure 3.3

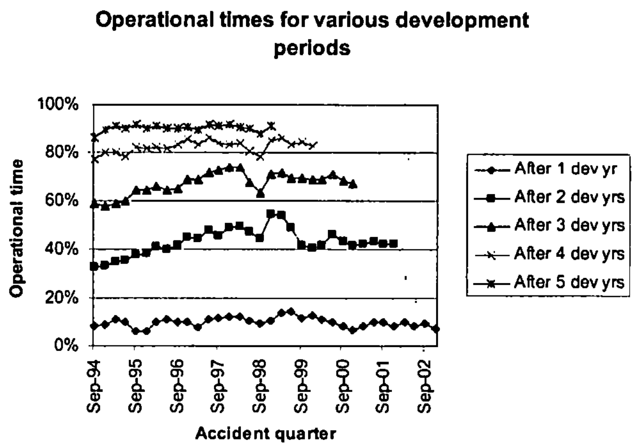


Figure 3.4 superimposes the plot of the quarterly age-to-age factor 3:2 on that of operational time at the end of development quarter 3. Figures 3.5 and 3.6 make the corresponding comparisons for age-to-age factors 7:6 and 11:10 respectively. In the first two of these cases, increases in age-to-age factors appear to coincide with increase in operational time, though the correlation is far from perfect.

Figure 3.4

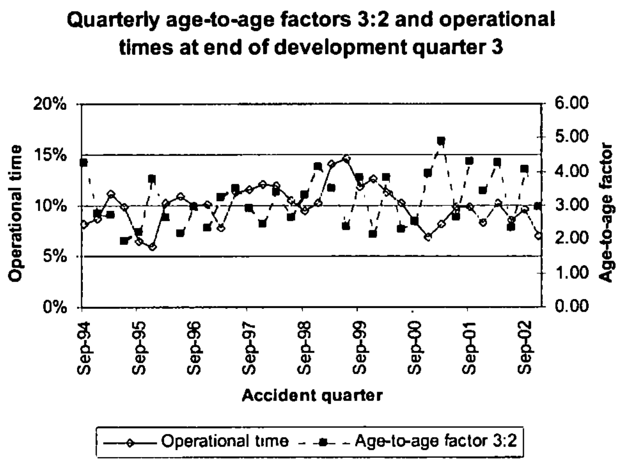


Figure 3.5

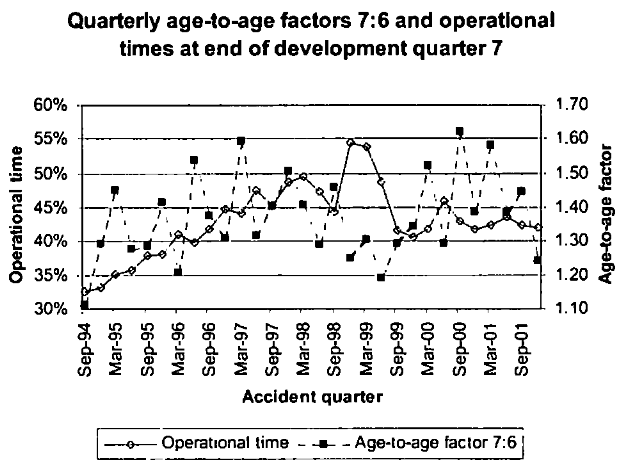
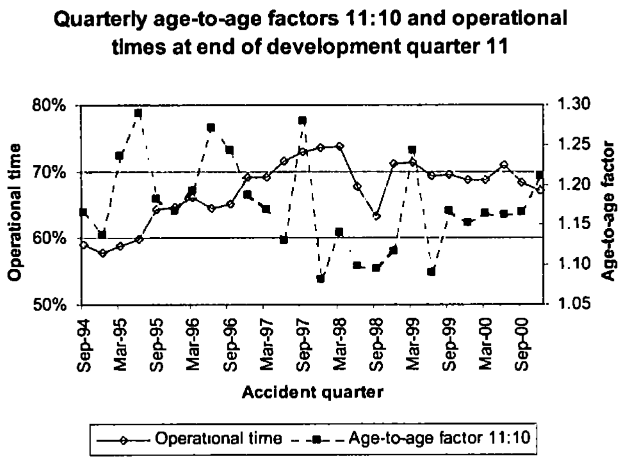


Figure 3.6



An alternative means of controlling for changing operational times is to replace cumulative payments by cumulative average claim sizes in the analysis. The cumulative average claim size (of finalised claims) associated with the (i,j) cell, given by (2.3), may be expressed by means of (3.3) in the alternative form:

$$T_{ij} = [C^F_{ij} / t_{ij}] / N_i. \tag{3.4}$$

This shows that cumulative average claim size is a multiple of cumulative claim payments per unit of operational time. Such claim sizes might be more stable than payment based age-to-age factors in the presence of changing operational times.

Figure 3.7 plots the cumulative average claim sizes to the end of development quarter 3, for the various accident quarters, against the corresponding operational times. It is found that average claim sizes are not in fact insensitive to variations in operational time, but appear to display a better correlation with operational times than do age-to-age factors.

It will be seen later that this occurs because the claim sizes associated with a particular accident quarter tend to increase with increasing operational time.

A similar improvement in correlation is obtained for development quarter 7, as displayed in Figure 3.8. The corresponding results for development quarter 11 are displayed in Figure 3.9.

Figure 3.7

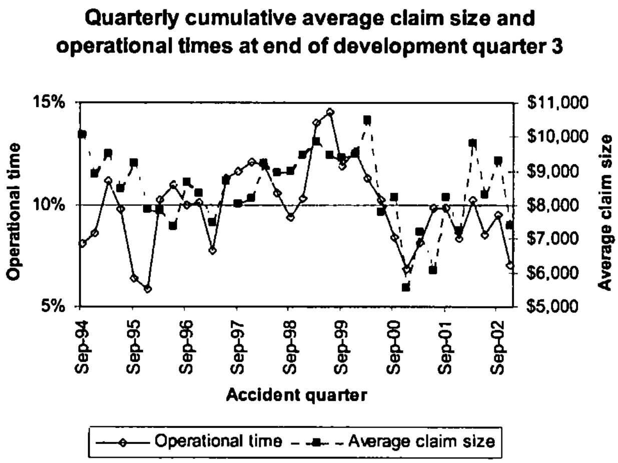


Figure 3.8

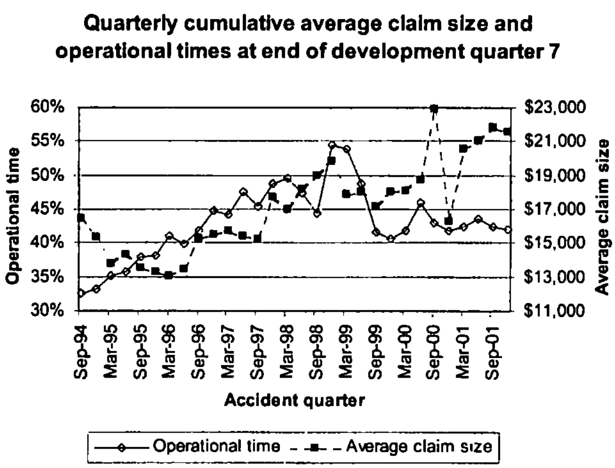
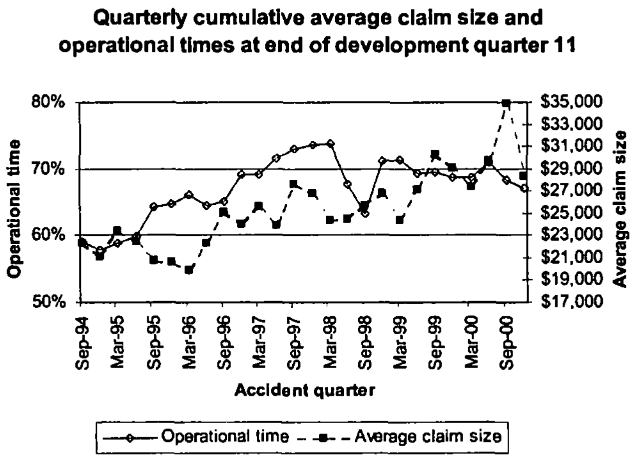


Figure 3.9



4. Exploration of triangular data on average claim size

4.1 Claim development measured by development quarter

The observations made on Figures 3.7 to 3.9 suggest that an average claim size analysis might be preferable to chain ladder analysis. Figures 4.1 to 4.3 therefore explore certain trends in average claim size. Each plots log(average size of finalised claims) against some variable. The triangular form of data is retained.

Figure 4.1 plots log(average size of finalised claims) against development quarter. This could have been carried out as a routine averaging process, but it proved efficient, and in fact more integrated with later sections, to obtain these averages through a modelling process.

Consider the model:

$$\log S_{ij} = \beta_j + \varepsilon_{ij}, \tag{4.1}$$

where

$$\varepsilon_{ij} \sim N(0, \sigma), \tag{4.2}$$

the ε_{ij} are stochastically independent, and the β_j, σ are constants.

Equivalently,

$S_{ij} \sim \log N(\beta_j, \sigma)$ (4.3)

For this model, simple regression estimates of the β_j are equal to the arithmetic means (taken over i) of the observed values of the log S_{ij}. Figure 4.1 could have been derived in this way. EMBLEM software (see also Section 6) has been applied to fit the regression model (4.1) and (4.2) to the data, and the resulting estimates of the β_j plotted against j (see Figure 4.1). The same software is used to produce the remaining plots in this paper.

Figure 4.1
Average claim size by development quarter

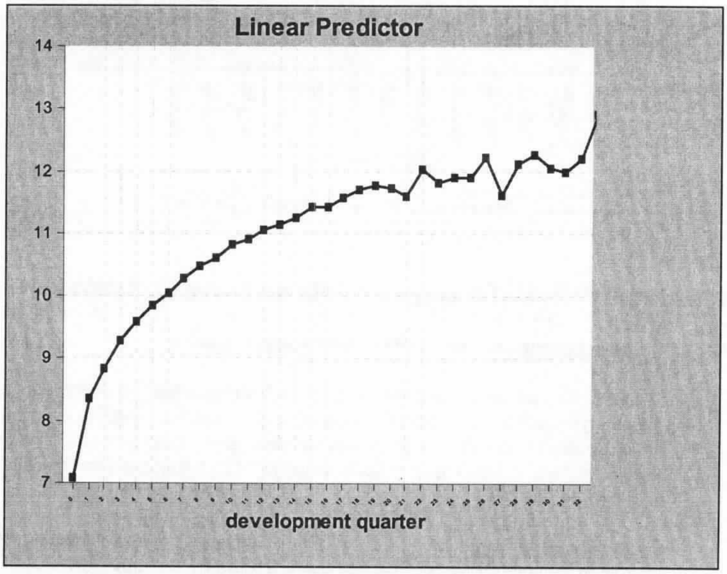


Figure 4.1 shows quite clearly how the average size of finalised claims increases with development quarter, as foreshadowed in Section 3.3.

Figures 3.7 to 3.9 illustrated how (cumulative) average sizes of finalised claims have varied with accident period. Any such effect can be incorporated in the model represented by (4.1) and (4.2) by extending it to the following:

$\log S_{ij} = \beta^d_j + \beta^a_i + \varepsilon_{ij}$, (4.1a)

where the β_j in (4.1) are now denoted β^d_j (the superscript d signifying that these coefficients relate to development quarters), and the accident quarter coefficients β^a_i have also been introduced. The relation (4.2) is retained.

It is worth noting in passing that exponentiation of (4.1a) yields

$$E[S_{ij}] = K \exp \beta^d_j \, . \exp \beta^a_i, \tag{4.4}$$

where K is the constant, $E[\exp \varepsilon_{ij}]$.

This is a model with multiplicative row and column effects, and hence is very closely related to the chain ladder. It is the same as the stochastic chain ladder of Hertig (1985) except that Hertig assumed the following in place of (4.2):

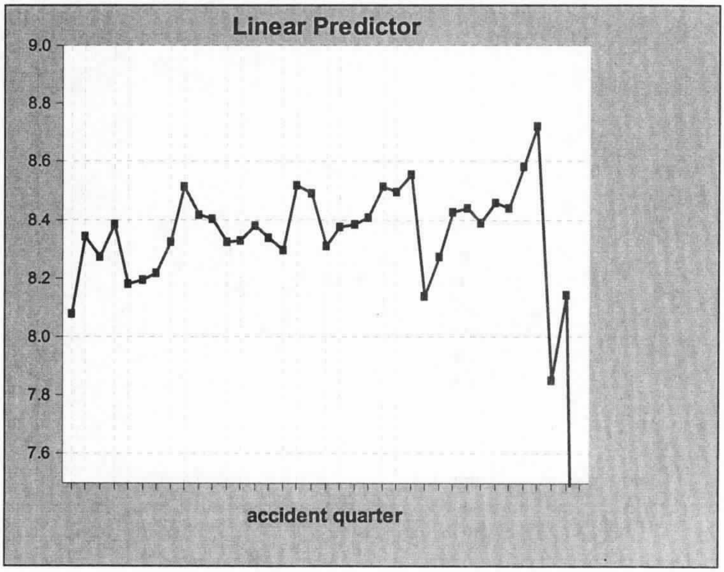
$$\varepsilon_{ij} \sim N(0, \sigma_j). \tag{4.2a}$$

Though related to the chain ladder of the type discussed in Section 3, models of this type differ from it, as was established by the exchange between Mack (1993, 1994), Mack (2000), Verrall (2000) and England and Verrall (2000).

Stochastic versions of the chain ladder have received extensive treatment in the literature (England and Verrall, 2002; Mack, 1993; Mack and Venter, 2000; Murphy, 1994; Renshaw, 1989; Verrall, 1989, 1990, 1991a, 1991b, 2000).

The coefficients β^d_j and β^a_i are no longer obtainable by simple averaging, but they are obtainable from simple (i.e. unweighted least squares) regression. Figure 4.2 gives the plot of the β^a_i against i.

Figure 4.2
Regression estimate of trend in average claim size by accident quarter



The plotted values become less reliable as one moves from left to right across the figure, because one is considering steadily less developed accident quarters. Hence the downward plunge at the right of the plot can be ignored. The indication is then that, when allowance for a development quarter trend of the type illustrated in Figure 4.1 is made, there remains an increasing trend in claim sizes over time.

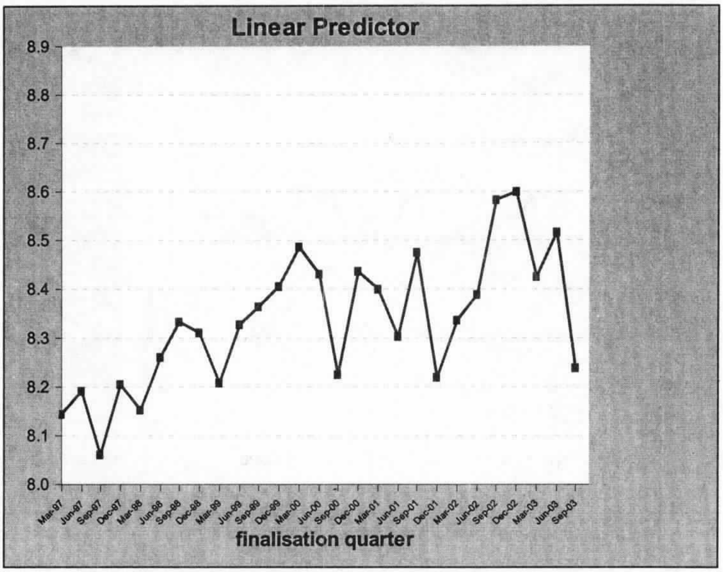
The possibility of a time trend has been incorporated in the model in the form (4.1a), in which the specific time dimension to which it is related is accident quarter, i.e. a row effect. It is possible, however, that the trend occurs over finalisation quarter, i.e. a diagonal effect, represented as follows:

$$\log S_{ij} = \beta^d_j + \beta^f_k + \varepsilon_{ij}, \tag{4.1b}$$

where $k = i+j$ = calendar quarter of finalisation, and (4.2) is still assumed to hold.

Fitting this model to the data yields Figure 4.3 as the plot of the β^f_k against k. This also indicates a time trend. Adjudication on which of (4.1a) and (4.1b) provides the more appropriate representation of the trend may not be easy. This question will be deferred until Section 7 when rather more modelling apparatus is in place.

Figure 4.3
Regression estimate of trend in average claim size by finalisation quarter



4.2 Claim development measured by operational time

The use of operational time as a measure of claim development was introduced in Section 3.3. The models of Section 4.1 may be re-formulated on the basis of it.

The operational time defined in (3.3) related to the end-point of time represented by the (i,j) cell. This was appropriate to the context of average claim sizes that were cumulative to that point. In the context of non-cumulative averages, as currently, the mid-value of operational time for the cell is more appropriate. This is

$$\begin{aligned}\bar{t}_{ij} &= \tfrac{1}{2}\left[t_{ij} + t_{i,j-1}\right] \\ &= \tfrac{1}{2}\left[G_{ij} + G_{i,j-1}\right]/N_i\end{aligned} \tag{4.5}$$

with the convention in the case j=0 that $t_{i,-1} = G_{i,-1} = 0$.

The quantity $\bar{t}_{ij}$ is a continuous variate in the sense that it may take any value on the continuum [0,1]. It will be convenient, to convert it to a categorical variate by recognising ranges of values in which it might lie.

For the present example, the interval [0,1] has been divided into 50 sub-intervals, [0%,2%), [2%,4%),...,[98%,100%], labelled by the values 1,2,...,50. Then each cell average size S_{ij} may be written in the alternative notation S_{it}, where t is the label corresponding to the mid-quarter operational time $\bar{t}_{ij}$.

Then the re-formulation of model (4.1) in which j is replaced by $\bar{t}_{ij}$ as a measure of development is as follows:

$$\log S_{it} = \beta_t + \varepsilon_{it}, \tag{4.6}$$

with

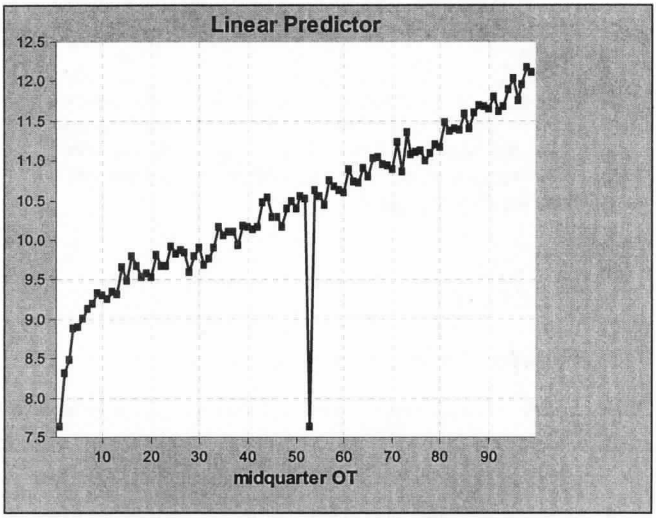
$$\varepsilon_{it} \sim N(0, \sigma). \tag{4.7}$$

the corresponding re-formulations of (4.1a) and (4.1b) are as follows:

$$\log S_{it} = \beta^d_t + \beta^a_i + \varepsilon_{it} \tag{4.6a}$$
$$\log S_{it} = \beta^d_t + \beta^f_k + \varepsilon_{it}. \tag{4.6b}$$

The three models (4.6), (4.6a) and (4.6b) produce the plots in Figures 4.4 to 4.6 in place of 4.1 to 4.3.

Figure 4.4
Regression estimate of trend in average claim size by operational time



Note: The observation at operational time 53 should be ignored as it relates to a point with no data.

Figure 4.5
Regression estimate of trend in average claim size by accident quarter

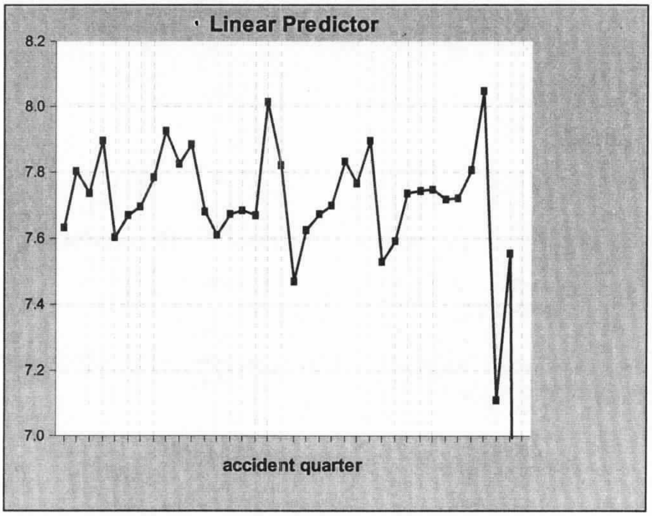
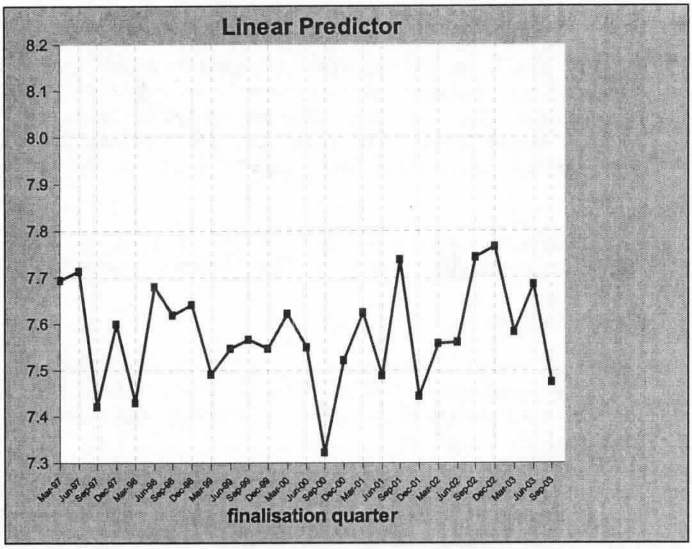


Figure 4.6
Regression estimate of trend in average claim size by finalisation quarter



It is interesting to note, in connection with Figure 4.4, that the use of operational time appears also to have simplified the relation between average claim size and the measure of development of an accident quarter. Indeed, average claim size appears closely approximated by an exponential function of operational time over the interval of roughly [10%,100%].

The actuary responsible for loss reserving against the example data set will by now have reached the following position:

- Any conventional application of a paid loss CL is dubious (Section 3.2).
- It appears that analysis of average claim sizes may be preferable (Section 4.1).
- It may also be desirable to take operational time into account somehow (present sub-section).
- The incorporation of a paid loss development pattern (as a function of operational time) together with the simultaneous identification of a time trend was achieved in Figures 4.4, 4.5 and 4.6 by means of regression.

Further progress by means of modification of a CL model appears difficult in the face of these observations.

5. Modelling individual claim data

5.1 Regression models

If one is impelled toward some form of regression modelling such as in Section 4.2, there is an argument that the regression may as well be carried out by reference to individual claim data as to the triangular summaries used there. The same models as applied in Section 4.2 can be formulated in terms of individual claims, and the use of data summaries then seems unnecessary and artificial.

As a preliminary to this, it will be useful to express (4.6) and its variants in a form more conventional for regression. Thus, (4.6) may be written as:

$$\log S_{it} = X_{it}\, \beta + \varepsilon_{it}, \tag{5.1}$$

where β is the vector of quantities β_t, viz. $(\beta_1, \beta_2, \ldots, \beta_{50})^T$, with the superscript T denoting matrix transposition, and X_{it} is the row vector $(X_{it1}, X_{it2}, \ldots, X_{it50})$ with $X_{itm} = 1$ if operational time label m is associated with S_{it}, and $X_{itm} = 0$ otherwise.

Thus the operational time variate in (4.6) is represented as a 50-vector of binary components. Regression variates of this type are often referred to as **class variates, or factor variates**. The numerical values corresponding to the binary components are called **levels**. Factor variates enable further simplification of the regression equation, with (5.1) being written as:

$$\log S = X\, \beta + \varepsilon, \tag{5.2}$$

where log S is (with a slight abuse of notation) the column n-vector of all observations $\log S_{it}$, taken in any convenient order, X is the nx50 matrix formed by stacking the n row vectors X_{it}, taken in the same order as the $\log S_{it}$, and ε is the n-vector of the ε_{it}, also taken in the same order.

Let Y_r denote the size of the r-th finalised claim. This claim will have associated values of i, j and k=i+j=calendar quarter of finalisation. It will also have an associated value of t=operational time at finalisation. Let this collection of observations on the r-th claim be denoted i_r, j_r, k_r, t_r.

The quantity t_r may denote operational time specifically, or it may be converted to the categorical form described in Section 4.2. The latter is chosen for the purpose of the present paper.

The model described by (4.6) and (4.7) requires very little modification for application to individual claims. Expressed in the form (5.1), it becomes:

$$\log Y_r = X_r\, \beta + \varepsilon_r, \tag{5.3}$$

with

$\varepsilon_r \sim N(0, \sigma)$ (5.4)

where X_r is the value of the operational time class variate applicable to the r-th claim and ε_r is the stochastic error term ε_{it} associated with it.

Just as (5.1) was notationally contracted to (5.2), so (5.3) may be abbreviated to:

$\log Y = X\,\beta + \varepsilon,$ (5.5)

The general idea underlying the models of Section 4.2 is that Y_r takes the form:

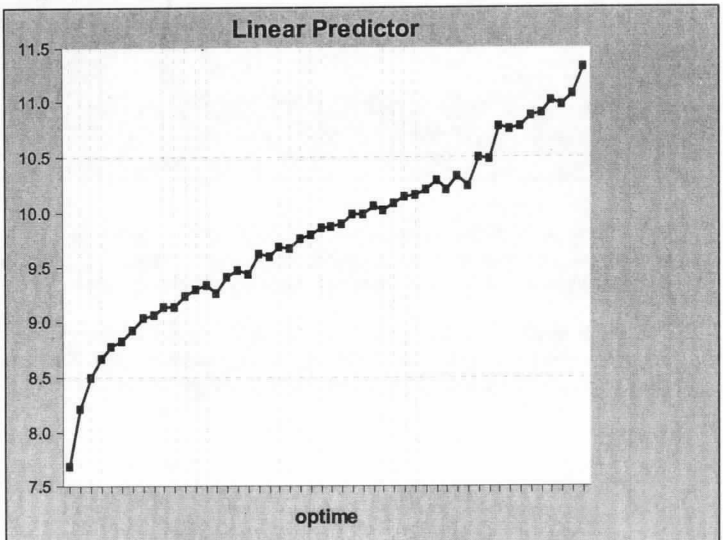
$\log Y_r = \text{function}(i_r, j_r, k_r, t_r) + \text{stochastic error}$ (5.6)

and that this may be written in the linear form (5.3), and hence (5.5), with X_r denoting a row composed of variates derived from i_r, j_r, k_r, t_r. These may or may not be factor variates.

5.2 Basic trends

Consider the model represented by (5.3) and (5.4), with X_r denoting the operational time factor variate discussed there. Ordinary least squares regression estimation of β yields Figure 5.1, which plots the components β_1, $\beta_2, \ldots, \beta_{50}$ of β against their associated midpoint operational times $1, 3, \ldots, 99$.

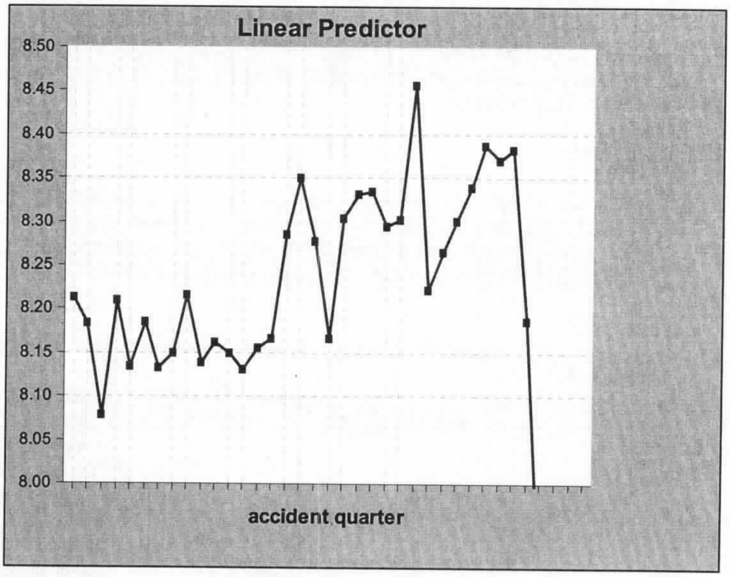
Figure 5.1
Individual claim regression estimate of trend in average claim size by operational time



Not surprisingly, Figure 5.1 closely resembles Figure 4.4, although Figure 5.1 exhibits greater smoothness due to the fact that it is based on about 60,000 observations, compared with ½x38x39=741 in the case of Figure 4.4.

The other models of Section 4.2, namely (4.6a) and (4.6b), may also be adapted to the form (5.3) and (5.4). The adaptation of (4.6a), for example, yields a version of (5.3) in which X_r comprises factor variates for operational time and accident quarter respectively. Figure 5.2 plots the components of the parameter vector β relating to accident quarter.

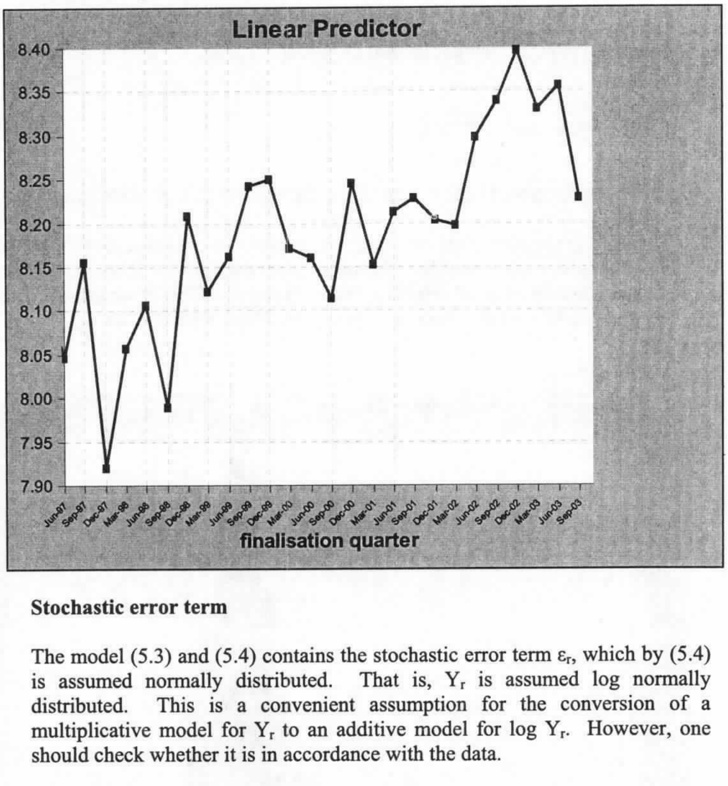
Figure 5.2
Individual claim regression estimate of trend in average claim size by accident quarter



The adaptation of (4.6b) is similar but with X_r comprising factor variates for operational time and finalisation quarter respectively. Figure 5.3 plots the components of the parameter vector β relating to finalisation quarter.

The trends displayed in Figures 5.2 and 5.3 differ somewhat from those in Figures 4.5 and 4.6. Presumably, the additional information included in the regression through the use of individual claims has improved their estimation.

Figure 5.3
Individual claim regression estimate of trend in average claim size by finalisation quarter



5.3 Stochastic error term

The model (5.3) and (5.4) contains the stochastic error term ε_r, which by (5.4) is assumed normally distributed. That is, Y_r is assumed log normally distributed. This is a convenient assumption for the conversion of a multiplicative model for Y_r to an additive model for log Y_r. However, one should check whether it is in accordance with the data.

This question may be investigated by means of residual plots. The residuals naturally adapted to the normal distribution are the **Pearson residuals**, defined as follows.

Consider the general model (5.5) and let $\hat{\beta}, \hat{\sigma}$ denote the regression estimates of β, σ respectively. Define

$$\mu = E[\log Y] = X\beta \tag{5.7}$$

and

$$\hat{\mu} = X\hat{\beta}, \tag{5.8}$$

the estimate of μ, and hence the **fitted value** corresponding to Y.

The Pearson residual associated with observation Y_r is

$$R_r^P = (\log Y_r - \hat{\mu}_r) / \hat{\phi}^{\frac{1}{2}} \tag{5.9}$$

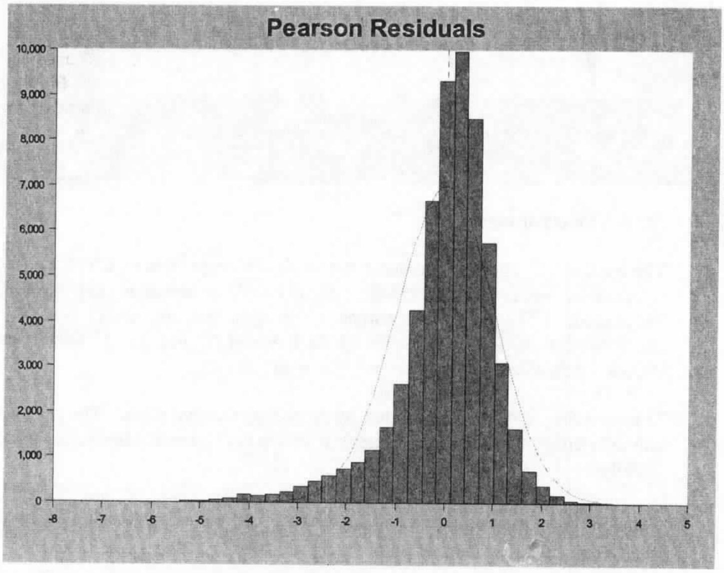
where $\hat{\phi}$ is the following estimator of $V[R^P_r]$:

$$\hat{\phi} = \sum_{r=1}^{n} (\log Y_r - \hat{\mu}_r)^2 / (n-p) \tag{5.10}$$

with p the dimension of the vector β, i.e. the number of regression parameters.

The Pearson residuals should be approximately unit normal distributed for large samples subject to (5.4). Figure 5.4 plots them for the model underlying Figure 5.3, indicating substantial negative skewness. This is confirmed by the alternative views of the residuals presented in Figures 5.5 and 5.6.

Figure 5.4



This suggests that the logarithmic transformation has over-corrected for the long tail of the Y_r, i.e. these observations, while right skewed, are shorter tailed than log normal. In this event, the choice of working with log transformed data, as in (5.5) is a poor one.

Figure 5.5

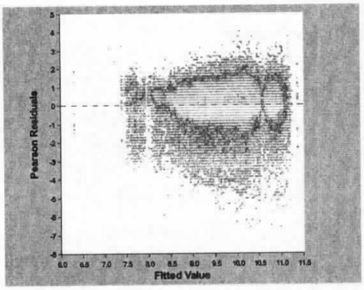
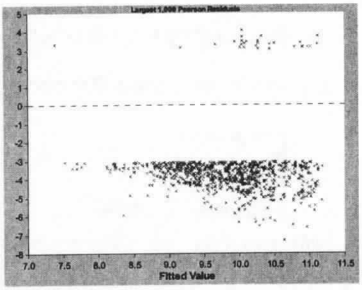


Figure 5.6



6. The exponential dispersion family and generalised linear models

6.1 The exponential dispersion family

One actually requires a distribution of the ε_r that lies between normal and log normal in terms of long-tailedness. The **exponential dispersion family (EDF)** of likelihoods (actually quasi-likelihoods) provides a comprehensive family within which to search for a distribution with suitable tail length.

The EDF comprises the following family of quasi-likelihoods (Nelder and Wedderburn, 1972):

$$f(y;\theta,\lambda) = a(\lambda,y) \exp \lambda\, [y\theta - b(\theta)] \quad (6.1)$$

where θ,λ are parameters and a(.) and b(.) are functions characterising the member of the family.

It may be shown that, for this distribution,

$$E[Y|\theta,\lambda] = b'(\theta) \quad (6.1)$$
$$Var[Y|\theta,\lambda] = b''(\theta)/\lambda \quad (6.2)$$

Denote $b'(\theta)$ by $\mu(\theta)$ whence, provided that $\mu(.)$ is one-one,

$$Var[Y|\theta,\lambda] = V(\mu)/\lambda \quad (6.3)$$

for some function V(.) called the **variance function**.

Many applications of the EDF restrict the form of the variance function thus:

$$V(\mu) = \mu^p \quad (6.4)$$

for some constant $p \geq 0$. This likelihood will be referred to as **EDF(p)**.

the quantity $\varphi = 1/\lambda$ is called the **scale parameter**.

Special cases of the EDF are:

p=0: normal
p=1: Poisson
p=2: gamma
p=3: inverse Gaussian.

6.2 Generalised linear models

Now let Y be a random n-vector, as in Section 5. Suppose $Y_1, Y_2, \ldots, Y_n$ to be stochastically independent drawings from the EDF likelihoods

$$f(y_r;\theta_r,\lambda) = a(\lambda,y_r) \exp \lambda \, [y_r\theta_r - b(\theta_r)] \tag{6.5}$$

where the same λ, a(.) and b(.) apply to all r.

Suppose further that $\mu(\theta_r)$ takes the form

$$\mu(\theta_r) = h^{-1}(X_r\beta) \tag{6.6}$$

for some one-one function h(.), called the **link function**, row p-vector X_r and column p-vector β.

With the same slight abuse of notation as occurred in connection with (5.2), the n relations (6.6) may be stacked into the form

$$\mu(\theta) = h^{-1}(X\beta) \tag{6.7}$$

where θ is the column n-vector with r-th component θ_r and X is an nxp **design matrix**. The n-vector $X\beta$ is called the **linear response**.

This specification of the vector Y is called a **Generalised Linear Model (GLM)** (Nelder and Wedderburn, 1972). GLMs are discussed by McCullagh and Nelder (1989). Note that the general linear model arises as the special case of a GLM with normal error term and identity link function.

The parameter vector β may be estimated by maximum likelihood. Generally, closed form solutions are not available, but various software products perform the estimation, e.g. SAS, S-Plus, EMBLEM. This paper uses the last of these, an interactive package produced by EMB Software Ltd of the UK.

Maximisation of the likelihood $L[Y|\theta,\lambda]$ is equivalent to minimisation of the so-called **deviance** $D[Y|\theta,\lambda]$ where

$$D[y \mid \theta, \lambda] = -2 \log L[y \mid \theta, \lambda]$$
$$= -2 \sum_{r=1}^{n} \{ \lambda [y_r \theta_r - b(\theta_r)] + \log a(\lambda, y_r) \} \tag{6.8}$$

6.3 Residuals

In the more general setting of a GLM, the Pearson residual (5.9) becomes

$$R_r^P = (Y_r - \hat{\mu}_r) / [\hat{\phi} V(\hat{\mu})]^{\frac{1}{2}} \tag{6.9}$$

where the observations are now the Y_r instead of the log Y_r, $\hat{\beta}$ is the estimated value of β, $\hat{\mu} = h^{-1}(X\hat{\beta})$ is now the fitted value defined in parallel with (5.8), with $X\hat{\beta}$ now called the **linear predictor**, and

$$\hat{\phi} = D[Y \mid \hat{\phi}, \hat{\lambda}] / (n - p). \tag{6.10}$$

Note that, for the identity link and normal error, (5.10) and (6.10) are the same. Then (5.9) and (6.9) are also the same since, for the normal case, $V(\mu) = \mu^0 = 1$.

Interpretation of Pearson residuals may be difficult for non-normal observations. Since the residual is just a linear transformation of the observation, any feature of non-normality, such as skewness, will be carried directly from one to the other.

An alternative form of residual is often helpful in these circumstances. Note that the deviance (6.8) may be written in the form (argument suppressed for brevity)

$$D = \sum_{r=1}^{n} d_r \tag{6.11}$$

where

$$d_r = -2 \log L_r \tag{6.12}$$

with log L_r the contribution of Y_r to log L.

Now define the **deviance residual**

$$R_r^D = \operatorname{sgn}(Y_r - \hat{\mu}_r) d_r^{\frac{1}{2}} \tag{6.13}$$

The advantage of deviance residuals is that they tend to be closer to normal than Pearson in their distribution. A variant is the **studentised standardised deviance residual**

$$R_r^{SSD} = R_r^D / \left[\hat{\phi}(1 - z_r)\right]^{\frac{1}{2}} \quad (6.14)$$

where z_r is the r-th diagonal element of the nxn matrix $X(X^TX)^{-1}X^T$. These residuals tend to have a distribution close to unit normal.

7. Application of GLM to data set

7.1 Loss reserving with GLMs

Although the use of GLMs in loss reserving is not widespread, it is also not new.

The use of general (as distinct from generalised) linear models can be seen in Taylor and Ashe (1983), Ashe (1986) and Taylor (1988). These two authors were in fact using GLMs for loss reserving consulting assignments during the 1980's.

The general linear model is also inherent in the loss reserving of De Jong and Zehnwirth (1983), based on the Kalman filter, and the related ICRFS software (Zehnwirth, 2003), marketed since the late1980's.

Wright (1990) gave a comprehensive discussion of the application of GLMs to loss reserving. Taylor, McGuire and Greenfield (2003) also made use of them.

All of these models other than in the last reference were applied to summary triangles of claims data, such as used in Section 4, rather than individual claims.

7.2 Choice of error distribution

As suggested at the start of Section 6.1, one requires an error distribution that lies between normal and log normal in terms of long-tailedness. Experimentation might begin with a gamma distribution. This is a more realistic distribution of claim sizes than normal, its density having strictly positive support and positive skewness. It is, however, considerably shorter tailed than log normal.

Consider the gamma (i.e. EDF(2)) GLM corresponding to (5.5). It has the same X and β, but observations are Y_r instead of log Y_r, and the link function is log. For example, the particular form of this model adapted to (4.6b) is as follows:

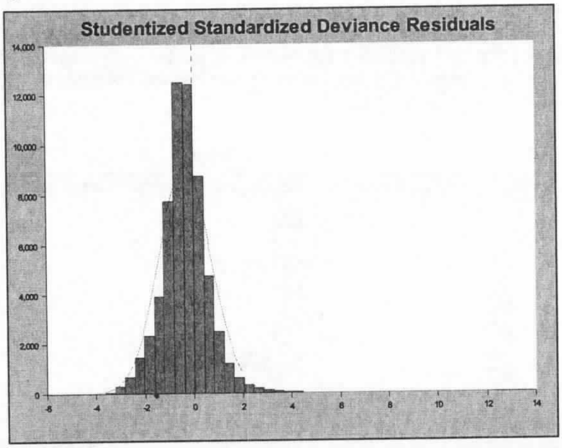
$$Y_r \sim EDF(2) \quad (7.1)$$

$$E[Y] = \exp X\beta = \exp [X^d\beta^d + X^f\beta^f] \quad (7.2)$$

where X^d and X^f are factor variates for operational time and finalisation quarter respectively.

Fitting this model to the data set yields the residual plots set out in Figure 7.1.

Figure 7.1



Comparison of Figure 7.1 with 5.4 reveals that the use of a gamma rather than log normal error has corrected the most obvious left skewness of the residuals. However, Figures 7.2 and 7.3 give more detail of the residuals and indicate that they are not altogether satisfactory.

Figure 7.2

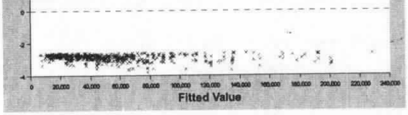


Figure 7.3

The studentised standardised residuals are expected to resemble standardised unit normal residuals. The largest 1,000 of these (from 60,050 observations) would numerically exceed 2.4. Figure 7.2 conforms reasonably well with this requirement, displaying residuals numerically exceeding a threshold value of roughly 2.6.

However, extreme values, up to 12, appear, indicating a much longer tail than normal. This abnormality in the residual plot is emphasised in Figure 7.3, which displays the largest 100 residuals. The unit normal range for these has a threshold value of about 3.1. the observed threshold exceeds 4, and all 100 residuals are positive.

These properties of the residual plots indicate that the distribution of claims sizes is longer tailed than gamma. As indicated by (6.3) and (6.4), a larger EDF exponent p will generate a longer tail. Therefore, one experiments with values of p>2 (gamma). Figures 7.4 to 7.6 are the residual plots for EDF(2.3) corresponding to Figures 7.1 to 7.3.

Figure 7.4

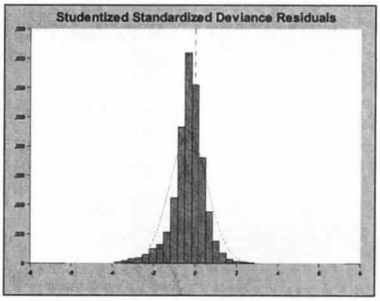


Figure 7.5

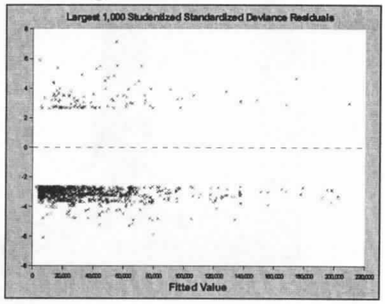


Figure 7.6

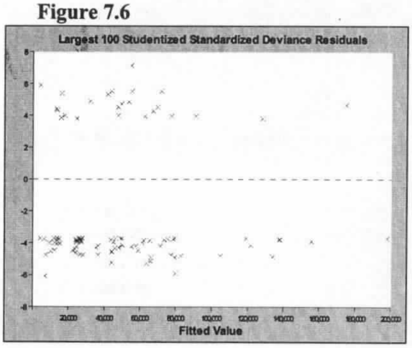


Figure 7.7

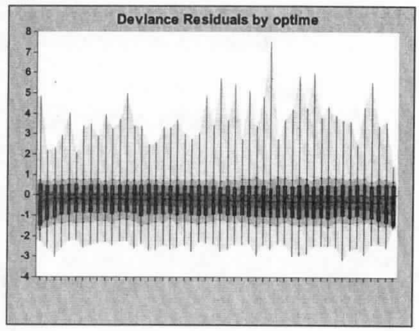


Figure 7.4 shows that the shift to the longer tail of EDF(2.3) has over-compensated somewhat for the right skewness, producing a degree of left skewness. Figure 7.5 shows little change in the threshold value of the largest 1,000 residuals. However, Figure 7.6 shows considerable improvement in the treatment of the extreme tail.

The final choice of claim size distribution needs to balance these observations. Generally, the improved treatment of the tail would be expected to improve robustness of the parameter estimation such that this more than offsets the unwanted skewness near the centre of the distribution. The choice of EDF(2.3) will be retained for the remainder of this paper.

There is a practice, common among actuaries, of separately analysing "small" and "large" claims, however defined, on the ground that the latter group are liable to distort the averaging processes inherent in modelling. It is worth remarking that the explicit incorporation of a (relatively) long tailed error distribution in the model (such as EDF(2.3) as above), and the adoption of a procedure for parameter estimation that is consistent with this distribution, may eliminate the need for this practice.

Figure 7.7 displays a further residual plot in which residuals are plotted in box-whisker form against operational time. The boxes correspond to the range between 10- and 90-percentiles, and the markers on the whiskers are placed at the 5- and 95-percentiles.

Once a tentative choice of claim size distribution has been made, it is necessary to examine plots of this type against each independent variate. These examinations seek two things:

- Trendlessness from left to right (horizontality of the box centres)
- Rough equality of dispersion (boxes all of about the same size).

Violation of the first requirement indicates some dependency of the dependent variable on the independent variate, not already accounted for in the model. The second requirement checks for homoscedasticity, i.e. that (6.3) holds for a value of φ that is constant over the entire range of the independent variate under scrutiny.

7.3 Refinement of the model design

7.3.1 Operational time

The model discussed in Section 7.2 still has the very elementary form set out in (7.1) and (7.2). The factor variate X^d, defined in Section 5.1, has 50 levels, which means that β^d contributes 50 parameters to the model. Inspection of Figure 5.1 indicates, however, these 50 parameters can be closely represented as linearly related to operational time over much of the latter's range.

Write (7.2) in the form:

$$E[Y_r] = \exp X_r\beta = \exp [X^d_r\beta^d + X^f_r\beta^f] \qquad (7.3)$$

where X^d_r and X^f_r are the values of the factor variates X^d and X^f assumed by the r-th observation.

Now replace this by the form:

$$E[Y_r] = \exp X_r\beta = \exp\,[\beta^d_1\, t_r + \beta^d_2 \max(0,10\text{-}t_r) + \beta^d_3 \max(0,t_r\text{–}80) + X^f_r\beta^f\,] \tag{7.4}$$

where t_r is the value of operational time applying to the r-th observation, and β^d_1, β^d_2 and β^d_3 are scalar parameters.

This is equivalent to representing the operational time trend in Figure 5.1 as a piecewise linear trend with breaks in gradient at operational times 10 and 80. The factor variate has been replaced by a set of **continuous variates.**

This enables operational time to be accommodated in the model by means of just 3 parameters, rather than 50. The factor variate representation of finalisation quarter is retained for the time being.

If the model (7.4) is fitted to the data, with error term EDF(2.3), as suggested by Section 7.2, the operational time component of (7.4) is as shown by the piecewise linear plot in Figure 7.8. It is superimposed on the factor variate plot in the figure. The correspondence between the two representations is seen to be quite good, indicating that the 3-parameter representation captures essentially all the information of the 50-parameter one.

7.3.2 Superimposed inflation

Similar economies in the representation of finalisation quarter can be made. Figure 7.9 shows the plot of the parameter vector β^f in the case of a factor variate fitted in the presence of the continuous representation of operational time, as in (7.4).

Figure 7.8
Continuous operational time variate

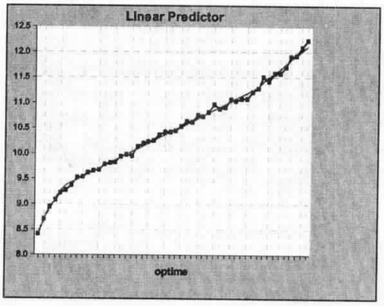
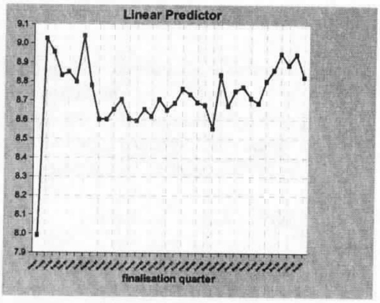


Figure 7.9
Factor variate representation of finalisation quarter



The trend displayed in the left portion, especially the left-most point, may be discounted, since the finalisation quarters here relate to the top left diagonals of the data triangles in Appendix A and contain comparatively little data. As might have been expected, Figure 7.9 is similar to Figure 5.3 over the range of finalisation quarters common to them.

One possibility would be to fit a linear trend from the beginning of 1997. An appropriate choice of model for the earlier finalisation quarters is unclear but, in view of the small quantity of data represented here and its antiquity, the model chosen is unlikely to affect estimation of a loss reserve unduly.

Consequently, Figure 7.10 relates to a model in which the linear trend assumed to apply to finalisation quarters from 1997 onwards is cavalierly assumed to apply to the earlier ones also, though with a step in claim sizes occurring at the start of 1997.

In this case, (7.4) is replaced by:

$$E[Y_r] = \exp[\alpha + \beta^d_1 t_r + \beta^d_2 \max(0,10\text{-}t_r) + \beta^d_3 \max(0,t_r\text{–}80) + \beta^f_1 k_r + \beta^f_2 I(k_r<97Q1)] \quad (7.5)$$

where k_r is the number of the finalisation quarter applying to the r-th observation, α, β^f_1 and β^f_2 are scalar parameters, and generally I(.) is the indicator function defined as follows:

$$I(c) = 1 \text{ if condition c holds;} \quad = 0 \text{ if it does not.} \quad (7.6)$$

The constant α now becomes necessary, having previously been absorbed into β^f.

Figure 7.10
Continuous finalisation quarter variate

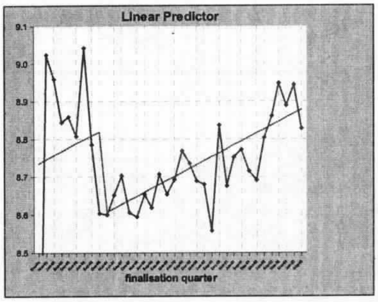
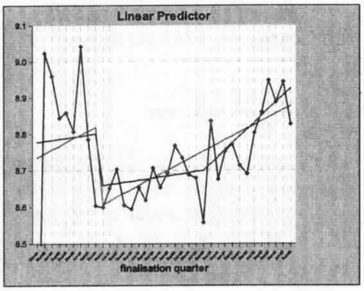


Figure 7.11
Additional break in the finalisation quarter trend



The comparison in Figure 7.10 between the trend of constant gradient over finalisation quarter and the corresponding factor variate hints at an increase in gradient over the more recent finalisation quarters. Figure 7.11 therefore represents an alternative model in which the gradient changes at the end of the September 2000 quarter.

Formally, the model (7.5) is replaced by:

$$E[Y_r] = \exp[\alpha + \beta^d_1 t_r + \beta^d_2 \max(0,10-t_r) + \beta^d_3 \max(0,t_r-80) + \beta^f_1 k_r + \beta^f_2 \max(0,k_r - 2000Q3) + \beta^f_3 I(k_r<97Q1)]. \quad (7.7)$$

One will need to make a choice between models (7.4), (7.5) and (7.7), and possibly others. The choice can be made on the basis of the so-called information criteria, which reward goodness-of-fit but penalise additional parameters. For example, the **Akaike Information Criterion (AIC)** (Akaike, 1969) is defined as:

$$AIC = D + 2p \quad (7.8)$$

where D denotes deviance and p number of parameters. Models with low values of the AIC are to be preferred.

Table 7.1 gives values of the AIC for the three models under consideration, showing that:

- The factor variate model is dramatically inferior to the two involving continuous finalisation quarter variates; and
- Model (7.7), allowing for a change in gradient of the trend is the best of the three.

Table 7.1
AIC for different models of finalisation quarter effect

Model of finalisation quarter effect	AIC
Factor variate (7.4)	-14,517.6
Constant gradient trend (7.5)	-14,566.6
Change in gradient of trend (7.7)	-14,567.1

7.3.3 Interaction terms

The trend over finalisation quarter measures the increase in claim sizes in real terms over calendar time, and may therefore be interpreted as SI. Figure 7.11 indicates that the preferred model estimates the factor of increase as about exp(0.22) over the 3 years from September 2000 to September 2003, or equivalently more than 7% per annum.

While it is quite possible for smaller bodily injury claims to inflate at this rate, it is less usual for the larger and catastrophic claims. A question arises, therefore, as to whether larger and smaller claims might be subject to differing rates of SI.

If operational time is adopted as a proxy for distinguishing between large and small claims, then one might investigate whether different operational times are subject to different rates of SI. This is done by searching for statistically significant interaction effects between operational time and finalisation quarter.

For this purpose, the 0-100 range of operational time is divided into the following 7 bands: 0-6, 6-14, 14-22, 22-40, 40-60, 60-80, 80-100, denoted $b_1,\ldots b_7$ respectively. Let X^{bt} denote the banded operational time factor variate, and let X^{bt}_r be its value for the r-th observation.

The following model is then fitted:

$$E[Y_r] = \exp\,[X^{ct}_r\beta^{ct} + X^{bt\otimes cf}_r\beta^{bt\otimes cf}] \tag{7.9}$$

where X^{ct} represents the set of three continuous operational time variates appearing in (7.7), X^{cf} represents the set of three continuous finalisation quarter variates in the same expression, and $X^{bt\otimes cf}$ denotes the 21-component vector of variates formed as the cartesian product of the 7-component X^{bt} and 3-component X^{cf}. Cartesian products of this type are called **interaction variates** in GLM parlance.

Model (7.9) may be written in the equivalent form:

$$E[Y_r] = \exp\,\{\alpha + \beta^d_1\, t_r + \beta^d_2 \max(0,10\text{-}t_r) + \beta^d_3 \max(0,t_r\text{–}80) +$$

$$\sum_{m=1}^{7} I(t_r \otimes b_m)\,[\beta^f_{m1}\, k_r + \beta^f_{m2} \max(0,k_r - 2000Q3) + \beta^f_{m3}\, I(k_r<97Q1)] \tag{7.10}$$

whose square bracketed member retains the same functional dependency on finalisation quarter as in (7.7), but separately for each operational time band. Note that the coefficients β^f_{m1}, β^f_{m2}, β^f_{m3} represent SI in operational time band b_m.

Figure 7.12 provides a display of the interaction term when (7.9) is fitted to the data. Here "opband7(m)" denotes band b_m. For each of these bands, the model's linear predictor, as defined in Section 6.2, is plotted for t_r=0. Features of the plot are as follows:

- The general level of claim size is seen to increase with increasing operational time band (as in Figure 7.8)
- While Figure 7.11 indicated the period since September 2000 to be subject to an increased rate of SI, it is now seen that this is confined to the operational time bands b_2, b_3, and b_4, which cover operational times 6-40. As hinted at the start of the present sub-section, the increased SI does not apply to the larger claims settled at the high operational times.
- The rate of SI over recent periods, which is measured by the gradients of the paths appearing in Figure 7.12, peaks in operational time bands b_3 and b_4, i.e. in the range 14-40.

The last remark suggests that the interaction terms represented by the summation in (7.10) can be simplified by means of continuous variates. An example of such a simplification is the following:

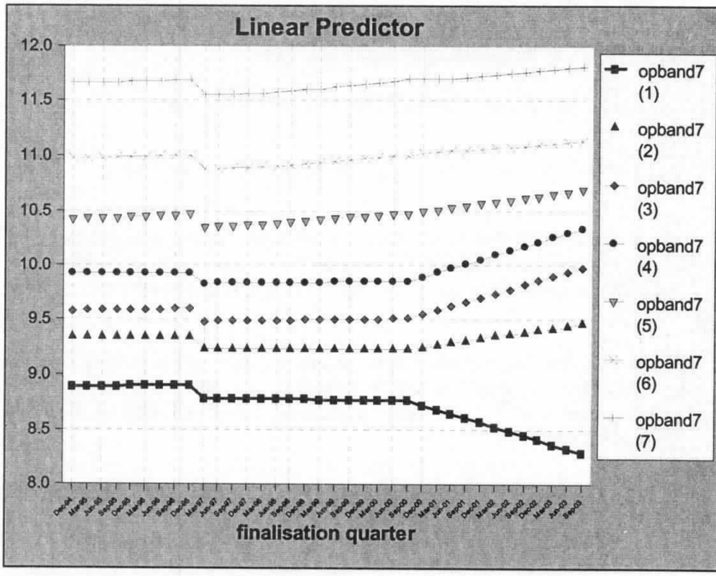
$$
\begin{aligned}
E[Y_r] = \exp \{ & \alpha + \beta^d_1\, t_r + \beta^d_2 \max(0,10\text{-}t_r) + \beta^d_3 \max(0,t_r-80) \\
& + \beta^f_1\, k_r + \beta^f_2 \max(0,k_r - 2000Q3) + \beta^f_3\, I(k_r<97Q1) \\
& + \gamma(t_r)\, [\beta^{tf}_1 + \beta^{tf}_2 \max(0,k_r - 2000Q3)]\}
\end{aligned} \qquad (7.11)
$$

where

$$
\gamma(t) = \min(15,\max(0,t\text{-}10)) - \min(15,\max(0,t\text{-}25)) \qquad (7.12)
$$

i.e. $\gamma(t)$ describes a function that is zero everywhere on the interval [0,100] except on the sub-interval (10,40), where it describes an isosceles triangle of height 15.

Figure 7.12
Interaction between SI and operational time



It can be seen that (7.11) comprises (7.7) plus a further term representing additional SI in the operational time range 10-40, at a rate that increases steadily from 0 at operational time 10 to a peak at operational time 25, and then declines steadily to 0 at operational time 40.

Fitting this model to the data produces the SI profile illustrated in Figure 7.13. Figure 7.14 provides the same type of display of model (7.11) as appears in Figure 7.12, and facilitates the comparison of model (7.11) with model (7.10). Here "opband7(m)" is as in the earlier figure, and "+opband7(m)" denotes the corresponding plot for the continuous model (7,11), i.e. the plot of the average linear predictor against k for t_r=0 and $t_r \otimes b_m$.

Figure 7.13
Profile of SI allowing for SI x operational time interaction

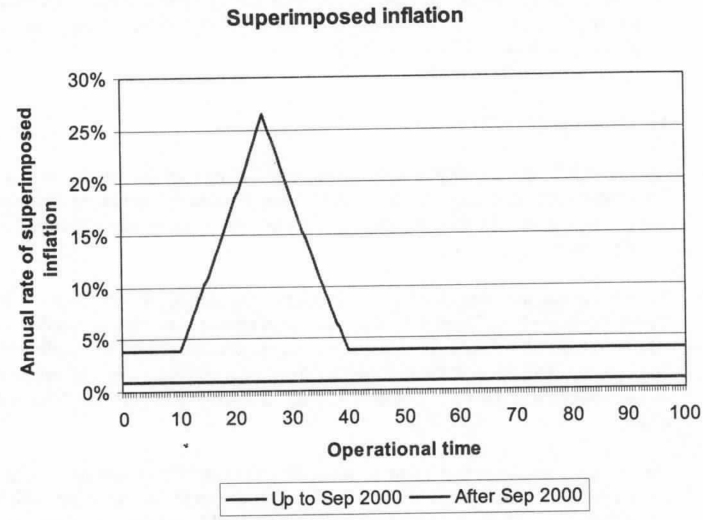
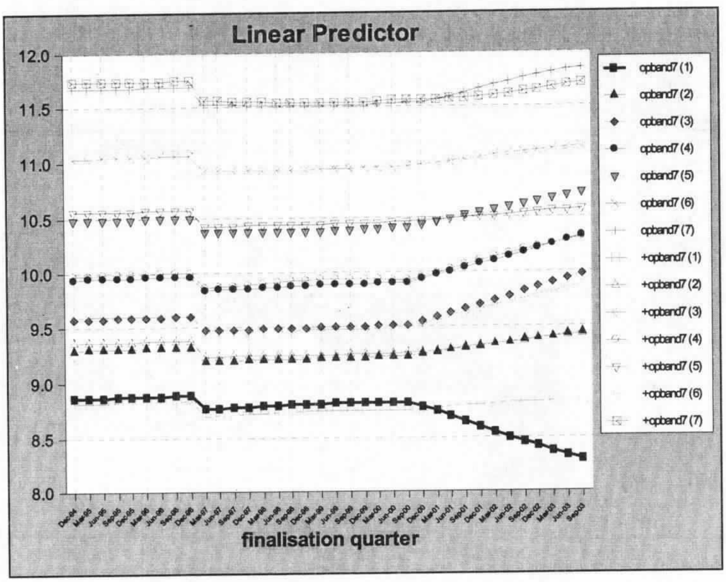


Figure 7.14
Interaction between continuous SI and operational time variates



The simplified model (7.11) is seen to produce a reasonable fit to the more elaborate (7.10). It would not be acceptable as it stands, as there are systematic discrepancies, particularly in relation to opband7(1). However, certain aspects of this model will be superseded in Section 7.3.4, and so detailed improvement of it is not pursued here.

7.3.4 Accident quarter effects

Section 7.3.3 has already noted the change in rate of SI at the end of September 2000, and how the rate changed much more at the low operational times than others. In fact, the legislation governing the scheme changed at precisely this date.

All subsequent accident periods were subject to limitations on payment of plaintiff costs, whose expected effect was to eliminate a certain proportion of smaller claims in the system. Larger claims were expected to be unaffected. The scheme of insurance, as modified by these changed rules, will be referred to as "**the new scheme**". Prior accident quarters make up the "the old scheme".

This strongly suggests that some or all of the SI observed at low operational times after September 2000 might constitute an accident quarter (row) effect rather than finalisation quarter (diagonal) effect. In this connection, it is noted from Figure 3.3 that virtually all of the exceptional operational times (<40) after September 2000 relate to the new scheme.

It is worthwhile returning to the average claim size data in respect of the new scheme. This is done in Table 7.2.

Table 7.2
Average sizes of claim finalisations for old and new schemes

Accident quarter	Average claim sizes (in 30/09/03 values) in development quarter 0	1	2	3	4	5	6	7
	$	$	$	$	$	$	$	$
Dec-99	547	6,035	8,934	11,699	18,397	18,062	26,086	32,139
Mar-00	5,050	5,185	6,958	14,904	13,504	20,746	22,489	27,879
Jun-00	2,910	4,177	7,433	10,275	13,895	18,916	26,206	32,897
Sep-00		6,512	7,116	9,917	14,163	24,034	27,392	41,851
Dec-00	221	2,977	4,175	7,571	10,869	17,505	24,393	29,700
Mar-01	792	2,498	4,605	10,000	11,581	20,672	29,574	39,969
Jun-01	1,271	3,342	5,683	7,936	16,207	21,294	34,237	40,814
Sep-01	1,258	3,516	5,127	12,012	21,726	25,997	26,019	38,150
Dec-01	1,355	2,623	5,225	11,374	19,439	22,548	35,709	28,963
Mar-02	1,594	2,658	7,018	14,700	16,768	26,827	26,851	
Jun-02	1,017	3,641	8,669	12,905	17,750	25,063		
Sep-02	3,484	3,303	5,982	14,379	18,852			
Dec-02	8,102	3,118	6,493	10,714				
Mar-03	1,182	2,454	2,931					
Jun-03	2,327	1,568						
Sep-03	103							

The heavy horizontal line in the table marks the passage from old to new scheme. Claim sizes are seen to decline instantaneously and substantially on introduction of the new scheme.

The shaded area marks one in which the reduction in claim size is maintained. Below this shaded area, however, claim sizes increase rapidly, and by the December 2002 finalisation quarter (the fourth last diagonal) are in excess of their old scheme counterparts.

The immediate reduction in claim sizes by the new scheme is certainly a row effect, and needs to be modelled as such. The subsequent increase in claim sizes can be viewed as either:

- a diagonal effect limited to low operational times (as in Section 7.3.3); or
- a row effect limited to low operational times.

In view of its likely origin in the new scheme, it is perhaps better regarded as the latter. This is the view taken in this paper, and reflected in the final model fitted to the data in Section 7.4. Details of the trend identification are similar to the examples dealt with above, and are not given here.

7.4 Final model

The final model fitted to the data set takes into account the issues discussed in Sections 7.1 and 7.2, and also includes a seasonal effect whereby the sizes of claims finalised in the March quarter tend to be slightly lower than in other quarters. It takes the following form:

$$
\begin{aligned}
E[Y_r] = \exp \{\alpha &+ \beta^d_1 t_r + \beta^d_2 \max(0,10\text{-}t_r) \\
&+ \beta^d_3 \max(0,t_r-80) + \beta^d_4 I(t_r < 8) && \textit{[Operational time effect]} \\
&+ \beta^s I(k_r\text{=March quarter}) && \textit{[Seasonal effect]} \\
&+ \beta^f_1 k_r + \beta^f_2 \max(0,k_r - \text{2000Q3}) \\
&+ \beta^f_3 I(k_r<\text{97Q1}) && \textit{[Finalisation quarter effect]} \\
&+ k_r [\beta^{tf}_1 t_r + \beta^{tf}_2 \max(0,10\text{-}t_r)] && \textit{[Operational time x finalisation quarter interaction]} \\
&+ \max(0,35\text{-}t_r) [\beta^{ta}_1 + \beta^{ta}_2 I(i_r > \text{2000Q3})]\} && \textit{[Operational time x accident quarter interaction]}
\end{aligned}
\qquad (7.13)
$$

where i_r is the accident quarter applying to the r-th observation.

The model form (7.13) is set out in a series of components that isolate the different types of effects, labelled in italics on the right.

Comparison of it with (7.11) shows that:

- It retains the concept of an operational time x finalisation quarter interaction, though this now:
  - o has its peak rate of SI shifted from operational time 25 to 10; and
  - o this profile of SI applies to all finalisation periods, not just those that fall within the new scheme.
- There is heightened SI in the new scheme, but affecting all operational times, not just the low range.
- A part of what previously appeared as heightened SI in the new scheme is now accounted for as an accident period effect, with a one-off shift in claim size at introduction of the new scheme, the size of the shift being largest at the low operational times and gradually decreasing with increasing operational time, until petering out at operational time 35.

Table 7.3 compares the AIC for model (7.7) with the final model, showing a considerable improvement achieved by the latter.

Table 7.3
AIC for final model and model (7.7)

Model of finalisation quarter effect	AIC
Model (7.7)	-14,567.1
Final model (7.13)	-14,588.9

7.5 Validation of final model

While (7.13) may appear the best model achievable, it needs to satisfy a number of routine tests before its final acceptance. These are concerned with the properties of residuals, and are illustrated in Figures 7.15 to 7.20.

Figure 7.15

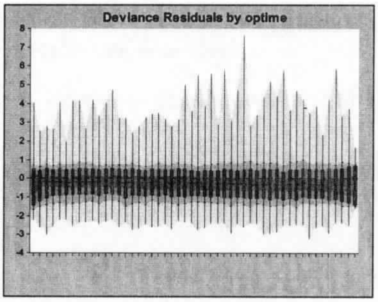
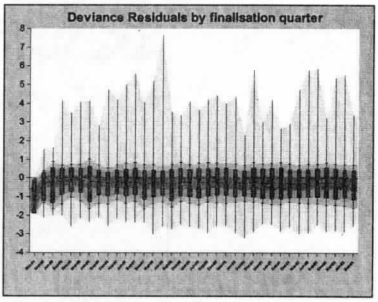


Figure 7.16



Figures 7.15 to 7.17 test for two things:

- Trendlessness, from left to right, with respect to the major variates, checking that no systematic trend in the data remains uncaptured by the model; and
- **Homoscedasticity**, i.e. constant dispersion from left to right.

Both of these tests are concerned just with trends rather than with the magnitude of the residuals. Hence standardisation is unnecessary (though it would do no harm), and just deviance residuals are displayed.

The possible trend at the extreme right of Figure 7.17 is, of course, based on very little data, as it relates to just the last three accident quarters. It has been ignored for the purposes of the present paper.

Figure 7.17

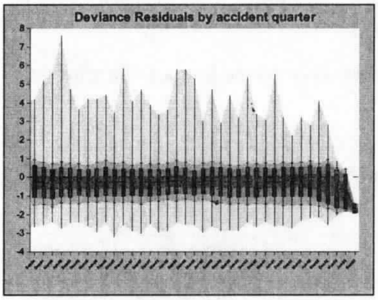


Figure 7.18

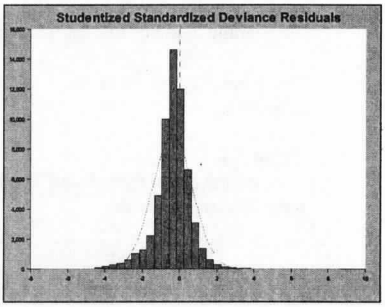


Figure 7.19

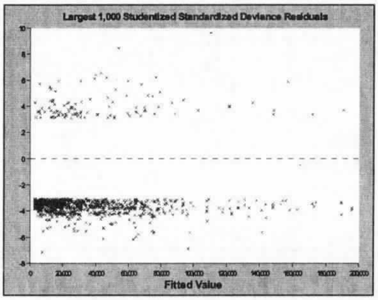
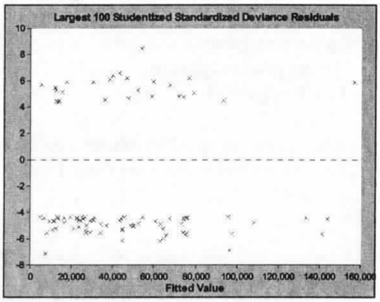


Figure 7.20



Figures 7.18 to 7.20 are concerned with the distribution of the residuals, with the same considerations as discussed in relation to Figures 7.4 to 7.6. Indeed, there is little difference to the naked eye between the two sets of graphs, showing that, once the EDF(2.3) error structure has been chosen, the rather extreme change in model from (7.2) to (7.13) has had little effect on the distribution of residuals.

7.6 Forecast of final model

Table 7.4 repeats Table 3.3, but supplemented by the loss reserve forecast by model (7.13). The following assumptions are made for the purpose of this forecast:

- The experience of finalised claims of an accident period is indicative of its ultimate average claim size.
- Future SI is as experienced to date in the new scheme.
- Future rates of claim finalisation are about the same as experienced over the most recent 8 quarters.

The first of these assumptions is fundamental to the forecasting methodology. It might be violated if, for example, at specific operational times, one observed a trend over time in the ratio of average amount paid to date on open claims to the average paid on finalised claims.

The second assumption has a major influence on the forecast, the third little influence.

Table 7.4
Loss reserves corrected and uncorrected for low September 2003 quarter paid loss experience

Averaging period	Loss reserve at 30 September 2003 (excluding September 2003 accident quarter)	
	Uncorrected	Corrected
	$B	$B
Chain ladder models:		
All experience quarters	1.61	
Last 8 experience quarters	1.68	
All experience quarters except September 2003	1.78	1.94
Last 8 experience quarters except September 2003	1.92	2.35
GLM (7.13)	2.23	

The GLM (7.13) generates a loss reserve near the top of the range of CL results. While there is reasonable agreement with the CL version derived from the experience of the last 8 quarters but one and corrected for the anomalous experience of the last quarter, this is a very detailed choice, and one has no means of determining this model to be superior to many other contenders.

For example, why 8 quarters? Why not 6? Or 10? Why correct for just the last quarter of experience? Why not the last 2? In any event, Table 7.5 shows that, while this version of the CL may produce a total reserve similar to that of the GLM, its composition by accident quarter is very different.

The former produces a reserve for the last accident year that is 19% higher than the GLM. This would lead to much higher estimates of average claim size, and hence to quite different pricing decisions for future underwriting periods.

Table 7.5
GLM and CL loss reserves by accident quarter

Accident quarter	Loss reserve at 30 September 2003 (excluding September 2003 accident quarter)	
	GLM (7.13)	CL based on last 8 experience quarters except the last - corrected
	$M	
Sep 94 – Dec 98	283	200
Mar 99 – Mar 02	1,122	1,174
Jun 02	154	183
Sep 02	159	199
Dec 02	160	201
Mar 03	173	206
Jun 03	179	192
Total	2,229	2,354

The validation devices represented in Figures 7.15 to 7.17 have the common feature that they are all 1-dimensional summaries of residuals. While the residuals may be trendless over the single dimension, finalisation quarter, and may also be trendless over the single dimension, accident quarter, it is possible that there are pockets of cells within the 2-dimensional triangle in which they tend to be systematically of the one sign.

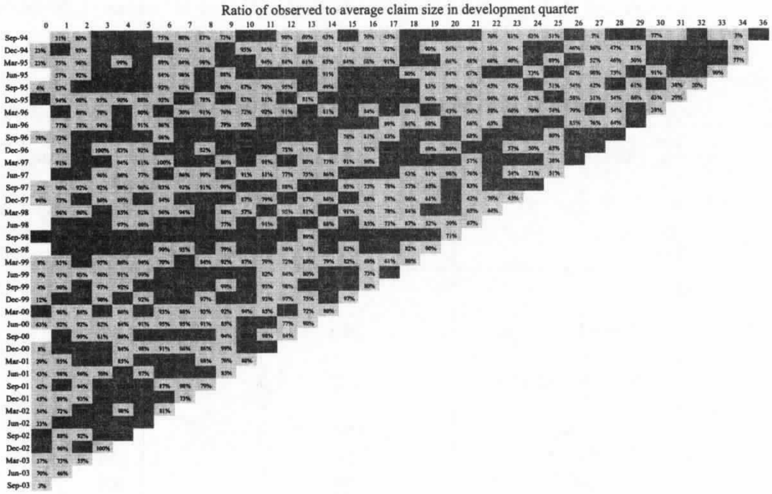
Figure 7.21 provides a simple test of such an eventuality. For each cell of the accident quarter/development quarter triangle, it records the ratio:

Observed average size of claim finalisation / GLM fitted average size.

These ratios are colour coded: red if greater than 100%, blue if less. The fact that the numerical values of the ratios are too small to be legible in the figure as reproduced here does not detract from its value A cursory examination of its colour patterns indicates a generally random scatter of red and blue.

There is no apparent congregation of cells of one or other colour in particular locations within the triangle. This confirms the trendlessness of the residuals over the whole of the 2-dimensional array.

Figure 7.21
Colour coded ratios of observed to fitted average claim sizes



8. Conclusions

The foregoing sections have dealt with a case study involving a loss triangle of obvious complexity. It contains multiple trends.

The triangle has been approached initially from the viewpoint of one with a predisposition to application of the CL. The trends then manifest themselves in the form of non-constancy of age-to-age factors over accident periods.

The complexity of the data set is reflected in the model of claim sizes fitted to it, which includes the following, in addition to the expected variation with operational time:

- a **seasonal effect**;
- **SI** whose rate varies with operational time, and also passes though one change-point;
- recognition of a **new scheme affecting accident periods** after its introduction, but with an effect that varies with operational time.

It is extremely difficult to accommodate such trends within the CL structure and estimate them efficiently. However, the GLM (7.13) adopted here does so **parsimoniously**, using just 13 parameters. This compares with the 73 parameters implicit in a CL applied to a triangle of dimension 37 even before the recognition of any trends.

The GLM is one example of a model with a **fully stochastic specification**, as opposed to the CL which is usually approached in practice as an algorithm (though the stochastic formulations mentioned in Section 4.1 may be noted). The stochastic framework provides a set of diagnostics that may be used to **compare candidate models** in a formal and organised manner, and to **validate the model** finally selected.

The stochastic framework also allows a choice of the **distributional form** from which observations are assumed drawn. This enables an informed treatment of **outliers**.

These properties of the GLM are seen to be more than academic as this model generates a loss reserve that differs vastly from some CL applications. While one CL model is found to produce a somewhat similar reserve (Section 7.6), there is no apparent reliable basis for distinguishing that model as superior to other CL models.

In any event, though the CL model in question appears to produce a total loss reserve that is approximately correct, its dissection by accident period appears quite wrong. Specifically, it over-estimates average claim sizes of recent accident periods by margins approaching 20%. Such estimates, if incorporated in the business process, would be liable to lead to quite **incorrect pricing decisions** for the ensuing underwriting periods.

Finally, but not of least significance, one emerges from the GLM fitting process described in Section 7 with a greatly **enhanced understanding** of one's data. Data exploration forms an integral part of the process, and the GLM provides the framework within which such exploration can be carried out efficiently.

The CL on the other hand provides a sausage machine, a rigid and unenquiring algorithm. This is an advantage in terms of required resources. Only relatively low-skilled resources are required to apply it in its unmodified form. A serious disadvantage to be set against this is that it may produce a totally wrong result, that it may give **precedence to process over substance**.

The CL model may be described as a multiplicative model with categorical accident and development period effects. This is a very simple design, which is highly convenient if justified. It is, however, a design that relies on an assumption of an identical process affecting every accident period.

Beyond this, it is phenomenological in the sense that there is no specification of what that process is. If evidence appears that the CL design is invalid, the lack of process specification leaves one with no indication of how the design should be modified.

One may attempt modification on some empirical basis, such as trending age-to-age factors, but the empiricism itself is a recognition of the lack of understanding of the process. Indeed, because of this, there is in our view a strong case for abandonment of the CL immediately its simple design is found to be violated. One is likely to be better served in this case by an attempt to build understanding of the process and then select the model design accordingly.

These arguments are presented not in the spirit of an anti-CL diatribe, but rather in recognition of the fact that, when the CL (or indeed any other highly standardised model design) turns out to be a poor device in practice, alternatives are required and use of a GLM may well be an effective alternative.

Appendix A
Paid loss data

A.1 Incremental paid losses

accident	development quarter ($000)									
quarter	0	1	2	3	4	5	6	7	8	9
Sep-94	1	61	273	934	1,320	1.017	492	393	1,111	2.096
Dec-94	40	416	1,362	2.348	3 671	2,823	2,207	3.031	5,083	4.987
Mar-95	30	581	1.352	2.452	1,678	1,704	2 603	4.747	3,078	3,868
Jun-95	24	493	1,641	1,504	1,972	3.581	3,318	3.248	4,805	5,714
Sep-95	28	689	876	1,973	2,639	3,823	2.588	4.270	5.290	7.363
Dec-95	59	239	751	1,698	2,526	2.209	3,319	4 812	4.316	4 181
Mar-96	30	268	1.300	2,016	2,732	3,036	3.317	4,058	3 614	3.978
Jun-96	27	488	1,444	1,715	2,492	3.405	3 534	3 471	4,759	8,035
Sep-96	19	459	1,188	2.383	3 485	3.097	3 346	5,426	6,796	6,364
Dec-96	7	315	1.439	2,278	3 213	2 900	5,411	4.532	4,548	5,868
Mar-97	56	381	1.216	2,615	2,290	3.195	5.206	6.497	4.561	7,066
Jun-97	7	486	1,813	2,054	2,970	3.433	5.971	4.222	6.311	4 334
Sep-97	45	557	1.270	2,763	2,714	4.640	3,783	5,336	6.592	10 646
Dec-97	45	447	1,734	2,767	4,107	3,660	5,290	8,830	7,564	6 157
Mar-98	17	385	1,593	3,050	3 344	4.132	5,526	5.433	4,802	5,677
Jun-98	29	746	1,830	3,100	3 599	5,265	7 271	4,743	6,868	4,533
Sep-98	100	678	1,582	3 172	4,391	5,865	5 132	8,321	9,431	7,880
Dec-98	54	533	1,599	4 207	6 823	8.897	10.541	7,628	5,492	5.131
Mar-99	28	721	2.393	4,798	5,052	7,237	6 378	5.879	4,394	6,118
Jun-99	92	725	2,517	3,238	5 455	5,472	7,317	4,549	8.027	6.979
Sep-99	65	649	1.419	3,913	3 531	6.699	5,169	7.277	7,891	16,651
Dec-99	55	740	2,094	2,694	5.952	3,925	6,103	6.790	11,315	7,334
Mar-00	75	666	1,364	3,879	2,758	5,350	6.112	7,328	6,486	7,222
Jun-00	60	571	1.527	2.133	4,521	5,852	8 414	6.501	9,512	6.807
Sep-00	76	810	1,156	2.825	3,602	8,354	7,015	10,612	9,707	9 489
Dec-00	40	476	762	1.576	3 394	3,905	5 806	6.412	8,394	8,060
Mar-01	42	382	950	2.411	3 240	5,281	6 840	10,038	7,674	8,413
Jun-01	71	629	1.203	1,857	4,116	5,433	9,705	7,721	10,723	6,983
Sep-01	63	999	1,180	3,101	4,923	7,240	7,068	8.900	6,862	
Dec-01	59	635	1,209	2,517	5,749	5,112	10.178	7,201		
Mar-02	54	687	1,164	3.445	2.814	7,077	5,729			
Jun-02	134	762	1,513	2,062	4,099	5,285				
Sep-02	67	719	1,316	2,630	3,243					
Dec-02	94	475	978	1 650						
Mar-03	71	473	689							
Jun-03	56	450								
Sep-03	45									

accident	development quarter ($000)									
quarter	10	11	12	13	14	15	16	17	18	19
Sep-94	1 101	1,413	1,839	1,170	1,493	805	2,153	932	1,865	730
Dec-94	4,569	6,094	8,931	4,781	6,972	3,183	6,695	5,344	3,563	1,667
Mar-95	6,165	6,640	2,973	4,302	5,603	5,982	5,248	4,287	3,473	5,550
Jun-95	12,655	5 078	5,780	6,620	7,086	8,035	5,216	3,932	5,322	2,935
Sep-95	4,589	4,753	6,304	6,085	6,043	5,016	10,251	5,847	4,274	2,830
Dec-95	7,169	6,308	8,881	4,183	4,446	5,274	4,247	3,703	4,917	2,656
Mar-96	4,491	5,847	5 015	6,081	5,736	4,635	4,857	4,756	3,793	3,224
Jun-96	5,366	5,246	6,932	7,495	5,589	4,762	9,615	3,532	3,362	2,067
Sep-96	6,984	6,170	5,031	9,244	5,783	4 996	4,842	3,730	2,297	4,424
Dec-96	5,934	6,767	8,576	4,098	7,389	2 687	3,886	1,880	4,534	7,378
Mar-97	5,654	6,678	5,797	4,207	4,167	5,396	3 236	5,807	12,137	3,909
Jun-97	5,225	3,730	7 353	3,374	5,833	2,744	3,950	3,817	2,499	2,694
Sep-97	3,815	10,341	4,479	5,755	3,072	5,046	3,969	2,822	2,666	3,847
Dec-97	6,880	4,670	4,775	4,734	3,146	4,016	5,570	2,002	2,779	2,021
Mar-98	4,215	6 045	3,188	6,368	3,316	3,345	4,198	3,334	2,685	4,675
Jun-98	5,476	5,212	7,386	4,765	7,866	4,308	6,153	3 455	5,819	1,793
Sep-98	4,992	6,735	7,242	7,403	9,829	8,446	7,969	6,711	7,192	2,693
Dec-98	6,237	6,806	10,558	5,085	6,570	4,882	5,377	2,669	4,702	3,006
Mar-99	8,260	6,386	5,277	7,161	4,647	3,459	4,264	4,344	2,455	
Jun-99	8,429	4,465	6,050	7,378	12,514	5,076	5,091	4 303		
Sep-99	8,427	6,730	7,886	9,256	5,401	7,277	5,676			
Dec-99	7,274	7,858	9 303	5,688	5 800	6,527				
Mar-00	7,803	11,137	11,257	5 040	5,261					
Jun-00	9,162	8,265	7,600	5,807						
Sep-00	10,347	8 534	8,310							
Dec-00	8,487	9,557								
Mar-01	6,164									
Jun-01										

accident	development quarter ($000)									
quarter	20	21	22	23	24	25	26	27	28	29
Sep-94	1,708	1,866	314	777	176	281	1,566	124	505	253
Dec-94	2,587	3,694	2,678	3,154	1,827	430	222	1,296	749	542
Mar-95	1,915	1,441	366	1,878	364	1,244	304	594	638	1,745
Jun-95	4,419	2,653	3,034	799	332	597	1,635	611	2,043	3,811
Sep-95	1,780	2,542	1,305	829	1,587	1,317	758	1,366	583	1,473
Dec-95	2,843	764	761	297	1,361	2,814	512	745	1,276	149
Mar-96	896	1,278	1,652	2,242	4,731	682	1,331	1,229	821	1 114
Jun-96	1,882	1,755	7,216	2,366	3,323	861	1,768	712	144	98
Sep-96	3,733	2,530	7,858	2 628	1,218	1,103	3,441	783	694	
Dec-96	972	1 594	2,057	1 644	1,051	1,149	1,858	105		
Mar-97	1,488	4,174	1 330	3 695	410	976	641			
Jun-97	2,406	2,387	2,706	1,725	2,431	785				
Sep-97	2,585	5 581	1,455	1 868	1 740					
Dec-97	3 221	5 013	887	1,711						
Mar-98	2,529	2 058	1,413							
Jun-98	2,426	3,088								
Sep-98	5,601									
Dec-98										

accident	development quarter ($000)						
quarter	30	31	32	33	34	35	36
Sep-94	522	1	-63	108	1	2	92
Dec-94	1,147	145	2,272	400	74	557	
Mar-95	1,892	2,062	88	191	676		
Jun-95	444	3,270	190	20			
Sep-95	1 082	2,675	41				
Dec-95	190	947					
Mar-96	541						
Jun-96							

A.2 Incremental paid losses in respect of finalised claims

accident quarter	development quarter of finalisation ($000)									
	0	1	2	3	4	5	6	7	8	9
Sep-94	0 0	14	145	524	1,254	771	429	351	707	1,852
Dec-94	3 5	277	552	1,474	3.334	2.404	1,125	2,683	4,341	4,203
Mar-95	3 3	211	850	1.834	1,320	1,101	2,158	3.360	2,341	4,804
Jun-95	0 0	197	906	1 032	1,122	2,302	3,466	2,519	4.032	3 352
Sep-95	0 9	293	423	862	2,141	3.461	2,323	2,710	4,087	3,792
Dec-95	54 4	120	212	1,081	2,000	2,055	2,594	3,368	2,878	6,206
Mar-96	0 0	105	794	1,466	2,345	2,280	2,987	2.049	4,942	3,889
Jun-96	0 0	178	869	1.209	1,760	2 353	1,953	4,481	4,497	3,498
Sep-96	5 3	145	743	1,741	1,963	2 497	3,941	4,155	5,150	5,827
Dec-96	0.0	127	910	1,367	1 559	3,490	4,873	3,801	4,398	4.188
Mar-97	0 0	96	447	1,216	2,738	2,725	2,883	6,002	4,586	4,830
Jun-97	0 0	133	762	2,239	2,617	2,446	4,554	4,041	6,119	5,324
Sep-97	0 4	77	895	1,881	2,285	3,567	3,319	4,841	6,014	7,102
Dec-97	10.0	172	1,063	1,785	3,062	3,647	4,147	7,040	8 524	6,175
Mar-98	0 0	134	820	2,298	2,288	4,212	4,079	5,667	5,645	6,282
Jun-98	0 0	201	1,010	1,987	3,540	3,935	7,108	5,173	6,683	3,595
Sep-98	5 8	157	838	2,314	3 376	5,839	4,765	7,974	5,220	5,438
Dec-98	0.0	104	859	3,027	6,470	6,290	8,646	6,389	8,235	3,714
Mar-99	0 4	215	1,327	3,884	4,278	7,361	4,166	6,488	3,916	3,600
Jun-99	0 2	192	1,798	2,708	4,636	5,046	5.928	3,868	5,073	5,491
Sep-99	0 2	231	861	3,100	3,046	4,407	3,779	4,531	7,213	12.158
Dec-99	1 6	368	1,581	2,234	4,581	2,727	4,513	5,496	10,136	8,289
Mar-00	15 1	311	724	2,966	1,877	3,610	4,475	7,277	5,305	8,413
Jun-00	5 8	192	959	1,500	2,626	4,407	8,700	5,428	9,670	6,131
Sep-00	0.0	339	612	1,438	2,294	7,234	5,698	10,923	7,560	7,947
Dec-00	0 4	71	259	977	2,511	3 448	5,806	5,079	6,537	6,609
Mar-01	0 8	62	387	1,750	2,478	5,230	6,033	9,273	7,673	7,299
Jun-01	3 8	217	574	1,317	3,501	4,791	8.593	7,265	9,867	6,866
Sep-01	6 3	176	502	2,258	4,280	6.135	5.126	8,279	5,131	
Dec-01	1 4	121	502	1,524	4,918	4,307	9,820	5,098		
Mar-02	11 2	141	632	2,558	2,280	6,599	4,457			
Jun-02	6 1	189	763	1,265	3,337	3,860				
Sep-02	7.0	175	526	2 171	2.375					
Dec-02	32.4	128	383	1,061						
Mar-03	7.1	96	111							
Jun-03	9 3	39								
Sep-03	0 4									

Note: Paid losses in finalisation quarter x include all amounts paid in quarters up to and including x for claims finalised in x.

accident quarter	development quarter of finalisation ($000)									
	10	11	12	13	14	15	16	17	18	19
Sep-94	1,169	1,192	1,381	1,065	1,149	1,437	1,250	481	926	2,628
Dec-94	3,439	3 270	5,306	10,651	7,187	4,929	4,616	4,471	8,490	3,100
Mar-95	5,531	5,555	5,758	3,769	3,443	5,781	3,887	6,597	5,242	5,367
Jun-95	3,898	6,602	11 973	6,055	4,933	6,079	7,011	7,515	4 888	5,306
Sep-95	5,332	4,648	5,253	8 834	2,824	6,063	8,382	7,525	6,979	2,821
Dec-95	4,295	4,173	7 276	4,211	7,421	5,877	7,486	3,928	5,070	3 583
Mar-96	3,039	4,596	5,485	6,140	3,394	7,740	3 876	8,296	2,885	4,328
Jun-96	4,438	6 842	7,675	5,985	5,869	7,775	6,455	3,315	3,505	897
Sep-96	4,038	7,355	6,985	9,914	7,170	4,608	3,632	3,378	3,166	1,857
Dec-96	6,361	5 805	6,119	4,438	8 435	3,231	2,410	2,775	3,280	3,050
Mar-97	7,444	5,571	6,903	4,754	2,866	3,287	2,015	3,962	5,238	5,091
Jun-97	4,742	4,314	7,397	3,176	3,282	4,055	3,707	2,844	3 510	2,622
Sep-97	6,485	10,205	4 452	6,501	3,640	2,103	2,039	4,868	2,341	5,025
Dec-97	6,292	3,413	7,127	2,846	2,826	4,147	4,940	5,124	4,838	1,528
Mar-98	2,823	4 810	3,227	2,481	5 889	5,258	2,633	3 344	2,715	4,699
Jun-98	5,203	3,783	4 084	5,255	7,258	7,690	6,548	4,481	5,312	2,087
Sep-98	5,117	3,893	7,186	7,966	5,599	11,969	7,303	7,723	7,486	10,009
Dec-98	4,587	5,634	9 425	5,373	8,626	4,608	6,539	4,038	4,868	6,188
Mar-99	4,916	9,749	5,366	8 804	5,391	3,899	3,736	4,402	2,483	
Jun-99	11,923	4,247	7,727	4 678	9,901	6,165	4,279	7,077		
Sep-99	9,317	8,123	8,999	7,495	6,069	8,988	4,428			
Dec-99	9,088	7,461	8,498	4,853	7,232	6,023				
Mar-00	6,589	6,830	11,414	5,911	4,560					
Jun-00	8,683	7,855	7,845	5,033						
Sep-00	10,856	9,088	7,014							
Dec-00	8,466	8,389								
Mar-01	6,256									
Jun-01										

accident quarter	development quarter of finalisation ($000)									
	20	21	22	23	24	25	26	27	28	29
Sep-94	1,164	1,956	1,219	970	672	252	712	9	1,265	1,414
Dec-94	2,619	4,602	2,271	2,610	1,556	2,682	253	625	610	1,318
Mar-95	1 957	1,618	1,326	658	1,033	741	4,524	675	421	472
Jun-95	3,270	1,673	6,170	2,822	850	1,295	1,362	2,958	1,286	1,870
Sep-95	1,469	3,770	426	2,141	1,934	1,547	1,183	631	7,640	816
Dec-95	2,073	2,000	1,702	201	2,263	3,465	1 538	358	311	738
Mar-96	510	1,095	1,137	2,827	1,604	1,265	722	2,736	1,011	4 683
Jun-96	6,680	1,443	3,234	7,912	3,951	1,476	2,503	1,831	500	809
Sep-96	4,437	2,836	3,828	4,531	2,256	1,533	1,817	4,079	1,814	
Dec-96	2,642	6,086	6,398	1,682	1,136	1,169	3,231	2,130		
Mar-97	5,736	2,574	13 854	2,865	2,180	466	2,401			
Jun-97	4,744	1,863	3,693	814	1,772	697				
Sep-97	3,184	2,226	6 450	3 056	1,862					
Dec-97	3,744	1,581	2,566	929						
Mar-98	3,518	2,732	1,189							
Jun-98	1,592	2,262								
Sep-98	3,478									
Dec-98										

accident quarter	development quarter of finalisation ($000)						
	30	31	32	33	34	35	36
Sep-94	140	0	1 009	0	6	0	634
Dec-94	1,147	1 935	1,076	1,827	1,165	0	
Mar-95	2,932	1 329	298	1,787	1,156		
Jun-95	1,398	1,603	914	963			
Sep-95	1,143	327	84				
Dec-95	862	397					
Mar-96	147						
Jun-96							

A.3 Numbers of claim finalisations

accident quarter	development quarter of finalisation									
	0	1	2	3	4	5	6	7	8	9
Sep-94	0	6	26	36	53	37	32	22	35	73
Dec-94	2	37	69	151	200	130	52	131	192	115
Mar-95	2	39	101	163	102	67	141	173	99	125
Jun-95	0	47	110	95	53	147	226	130	150	126
Sep-95	2	51	51	67	189	216	155	171	126	139
Dec-95	6	21	32	127	185	184	173	135	135	176
Mar-96	0	16	113	173	174	185	139	122	184	133
Jun-96	1	37	126	143	148	177	128	191	147	126
Sep-96	1	33	103	167	150	171	222	148	149	136
Dec-96	0	32	115	141	159	246	193	154	157	105
Mar-97	2	22	68	143	246	205	149	187	123	139
Jun-97	0	21	99	240	215	180	176	158	166	116
Sep-97	5	19	140	191	175	217	170	190	181	161
Dec-97	2	46	125	197	242	188	205	178	181	126
Mar-98	0	33	122	198	196	239	171	187	143	146
Jun-98	0	40	130	188	256	220	264	163	166	110
Sep-98	1	27	113	228	227	270	208	257	138	119
Dec-98	0	20	129	272	381	302	306	190	147	98
Mar-99	1	54	160	335	304	338	196	164	109	79
Jun-99	2	44	225	226	307	236	193	108	116	103
Sep-99	2	55	116	273	214	201	148	152	162	279
Dec-99	3	65	180	193	253	155	173	173	282	170
Mar-00	3	69	107	204	140	179	202	268	155	192
Jun-00	3	49	138	150	192	238	333	170	242	134
Sep-00	0	55	89	146	167	307	215	264	168	164
Dec-00	3	29	68	135	240	203	255	182	185	138
Mar-01	2	28	91	184	219	260	208	237	186	184
Jun-01	3	71	102	173	225	232	260	181	198	157
Sep-01	7	53	103	195	202	242	205	221	145	
Dec-01	2	49	101	145	259	204	278	182		
Mar-02	7	58	96	180	148	252	167			
Jun-02	6	55	96	110	192	162				
Sep-02	5	57	94	154	130					
Dec-02	4	44	63	106						
Mar-03	7	40	42							
Jun-03	4	28								
Sep-03	7									

accident	development quarter of finalisation									
quarter	10	11	12	13	14	15	16	17	18	19
Sep-94	30	26	32	26	32	18	19	13	9	10
Dec-94	104	100	142	136	104	68	49	48	61	27
Mar-95	96	135	137	100	75	84	60	63	39	38
Jun-95	95	134	118	77	77	81	72	64	53	48
Sep-95	157	126	98	107	77	68	56	61	39	26
Dec-95	126	104	111	79	79	62	57	37	41	31
Mar-96	101	96	109	83	56	81	49	54	34	20
Jun-96	111	101	126	89	75	64	61	37	35	11
Sep-96	95	121	109	117	88	66	44	36	26	13
Dec-96	136	99	124	69	81	56	24	24	15	33
Mar-97	112	108	98	82	49	40	23	28	31	39
Jun-97	98	95	141	54	42	37	27	20	42	24
Sep-97	117	122	77	57	41	28	27	55	36	42
Dec-97	129	71	80	45	39	41	67	55	36	20
Mar-98	91	76	53	43	46	61	42	35	25	27
Jun-98	111	79	64	69	114	72	80	55	49	30
Sep-98	93	72	101	144	75	89	80	61	53	46
Dec-98	78	89	165	74	74	53	50	33	44	46
Mar-99	106	197	105	116	67	42	45	54	21	
Jun-99	225	89	135	75	78	64	52	50		
Sep-99	138	136	130	90	64	65	47			
Dec-99	162	130	122	87	81	64				
Mar-00	123	124	113	106	61					
Jun-00	132	112	131	65						
Sep-00	116	141	116							
Dec-00	144	114								
Mar-01	127									
Jun-01										

accident	development quarter of finalisation									
quarter	20	21	22	23	24	25	26	27	28	29
Sep-94	9	10	12	8	7	3	1	1	4	5
Dec-94	33	34	25	18	9	8	3	6	7	9
Mar-95	20	23	12	10	6	5	10	8	5	6
Jun-95	27	17	15	8	7	3	13	17	10	5
Sep-95	20	26	6	14	10	17	12	9	6	7
Dec-95	22	22	12	2	22	15	15	6	3	7
Mar-96	9	13	15	29	14	10	5	12	11	6
Jun-96	18	16	33	26	21	7	16	13	4	4
Sep-96	16	29	25	21	10	11	7	12	5	
Dec-96	24	29	15	18	13	10	10	6		
Mar-97	34	30	20	15	12	9	8			
Jun-97	31	15	18	14	15	8				
Sep-97	20	17	10	17	9					
Dec-97	18	24	19	15						
Mar-98	22	28	16							
Jun-98	29	21								
Sep-98	37									
Dec-98										

accident	development quarter of finalisation						
quarter	30	31	32	33	34	35	36
Sep-94	1	0	5	0	1	0	1
Dec-94	6	7	2	3	7	1	
Mar-95	8	5	1	3	7		
Jun-95	8	5	4	5			
Sep-95	4	4	2				
Dec-95	7	7					
Mar-96	2						
Jun-96							

A.4 Incremental average sizes of finalised claims

accident	development quarter of finalisation									
quarter	0	1	2	3	4	5	6	7	8	9
Sep-94		2,382	5,594	14,548	23,662	20,845	13,393	15 952	20,187	25,363
Dec-94	1,735	7,483	8,005	9,761	16,670	18,494	21,625	20,482	22,610	36,551
Mar-95	1,636	5,401	8,415	11,250	12 939	16,427	15,306	19,423	23,650	38,433
Jun-95		4,201	8,235	10,865	21,174	15,658	15,338	19,380	26,883	26,601
Sep-95	433	5,741	8,290	12,863	11,326	16,024	14,984	15,849	32,440	27,282
Dec-95	9,060	5,734	6,634	8,514	10,810	11,168	14,994	24,945	21,316	35,263
Mar-96		6,532	7,028	8,476	13,478	12,324	21,493	16,797	26,858	29,239
Jun-96		4,820	6,896	8,456	11,891	13,291	15,259	23,460	30,592	27 762
Sep-96	5,307	4,384	7,214	10,427	13,090	14,603	17,752	28,077	34,566	42,847
Dec-96		3,967	7,915	9,696	9,805	14,188	25,250	24,684	28,015	39,882
Mar-97		4,351	6,578	8,504	11,132	13,294	19,350	32,097	37,282	34,748
Jun-97		6,340	7,701	9,328	12,174	13,587	25,875	25,577	36,860	45,901
Sep-97	73	4,063	6,393	9,849	13,056	16,439	19,525	25,478	33,226	44,113
Dec-97	5,013	3,749	8,501	9,059	12,652	19,397	20,228	39,553	47,096	49,009
Mar-98		4,069	6,720	11,608	11,671	17,624	23,852	30,306	39,476	43,025
Jun-98		5,032	7,769	10,571	13,827	17,887	26,926	31,734	40,262	32,682
Sep-98	5,828	5,832	7,420	10,149	14,871	21,627	23,007	31,026	37,829	45 701
Dec-98		5,181	6,660	11,127	16,982	20,827	28,255	33,628	56,021	37,895
Mar-99	401	3,986	8,292	11,595	14,073	21,779	21,256	39,558	35,930	45,572
Jun-99	111	4,363	7,990	11,984	15 102	21,380	30,718	35,818	43,731	53,315
Sep-99	97	4,207	7,420	11,354	14,234	21,926	25,532	29,806	44,528	43,578
Dec-99	547	5,663	8,785	11,578	18,106	17,596	26,086	31,767	35,942	48,759
Mar-00	5,050	4,509	6,763	14,539	13,408	20,166	22,155	27,151	34,228	43 820
Jun-00	1 940	3,922	6,948	10,001	13,678	18,518	26,127	31,930	39,958	45,756
Sep-00		6,157	6,876	9,850	13,739	23,564	26,500	41,375	45,000	48,456
Dec-00	147	2,464	3,807	7,235	10,462	16,988	22,767	27,905	35,336	47,889
Mar-01	396	2,231	4,251	9,510	11,317	20,115	29,005	39,125	41,250	39,670
Jun-01	1,271	3,060	5,628	7,615	15,559	20,652	33,051	40,138	49,832	43,731
Sep-01	898	3,317	4,878	11,581	21,188	25,352	25,003	37,460	35,387	
Dec-01	678	2,463	4,966	10,511	18,989	21,111	35,324	28,008		
Mar-02	1,594	2,429	6,579	14,210	15,408	26,188	26,690			
Jun-02	1,017	3,443	7,947	11,497	17,380	23,825				
Sep-02	1,394	3,072	5,600	14,098	18,272					
Dec-02	8,102	2,905	6,081	10,007						
Mar-03	1,013	2,392	2,652							
Jun-03	2,327	1,400								
Sep-03	59									

Note: Each entry is calculated as the quotient of the corresponding entries in Appendices A.2 and A.3.

accident	development quarter of finalisation									
quarter	10	11	12	13	14	15	16	17	18	19
Sep-94	38,981	45,863	43,155	40,948	35,908	79,834	65,785	37,000	102,906	262,773
Dec-94	33,065	32,703	37,365	78,315	69,108	72,480	94,207	93,147	139,178	114,821
Mar-95	57,613	41,147	42,031	37,687	45 913	68,820	64,777	104,722	134,404	141,225
Jun-95	41,032	49,272	101,468	78,637	64 070	75,045	97,381	117 429	92,234	110,545
Sep-95	33,960	36,892	53,608	82,564	36,676	89 167	149,679	123 365	178,941	108,499
Dec-95	34,086	40,123	65,550	53,308	93,931	94,785	131,331	106,164	123,663	115,566
Mar-96	30,093	47,875	50,320	73,979	60,599	95,555	79,094	153,636	84,868	216,397
Jun-96	39,983	67,740	60,913	67,250	78,248	121,485	105,823	89,606	100,151	81,515
Sep-96	42,509	60,787	64,080	84,732	81,474	69,821	82 547	93,824	121 783	142,849
Dec-96	46,771	58,639	49,350	64,319	104,134	57,695	100,423	115,642	218,638	92,438
Mar-97	66,464	51,580	70,439	57,971	58,495	82,164	87,609	141,490	168,964	130,546
Jun-97	48,392	45,413	52,464	58,820	78,146	109,606	137,301	142,201	83,567	109,264
Sep-97	55,427	83,651	57,813	114,059	88,783	75 094	75,506	88,514	65,035	119 652
Dec-97	48,775	48,073	89,093	63,252	72,469	101,136	73,730	93,165	134,379	76,378
Mar-98	31,026	63 288	60,886	57,705	128,024	86 192	62,702	95,530	108,612	174,022
Jun-98	46,875	47,891	63,811	76,160	63,667	106,801	81,854	81,477	108,414	69,581
Sep-98	55,025	54,074	71,146	55,322	74,652	134,479	91,283	126,607	141,252	217,591
Dec-98	58,810	63,307	57,124	72,602	116,566	86,935	130 781	122,363	110,631	134,531
Mar-99	46 377	49,490	51,105	75,895	80,459	92,833	83,014	81,526	118,238	
Jun-99	52,992	47,720	57,234	62,371	126,934	96,335	82 298	141,547		
Sep-99	67,518	59,729	69,221	83,279	94 824	138,272	94,214			
Dec-99	56,099	57,395	69,656	55,780	89,280	94,104				
Mar-00	53,568	55,077	101,005	55,766	74,750					
Jun-00	65,777	70,131	59,887	77,433						
Sep-00	93,585	64,453	60,465							
Dec-00	58,789	73,588								
Mar-01	49,258									
Jun-01										

accident	development quarter of finalisation									
quarter	20	21	22	23	24	25	26	27	28	29
Sep-94	129,349	195,633	101,579	121,257	95,978	84,158	712,264	8,764	316,169	282,728
Dec-94	79,351	135,363	90,855	145,021	172,933	335,296	84,477	104,109	87 098	146 445
Mar-95	97,834	70,355	110,518	65,824	172,169	148,164	452,372	84,431	84,166	78,663
Jun-95	121,116	98,383	411,325	352,690	121,433	431,789	104,799	174,005	128 565	374,070
Sep-95	73,437	144,988	71,009	152,943	193,406	91,006	98,609	70,139	1,273 300	116,584
Dec-95	94,241	90 905	141,854	100,275	102 849	230,998	102,517	59,253	103,639	105,413
Mar-96	56,632	84 235	75,801	97,496	114 575	126,450	144,496	227,986	91,890	780,581
Jun-96	371,135	90,167	98,013	304 316	188 162	210,869	156,466	140,817	124 941	202,313
Sep-96	277,303	97,799	153,110	215,742	225,600	139,336	259,511	339,882	362,793	
Dec-96	110,092	209,851	426,546	93,455	87 416	116,867	323,074	355,008		
Mar-97	168,714	85,802	692,725	190,969	181 670	51,736	300,118			
Jun-97	153,029	124,170	205,154	58,125	118,166	87,164				
Sep-97	159,195	130,970	644,990	179,757	206,863					
Dec-97	207,985	65,860	135,046	61,946						
Mar-98	159,924	97,588	74,291							
Jun-98	54,887	107,701								
Sep-98	94,013									
Dec-98										

accident	development quarter of finalisation						
quarter	30	31	32	33	34	35	36
Sep-94	139,507		201,849		6,200		633,545
Dec-94	191,107	276,459	537,824	608,937	166,449		
Mar-95	366,509	265,796	297,888	595,605	165,077		
Jun-95	174,706	320,567	228,614	192,673			
Sep-95	285,658	81,822	41,975				
Dec-95	123,129	56,756					
Mar-96	73,749						
Jun-96							

A.5 Cumulative average sizes of finalised claims

accident	development quarter of finalisation									
quarter	0	1	2	3	4	5	6	7	8	9
Sep-94		2.382	4,992	10,051	16,013	17,144	16,512	16,454	16,983	18 895
Dec-94	1,735	7,188	7,710	8,906	12,289	13,659	14 305	15,353	16,798	18,904
Mar-95	1,636	5 218	7,492	9,500	10,362	11,219	12,156	13,752	14,856	17,769
Jun-95		4,201	7,027	8 474	10,681	12,300	13,312	14,289	16,261	17,463
Sep-95	433	5,540	6,889	9 229	10,330	12,465	12,999	13,539	15,856	17,217
Dec-95	9,060	6.473	6,560	7.894	9.348	9.951	11,150	13,308	14,391	17,520
Mar-96		6 532	6,967	7,831	9 896	10,575	12,472	13,044	15.342	16,834
Jun-96	0	4,693	6.386	7,350	8 827	10.077	10,950	13,462	15,756	16,992
Sep-96	5,307	4,411	6.518	8 666	10.127	11.352	13,030	15,268	17,781	20,444
Dec-96		3.967	7,056	8,348	8.866	10.755	13,913	15,508	17,148	18,982
Mar-97	0	3,988	5,902	7,485	9,350	10,529	12,103	15,761	18,073	19,878
Jun-97		6,340	7.463	8,706	10.003	10 857	13,696	15.420	18 256	20,595
Sep-97	73	3,232	5 930	8,039	9,695	11,654	13,113	15,236	17,764	20,691
Dec-97	5,013	3,802	7,197	8.189	9.954	12,173	13,816	17,689	21,591	23,909
Mar-98		4.069	6,156	9,214	10.091	12.378	14,422	17,014	19.506	21.899
Jun-98		5,032	7.125	8 934	10.974	12.798	16,195	18,203	20,769	21,622
Sep-98	5.828	5,832	7,105	8,986	11,227	14,470	16,123	19,001	20,769	22,638
Dec-98		5.181	6,462	9.476	13.042	15,172	18.011	19,865	22.908	23.704
Mar-99	401	3,921	7,174	9,867	11,364	14,317	15.297	17,861	19.046	20 251
Jun-99	111	4,178	7.343	9,453	11,610	13,827	16,471	18,029	20,075	22 270
Sep-99	97	4,083	6,314	9,399	10.966	13.525	15.286	17,187	20.535	24 548
Dec-99	547	5,438	7,867	9,491	12 632	13,538	15,662	17,994	21,420	24,242
Mar-00	5,050	4,532	5.866	10,485	11,268	13.537	15,462	18,135	20.015	23,024
Jun-00	1,940	3,807	6,088	7,815	9.931	12 585	16,673	18.711	22,105	24 027
Sep-00		6,157	6,601	8,237	10.247	15.598	17,993	22,959	25,583	27,965
Dec-00	147	2,247	3,308	5,564	8.038	10,718	14,011	16,279	18,991	21,764
Mar-01	396	2,108	3,719	7,213	8,928	12,638	16,070	20,516	23,242	25.132
Jun-01	1,271	2,987	4,517	6,053	9,779	12,909	17,822	21.061	25,003	26,839
Sep-01	898	3,035	4,199	8,220	12,898	16,656	18,355	21,793	23.229	
Dec-01	678	2.393	4,103	7,231	12,708	14.964	20.417	21,549		
Mar-02	1,594	2,339	4,867	9,799	11,496	16,493	18,368			
Jun-02	1,017	3,204	6,104	8,326	12,113	15,168				
Sep-02	1,394	2,936	4,541	9,289	11,943					
Dec-02	8,102	3,338	4,895	7,392						
Mar-03	1,013	2,187	2,406							
Jun-03	2,327	1,516								
Sep-03	59									

Note: Each entry is calculated as the quotient of the corresponding entries in the cumulative versions of Appendices A.2 and A.3.

accident	development quarter of finalisation									
quarter	10	11	12	13	14	15	16	17	18	19
Sep-94	20,617	22,362	23,993	25 009	25,757	27,768	29,204	29,401	30,661	34,999
Dec-94	20,149	21,127	22,745	27,587	30,180	31,840	33,555	35,118	38,475	39,549
Mar-95	21,221	23,385	25,236	26,077	27,034	29,176	30,433	33,089	35,283	37,472
Jun-95	19,362	22,414	28,933	31,471	33,054	35,096	37,676	40,509	41,987	43,716
Sep-95	19,202	20,740	22,820	26,683	27,127	29,471	33 098	35,970	38,821	39,735
Dec-95	19,126	20,681	23,969	25,423	28,658	31,021	34 211	35,667	37,596	38,868
Mar-96	17,833	19,842	21,992	24,643	25,838	29,038	30,390	33,953	34,863	36,752
Jun-96	18,903	22,338	25,450	27,703	29,900	33,174	35,568	36,627	37,783	38,032
Sep-96	21,969	25,109	27,755	31,627	34,050	35,309	36,391	37,448	38,554	39,234
Dec-96	21,610	23,995	25,888	27,421	30,852	31,657	32,530	33,571	35,010	35,975
Mar-97	23,616	25,624	28,365	29,807	30,618	31,781	32,496	34,169	36,422	38,360
Jun-97	22,449	23 844	26,211	27,212	28,400	30,034	31,587	32,761	33,868	34,796
Sep-97	23,287	27,649	28,965	31,627	32,885	33,510	34,101	35,618	36,145	37,855
Dec-97	25,891	26,823	29,637	30,471	31,354	32,864	34,259	35,866	37,593	37,967
Mar-98	22,443	24,381	25,550	26,364	29,045	30,977	31,698	32,885	33,878	35,834
Jun-98	23,323	24,447	25,853	27,720	29,796	32,505	34,362	35,550	37,151	37,582
Sep-98	24,430	25,647	28,126	30,086	31,698	35,929	37,904	40,254	42 526	45,879
Dec-98	25,128	26,817	29,114	30 543	33,281	34,477	36,460	37,612	38,895	40,619
Mar-99	21,751	24,426	25,730	28,300	29,799	30,915	31,885	32,969	33,687	
Jun-99	26,143	27,167	29,188	30,382	33,865	35,661	36,726	38,977		
Sep-99	27,956	30,259	32,784	34,953	36,727	39,693	40,821			
Dec-99	27,095	29,127	31,526	32,508	34,571	36,233				
Mar-00	25,312	27,402	31,828	33,107	34,348					
Jun-00	27,121	29,666	31,622	33,047						
Sep-00	32,466	34,928	36,449							
Dec-00	25,134	28,391								
Mar-01	26,907									
Jun-01										

accident	development quarter of finalisation									
quarter	20	21	22	23	24	25	26	27	28	29
Sep-94	36,560	39,432	40,749	41,871	42,523	42 737	43,882	43,822	45,668	47,660
Dec-94	40 223	41,852	42,462	43,372	43,944	45 084	45,142	45,314	45,456	45,895
Mar-95	38 121	38,515	38,972	39,113	39,531	39,814	41,957	42,133	42,242	42,354
Jun-95	44,799	45,267	48,065	49,302	49,558	50,137	50,493	51,538	51,919	52,715
Sep-95	40,072	41,417	41,504	42,266	43,000	43,393	43,711	43,824	47,334	47,564
Dec-95	39,501	40,083	40,707	40,768	41,457	42 881	43,326	43,374	43,463	43,677
Mar-96	36,844	37,161	37,457	38,333	38,866	39,302	39,562	40,677	40,953	43,122
Jun-96	41,104	41,503	42,435	45,794	47,254	47,811	48,650	49,225	49,370	49,663
Sep-96	41,128	41,934	43 280	45,016	45,878	46,365	47,071	48,724	49,461	
Dec-96	36,870	39,359	42,218	42,668	42,950	43,307	44,651	45,543		
Mar-97	40 659	41,350	47,936	49,013	49,807	49,816	50,806			
Jun-97	36 645	37,303	38 802	38,935	39,517	39,702				
Sep-97	39,027	39,776	42,661	43,762	44,454					
Dec-97	39,431	39,731	40,579	40,729						
Mar-98	37,230	38,082	38,372							
Jun-98	37 801	38,437								
Sep-98	46 609									
Dec-98										

accident	development quarter of finalisation						
quarter	30	31	32	33	34	35	36
Sep-94	47,814	47,814	49,096	49,096	49,025	49,025	49,994
Dec-94	46,315	47,088	47,559	48 366	48,760	48,737	
Mar-95	43,683	44,250	44 380	45 223	45,649		
Jun-95	53,195	53,851	54,193	54 531			
Sep-95	48,014	48,078	48,073				
Dec-95	43,950	43,994					
Mar-96	43,152						
Jun-96							

Appendix B
Age-to-age factors

B.1 Age-to-age factors based on paid losses in respect of finalised claims

accident quarter	development quarter of finalisation									
	0	1	2	3	4	5	6	7	8	9
Sep-94			11.18	4.28	2 83	1.40	1 16	1.11	1.20	1.44
Dec-94		80 79	2.97	2.77	2 45	1 43	1 14	1 29	1.37	1.26
Mar-95		65 36	4.97	2.72	1 46	1 26	1.41	1.45	1.22	1 36
Jun-95			5.59	1.94	1 53	1 71	1.62	1 28	1.35	1 22
Sep-95		339 23	2.44	2.20	2 36	1.93	1 32	1 29	1 33	1.23
Dec-95		3 22	2.21	3 79	2 36	1.59	1.47	1 41	1 25	1.43
Mar-96			8 60	2 63	1 99	1.48	1.43	1 21	1 41	1 23
Jun-96			5.87	2 15	1 78	1.59	1 31	1 54	1 35	1.20
Sep-96		28.26	5 95	2 95	1 75	1.54	1 56	1.38	1.34	1.29
Dec-96			8 17	2.32	1.65	1 88	1 65	1 31	1 27	1 20
Mar-97			5 67	3 24	2 56	1 61	1 40	1 59	1.28	1 23
Jun-97			6.73	3.50	1 84	1 43	1.56	1 32	1.36	1 23
Sep-97		212 34	12 54	2.93	1 80	1 69	1.38	1 40	1.36	1.31
Dec-97		18.20	6 82	2.43	2.01	1 60	1.43	1 51	1 41	1.21
Mar-98			7.11	3.41	1 70	1 76	1.42	1.41	1 29	1 25
Jun-98			6.02	2.64	2 11	1.58	1.67	1 29	1.29	1 12
Sep-98		28 02	6 13	3.31	2 02	1 87	1.38	1 46	1.21	1 18
Dec-98			9.29	4 14	2 62	1 60	1 52	1 25	1.26	1 09
Mar-99		538.07	7 15	3 52	1 79	1 76	1 24	1 31	1.14	1.11
Jun-99		863.69	10 35	2.36	1 99	1 54	1.41	1.19	1.21	1 19
Sep-99		1,195 03	4 72	3 84	1.73	1 61	1 32	1 29	1 36	1 45
Dec-99		225.16	5 28	2.15	2 09	1 31	1.39	1 34	1 47	1 26
Mar-00		21.54	3 22	3.82	1.47	1 61	1 47	1.52	1 25	1 32
Jun-00		34 02	5 84	2.30	1.99	1 83	1 90	1.30	1 41	1 18
Sep-00			2.81	2.51	1 96	2.54	1.48	1 62	1 26	1 22
Dec-00		162 87	4 60	3.95	2.92	1.90	1.80	1.39	1 36	1.27
Mar-01		79 90	7.12	4 89	2 13	2.12	1 61	1 58	1 30	1 22
Jun-01		57 97	3.60	2.66	2 66	1 85	1 83	1 38	1.38	1 19
Sep-01		28 96	3 76	4 30	2 45	1 85	1.38	1.45	1.19	
Dec-01		90.03	5 11	3 44	3 29	1 61	1 86	1.24		
Mar-02		13.62	5 15	4 26	1 68	2 17	1 36			
Jun-02		32.02	4 90	2 32	2.50	1 69				
Sep-02		26 13	3 89	4 06	1.82					
Dec-02		4.94	3 39	2.95						
Mar-03		14.49	2.08							
Jun-03		5 21								
Sep-03										
last 1 year		8 85	3.78	3 36	2 22	1 82	1 59	1 40	1.30	1.22
last 2 years		14 19	4 03	3 51	2.34	1.94	1 63	1.43	1 32	1 26
last 3 years		17 51	4 12	3 30	2.14	1.79	1 55	1.38	1.29	1 22
last 4 years		24 20	4 59	3 10	2.09	1 76	1 51	1 37	1 29	1 22
all years		32.18	5 16	2 97	2.02	1 69	1 48	1 37	1 30	1.23

accident quarter	development quarter of finalisation									
	10	11	12	13	14	15	16	17	18	19
Sep-94	1.19	1.17	1.16	1.11	1.11	1.12	1.09	1.03	1.06	1.16
Dec-94	1.17	1.14	1.20	1.33	1.17	1.10	1.08	1.07	1.13	1.04
Mar-95	1.31	1.24	1.20	1.11	1.09	1.14	1.08	1.13	1.09	1.08
Jun-95	1.21	1.29	1.41	1.15	1.10	1.12	1.12	1.11	1.07	1.07
Sep-95	1.27	1.18	1.17	1.25	1.06	1.13	1.16	1.12	1.10	1.04
Dec-95	1.21	1.17	1.25	1.12	1.18	1.12	1.14	1.06	1.08	1.05
Mar-96	1.15	1.19	1.19	1.18	1.08	1.18	1.08	1.15	1.05	1.07
Jun-96	1.21	1.27	1.24	1.15	1.13	1.15	1.11	1.05	1.05	1.01
Sep-96	1.15	1.24	1.19	1.22	1.13	1.07	1.05	1.05	1.04	1.02
Dec-96	1.26	1.19	1.17	1.10	1.18	1.06	1.04	1.05	1.05	1.05
Mar-97	1.29	1.17	1.18	1.10	1.06	1.06	1.04	1.07	1.08	1.08
Jun-97	1.17	1.13	1.20	1.07	1.07	1.08	1.07	1.05	1.06	1.04
Sep-97	1.22	1.28	1.10	1.13	1.06	1.03	1.03	1.07	1.03	1.07
Dec-97	1.18	1.08	1.16	1.05	1.05	1.07	1.08	1.08	1.07	1.02
Mar-98	1.09	1.14	1.08	1.06	1.13	1.10	1.05	1.06	1.04	1.07
Jun-98	1.16	1.10	1.10	1.11	1.14	1.13	1.10	1.06	1.07	1.03
Sep-98	1.14	1.09	1.16	1.15	1.09	1.18	1.09	1.09	1.08	1.10
Dec-98	1.10	1.12	1.17	1.08	1.13	1.06	1.08	1.05	1.05	1.06
Mar-99	1.14	1.24	1.11	1.16	1.08	1.06	1.05	1.06	1.03	
Jun-99	1.34	1.09	1.15	1.08	1.16	1.08	1.05	1.08		
Sep-99	1.24	1.17	1.16	1.11	1.08	1.11	1.05			
Dec-99	1.23	1.15	1.15	1.07	1.10	1.08				
Mar-00	1.19	1.16	1.24	1.10	1.07					
Jun-00	1.22	1.16	1.14	1.08						
Sep-00	1.25	1.17	1.11							
Dec-00	1.27	1.21								
Mar-01	1.16									
Jun-01										
last 1 year	1.22	1.17	1.15	1.09	1.10	1.08	1.06	1.07	1.08	1.07
last 2 years	1.23	1.17	1.15	1.10	1.11	1.10	1.07	1.07	1.05	1.06
last 3 years	1.20	1.15	1.14	1.10	1.10	1.09	1.06	1.06	1.06	1.05
last 4 years	1.19	1.15	1.15	1.10	1.10	1.09	1.07	1.07	1.06	1.05
all years	1.20	1.17	1.17	1.12	1.11	1.10	1.08	1.07	1.06	1.05

accident quarter	development quarter of finalisation									
	20	21	22	23	24	25	26	27	28	29
Sep-94	1.06	1.10	1.06	1.04	1.03	1.01	1.03	1.00	1.05	1.05
Dec-94	1.03	1.06	1.03	1.03	1.02	1.03	1.00	1.01	1.01	1.01
Mar-95	1.03	1.02	1.02	1.01	1.01	1.01	1.06	1.01	1.01	1.01
Jun-95	1.04	1.02	1.07	1.03	1.01	1.01	1.01	1.03	1.01	1.02
Sep-95	1.02	1.05	1.01	1.03	1.02	1.02	1.01	1.01	1.08	1.01
Dec-95	1.03	1.03	1.02	1.00	1.03	1.04	1.02	1.00	1.00	1.01
Mar-96	1.01	1.02	1.02	1.04	1.02	1.02	1.01	1.03	1.01	1.06
Jun-96	1.09	1.02	1.04	1.09	1.04	1.02	1.03	1.02	1.00	1.01
Sep-96	1.06	1.03	1.04	1.05	1.02	1.02	1.02	1.04	1.02	
Dec-96	1.04	1.08	1.08	1.02	1.01	1.01	1.04	1.02		
Mar-97	1.08	1.03	1.17	1.03	1.02	1.00	1.02			
Jun-97	1.07	1.03	1.05	1.01	1.02	1.01				
Sep-97	1.04	1.03	1.08	1.03	1.02					
Dec-97	1.05	1.02	1.03	1.01						
Mar-98	1.05	1.04	1.02							
Jun-98	1.02	1.03								
Sep-98	1.03									
Dec-98										
last 1 year	1.04	1.03	1.04	1.02	1.02	1.01	1.03	1.03	1.01	1.02
last 2 years	1.05	1.04	1.06	1.04	1.02	1.02	1.02	1.02	1.02	1.02
last 3 years	1.05	1.03	1.05	1.03	1.02	1.01	1.02	1.02	1.01	
last 4 years	1.04	1.02	1.06	1.01	1.01					
all years	1.04									

accident quarter	development quarter of finalisation						
	30	31	32	33	34	35	36
Sep-94	1.00	1.00	1.04	1.00	1.00	1.00	1.02
Dec-94	1.01	1.02	1.01	1.02	1.01	1.00	
Mar-95	1.04	1.02	1.00	1.02	1.01		
Jun-95	1.01	1.01	1.01	1.01			
Sep-95	1.01	1.00	1.00				
Dec-95	1.01	1.00					
Mar-96	1.00						
Jun-96							
last 1 year	1.01	1.01	1.01	1.01	1.01	0.00	0.00
last 2 years	1.00	1.01	1.01				
last 3 years							
last 4 years							
all years							

B.2 Age-to-age factors based on average sizes of finalised claims

accident quarter	development quarter of finalisation									
	0	1	2	3	4	5	6	7	8	9
Sep-94			2.10	2.01	1.59	1.07	0.96	1.00	1.03	1.11
Dec-94		4.14	1.07	1.16	1.38	1.11	1.05	1.07	1.09	1.13
Mar-95		3.19	1.44	1.27	1.09	1.08	1.08	1.13	1.08	1.20
Jun-95			1.67	1.21	1.26	1.15	1.08	1.07	1.14	1.07
Sep-95		12.80	1.24	1.34	1.12	1.21	1.04	1.04	1.17	1.09
Dec-95		0.71	1.01	1.20	1.18	1.06	1.12	1.19	1.08	1.22
Mar-96			1.07	1.12	1.26	1.07	1.18	1.05	1.18	1.10
Jun-96			1.36	1.15	1.20	1.14	1.09	1.23	1.17	1.08
Sep-96		0.83	1.48	1.33	1.17	1.12	1.15	1.17	1.16	1.15
Dec-96			1.78	1.18	1.06	1.21	1.29	1.11	1.11	1.11
Mar-97			1.48	1.27	1.25	1.13	1.15	1.30	1.15	1.10
Jun-97			1.18	1.17	1.15	1.09	1.26	1.13	1.18	1.13
Sep-97		44.24	1.84	1.36	1.21	1.20	1.13	1.16	1.17	1.16
Dec-97		0.76	1.89	1.14	1.22	1.22	1.13	1.28	1.22	1.11
Mar-98			1.51	1.50	1.10	1.23	1.17	1.18	1.15	1.12
Jun-98			1.42	1.25	1.23	1.17	1.27	1.12	1.14	1.04
Sep-98		1.00	1.22	1.26	1.25	1.29	1.11	1.18	1.09	1.09
Dec-98			1.25	1.47	1.38	1.16	1.19	1.10	1.15	1.03
Mar-99		9.78	1.83	1.38	1.15	1.26	1.07	1.17	1.07	1.06
Jun-99		37.55	1.76	1.29	1.23	1.19	1.19	1.09	1.11	1.11
Sep-99		41.93	1.55	1.49	1.17	1.23	1.13	1.12	1.19	1.20
Dec-99		9.93	1.45	1.21	1.33	1.07	1.16	1.15	1.19	1.13
Mar-00		0.90	1.29	1.79	1.07	1.20	1.14	1.17	1.10	1.15
Jun-00		1.98	1.60	1.28	1.27	1.27	1.32	1.12	1.18	1.09
Sep-00			1.07	1.25	1.24	1.52	1.15	1.28	1.11	1.09
Dec-00		15.27	1.47	1.68	1.44	1.33	1.31	1.16	1.17	1.15
Mar-01		5.33	1.76	1.94	1.24	1.42	1.27	1.28	1.13	1.08
Jun-01		2.35	1.51	1.34	1.62	1.32	1.38	1.18	1.19	1.07
Sep-01		3.38	1.38	1.96	1.57	1.29	1.10	1.19	1.07	
Dec-01		3.53	1.71	1.76	1.76	1.18	1.36	1.06		
Mar-02		1.47	2.08	2.01	1.17	1.43	1.11			
Jun-02		3.15	1.91	1.36	1.45	1.25				
Sep-02		2.11	1.55	2.05	1.29					
Dec-02		0.41	1.47	1.51						
Mar-03		2.16	1.10							
Jun-03		0.65								
Sep-03										
last 1 year		0.78	1.54	1.71	1.39	1.29	1.23	1.17	1.14	1.10
last 2 years		1.23	1.59	1.72	1.42	1.34	1.24	1.18	1.14	1.12
last 3 years		1.50	1.51	1.63	1.35	1.28	1.21	1.16	1.14	1.10
last 4 years		1.74	1.51	1.53	1.32	1.27	1.20	1.16	1.14	1.10
all years		2.65	1.43	1.39	1.27	1.21	1.16	1.15	1.14	1.11

accident quarter	development quarter of finalisation									
	10	11	12	13	14	15	16	17	18	19
Sep-94	1.09	1.08	1.07	1.04	1.03	1.08	1.05	1.01	1.04	1.14
Dec-94	1.07	1.05	1.08	1.21	1.09	1.05	1.05	1.05	1.10	1.03
Mar-95	1.19	1.10	1.08	1.03	1.04	1.08	1.04	1.09	1.07	1.06
Jun-95	1.11	1.16	1.29	1.09	1.05	1.06	1.07	1.08	1.04	1.04
Sep-95	1.12	1.08	1.10	1.17	1.02	1.09	1.12	1.09	1.08	1.02
Dec-95	1.09	1.08	1.16	1.06	1.13	1.08	1.10	1.04	1.05	1.03
Mar-96	1.06	1.11	1.11	1.12	1.05	1.12	1.05	1.12	1.03	1.05
Jun-96	1.11	1.18	1.14	1.09	1.08	1.11	1.07	1.03	1.03	1.01
Sep-96	1.07	1.14	1.11	1.14	1.08	1.04	1.03	1.03	1.03	1.02
Dec-96	1.14	1.11	1.08	1.06	1.13	1.03	1.03	1.03	1.04	1.03
Mar-97	1.19	1.09	1.11	1.05	1.03	1.04	1.02	1.05	1.07	1.05
Jun-97	1.09	1.06	1.10	1.04	1.04	1.06	1.05	1.04	1.03	1.03
Sep-97	1.13	1.19	1.05	1.09	1.04	1.02	1.02	1.04	1.01	1.05
Dec-97	1.08	1.04	1.10	1.03	1.03	1.05	1.04	1.05	1.05	1.01
Mar-98	1.02	1.09	1.05	1.03	1.10	1.07	1.02	1.04	1.03	1.06
Jun-98	1.08	1.05	1.06	1.07	1.07	1.09	1.06	1.03	1.05	1.01
Sep-98	1.08	1.05	1.10	1.07	1.05	1.13	1.05	1.06	1.06	1.08
Dec-98	1.06	1.07	1.09	1.05	1.09	1.04	1.06	1.03	1.03	1.04
Mar-99	1.07	1.12	1.05	1.10	1.05	1.04	1.03	1.03	1.02	
Jun-99	1.17	1.04	1.07	1.04	1.11	1.05	1.03	1.06		
Sep-99	1.14	1.08	1.08	1.07	1.05	1.08	1.03			
Dec-99	1.12	1.07	1.08	1.03	1.06	1.05				
Mar-00	1.10	1.08	1.16	1.04	1.04					
Jun-00	1.13	1.09	1.07	1.05						
Sep-00	1.16	1.08	1.04							
Dec-00	1.15	1.13								
Mar-01	1.07									
Jun-01										
last 1 year	1.13	1.09	1.09	1.05	1.07	1.06	1.04	1.05	1.04	1.05
last 2 years	1.13	1.09	1.08	1.05	1.07	1.07	1.04	1.04	1.04	1.04
last 3 years	1.11	1.08	1.08	1.05	1.06	1.06	1.04	1.04	1.04	1.04
last 4 years	1.10	1.08	1.08	1.06	1.07	1.06	1.04	1.05	1.04	1.04
all years	1.11	1.09	1.09	1.07	1.06	1.06	1.05	1.05	1.05	1.04

accident quarter	development quarter of finalisation									
	20	21	22	23	24	25	26	27	28	29
Sep-94	1.04	1.08	1.03	1.03	1.02	1.01	1.03	1.00	1.04	1.04
Dec-94	1.02	1.04	1.01	1.02	1.01	1.03	1.00	1.00	1.00	1.01
Mar-95	1.02	1.01	1.01	1.00	1.01	1.01	1.05	1.00	1.00	1.00
Jun-95	1.02	1.01	1.06	1.03	1.01	1.01	1.01	1.02	1.01	1.02
Sep-95	1.01	1.03	1.00	1.02	1.02	1.01	1.01	1.00	1.08	1.00
Dec-95	1.02	1.01	1.02	1.00	1.02	1.03	1.01	1.00	1.00	1.00
Mar-96	1.00	1.01	1.01	1.02	1.01	1.01	1.01	1.03	1.01	1.05
Jun-96	1.08	1.01	1.02	1.08	1.03	1.01	1.02	1.01	1.00	1.01
Sep-96	1.05	1.02	1.03	1.04	1.02	1.01	1.02	1.04	1.02	
Dec-96	1.02	1.07	1.07	1.01	1.01	1.01	1.03	1.02		
Mar-97	1.06	1.02	1.16	1.02	1.02	1.00	1.02			
Jun-97	1.05	1.02	1.04	1.00	1.01	1.00				
Sep-97	1.03	1.02	1.07	1.03	1.02					
Dec-97	1.04	1.01	1.02	1.00						
Mar-98	1.04	1.02	1.01							
Jun-98	1.01	1.02								
Sep-98	1.02									
Dec-98										
last 1 year	1.02	1.02	1.04	1.01	1.01	1.01	1.02	1.02	1.01	1.02
last 2 years	1.03	1.02	1.05	1.03	1.02	1.01	1.01	1.02	1.01	1.02
last 3 years	1.03	1.02	1.04	1.02	1.02	1.01	1.02	1.01	1.02	
last 4 years	1.03	1.02	1.04	1.02	1.02					
all years	1.03									

accident quarter	development quarter of finalisation						
	30	31	32	33	34	35	36
Sep-94	1 00	1 00	1 03	1.00	1 00	1 00	1 02
Dec-94	1 01	1 02	1.01	1 02	1 01	1.00	
Mar-95	1.03	1 01	1.00	1 02	1 01		
Jun-95	1 01	1 01	1.01	1.01			
Sep-95	1.01	1 00	1 00				
Dec-95	1.01	1 00					
Mar-96	1.00						
Jun-96							
last 1 year	1 01	1.01	1 00	1 01	1 01	1 00	1 02
last 2 years	1 01	1 01	1 01				
last 3 years							
last 4 years							
all years							

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The Case of the Medical Malpractice Crisis: A Classic Who Dunnit

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The Case of the Medical Malpractice Crisis: A Classic Who Dunnit

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We actuaries, detectives of the first order, are presented with a most intriguing case: numerous, grisly bodies of dead insurance companies and physicians' practices in public view, various signs of intrigue and foul play abound, suspects galore, an abundance of alibis, and an endless supply of opinions on how the culprit(s) must repay their debt to society. This case of the medical malpractice crisis is complicated because there is not only no consensus on "who dunnit?" but not even an agreement on "what happened?" This is the situation we are currently faced with in the medical malpractice insurance industry. There is evidence scattered all over the medical malpractice insurance landscape, but there is no agreement at all on the cause, the culprit, the motives or the appropriate sentence.

The Suspects

First, let's identify some of the suspects. Like any good murder mystery, this case presents an abundance of suspects. First, there's the stereotypical bad guy: the trial attorneys. Despised by many members of the (insurance) community (except the claimants) they stand accused of causing run away large losses due to out of control juries encouraged by their wily tactics. Their very livelihood is being threatened by caps on non-economic damages and even worse (gasp!) caps on attorney contingency fees.

The medical malpractice insurance industry's hat is only slightly less black than the attorneys. Their reputation for poor investment strategies, destructive price competition, a preference to pay defense attorneys rather than patients with negative treatment outcomes, grossly inaccurate reserve estimates, and general mismanagement is widely held and is leading to a variety of lawsuits against medical malpractice insurance company directors, officers, and managers, both insolvent and not. Insurance companies are threatened with constraints on their ability to adjust rates as well as increased solvency regulation if convicted of their accused crimes.

The doctors and other healthcare providers are accused of crying wolf over increased costs trends that are actually less than the rate at which health care inflation is increasing their revenues.

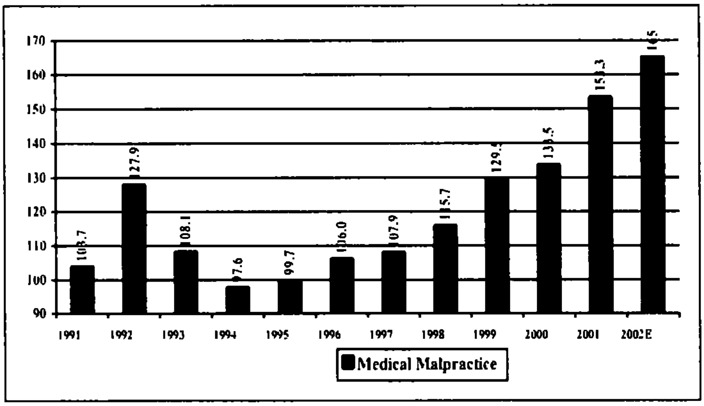
They are also accused of being the victims that are at the root of the crime. They are accused of not doing enough to reduce the negative patient outcomes. Then when the negative outcomes occur, they protect one another through self-governing mechanisms such as medical panels and review boards. Their resistance to reporting malpractice events to public, national, or state databases is also sometimes characterized as "protectionism." The call for damage caps without corresponding loss prevention initiatives is viewed quite negatively by some investigators of the matter. In the eyes of those who accuse the health care providers, the increased premiums they are experiencing are not only appropriate it is the logical result of their behavior.

The list of potential suspects and accessories also includes state insurance regulators, rating agencies, and even the public at large (that's right everybody did it!). State regulators are accused of not monitoring rate adequacy and solvency sufficiently. Rating agencies threatened downgrades to single state medical malpractice insurers that didn't diversify or expand geographically. The logic for accusing all of us is that the current sense of entitlement in American culture is enabling the runaway jury awards.

Signs of Intrigue

The most obvious signs of intrigue and wrongdoing are related to the widespread lack of available or affordable medical malpractice coverage around the country. From an insurance perspective, this problem began with an extreme deterioration of medical malpractice insurance underwriting results. As can be seen in the graph below, the combined operating ratio for the medical malpractice insurance has deteriorated almost 70 points in 8 calendar years.

Table 1. Medical Malpractice Calendar Year Combined Operating Ratios



Companies experiencing underwriting results of this magnitude basically had two responses: exit the line or increase rates. As can be seen in the table below, some insureds have seen significant rate increases in the last five years. General surgeons in Broward County, Florida for this carrier have had their rates triple since 1998.

Table 2. Example Annual Rate Changes – Broward County, FL General Surgeons

Year	Premium	Annual Change
2003	213,763	30.0%
2002	164,437	40.5%
2001	117,049	12.7%
2000	103,859	27.0%
1999	81,765	17.2%
1998	69,786	12.1%
1997	62,259	-7.3%
1996	67,141	+5.7%
Annual Average		16.7%

Source: Medical Liability Monitor

As previously mentioned, another option available to insurers was an exit from the market – either voluntarily or not. Some carriers did not survive the dramatic balance sheet impact of operating results similar to the results shown in Table 1. A total of almost 8% of the 1998 countrywide direct written premium for medical malpractice coverage has left the marketplace due to insolvencies. The table below highlights several major medical malpractice insurer insolvencies over the last five years. These involuntary departures by themselves would have been more than enough to materially impact the availability of any insurance market; but there were other market dislocations.

Table 3. Major Medical Malpractice Insurer Insolvencies

Company	1997 AM Best Rating	1997 Countrywide Market Share
Frontier	A-	2.5%
PHICO	A-	2.5%
Reliance	A-	0.8%
Reciprocal Group	A	0.8%
Fremont General	A-	0.6%
Legion	A-	0.5%

Operating results, adverse reserve development, and the inability to achieve acceptable rate levels caused several companies to voluntarily withdraw either from specific states, specific specialties (e.g. OB/GYNs or emergency room physicians) or from the medical malpractice market entirely. The three most significant of these countrywide departures, St. Paul Companies, Zurich/Farmers and MIIX, accounted for over 15% of the countrywide medical malpractice premium in 1998.

Table 4. Major Medical Malpractice Insurer Exits

Company	1997 AM Best Rating	1997 Countrywide Market Share
St. Paul Companies	A+	6.8%
MMI Companies (merged w/St. Paul)	A	2.8%
MIIX	A	2.8%
Zurich Group	A+	1.8%
Farmers Group	A-	1.1%

As if the insolvencies and voluntary exits did not reduce capacity enough, the medical malpractice industry is highly sensitive to the ratings assigned by industry rating agencies, particularly A.M. Best Company. This is the result of many hospital corporate by-laws requiring both physicians with hospital privileges and the hospital itself to maintain malpractice coverage with an insurer rated A- or better. Therefore, when A.M. Best downgrades a medical malpractice insurer below A-, the downgrade creates an impediment that can severely impair the insurer's ability to provide ongoing malpractice coverage to many insureds. This is not to imply that non A-rated carriers do not write medical malpractice insurance. Rather there is a great deal of time and expense that must be expended to address the concerns of insureds, sometimes including the additional expense of engaging an A-rated fronting carrier, when a company loses its "A" rating. A number of previously "A-rated" malpractice insurers are currently rated below A-. These carriers include: AP Capital, SCPIE, MLMIC Group (including OHIC), Florida Physicians Group, and Connecticut Medical Insurance Company. Together, the downgraded companies account for about 20% of the 1997 market share.

These three groups, the liquidated, the voluntary exits, and the downgraded, combined suggest that over 40% of the 1997 medical malpractice insurance market share has either exited the industry or had their ability to grow and compete for business impaired due to changes in their Best rating. In states where one or more of the exited carriers, such as St. Paul, PHICO, or MIIX, had a commanding market share the market dislocation effect has been even more severe, sometimes up to 70% or 80% of the total market.

Like a cascade of dominos, these significant increases in medical malpractice insurance costs and decreases in coverage availability have caused health care providers to respond in a variety of ways to the increased costs of doing business. The American Medical Association, insurance trade press, and even the national media at times have widely publicized some of these responses which have included significant reduction in coverage limits, the discontinuation of risky procedures, relocations of physician practices to neighboring states with more favorable malpractice laws, early retirements by physicians, and in more extreme cases to hospital closures and marches on state capitals.

Problems with the Available Evidence

Why do we have so much trouble definitively identifying a cause for the current problems of the medical malpractice insurance industry? One of the biggest complications is the lack of a robust, countrywide experience database. Insurance Services Office, Inc., a national statistical agent and rating bureau for medical malpractice, does not have the market share in medical malpractice they have in personal lines and some standard commercial lines. One main cause of this reduced credibility in the ISO data is the flight of medical malpractice from standard insurance carriers to alternative market mechanisms and other non-ISO reporting companies. These programs include captive insurance companies, risk retention groups, and other self-insurance programs.

Other malpractice databases such as the National Practitioner Data Bank (different reporting standards by state) and Jury Verdict Research (claims settled by trial only; incomplete geographic coverage) have significant limitations that reduce their effectiveness as diagnostic tools for examining causes of the current market emergency.

Another issue contributing significantly to the complexity of the problem is the significant degree of variability in results and environmental changes by state over time. In some cases, numerous changes were implemented closely enough to one another in time that advocates of each reform claim validation. A classic example is in California where the sweeping medical malpractice reforms of MICRA and the vast insurance rate regulatory changes of Proposition 103 are both credited with the successes in the state by advocates of the competing reforms. In other states, particularly some of the more troubled states, there have been so many insolvencies, government programs, and other changes it's hard to say any of them have truly succeeded. It is equally difficulty to confidently say that a particular measure wouldn't have succeeded under different conditions.

There are a number of factors that can impact the claims characteristics of a state's medical malpractice system including the presence and details of joint underwriting authorities, patient compensation funds, caps on non-economic damages, restrictive rate regulatory approaches, domestic healthcare provider-owned mutual companies, and many others. The presence, specifics, and timing of these factors are different for each state which makes it very difficult to transfer the results from one state to another.

Exacerbating these problems is the reach and strength of the influence of two of the leading suspects: the health care providers and the trial bar. Health care providers (through associations such as the American Medical Association and the American Hospital Association) have a tremendous stake not just in the determination of the cause, but also the solution. It is an oversimplification of the problem, but if the primary cause of the "crisis" is the trial bar, "runaway" juries, or any other external factors, then the likelihood of reforms, including caps on damages (especially non-economic damages) is increased.

Because of the close relationship between health care providers and their malpractice insurers (partly due to the growing importance and market share of provider-owned insurers), their interests are closely aligned. If damage caps and other cost controlling measures such as medical review boards, arbitration, patient compensation funds, and caps on attorney contingency fees are implemented, insurer rates decrease and more importantly the potential for profitability increases.

On the other hand, if the allegations of the trial bar such as destructive price competition by insurers (with the contribution of poor regulatory oversight), irresponsible investment strategies, poor loss reserving discipline, increasingly risky medical procedures, and a lagging focus on reducing adverse patient outcomes are found to be significant contributors, then reforms that do not impact patient recoveries (and lawyers' fees) will be given more consideration. These reforms might include more reporting requirements of adverse patient outcomes and tighter rate regulatory requirements. Both groups have tremendous income at stake in the "medicine" the industry takes for its problems. They are both exerting tremendous political and media pressure to influence the perception of the cause and the remediation states choose to implement.

The Approach

Because of the concerns stated above regarding countrywide data and the appropriateness of data from one state, the approach taken in this study is to focus on a large database for a single state in an attempt to make some inferences about the medical malpractice insurance market in that state. Another goal of the study is to demonstrate how a Generalized Linear Modeling (GLM) approach can assist in evaluating claims trends for commercial lines of insurance more generally. One state with robust, readily available medical malpractice claims data is the state of Florida. Our approach will be to look at an industry-wide Florida database that contains a variety of health care provider and claim characteristics. The analysis will use GLM to reflect the impact of each factor when all factors included in the model are reviewed simultaneously and also to identify any interactions between characteristics.

A general discussion of GLM is outside the scope of this paper, and the existence of several excellent writings in this area makes an effort along these lines on my part unnecessary. For the purposes of this paper, it will suffice to say that GLM is a statistical approach to developing a model that explains how a group of explanatory variables can be used to estimate or predict a dependent or response variable. For this analysis a number of claim and health care provider characteristics will be used to predict closed claim severities and a couple of additional response variables. In most ratemaking exercises, GLM takes ratemaking or underwriting characteristics (e.g. territory, class, credit) and uses them to predict claim frequencies, severities and pure premiums. GLM also provides the capabilities to fit polynomial curves, manually override indicated factors, and regroup explanatory variables (e.g. zip codes, credit scores).

The greatest advantage GLM has over traditional one-way loss ratio analyses is the reflection of interactions between explanatory characteristics. As a simple example, consider the following fictional one-way severity results:

Table 5. Example State F Average Severity by Territory

Territory	Average Severity
Metropolitan	$27,600
Urban	$32,400

There would a natural tendency to assume that something was different about the urban territory. Similarly, a one-way severity study was conducted for the two classes insured by a company:

Table 6. Example State F Average Severity by Class

Class	Average Severity
Tree Surgeon	$24,000
Shrub Doctor	$36,000

So, based on this information there is also apparently a severity problem with the shrub class.

These one-way analyses make the assumption that the distribution by class and territory are uniform. Assume the actual results looked as follows:

Table 7. Example State F Average Severity by Territory

	Territory				
	Metro		Urban		
Class	**Severity**	**Weight**	**Severity**	**Weight**	**Total**
Tree	24,000	70%	24,000	30%	24,000
Shrub	36,000	30%	36,000	70%	36,000
Total	27,600		32,400		30,000

Because of the distributional bias between territories by class, the identical severities by class, by territory disguised themselves as a territory problem. Imagine if this example had been pure premiums instead of severities. The pricing actuary relying on one-way analyses would have imposed territory and class relativities that in concert would have significantly over-priced urban, shrub doctors. Please don't miss the other problem in this simple example: if the rates in total were adequate, the tree surgeons in the Metro territory would be under-priced by a similar percentage!

It would be natural to ask, "Why hasn't this GLM approach been used more commonly in medical malpractice?" The initial focus of GLM in both Europe and North America has been on personal lines pricing. There are at least three key reasons for this emphasis. Personal lines rating plans are more complex and thus more in need of an understanding of the interactions between variables, particularly factors such as credit that are highly correlated to other factors. Second, personal lines, especially personal automobile, have more premium, more policies, and more claims than commercial lines and thus provide more data. Finally, the impact that GLM can have on personal lines has already been demonstrated by companies that have used this type of analysis to create a sustainable competitive advantage and profitable growth. Now that GLM for personal lines pricing is quickly becoming an industry standard approach, new applications of GLM are constantly emerging for such applications as agency management, claims analysis, utilization review, and commercial lines class plan analysis for such lines as business owners (BOP), workers compensation, commercial automobile, and medical malpractice.

Our Evidence

Starting in 1974, the Florida Department of Financial Services Regulation – Office of Insurance Regulation, has maintained a Medical Professional Liability Closed Claim database. This comprehensive database is readily available to regulators, insurers, trial attorneys, health care providers, and other parties interested in the current medical malpractice crisis. In July 1999, the original database was replaced with a new closed claims database with a slightly different format. As a result, some fields contained in one database are not continued in the other, some fields use slightly different entries for the same information, and some fields change definition slightly even though they look the same (e.g. county fields). As a result, there were several fields that required some coordination of similar fields, and others that resulted in an entry of "NO RESPONSE" from one source or the other. A total of almost 65,000 claims were ultimately used in the analysis.

The new version of the Florida database contains the following fields that were identified as potentially useful in the analysis. The table also contains an example from the database and some useful notes on the fields. More information on the database is available from the Florida Office of Insurance Regulation and is also provided when the data is produced for a nominal fee.

Table 8. Florida Database Fields

	Fields	Example(s)	Notes
1.	Injury Location	Hospital Inpatient Facility	Individual Facility named as well
2.	Injury Location Detail	Labor and Delivery Room	More detail for hospitals
3.	Occurrence Date	6/5/1981	
4.	Report Date	10/24/1984	
5.	Patient Date of Birth	6/5/1981	
6.	Injured Patient Sex	F	
7.	Severity of Injury	Permanent: Major	Emotional Only, Death, and several classes of temporary and permanent
8.	Suit Date	10/24/1984	Can be used as an indicator for suits
9.	County of Suit	Dade	These two were combined to assess
10.	Insured County	Dade	geographic differences using county of suit as primary, if provided.
11.	Fin_Meth_Desc	Settled by parties	Also reflects court and arbitration
12.	Stage_of_Desc	More than 90 days after suit	
13.	Final_Date_Disp	12/21/2001	
14.	Court_Desc	No Court Proceedings.	Reflects settlements for plaintiffs and Defendants and directed verdicts.
15.	Arbitration indicator	Not subject to Arbitration.	Also shows the results for arbitration eligible claims
16.	Insurer Type	Primary	Actual Insurer Named as well

Table 8. Florida Database Fields (cont.)

Fields	Example(s)	Notes
17. Insured Type	Physicians and Surgeons	e.g. Hospital, Dentist, Podiatrist, HMO, Corporation, Ambulatory Surgical Centers
18. Provider Specialty	80267- PEDIATRICS	This is a field that required scrubbing
19. Insured Occ. Limits	$100,000	Aggregate limit available as well
20. Indemnity Paid	$100,000	Additional detail between medical loss, wage loss and other; both paid to date and future is available
21. Loss Adjustment	$184,549	Defense Costs
22. Loss Adjustment Other	$57,001	Other ALAE
23. Non-Economic Loss	$100,000	This field was only partially utilized and required some modification

There is a tremendous amount of specific detail related to the facility and individual health care provider involved in the claim in the database. This data is not relevant to this analysis. Although using the physician detail to assess the impact of "repeat offenders" on overall costs is certainly a conceivable use of the database. There is also more detail related to the patients' conditions and diagnoses that adds to the robustness of the database, but was viewed as too granular to be useful for our purposes.

Several additional values were computed to simplify our analysis. First, a report lag was computed as the number of years from the occurrence of a claim to the first report of the incident. The time from occurrence to the filing of a lawsuit was also computed. Similarly, the settlement lag was calculated as the time from occurrence to settlement of a claim. Two additional response variables were also computed.

This study focused on examining three questions with three response variables:

1) What factors influence overall claim severity? (measured using total loss and ALAE),
2) What factors increase the proportion of non-economic damages in a loss? (measured as the ratio of non-economic damages to total pure losses (excluding ALAE)),
3) What factors increase the proportion of defense costs in a claim? (measured as the ratio of ALAE to total damages)

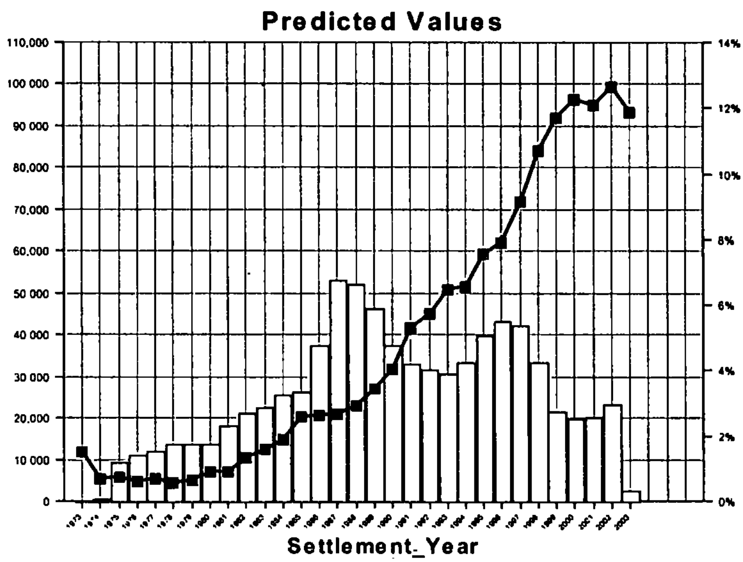
These are by no means the only response variables that could be used for this data. Other severity metrics (total non-economic damage dollars, total ALAE or just defense cost dollars, medical losses only, wage loss, etc.) could certainly be modeled easily using this data. Given the

AMA's fascination with claims over $1,000,000, data looking at the frequency and severity for these claims could also be constructed with meaningful results. Losses above this or any limit (adjusted for inflation or not depending on the analysis goal) could be modeled effectively from this data to examine large loss propensities or to compare them to overall claim characteristics.

The Results of the Investigation

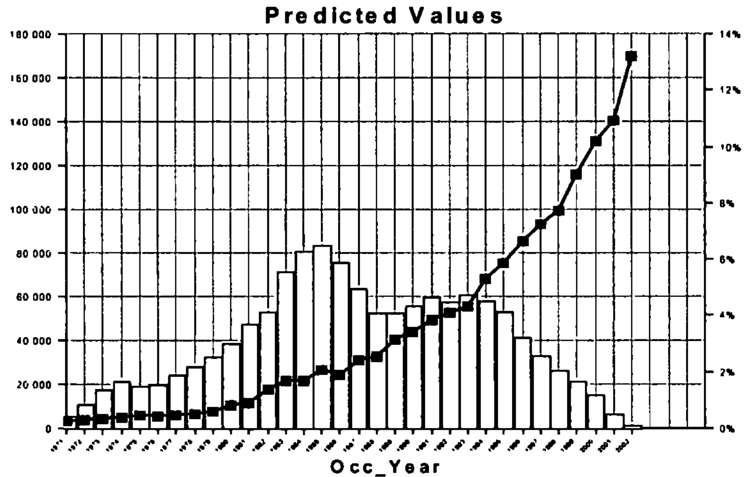
The first widely disputed question is, "How much are severities increasing annually?" The Florida database contains two pertinent dates: occurrence date and settlement date. Both of these dates were converted to calendar years for the purpose of our study. The results by settlement year show a tremendous increase during the 1990s as can be seen in the following table which shows the distribution of claims and predicted severities from the GLM.

Table 9. Predicted Severities by Settlement Year



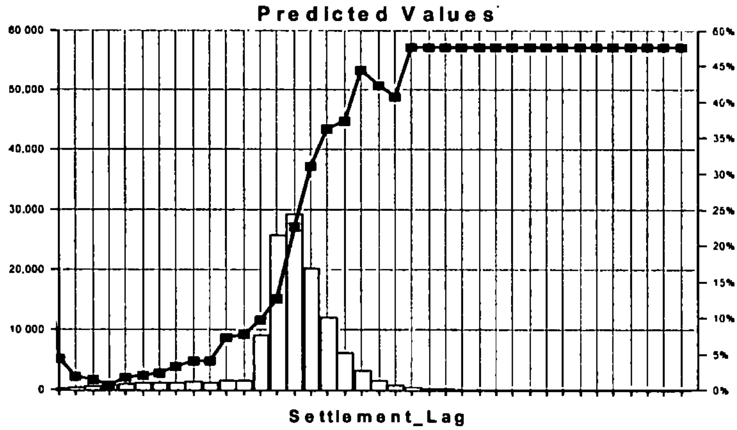
The overall severities appear to have leveled off somewhat in the last few years after dramatic growth in the 1990s. Even more alarming are the results by occurrence year. Part of the cause for alarm in the following table is that most of the claims for the more current occurrence years remain open and are not in the database yet. A lower average severity for closed claims to date would have been expected for the more recent occurrence years due to the larger average severity for open claims with longer settlement lags.

Table 10. Predicted Severities by Occurrence Year



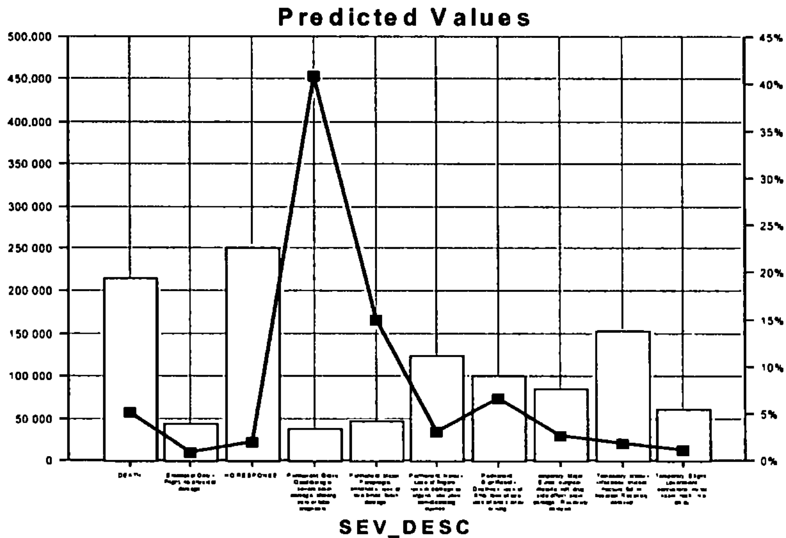
The higher severity by settlement lag described earlier can also be seen in our GLM results. Results of this kind could be effectively utilized in loss reserve analyzes examining the impact of inflation on closed claims over an extended settlement period. You will also notice that all settlement periods over 10 years were grouped together due to the sparsity of the data.

Table 11. Predicted Severities by Settlement Lag



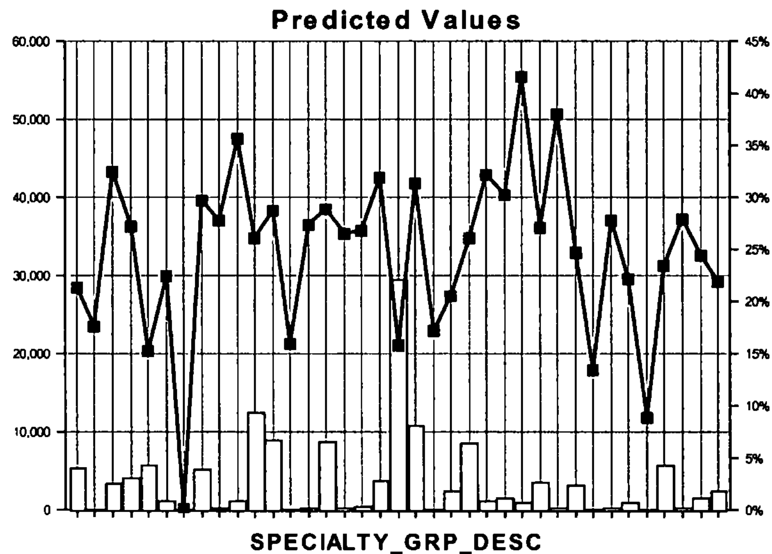
Next, the consistency of this behavior across different types of injury was investigated. As you can see, the permanent claim classes; grave, major and significant; all predict substantially higher claim severities than the other injury types. These results by severity of injury were found to be consistent by type of health care provider and location of injury.

Table 12. Predicted Severities by Claim Type



How about different categories of health care provider? The next table shows a significant amount of variation in average severity by physician type. In particular, classes such as cardiovascular disease, neoplastic diseases, emergency room physician, gastroenterology, neurology, obstetrics/gynecology, pediatrics and pathology all produced higher predicted severities while dentistry, allergy, diabetes treatment and podiatry all predicted much lower severities than average.

Table 13. Predicted Severities by Specialty



A number of critical concerns related to claim settlement environment can be addressed by the results of a GLM analysis of the Florida data including:

- the potential impact different courts geographically (by county),
- the impact on severities caused by a claim going to suit, and
- the impact on severities caused by the use of arbitration.

The issue of geographic differences presented some fascinating results, while the results by type of claim resolution and arbitration impact (shown in tables 14 and 15) were reasonably intuitive with severities of claims resulting from judgments for plaintiffs three to four times those with judgments for defendants. Also, court disposals had lower severities than arbitration cases which were in turn lower than settlements agreed to by the parties. This behavior is also extremely stable by settlement year.

Table 14. Predicted Severities by Claim Resolution

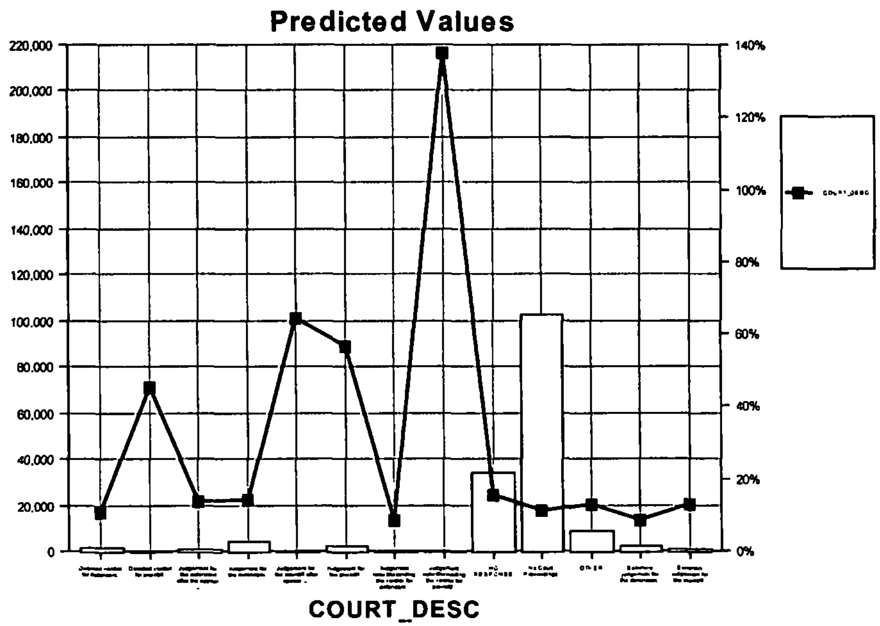
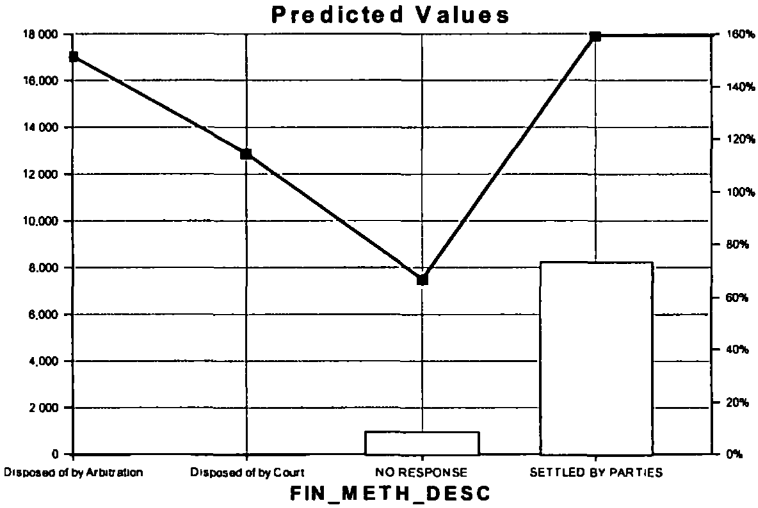
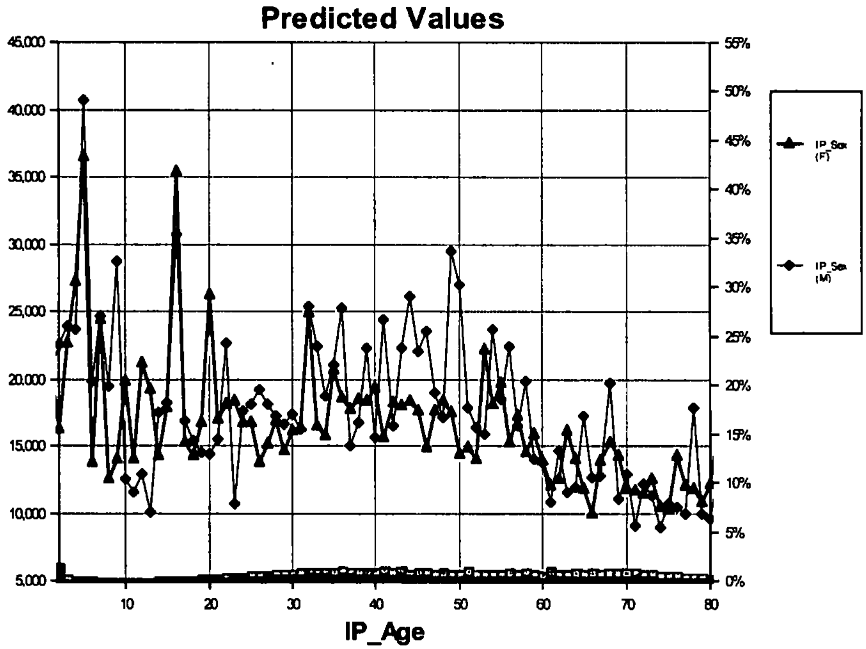


Table 15. Predicted Severities by Arbitration Impact



While not a key discussion point at this stage in the debate, variances by sex and age, along with their interactions were investigated to identify any differences in severities. Particularly, note the slightly higher average severities for male patients between the ages of 30 and 60. The theory that this could be the result of higher wage loss payments for men versus women in this age segment is currently being investigated.

Table 16. Predicted Severities by Age and Sex



Two of the interesting results from, the ALAE to loss + ALAE ratio analysis relate to the dramatic differences by severity of claim and by settlement lag. The results are summarized in Tables 17 and 18. Table 17 shows that emotional and temporary claims result in a much higher ratio of ALAE to Loss and ALAE. Table 18 shows that the ratio of ALAE to Loss and ALAE increases as the settlement lag increases and exceeds 80% by the time the claim has been open 10 years.

Table 17. Predicted ALAE to Loss and ALAE Ratios by Claim Type

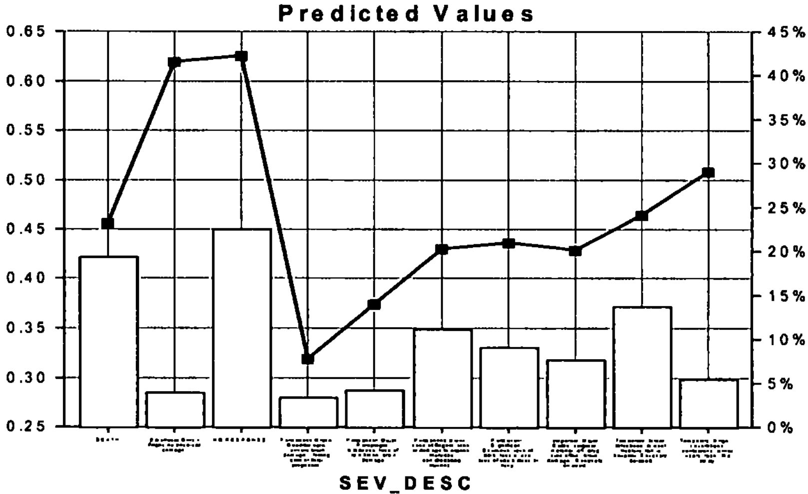
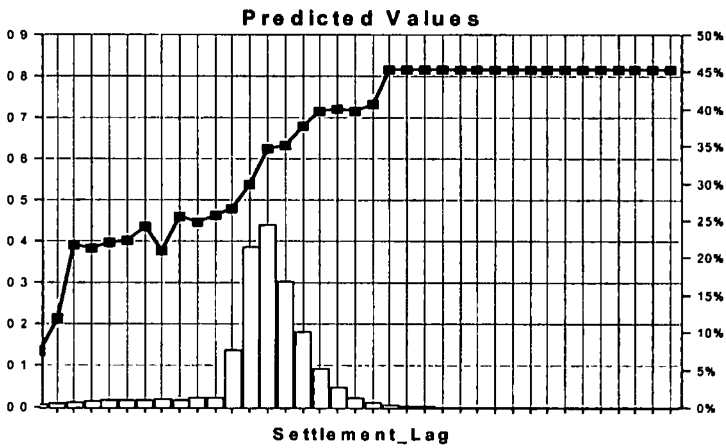
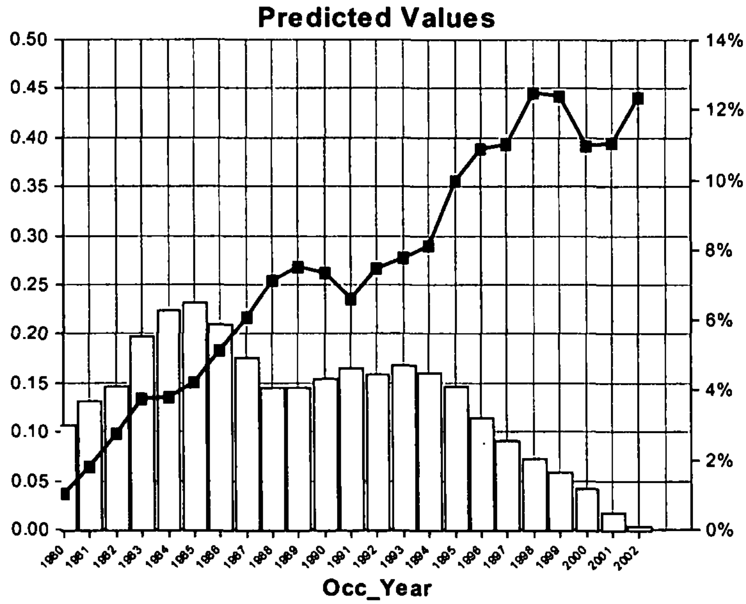


Table 18. Predicted ALAE to Loss and ALAE Ratios by Settlement Lag



Easily the most disconcerting element of the non-economic damages study was the skyrocketing of non-economic damages to total loss by occurrence year as is seen in table 19.

Table 19. Predicted Non-economic damages to total loss by Occurrence Year



Many additional insights can be gleaned from GLM model and analysis, but hopefully these exhibits have shown the highlights and demonstrated the usefulness of GLM for this type of analysis.

Enhancements

Licensed physician counts by year, county, and specialty are available for a nominal fee from the American Medical Association. The Florida Closed Claim database could easily be augmented with this exposure data to create an ideal data set for modeling frequency characteristics in a manner very similar to the approach shown in this paper.

Applications

A GLM severity analysis of the type shown in this paper could have multitude of potential applications. It could be used to assess the value added to the claims process by different claims offices, "managed care" operations offering provider networks for such services as auto glass repair, defense attorney services, and health services generally or just for prescription drugs or medical appliances. These types of "utilization review" GLM applications have some fascinating results when applied to workers compensation and commercial automobile programs. A more detailed understanding of changing severity trends is an obvious use of a GLM severity analysis. These analyses also have applications in pricing and reinsurance program design. Medical malpractice insurers can apply this type of analysis to the development of enhanced classification and territory relativities or underwriting guidelines. Claims departments can apply the results of this type of analysis to change their approach to different types of claims and reserving actuaries can use some elements of this type of study in some loss reserving methods, such as the Berquist-Sherman method.

Conclusion

In retrospect, the original goal of the paper, to identify "Who Dunnit?" was patently unachievable. No single analysis of a single database, no matter how rigorous, could hope to resolve this issue. Hopefully this paper has done something even more dangerous. It is my hope that this paper has demonstrated just one of the many applications of GLM. I have taken some liberties with all of the parties involved in this crisis. Please recognize this as an attempt to bring a little humor to a somewhat dark and emotionally-charged situation. As an actuary, I can think of no more noble a professional goal than to introduce an actuarial technique that can help prevent the pain, suffering, and misery that so many will have to endure in the <u>next</u> crisis. Hopefully, GLM will be one of the tools our professional can take into the future to accomplish just this end.

The Application of Fundamental Valuation Principles to Property/Casualty Insurance Companies

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Joy A. Schwartzman, FCAS, MAAA,
and Dov Siegman

Introduction

This paper explores the concepts underlying the valuation of an insurance company in the context of how other (non-insurance) companies are valued. Among actuaries, the value of an insurance company is often calculated as (i) adjusted net worth, plus (ii) the present value of future earnings, less (iii) the cost of capital. Among other financial professionals (e.g., chief financial officers, investment bankers, economists), value is often calculated as the present value of future cash flows. This paper will discuss both methods and explain under what circumstances the two methodologies derive equivalent value and under what circumstances the results of the two methods diverge. This paper also addresses recent developments in the insurance industry that could affect valuation, including the NAIC's codification of statutory accounting principles, fair value accounting, and the Gramm-Leach-Bliley Act of 1999.

The authors acknowledge David Appel, Ph.D., Director of Economics Consulting Practice for Milliman USA, for his extensive contributions to this paper.

SECTION 1 - Valuation Framework

Why Value a Company?

Valuation of a property/casualty insurance company is an important feature of actuarial work. Much of the work arises from merger, acquisition, and divestiture activity, although the need for valuation arises from other sources. An insurance company valuation might be prepared for lending institutions or rating agencies. It might be performed as part of a taxable liquidation of an insurance company, reflecting the value of existing insurance policies in force. A valuation might also be prepared for the corporate management of insurance companies in order to provide the clearest picture of value and changes in value of the company over a given time period.

The assumptions underlying the valuation and, therefore, the computed value may differ for different uses[1]. As such, the purpose of the valuation and the source of the assumptions should be clearly identified.

Basic Principles of Valuation

Before discussing valuation methodologies, we introduce some basic principles.

1. The value of any business has two determining factors.

    i. The future earnings stream generated by a company's assets and liabilities.
    ii. The risk of the stream of earnings. This risk is reflected in the cost to the entity of acquiring that capital, measured by the investors' required rate of return (i.e., the "hurdle rate").

2. For a given level of future risk, the greater the expected profits[2], the greater the value of the business.

3. For a given level of future profitability, the greater the volatility (and, therefore, the higher the hurdle rate), the lower the value of the business.

4. A company has value in excess of its invested capital only when future returns are in excess of the hurdle rate.

5. When a company is expected to produce an earnings stream that yields a return on invested capital that is less than the hurdle rate, the economic value of the required capital is less than its face value. In this case, the logical action would be to liquidate assets.

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[1] For example, in an acquisition, the purchaser may be able to lower expenses, grow a business faster because of the purchaser's current business, reduce the effective tax rate, or reduce the cost of capital for the acquired or target entity. These assumptions would serve to increase value of the target entity. These same assumptions may not be valid for valuing the target entity as a stand-alone business unit.

[2] Expected profits refer to the present value of the expected earnings stream.

Valuation Methodologies

There are two methodologies prevalent in valuation literature that form the basis of our discussion of insurance company valuation:

(i) Discounted Cash Flow ("DCF")
(ii) Economic Value Added ("EVA")

A DCF model discounts free cash flows to the equity holders at the hurdle rate. The starting capital of the entity is *not* a direct element in the valuation formula[3].

An EVA model begins with the starting capital of the entity and defines value as:

Value = Initial capital invested + PV of expected "excess returns" to equity investors

Sturgis[4] refers to two methods in his paper on valuation:

1. The discounted value of maximum stockholder dividends.
2. Current net worth[5] plus the discounted value of future earnings less cost of capital

The first method corresponds to DCF methodology. The second method is also discussed by Miccolis[6] and in other actuarial literature as.

ANW + PVFE – COC

Where:

ANW	=	adjusted net worth (statutory capital and surplus with a series of modifications)
PVFE	=	present value (PV) of future earnings attributable to in-force business and new business
COC	=	cost of capital = PV of [(hurdle rate × required starting capital for each period) – (investment earnings on capital excluded from future earnings)] [7]

[3] If the starting capital of the entity is higher (or lower) than capital required, it will generate a positive (or negative) cash flow to the investor at "time zero."
[4] Robert W. Sturgis, "Actuarial Valuation of Property/Casualty Insurance Companies"
[5] Throughout this paper, we use the terms capital, equity, net worth, and surplus interchangeably
[6] Robert S. Miccolis, "An Investigation of Methods, Assumptions, and Risk Modeling for the Valuation of Property Casualty Insurance Companies"
[7] If future earnings include investment income on capital, the cost of capital calculation will be modified to be equal to the present value of (hurdle rate × starting capital each period).

This second method is a form of the EVA model, in which *PVFE - COC* equals the present value of expected excess returns.

Discounted Cash Flow

A company's value may be determined by discounting free cash flows to the equity owners of the company[8] at the cost of equity, or the hurdle rate. Free cash flow is often defined as the after-tax operating earnings of the company, decreased by earnings that will be retained in the company, or increased by capital releases to maintain an appropriate level of capital to support ongoing business of the company.

After-tax operating earnings usually constitute changes in capital during a period, other than capital infusions or distributions. For property/casualty insurance companies, however, there are gains and losses in surplus due to "below the line" adjustments[9] that do not flow through statutory earnings. Capital changes associated with the change in unrealized capital gains or losses, the change in non-admitted assets, the change in statutory reinsurance penalties, the change in foreign exchange adjustment, and the change in deferred income tax must be considered along with after-tax operating earnings when evaluating free cash flows. For the valuation formulas discussed throughout this paper, after-tax operating earnings include these direct charges and credits to statutory surplus

A company creates value for its shareholders only when it earns a rate of return on invested capital ("ROIC") that exceeds its cost of capital or hurdle rate ROIC and the proportion of after-tax operating earnings that the company invests for growth drive free cash flow, which in turn drives value For some industries, regulatory or statutory restrictions create an additional consideration that limits dividendable free cash flow.

The DCF value of the business is often projected as two separate components: (a) the value of an explicit forecast period and (b) the value of all years subsequent to the explicit forecast period (the "terminal value"). Projections for the forecast period, which is usually five to ten years[10], typically include detailed annual earnings projections that reflect revenue projections, loss and expense projections, investment income projections, tax liabilities, after-tax operating earnings, assets, liabilities, initial capital and the marginal capital that needs to be invested in the company to grow the company at the expected annual growth rate.[11]

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[8] Free cash flows are released in the form of dividends or other capital releases to the equity owners

[9] "Below the line" refers to the Underwriting and Investment Exhibit in the statutory Annual Statement prescribed by the NAIC Direct charges and credits to surplus are shown below the line for Net Income, which is the starting point for regular taxable income

[10] Five to ten years is typical because beyond that period it is usually too speculative to project detailed financials. A long-term earnings growth rate and a corresponding capital growth rate are selected to derive value beyond the forecast period

[11] Section 4 – Sample Company Valuation addresses these earnings forecasts in detail and provides an example

The value of the **forecast period** is:

$$FC_0 + \sum_{x=1}^{n} \frac{OE_x - (C_{x-1} \times g_x)}{(1+h)^x}$$

Where:

n	= the number of years in the forecast period (usually 5 to 10 years)
OE_x	= after-tax operating earnings in year x (including gains and losses in capital that do not flow through earnings)
g_x	= expected growth rate of capital in year x
C_{x-1}	= capital at the end of year $x-1$; this equals capital at the beginning of year x
$C_{x-1} \times g_x$	= incremental capital required to fund future growth
h	= hurdle rate
FC_0	= free capital at time zero – this represents capital that may be either released from the company at the valuation date if the company is over-capitalized or infused into the company at the valuation date if the company is under-capitalized = $SC_0 - C_0$, the difference between SC_0, the starting capital of the entity, and C_0, the capital *needed* at the end of year zero/beginning of year 1

The value of the second component of DCF value is often referred to as the **terminal value**. The terminal value can be developed using a simplified formula based on (a) projected after-tax net operating profits in the first year *after* the forecast period, (b) the perpetual growth rate, and (c) the hurdle rate.

$$\textbf{Terminal value} = \sum_{x=n+1}^{\infty} \frac{OE_x - (C_{x-1} \times g)}{(1+h)^x}$$

$$= \frac{OE_{n+1} - (C_n \times g)}{(h-g)(1+h)^n}$$

Where.

n	= the number of periods in the forecast period
C_n	= the capital at the end of the last period of the forecast period
g	= the expected perpetual growth rate of capital
h	= the hurdle rate

OE_{n+1} = after-tax operating earnings in the period after the forecast period

$OE_{n+1} - (C_n \times g)$ = free earnings, equal to after-tax earnings less amounts needed to be retained in the company to grow the capital at rate g.

This terminal value calculation gives credit for earnings into the future in perpetuity. Sometimes a higher hurdle rate is used for the terminal value than for the forecast period to reflect the increased uncertainty associated with operating earnings many years in the future. A discussion of considerations related to the selection of the hurdle rate is provided in **SECTION 3 – Parameterizing the Valuation Model**.

The terminal value can be thought of as the present value of the free earnings (in the period after the forecast period) multiplied by a price to earnings ("P/E") ratio. The P/E ratio is determined by the hurdle rate, h, and the growth rate, g, and is equal to $\frac{1}{h-g}$[12].

If the hurdle rate is 15% and the growth rate is 5%, then the P/E ratio = $\frac{1}{.15-.05} = 10$.

In practice, the P/E ratio underlying the Terminal Value calculation can be selected by reviewing sale prices of recent insurance company transactions relative to earnings. Relating that P/E factor to an implicit growth rate and hurdle rate may make the price to earnings ratio more intuitive

Economic Value Added

The value of a company can be written as the sum of the equity invested and the expected *excess returns* to investors from these and future investments

Value = Initial capital invested + PV of expected "excess returns" to equity investors

The expected "excess returns" in each period are defined as:

(rate of return on capital invested – hurdle rate) × capital invested

= after-tax operating earnings – (hurdle rate × capital invested)

To calculate EVA, we need three basic inputs:

1. The level of capital needed for each period to support the investment, both initial capital invested and additional capital to support growth.

[12] The expected growth rate will typically be between 0% and the selected hurdle rate. If, however, the growth rate g were less than 0%, the resulting P/E ratio would decrease (as $h - g$ increases)

2. The actual return earned on that investment in each period, i.e., the after-tax operating earnings or ROIC. (Again, operating earnings include gains and losses in capital that do not flow through earnings.)

3. The selected hurdle rate

These are the same inputs as required for the DCF model.

To determine initial capital invested, we start with the book value of a company. The book value of an insurance company, however, is an amount that reflects the accounting decisions made over time on how to depreciate assets, whether reserves are discounted, and conservatism in estimating unrecoverable reinsurance, among other factors. As such, the book value of the company may be modified in the valuation formula to adjust for some of the accounting influence on assets and liabilities.

In valuing an insurance company, the initial capital invested is represented by the statutory capital and surplus[13] at the valuation date, modified with a series of adjustments discussed later in this paper. The surplus after modifications is often referred to as adjusted net worth (ANW). The capital needed to support growth is funded by retained earnings for the DCF model and reflected through the Cost of Capital calculation for the EVA model.

To evaluate the ROIC, an estimate of after-tax income earned by the firm in each period is needed. Again, the accounting measure of operating income has to be considered. For an insurance company valuation, this component represents the projection of future statutory earnings of the insurance entity, modified in consideration of initial valuation adjustments made to statutory capital, and inclusive of all direct charges and credits to statutory surplus. These earnings will include the runoff of the existing balance sheet assets and liabilities along with the earnings contributions from new and renewal business written. This component may also include investment income on the capital base.[14]

The earnings will reflect a specific growth rate (which could be positive, flat or negative) that must also be reflected in growth in capital needed to support the business. The ROIC represents the after-tax operating earnings in each period (including any "below the line" changes to capital during the period) as a ratio to the starting capital for the period.

The third and final component needed to estimate the EVA is the hurdle rate. Considerations in the determination of the hurdle rate are discussed in **SECTION 3 – Parameterizing the Valuation Model.**

For the EVA model, "excess returns" are represented by the excess of (a) the operating earnings in each period over (b) the product of the starting capital for each period and the

[13] The reasons for using statutory accounting values instead of GAAP or other accounting values are discussed in SECTION 3 – Parameterizing the Valuation Model

[14] If investment income on the capital base is excluded from earnings, the cost of capital calculation will be modified accordingly. This is discussed further in SECTION 2 – Valuation Results: EVA versus DCF

hurdle rate.[15] Recall that a company has value in excess of its capital invested only when ROIC exceeds the hurdle rate for the company. Therefore, a company has positive "excess returns" in a period only when the after-tax operating earnings for that period exceed the product of the hurdle rate and the required capital at the beginning of the period.

In the valuation formula:

$$ANW + PVFE - COC,$$

the term

$$PVFE - COC$$

represents these "excess returns."

"Excess returns" have positive value only when the future earnings exceed the "cost of capital." In this case, the "cost of capital" represents the present value of the (hurdle rate × starting capital) for each period for which earnings are projected. If investment earnings on the capital are excluded from future earnings, then the "cost of capital" calculation will be the present value of [(hurdle rate × starting capital) – investment earnings on the capital].

While the two calculations of excess returns should be mathematically equivalent, there are numerous practical advantages to including earnings on the capital in future earnings. First, the earnings projections will be more in line with historical earnings so one can review the reasonableness of the projections relative to past experience. Second, allocation of assets between capital and liabilities is unnecessary. Third, one does not need to allocate taxes, tax loss carryforwards and other factors between investment earnings on capital and all other earnings.

In **SECTION 4 – Sample Company Valuation**, this paper will demonstrate that the two methodologies, DCF and EVA, produce equivalent values when specific conditions hold.[16] These conditions are:

1. The starting capital and after-tax operating income that is used to estimate free cash flows to the firm for a DCF valuation should be equal to the starting capital and after-tax operating income used to compute EVA. (For insurance company valuations, after-tax operating income should include "below the line" gains and losses in capital that do not flow through earnings.)

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[15] If operating earnings exclude investment income on capital, then the investment income on capital will be subtracted from term (b).
[16] Aswath Damodaran, "Investment Valuation Tools and Techniques for Determining the Value of Any Asset," Second Edition

2. The capital invested that is used to compute excess returns in future periods should be the capital invested at the *beginning* of the period:

   $$\text{Excess Return}_t = \text{after-tax operating income}_t - (\text{hurdle rate} \times \text{capital invested}_{t-1})$$

3. Consistent assumptions about the value of the company after the explicit forecast period are required. That means that for both models, capital required, earnings growth rate, and the hurdle rate must be consistent in computing the terminal value.

4. The hurdle rate for the explicit forecast period must be the same as the hurdle rate after the explicit forecast period

Relative or Market Multiple Valuation

While the value of a company may be derived from the DCF or EVA valuation methodologies, there are other more simplistic methods that are often used to corroborate or supplement more sophisticated models. In relative valuation, one estimates the value of a company by looking at how similar companies are priced. Relative valuation methods are typically based on market-based multiples of balance sheet or income statement values such as earnings, revenues, or book value.

Comparable Companies

The first step in the market multiple approach is to identify a peer group for the subject company. To select insurers for the peer group, it is common to rely on data for publicly traded insurers that meet certain criteria based on premium volume, mix of business, asset size, statutory or GAAP equity, and regulatory environment These criteria are intended to assure that the peer group is reasonably comparable to the subject company. In selecting the criteria, however, it is important to balance precision and sample size. While the analysis could be restricted to only those insurers that were virtually identical to the subject company, the sample size would likely be too small to yield meaningful results.

Valuation Bases

The market multiple valuation method estimates the "market price" of the subject company by reference to the multiples of its peer group. For example, if the peer group average ratio of price to earnings per share is 15 0, and the subject company's most recent annual earnings are $10 million, then the estimated market value of the subject company is $150 million Typically, several alternative ratios will be used in performing a market multiple valuation. In most instances, the ratios employed include an operating multiple (such as the price/earnings ratio), a revenue multiple (such as price/premium or price/total revenues), and a balance sheet multiple (such as the price/book value ratio).

A relative valuation is more likely to reflect the current mood of the market because it is a measure of relative value, not intrinsic value.[17] While these methods serve a valuable purpose in the formulation of an opinion on the price the market may be willing to pay, they provide little guidance on the returns that will be achievable and the extent to which capital outlaid now can be repaid.

[17] Aswath Damodaran, "Investment Valuation Tools and Techniques for Determining the Value of Any Asset," Second Edition

SECTION 2 – Valuation Results: EVA versus DCF

Introduction

The following examples illustrate the DCF and EVA valuation methodologies and derive relevant conclusions related to use of the two methods. This section focuses on the mechanics and properties of the DCF and EVA valuation calculations. **SECTION 4 – Sample Company Valuation** will provide a property/casualty insurance company example.

We will demonstrate two equivalent forms of the EVA model. The first form, "**EVA(a)**" will follow the basic EVA formula structure in which.

excess returns = after-tax operating income – (hurdle rate × capital invested)

The second form, "**EVA(b),**" will use the following definition:

excess returns = after-tax earnings on insurance operations excluding investment income on capital - ((hurdle rate – average investment rate for capital) × capital invested)

Excess returns for EVA(a) and EVA(b) are equivalent in theory. However, while EVA(b) is discussed in actuarial literature on company valuation,[18] there are a number of advantages to using the EVA(a) model in practice. The advantages, previously disclosed in **SECTION 1**, are:

(1) The earnings projections will be more in line with historical earnings so one can review the reasonableness of the projections relative to past experience.
(2) It is not necessary to allocate assets between capital and liabilities.
(3) It is not necessary to allocate taxes, tax carryforwards and other factors between investment earnings on capital and all other earnings.

Basic Model Assumptions

We will use the following assumptions to demonstrate the basic calculations for the DCF and EVA models applied to a property/casualty insurer.

- The capital at time 0, just prior to projected year 1, is $100. For a property/casualty insurance company, this amount is the surplus.

- We tested expected growth rate values of g = 0% and g = 3%.

- Investment income return on capital is 4% per annum.

- The hurdle rate is 15% per annum.

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[18] Robert W. Sturgis, "Actuarial Valuation of Property Casualty Insurance Companies"

- Capital is determined based on a premium to capital ratio of 2:1.

- We separately identify total earnings as investment income on the capital and earnings from insurance operations.[19]

- The investment income on the capital component equals the product of the investment income percentage and the capital at the beginning of the year.

- We show the insurance operation earnings component as a percentage of premiums earned for the year. Premium-related earnings encompass underwriting profits and investment earnings associated with all non-capital assets.

For projection scenarios in which the hurdle rate is exactly achieved, earnings are 5.5% of earned premium[20] For projection scenarios in which the hurdle rate is not achieved, earnings are 5% of premium. When earnings exceed the hurdle rate requirement, this percentage is 6%.

We compiled projection scenarios using two time horizons. First, we estimated the company's value using a 10-year forecast period. We also estimated the continuing value using the present value of earnings beyond 10 years using the same model assumptions.

This time horizon is important in valuing an actual company The 10-year forecast period value will be based on detailed financial projections by line of business as shown in **SECTION 4 – Sample Company Valuation**. The terminal value will be based on the simplified assumptions with respect to (i) expected growth in earnings by future period and (ii) expected changes in capital required by future period.

Total Earnings Equal Hurdle Rate and the Company is Not Growing

Table 1 displays the company value results for the three models in which the annual total earnings relative to capital equals the hurdle rate and neither the company's capital nor business is growing. Exhibits 1A, 1B, and 1C show the calculations leading to these results.

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[19] A number of judgments regarding asset allocation and tax allocation must be made to do this in practice.

[20] 5.5% = 15% hurdle rate less 4% investment income on capital, yielding 11%, which is divided by premium-to-surplus ratio of 2.

Table 1
Valuation Results
Total Earnings Equal Hurdle Rate
No Growth

Model	10 Year Forecast Period[21]	Terminal Value	In Perpetuity (Total)
DCF	75.28	24.72	100.00
EVA(a)	100.00	0.00	100.00
EVA(b)	100.00	0.00	100.00

The *In Perpetuity* results are 100.00, equal to the starting capital of the company.

For the DCF model, the value calculation simplifies to $\frac{OE_1 - 0}{h - 0}$ = (100 × 15%) ÷ 15% = 100. For the EVA(a) model, Exhibit 1A shows that for each forecasted year the total earnings are exactly offset by the cost of capital. This result, of course, follows because both earnings and cost of capital are 15% of each year's starting capital of 100. The same progression is demonstrated by the EVA(b) model except earnings are only 100 × 11% (earnings on insurance operations only) offset by cost of capital of 100 × (15% – 4%).

As noted in the *Basic Principles of Valuation* section, a company has value in excess of its capital invested or hurdle rate only when future returns are in excess of the hurdle rate requirement In the DCF model, the present value of the perpetual cash flow is equal to the starting capital because annual earnings of 15% of capital, discounted at 15% annually, yields the starting capital In the EVA models, excess returns are always 0 and, therefore, the only contribution to value is the capital

Looking at the modeled time periods (10-year forecast period and terminal value) reveals a fundamental difference in the DCF and EVA models. The DCF model must be computed *in perpetuity* (forecast period plus terminal value) to capture the capital value in the company. The EVA models, however, recognize the value of the capital "immediately" as it incorporates the capital amount directly in the value computation. *Therefore, the EVA model will produce higher estimates of value than DCF when earnings are not valued in perpetuity.*

Total Earnings Equal Hurdle Rate and the Company is Growing

Table 2 displays the company value results for the three models in which the annual total earnings relative to capital equals the hurdle rate and the company's capital and earnings are growing by 3% per annum. Exhibits 2A, 2B and 2C show the calculations leading to these results.

[21] Excess earnings are 0, so value for the EVA methods is equal to the starting capital

Table 2
Valuation Results
Total Earnings Equal Hurdle Rate
Earnings (and Capital) Growing @ 3% per annum

Model	10 Year Forecast Period	Terminal Value	In Perpetuity (Total)
DCF	66.78	33.22	100.00
EVA(a)	100.00	0.00	100.00
EVA(b)	100.00	0.00	100.00

The results in Table 2 are nearly identical to the value results shown in Table 1 in which no business growth was modeled. Basically, the EVA models behave exactly the same – the earnings each year are exactly offset by the cost of capital. Incorporating growth into the model only changes the earnings and cost of capital amounts for each year, not the difference between the two values. However, this basic demonstration still emphasizes the relationship of earnings to hurdle rate as the determinant of value, positive or negative, in conjunction with starting capital.

The components of the DCF model result do change from a no-growth to growth assumption. The value amount for the 10-year forecast decreases and is exactly offset by an increase in the terminal value. The "total" *in perpetuity* amount, however, is not affected by growth because annual earnings are still equivalent to the hurdle rate. Growth, however, shifts more of the company's value to later projected years at the expense of earlier projected years. This "value shift" occurs because the DCF model accounts for capital growth via a reinvestment of a portion of annual earnings, thereby reducing free cash flows.

Funding Capital Growth: Comparing the DCF and EVA Models

The DCF and EVA models have different treatments of the costs associated with growing the capital base of the company. We think of the DCF model as a reinvestment for growth process and the EVA model as a capital borrowing process.

Exhibit 2A, Column (8) shows the annual capital reinvestment amount necessary for the DCF model to account for the 3% growth in capital. The capital reinvestment amount is taken from current year earnings to fund the following year starting capital - Column (2) equals Column (8) shifted one year. The DCF model fully funds capital growth, thereby reducing "free cash flows" for valuation.

In the EVA models, the cost for growing the capital is a part of the cost of capital calculation. For the EVA(a) model, Exhibit 2B, Columns (10a) and (10b) show the components of the cost of capital related to the initial and additional capital for growth, respectively. The growth-related earnings reduction equals the product of the hurdle rate and the cumulative additional capital amount beyond the initial capital. This increment can be thought of as the interest payment on "borrowed" capital used to fund business growth

Although the negative cash flows necessary to support capital growth are different for the DCF and EVA models, the present values of the cash flows are identical when considered *in perpetuity*. The DCF model reinvestment to grow the capital is a larger offset to earnings in early forecasted years than the EVA model required return on additional capital amounts. By the 9th forecasted year, though, the EVA model capital growth cost (Exhibit 2B, Column 10b) overtakes the DCF model reinvestment amount (Exhibit 2A, Column 8).

Total Earnings Not Equal to Hurdle Rate and the Company is Not Growing

Table 3 displays the company value results for the three models in the scenario in which the annual total earnings relative to capital *does not* equal the hurdle rate and the company is not growing Exhibits 3A, 3B, 3C, 4A, 4B and 4C show the calculations leading to these results.

Table 3
Valuation Results
Total Earnings Not Equal to Hurdle Rate
No Growth

Model	10-Year Forecast Period	Terminal Value	In Perpetuity (Total)
Earnings Less Than Hurdle Rate			
DCF	70.26	23.07	93.33
EVA(a)	94.98	(1 65)	93.33
EVA(b)	94.98	(1.65)	93.33
Earnings Greater Than Hurdle Rate			
DCF	80 30	26.37	106.67
EVA(a)	105.02	1.65	106.67
EVA(b)	105.02	1.65	106.67

Table 3 reaffirms the *in perpetuity* equivalency of the DCF and EVA models. Like the previous examples, the 10-year and terminal values are different between the DCF and EVA valuations but the *in perpetuity* valuations are equal. The equivalency of the DCF and EVA models in perpetuity will be shown on an algebraic basis in the Appendix.

When the earnings are not equal to the hurdle rate there is a marginal value (positive or negative) in addition to the initial capital As expected, when hurdle rate requirement exceeds earnings, the value of the company drops below the value of the starting capital ($100 in this example) Likewise, when earnings exceed the hurdle rate, there is additional value beyond the initial capital In Exhibits 3A, 3B, and 3C, the total annual earnings is 16% and the cost of capital is dictated by the hurdle rate, 15%, leaving an excess return on capital of 1% for each year in the future. The present value of the 1% marginal profit in return on capital of 100 is 6.67 *in perpetuity*. Referring to Exhibits 4A, 4B, and 4C, a 1% marginal loss in return on capital of 100 leads to a value decrease of 6.67.

Total Earnings Not Equal to Hurdle Rate and the Company is Growing

Table 4 displays the company value results for the three models in the scenarios in which the annual total earnings relative to capital *does not* equal the hurdle rate and the company's capital and earnings are growing by 3% per annum. Exhibits 5A, 5B, 5C, 6A, 6B, and 6C show the calculations leading to these results.

Table 4
Valuation Results
Total Earnings Not Equal to Hurdle Rate
Earnings and Capital Growing @ 3% per annum

Model	10 Year Forecast Period	Terminal Value	In Perpetuity (Total)
Earnings Less Than Hurdle Rate			
DCF	61.22	30.45	91 67
EVA(a)	94.43	(2.76)	91.67
EVA(b)	94 43	(2.76)	91.67
Earnings Greater Than Hurdle Rate			
DCF	72.35	35.99	108 33
EVA(a)	105.57	2.76	108.33
EVA(b)	105 57	2.76	108.33

The impact of growth on the company's value is to increase the portion of value contributed in the future. If the company's earnings are not achieving the hurdle rate, growing the business further lowers value. When earnings exceed the hurdle rate, growth produces increased value

The DCF model results show that capital growth, necessary to support business and earnings growth, reduces free cash flow in the short term in return for an increase in future earnings. Looking at the *Earnings Greater Than Hurdle Rate* scenario, the 10-year forecast period value with no growth is 80.30, dropping to 72.35 with 3% annual growth However, the comparable terminal value amounts increase from 26 37 to 35 99 yielding an *in perpetuity* gain in total value of 1 66 with growth (108.33 with 3% growth versus 106 67 with 0% growth). In the early projection years, the reinvestment earnings to grow the capital (thereby reducing free cash flows) exceed the marginal increase in earnings on the additional capital. This reverses itself in later projection years, resulting in higher terminal values.

Comparison of DCF and EVA Models

The parameterization of the DCF and EVA models presented in the paper cause the models to produce equal value if considered in perpetuity. The parameters selected to populate the models should be equivalent as they are independent of which model is used. For example, the appropriate hurdle rate does not depend on the model selected. The Appendix discusses the formula assumptions necessary to ensure the equivalency property The equivalency of these valuation methodologies is expected because each model is measuring the same value contributors, just using different formula structures.

In the DCF model, the starting capital is used only to determine free cash flow at time 0. The principle of a DCF valuation is that an investment, a company for our discussion, is worth the value of its future earnings. If the capital leads to future earnings (by investment and supporting profitable business), then value will emerge. If future earnings are less than the hurdle rate, then the capital invested in this entity is less than its face value[22].

The EVA model (both forms, EVA(a) and EVA(b)) includes the full starting capital for its determination of value, but at a cost represented by the Cost of Capital calculation. Column 10a in the EVA model calculations (Exhibits 1B, 2B, 3B, 4B, 5B, and 6B) shows the cost of the initial capital. The present value of this negative cash flow in perpetuity exactly offsets the value contributed by immediate recognition of the capital in the EVA formula. If the capital does not provide earnings equal to or greater than the hurdle rate in the form of excess profits, then the capital does not substantiate its value and is worth less than 100 cents on the dollar.

That the EVA model counts the initial capital amount as value and the DCF model does not leads to significant differences in value contributors between the forecast period value and the terminal value. Tables 1, 2, 3, and 4 all show that the 10-year forecast period results for the EVA model are close (and sometimes equal) to the *in perpetuity* time frame results. In the EVA model, therefore, excluding earnings beyond a certain time period does not have a material effect on value. In contrast, a significant portion of the value indicated by the DCF model is captured as terminal value. In these examples, in which the total earnings of the company are set close or equal to the hurdle rate, the EVA model approaches *in perpetuity* value faster

Table 5 shows model value results in which earnings related to operations are 0.0%.

Table 5
Valuation Results
Earnings on Operations =0.0%
Total Earnings = 4.0% (Investment Only)
No Growth

Model	10 Year Forecast Period	Terminal Value	In Perpetuity (Total)
DCF	20.08	6.59	26 67
EVA(a)	44.79	(18.13)	26 67
EVA(b)	44.79	(18.13)	26 67

[22] The value of capital is worth 100 cents on the dollar if you can release the capital at time zero Otherwise, the capital is worth the present value of the distributable earnings generated by the capital. If distributable earnings represent a return lower than the hurdle rate, then capital is worth less than 100 cents on the dollar

For a scenario in which the company's earnings potential is low, the DCF model produces value closer to the *in perpetuity* value in the 10-year period than the EVA model. The DCF model is not "fooled" by the value of the stated initial capital in the short term. The DCF model considers only the earnings potential of the capital, not the capital itself. The result is further exaggerated when growth is incorporated as shown in Table 6.

Table 6
Valuation Results
Earnings on Operations = 0.0%
Total Earnings = 4.0% (Investment Only)
Earnings and Capital Growing @ 3% per annum

Model	10 Year Forecast Period	Terminal Value	In Perpetuity (Total)
DCF	5.57	2.77	8.33
EVA(a)	38.78	(30.45)	8.33
EVA(b)	38.78	(30.45)	8.33

Comparison of EVA(a) and EVA(b)

We present two versions of the EVA model. EVA(a) and EVA(b). The EVA(a) version defines excess earnings as the difference in after-tax operating income and the cost of invested capital. After-tax operating income is recognized for the company as a whole; the amount is not segregated into investment versus operational earnings. Likewise, the cost of capital relies on the product of the "full" hurdle rate and the amount of capital.

The EVA(b) model formula defines earnings and cost of capital differently. The EVA(b) model formula does not include investment earnings related to the capital as earnings. In the context of a property/casualty insurer, earnings are only underwriting earnings from premium written and investment income on assets supporting the liabilities ensuing from writing insurance policies. Under EVA(b), earnings are lower, but so is the cost of capital. The cost of capital is the hurdle rate less the investment income rate the company will earn on its capital, in a sense, the shortfall in investment earnings relative to the hurdle rate.

From the basic valuation examples presented in this section, the two forms of the EVA produce identical results. EVA(a) follows from financial valuation fundamentals.[23] EVA(b) is often regarded as the "actuarial valuation method." Sturgis[24] describes the economic value of a property/casualty insurance company as composed of three parts: (i) current net worth, plus (ii) the discounted value of future earnings, less (iii) cost of capital, where future earnings and cost of capital are defined per our EVA(b) model.

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[23] McKinsey & Company, Inc., "Valuation Measuring and Managing the Value of Companies," Third Edition

[24] Robert S. Miccolis, "An Investigation of Methods, Assumptions, and Risk Modeling for the Valuation of Property/Casualty Insurance Companies"

Miccolis describes a computation similar to Sturgis to determine an insurer's economic value (i) adjusted surplus, plus (ii) discounted value of future earnings, less (iii) cost of capital. Miccolis, however, is unclear regarding the computation for the cost of capital.

We consider the EVA(a) model to be the preferred structure for applying the economic value added model. EVA(a) is more straightforward to apply and avoids potential complications It relies on financial estimates of earnings that are comparable to actual financial projections for a property/casualty insurer. To use the EVA(b) model, one must attempt to isolate the source of earnings between amounts earned from premium written and investment income on the capital. This approach further necessitates an allocation of invested assets between those supporting the liabilities and assets underlying the capital and surplus. In addition, splitting earnings into its "component" parts raises potential tax application questions that complicate the valuation process.

SECTION 3 – Parameterizing the Valuation Model

Accounting

Insurance companies in the United States use multiple forms of accounting. Statutory accounting principles (SAP) are used for reporting to state regulatory authorities and generally accepted accounting principles (GAAP) are used for reporting to the Securities and Exchange Commission and the public. Tax accounting underlies the computation of taxable income. SAP focuses on the current solvency of an insurance company and its ability to meet its obligations. Due to this focus on protection of policyholders, assets and liabilities are generally valued conservatively on the statutory balance sheet, although the result is dependent on specific company or financial conditions.

Historically, noteworthy differences between GAAP and SAP for property/casualty insurance companies related to:

1. Deferred acquisition costs ("DAC")
2. Deferred tax assets ("DTA") and liabilities ("DTL")
3. Premium deficiency reserve ("PDR")
4. Valuation of bonds

- Deferred acquisition costs

  The asset associated with DAC recognizes that the unearned premium reserve ("UEPR") may be overstated because it funds expenses (e.g., agents' commissions) that have are typically paid at the beginning of the policy and have already been incurred on the income statement. As the unearned premium reserve is earned, this overstatement disappears.[25] Statutory accounting does not permit recognition of the value of this asset until it materializes in future statutory earnings. In isolation, this difference in the treatment of the DAC asset would cause GAAP equity always to be greater than or equal to SAP equity.

- Deferred tax assets and liabilities

  Deferred tax assets and liabilities are created primarily from taxes calculated on earnings reflecting discounted reserves and tax liabilities related to unrealized gains (losses). For a growing company, the tax calculation results in an "overpayment" of taxes initially related to incurred losses, offset by a lower payment in subsequent years when claims are paid. This difference is solely a timing issue, as the total amount of taxes that will be paid for profits associated with a block of business or block of assets does not change. The prepayment of taxes (or tax credit for unrealized losses) is corrected as the business runs off or the assets are sold.

[25] For a going concern, we acknowledge that it is replaced by equity in the unearned premium reserves for the following year's business

With the introduction of DTA and DTL for statutory accounting, these assets and liabilities are now recognized on the balance sheet before the business runs off or the assets are sold. For many companies, this change increases their statutory capital

- Premium deficiency reserves

  The PDR is required when the unearned premium reserve is expected to be insufficient to fund the future loss and expense payments originating from those policies. This reserve will reduce statutory capital.

- Valuation of bonds

  In general, SAP requires bonds to be held at amortized cost (although bonds that are not "in good standing" are carried at market value). GAAP, on the other hand, uses amortized cost for only "held-to-maturity" bonds, which the company has both the intent and ability to hold to maturity. For those bonds in the company's active trading portfolio, GAAP requires market value treatment on the balance sheet

With the codification of statutory accounting principles, which became effective January 1, 2001, deferred tax assets, deferred tax liabilities and premium deficiency reserves were recognized on the statutory balance sheet. The most significant difference that remains relates to deferred acquisition costs.

As stated in Actuarial Standard of Practice (ASOP) No 19, for insurance companies, statutory (or regulatory) earnings form the basis for determining distributable earnings, since the availability of dividends to equity owners is constrained by the amount of accumulated earnings and minimum capital and surplus requirements. Both of these amounts must be determined on a statutory accounting basis. Distributable earnings consist of statutory earnings, adjusted as appropriate in recognition of minimum capital and surplus levels necessary to support existing business. Therefore, statutory accounting determines the earnings available to the equity owners.

While future earnings calculated according to GAAP or another basis will often be of interest to the user of an actuarial appraisal, the free cash flow calculations contemplated within the definition of actuarial appraisal in ASOP No. 19 should be developed in consideration of statutory earnings, rather than some other basis.

GAAP earnings and GAAP net worth, however, are often the basis of the relative valuation methods involving market multiples.

As the major difference between GAAP and SAP accounting is DAC, which may be recognized as an asset on the GAAP balance sheet immediately instead of through future earnings, GAAP net worth is typically higher than SAP net worth. SAP net worth may be greater, however, when the amortized value of bonds in the SAP asset portfolio is higher than the market value of bonds in the GAAP asset portfolio.

Estimating Free Cash Flows or Value Added

Estimating free cash flows for a DCF valuation or changes in value of the company in each period for an EVA valuation requires the use of after-tax operating earnings from accounting statements. However, accounting earnings may not represent true earnings because of limitations in accounting rules and the firms' own actions.

For a property/casualty insurance company, changes in the equity of the firm derive from not only (a) after-tax operating earnings (net income in the statutory income statement) and (b) capital infusions or distributions, but also from (c) "below the line" adjustments to capital. These adjustments represent items that do not flow through the statutory income statement for changes in unrealized capital gains/losses, changes in non-admitted assets, changes in provisions for reinsurance, change in foreign exchange adjustment and changes in deferred income taxes. To the extent that these adjustments increase (decrease) the equity of the firm, they also increase (decrease) free cash flows for the DCF valuation methodology and increase (decrease) excess returns for the EVA valuation methodology.

For a property/casualty insurer, estimating after-tax operating earnings (including "below the line" statutory adjustments to capital) typically requires rigorous analysis. For the purpose of analysis, the sources of future earnings can be sub-divided into two broad categories the runoff of the existing balance sheet and future written business.

Runoff of the Existing Balance Sheet

The runoff of the existing balance sheet produces earnings associated with (i) underwriting profit imbedded in the UEPR[26], (ii) investment income on the assets supporting (a) the loss reserves (inclusive of all loss, allocated loss adjustment expense and unallocated loss adjustment expense reserves) and (b) UEPR liabilities until all the associated claims are paid, and (iii) investment income on the capital base supporting the runoff of the business.[27]

The earnings associated with new (or renewal) business derives from (i) the underwriting profit generated by the business, (ii) the investment income on the assets generated by the premium, supporting loss reserves and UEPRs until all of the claims are paid, and (iii) investment income on the capital base supporting the writing of the new business.

Developing financial projections (income statements, balance sheets, and cash flows) related to running off the existing balance sheet liabilities, assuming no new or renewal business is written, will provide the basic elements for valuing the company in runoff. The key factors involved are (i) the payout of the loss reserves, (ii) the ultimate losses and expenses associated with the unearned premium reserve, (iii) the payout of the losses

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[26] Profit imbedded in the UEPR represents underwriting profit and profit associated with the prepaid expenses (corresponding to the deferred acquisition cost asset established for GAAP accounting).
[27] For an EVA valuation, if one projects earnings with a capital base of zero (an EVA(b) scenario), this component will be zero.

and expenses associated with the unearned premium reserve, (iv) the capital needed each year to support the company in runoff and (v) the investment yield earned on assets until all claims are paid and all capital is released In practice, when running off a company that writes personal lines business, renewals may be mandated for several years by the regulatory authorities. In those instances, running off the company might also reflect the writing of some renewal business

When it is important to understand the value associated with the runoff of the business separate from value associated with the writing of new (or renewal) business, we recommend the following approach. Value the company in runoff reflecting the level of capital required to run off the company. Then, value the company reflecting earnings and capital needs associated with maintaining the company as a going concern That is, earnings projections and capital needs are developed for the combination of running off the existing balance sheet and writing new business. The value of solely writing new business should be computed as the difference between the two valuations.

The suggested approach is beneficial on both a practical and theoretical basis. On a theoretical basis, the valuation of the runoff company relative to the going concern improves the determination of capital required for new business. On a practical basis, both valuations will use the same starting balance sheet

Future Written Business

For property/casualty insurance companies, in contrast with life insurance companies, the distinction between new and renewal business is often not meaningful for developing financial projections for future written business. For direct writers of personal lines business, however, for whom the initial cost of acquiring new business and the associated expected loss ratio differs substantially from the expenses and loss ratios associated with renewal business, the distinction between new and renewal business may be very important for developing financial projections

Financial projections are usually developed by line of business or business segment that corresponds to the detail in which the company being valued provides its premium forecasts. The key elements to be estimated by year are:

By line of business:

- Gross written premium
- Net written premium
- Accident year gross and ceded loss and loss expense ratios
- Gross commissions and ceding commissions
- Other overhead expenses (premium taxes, general and administrative expenses, other acquisition costs)
- Collection schedules for premium
- Payment schedules for commissions and other overhead expenses
- Payment pattern for gross and ceded accident year loss and loss adjustment expense

- Collection pattern for ceded reinsurance recoveries

For the book of business in total:

- Investment yield on investible assets
- Capital needed to support the entire book of business
- Federal income taxes applicable to earnings

The primary contributors to investment earnings are the timing differences between the collection of premium and the payment of claims and loss adjustment expense. For most lines of business, there is little delay in premium payment by the policyholder When premiums are paid in installments, however, or when audit premiums represent a significant portion of the ultimate collected premium, it is important to evaluate the lag because of the resulting impact on the investment income calculation Reinsurance recoveries may need to be projected on a contract-by-contract basis if the indemnification terms vary significantly.

In determining the future earnings from new and renewal business, projected loss and expense ratios are the most important components to be modeled. As Miccolis[28] and Ryan & Larner[29] note in their papers on valuation, issues to be considered in the projection of future loss and expense ratios include:

- Changes in price levels
- Trends in loss severity, claim frequency, and exposure base
- Historical industry results
- Underwriting cycles
- Target rates of return
- Expected future growth rates
- Degree of competition in market
- Regulatory environment
- Exposure to catastrophes
- Changes in ceded reinsurance (coverage, terms, pricing)

Present Value of Future Earnings

Once the future earning stream (including gains and losses in capital that do not flow through earnings) from running off the existing balance sheet and future written business has been estimated, it is discounted to present value at the selected hurdle rate For an EVA valuation, the future earnings stream is used directly without consideration of capital infusions or distributions. For a DCF valuation, the future earnings stream (a) less

[28]Robert S Miccolis, "An Investigation of Methods, Assumptions, and Risk Modeling for the Valuation of Property/Casualty Insurance Companies"
[29]J P Ryan and K.P W. Larner, "The Valuation of General Insurance Companies"

earnings retained for capital growth or (b) plus additional capital released represents free cash flows.

Adjusted Net Worth ("ANW")

In valuing a company, it is common practice to adjust the equity of the firm at time zero to consider value (positive or negative) associated with reserve deficiencies or redundancies, market value of assets, non-admitted assets, and statutory provisions for reinsurance, among other factors.

The adjustments to statutory equity in the computation of ANW for an EVA valuation (and free cash flow at time 0, "FC_0," for a DCF valuation) represent an effort to adjust the starting statutory balance sheet to its true market value. These adjustments described by Miccolis and Ryan & Larner and summarized below represent an attempt to recognize the market value of some items on the statutory balance sheet. For example, common adjustments include reflecting assets at market value and eliminating goodwill. In contrast, there are usually no comparable adjustments for liabilities. For loss reserves and unearned premium reserves, market value would reflect future investment income plus a provision for risk. Instead, any value associated with the liabilities (other than adjusting reserves to their actuarially indicated amount) is recognized through the present value of future earnings

Since statutory accounting determines free cash flows to investors, one could support the position that adjustments to the equity of the firm at time zero should be limited to tax-affected reserve adjustments (to bring carried reserves to the actuarial indicated level) and other changes that "true up" the statutory balance sheet. Adjustments to statutory capital to compute ANW that are not permitted under statutory accounting will not change statutory capital and, therefore, will not affect free cash flows. Many financial experts, however, insist that assets be adjusted to their market value at the date of valuation. Further, goodwill carried on the balance sheet is almost always eliminated for valuation, even though it is now a statutory asset. Experts continue to disagree on how these adjustments should be handled for valuation

Either way, if the net worth or the equity of the firm is adjusted to recognize non-admitted assets, or reflect the market value of all assets, then the firm's future earnings or changes in capital must be adjusted to prevent double counting this value. For example, if all assets are marked to market for the valuation, then future earnings of the firm must not reflect any realized gains or losses associated with assets unless the market values change. Further, if non-admitted assets are added back to the starting net worth of the firm, then any capital increases associated with the recognition of non-admitted assets must be eliminated from future financial projections.

Any adjustments to the starting capital to determine ANW will cause the EVA and DCF valuation results to diverge unless the same adjustments are made for both valuation methodologies. Otherwise, for DCF, these values will be recognized on a discounted basis through future earnings or "below the line" adjustments to equity. For EVA, they will be recognized at time zero, thereby reflecting no present value discount in the computation of value.

The common adjustments to the starting capital ("SC_0") for valuation are listed below. Only items 1 and 6 are consistent with statutory accounting principles and, therefore, will have the same effect on EVA and DCF valuations. The other adjustments to ANW, unless also assumed to impact SC_0 for DCF, thereby affecting FC_0, will cause the EVA and DCF valuation results to diverge. The direction (positive or negative) of the difference between the EVA and DCF valuation result will be dependent on the direction (positive or negative) of the tax-affected adjustments for items 2, 3, 4 and 5.

1 Loss reserve adequacy

For a property/casualty insurance company, policyholder claim obligations are usually the largest liability on the statutory balance sheet. As a result, it is critical to assess the reasonableness of the carried loss and loss adjustment expense (LAE) reserves as of the valuation date to meet unpaid claim obligations

Adjustments for the loss reserve position should be made directly against statutory equity as of the valuation date for both DCF and EVA valuations. Adjustments to the carried loss reserves will impact ANW for an EVA valuation and FC_0 for a DCF valuation

2. Market value of assets

Traditionally, the majority of property/casualty insurance companies' investment portfolios have been placed in bonds, especially U.S. Treasury or other federal agency instruments. SAP requires bonds "in good standing" to be valued at amortized cost. For the purpose of a valuation, however, bonds should be valued at market value in order to reflect what an independent buyer would actually pay to purchase the securities.

Common and preferred stocks, which represent the next largest portion of most property/casualty insurance companies' portfolios, are recorded at values provided by the NAIC's Securities Valuation Office. These values are typically equal to market value and thus are less likely to require an additional adjustment. Other investable assets should also be adjusted to market value, but are a much smaller component of the total portfolio and thus the adjustments are likely to have a smaller impact on the adjusted net worth.

3. Inclusion of non-admitted assets

Some states do not admit certain assets on the statutory balance sheet because they either do not conform to the laws and regulations of the state or are not readily convertible to liquid assets. Exclusion from the balance sheet results in a charge to statutory equity For the purpose of a valuation, however, one should include any portion of non-admitted assets that has financial value and may be convertible to cash.

Examples of non-admitted assets include:

- Agents' balances overdue by 90 days or longer
- Bills receivable that have not been taken for premium
- Furniture, equipment (other than electronic data processing [EDP] equipment and software), and supplies
- Leasehold improvements

In some cases, there may be overlap with the adjustment of assets to market value For example, when the market value of real estate is below its net book value, the excess of book over market value is recorded as a non-admitted asset while the admitted asset, which underlies the amount of statutory surplus, is equal to the market value Care should be taken to ensure that there is no double-counting.

4. Accounting goodwill

   SAP for purchases define *goodwill* as the difference between the cost of acquiring a subsidiary, controlled, or affiliated entity and the purchaser's share of the book value of the acquired entity. Positive goodwill exists when the cost of the acquired entity is greater than the purchaser's share of the book value. According to codified SAP, however, positive goodwill from all sources is limited in the aggregate to 10% of the parent's capital and surplus (adjusted to exclude any net positive goodwill, EDP equipment and software).

   Assets for goodwill are generally assumed to have zero value until such value emerges through future earnings.

5. Provision for reinsurance

   SAP produce a "provision for reinsurance" that is calculated in Schedule F of the NAIC Annual Statement and is carried forward to the statutory balance sheet as a liability. This provision is intended to be a measure of conservatism to reflect unsecured reinsurance placed with unauthorized companies and collectibility issues with all reinsurers.

   In a valuation, a more detailed review of collectibility issues is worthwhile in order to estimate any additions (or further reductions) to equity to reflect a more rigorous estimate of reinsurance recoverables.

6 Tax issues regarding all of the above

   Any adjustments to the statutory balance sheet may also have a corresponding impact on the company's federal income tax liability. The federal tax liability, or deferrable tax asset, is based on statutory net income and a series of adjustments. Any adjustments made to statutory equity for valuation should be tax-affected

In mergers or acquisitions, taxes are particularly difficult to address because one must consider the tax position of both parties.

Hurdle Rate

The hurdle rate used in a valuation should reflect the cost to the firm of acquiring the capital necessary to make the acquisition or perform the transaction in question Typically, this value will be provided by management based on its appraisal of the acquisition's relative risk and required return. When not provided by management, the hurdle rate can be estimated using a variety of security valuation methods.[10] In either case, when establishing the hurdle rate, it is important for the analyst to consider several issues including the following:

1. Risks attributable to business activities of the acquisition

   The risk attributable to the business activities of the acquisition determines the cost of the capital required to make the acquisition. This risk measure should not be confused with the risk associated with the acquiring entity, which may be different. The risk of a firm, in total, reflects a weighted average of the risks of its underlying business activities and the cost of capital of any particular activity may differ from that of the firm as a whole.

2. Consideration of multiple hurdle rates

   If the target acquisition in engaged in several activities (e.g., different lines of business) of varying risk, it may be appropriate to consider projecting several streams of free cash flow and discounting them at different rates. An alternative to this approach may be to allocate capital to business activity in such a way as to equalize risk across lines. If this approach is used, then a single discount rate for all cash flows may be appropriate.

   One reason to consider the latter approach is that one can generally observe the hurdle rate only for the firm as a whole, and not for its component parts. Thus, the hurdle rates reflect the average risk of the firm's activities and are not necessarily appropriate for any single business. If there were large samples of publicly traded firms specializing in particular lines of business, then it would be possible in theory to observe the hurdle rate for those specific activities. In practice, however, there are a limited number of publicly traded insurers and they tend to be multi-line firms involved in a wide variety of businesses (many of which have substantially different risk profiles). These considerations support using a single hurdle rate reflecting

[10] The most prominent models in widespread use are the capital asset pricing model (CAPM) and the dividend valuation model (sometimes known as the DCF or Gordon growth model). Both models are described in numerous sources, including Investment Valuation by Damodaran.

average risk activities, and then adjusting the amount of required capital so that the risk of the acquisition is equivalent to the average risk of the firm.

3. Method of financing the acquisition

   If the acquisition is to be financed with a mix of debt and preferred and common equity, then the appropriate hurdle rate should reflect the weighted average after-tax costs to the firm of acquiring capital through these vehicles. The capital structure underlying the acquisition, and not necessarily the existing capital structure of the acquiring entity, is the relevant issue. For example, if a firm is currently financed with a mix of debt and equity, but intends to pursue an acquisition financed solely by equity, then the relevant hurdle rate is the equity cost of capital.

4. Consistency with other assumptions

   The discount rate depends on relative risk, which in turn depends on several factors that may be related to other aspects of the valuation. For example, in addition to the intrinsic risk of its specific business activities, the cost of capital for a firm will depend, among other things, on the firm's leverage and mix of assets. Both of these factors, however, will have an impact on the projected free cash flow that forms the foundation of the valuation. There must be consistency between the assumptions used to develop the cash flows and those used to develop the discount rate[31].

Capital Needs

The capital required to support an insurance company is a key assumption in the valuation process

For the DCF methodology, capital requirements dictate the amount of capital to be retained in the company to support ongoing operations, thereby determining distributable earnings and associated value. For the EVA methodology, capital requirements dictate the capital that underlies the cost of capital calculation. The higher the capital requirement, the higher the cost of capital element of the valuation formula.

Property/casualty insurance companies are subject to statutory capital requirements. Statutory capital requirements are determinable through the property/casualty insurance industry's risk-based capital (RBC) requirements. The results can be viewed as minimum capital requirements. Often, larger capital investments are required to satisfy the financial rating agencies such as A.M. Best, Standard & Poor's, and Moody's in order to maintain desirable financial ratings All of these factors are considerations in determining capital requirements for valuation.

Premium-to-surplus ratios, loss reserves-to-surplus ratios, and multiples of RBC have been used in valuation to determine capital needs These are typically based on

[31] The discount rate is often viewed as the sum of a risk-free rate and a market risk premium (CAPM) The value of the market risk premium is a topic of debate among financial economists

comparable ratios for "peer companies," which are companies with premium volume and lines of business comparable to the subject company. In these instances, it is essential that the selected capital that match or exceed RBC requirements.

In actuarial and finance literature, there are many articles and papers related to capital requirements and capital allocation for insurers. Theories about capital requirements range from simplistic rules of thumb (e.g., maintenance of a premium-to-surplus ratio of 2.0) to intricate risk models.

In practice, it is common for insurance companies to maintain a level of capital that is sufficient for a desired financial rating.

Cost of Capital

We defined the cost of capital ("COC") as the present value of the starting capital in each period multiplied by the hurdle rate. The COC is used to measure excess returns in each period for the EVA valuation methodology. Excess returns are computed as the difference between operating earnings in each period (inclusive of gains and losses in capital that do not flow through earnings) and the COC. This concept is more thoroughly discussed in **SECTION 1 – Valuation Framework** and **SECTION 2 – Valuation Results: EVA versus DCF**.

Economists and other financial professionals equate the term cost of capital with the hurdle rate. Care should be taken in using and understanding the meaning of the term in a particular context.

SECTION 4 – Sample Company Valuation

This section presents a detailed example of valuing a property/casualty insurance company. The modeled valuation will focus on:

- Modeling aspects of a property/casualty insurer given current financial statements, investment assumptions, underwriting assumptions for current and future business, and loss and expense payment assumptions;

- Determination of future earnings from projected financial statements based on selected surplus and business volume constraints;

- Application of DCF and EVA valuation approaches using an existing balance sheet and projected financial statement amounts (balance sheet, income statement, and cash flow exhibits);

- Testing the sensitivity of indicated value to changes in key assumptions (risk-based capital-to-surplus requirement, loss ratios, investment yield, hurdle rate, growth rate).

Our objective is to provide a thorough and functional discussion of the valuation of a property/casualty insurance company valuation and a basic discussion of the development of earnings projections. The actuary or other professional preparing the valuation will, of course, undertake extensive analysis to develop premium, loss, and expense assumptions, investment yields, and other factors to project earnings. We present many assumptions "as given" without further explanation.

Valuation Estimates Based on Financial Model Results

The valuation results for the sample company, Primary Stock Insurance Company or "PSIC", rely on two basic assumption sets:

1. . Financial modeling assumptions underlying financial statement projections, and

2. Valuation assumptions underlying the application of the DCF and EVA methodologies yielding value estimates of PSIC based on the financial statement projections.

Exhibit 7 shows the value estimates for PSIC for each method and the principal components for applying the valuation formulae. The fundamental financial amounts entering the valuation calculations are current and future year-end surplus estimates and future total income estimates. Basic financial modeling assumptions will be discussed later in this section; the primary focus is the application of the valuation methodologies with the modeled surplus and income amounts given specific valuation assumptions

The valuation assumptions are:

1. A valuation date of December 31, 2001.

2. PSIC's risk-based capital ("RBC") indication at each year-end dictates the statutory surplus at the respective year-end. The example uses a surplus-to-RBC relationship of 2-to-1 where the RBC indication is the Company Action Level (100% of the RBC calculation).[32]

3. A hurdle rate of 15% per annum for all future years.

4. After the explicit forecast period ending December 31, 2011, we assume the surplus and total company income will increase at 2% per annum indefinitely.

For each valuation methodology, future valuation amounts are modeled in two distinct time periods: the explicit forecast period (10 years for the example, 2002 through 2011) and all subsequent years (2012 and later). For our sample company valuation, the explicit forecast period income and surplus estimates (via the RBC calculation) rely on financial modeling procedures. Valuation indications for all subsequent years were estimated using the respective method's value formulae starting one year after the explicit forecast period. For the DCF method, this calculation develops to the terminal value. For the EVA method this calculation develops the "continuing value added" after the explicit forecast period.

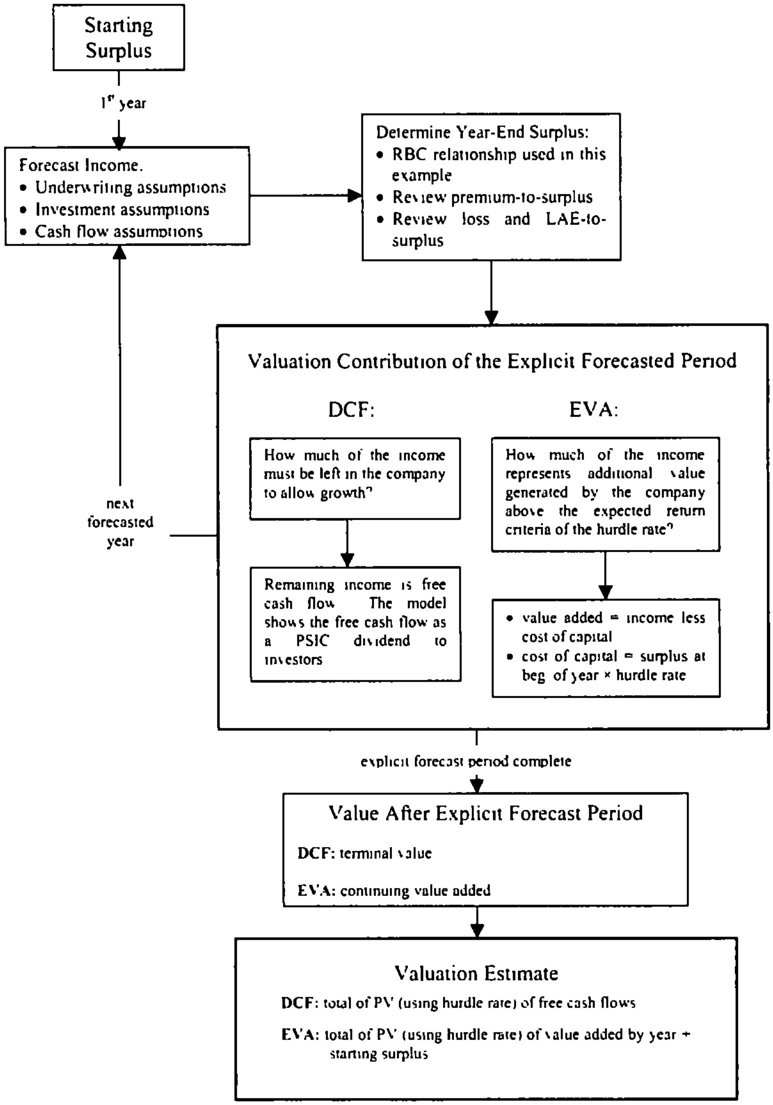
Both models yield value of approximately $88 million as of December 31, 2001. The comparison of the value components for the two methodologies parallels observations made in **SECTION 2 – Valuation Results: EVA versus DCF** about the scenario in which a company achieves more than the hurdle rate and is growing.

- The EVA method recognizes value amounts in the forecast process faster than the DCF method. As of the end of the explicit forecast period, through 2011, the EVA method value estimate is $73.9 million ($42.1 million surplus plus $31.8 million as the present value of future value added in years 2001 through 2011). The DCF method value estimate is $54.7 million representing the present value of free cash flow for years 2001 through 2011.

- The present value of the reinvestment cost (retained earnings) of $21.9 million (for all years) for DCF equals the present value of the cost of growth capital for EVA. The DCF reinvestment cost over the 10-year explicit forecast period ($18.97 million) is greater than the EVA cost of growth capital during the same period ($10.14 million). The difference is offset in modeled amounts for 2012 and subsequent – $2 96 million for DCF and $11.79 million for EVA.[33]

[32] Feldblum, "NAIC Property/Casualty Insurance Company Risk-Based Capital Requirements"
[33] DCF (18 97 + 2.96) = EVA (10 14 + 11 79) = 21 93

The following diagram shows the steps in the development of value presented in Exhibit 7.



The recorded statutory surplus for PSIC as of December 31, 2001 is \$45.00 million. However, this amount exceeds the selected capitalization standard result of 2.0 × \$21 07 million (the RBC indication at December 31, 2001) or \$42.13 million. The "excess" surplus is recognized as free cash flow/value added for both DCF and EVA at December 31, 2001 (time 0) and our valuation models begin with a statutory surplus of \$42.13 million. For the EVA model, the surplus of \$42.13 million is recognized immediately as value. It is also the basis of the Cost of Capital calculation in the first period. For the DCF model, the surplus of \$42 13 million contributes to value only through the investment income it earns in subsequent periods.

No other adjustments were made to the starting surplus for valuation. Carried reserves were assumed to be at the actuarially indicated amount. There was no difference between market value and book value of investments and no other adjustments were deemed warranted.

After establishing PSIC's adjusted net worth, the valuation process requires the total statutory income and RBC amounts for the first future projection year, 2002, from the financial model constructed for PSIC. Exhibit 8, Changes in Statutory Surplus, shows the estimated future income for PSIC during 2002 to be \$10.44 million. The PSIC valuation model includes income from two categories: statutory net income and changes in unrealized capital gains. Exhibit 9 shows the computation of statutory net income. Unrealized capital gains stem from increases in market value for preferred and common stock investments.

The projected RBC for year-end 2002 is \$23.25 million leading to a December 31, 2002 required surplus of \$46.5 million. Exhibit 12 shows PSIC's RBC calculation. During 2002, the required surplus increases by \$4.37 million, from \$42.13 million to \$46.51 million.

The DCF methodology determines value from free cash flow estimates; for 2002 free cash equals \$10.44 million of income less earnings retained to fund surplus growth of \$4.37 million. Exhibit 8 shows the \$6.07 million free cash flow (\$10.44 – \$4.37 = \$6 07) as a stockholder dividend. The contribution to value of the 2002 free cash flow is the PV of \$6.07 million using the 15% hurdle rate.

The EVA methodology values returns in excess of the cost of capital. For 2002, excess returns equal \$10.44 million of income less the cost of capital of \$6.32 million, or \$4.12 million. The cost of capital equals the surplus as of the end of the prior year, \$42.13 million, multiplied by the hurdle rate of 15% The contribution to value of the 2002 excess returns is the PV of \$4.12 million using the 15% selected hurdle rate, or \$3.59 million.

As shown on Exhibit 7, the application of the DCF and EVA methodologies given the total income, RBC, surplus projections, and valuation assumptions is repeated for each year in the 10-year explicit forecast period. The PV of free cash flow for the DCF method during the 10-year period is $54 69 million. The PV of excess returns for the EVA method through the 10-year period is $31.81. The PV of excess returns plus the starting surplus of $42.13 million yields the EVA indicated value through year 10 of $73.94 million.

The All Years value of PSIC under both valuation methods includes the PV contribution of value amounts beyond the explicit forecast period. The amount shown in the "Total '12 to ∞" column in Exhibit 7 rely on perpetuity formula calculations rather than annual detailed financial projections for 2012 and subsequent years Appendix and **SECTION 1 – Valuation Framework** show these formulae for both methods and the algebraic derivation. The key assumptions for these calculations are:

- The expected annual growth rate of surplus and total income after 2011 is 2% Thus, the implicitly projected surplus for 2012 is $77.86 million × 1.02 = $79.42 million and the income for 2012 is $18.71 million × 1 02 = $19 08 million.

- The hurdle rate is 15% for calculating the cost of capital for the EVA method and for determining the PV of 2012 and subsequent value amounts.

Both methods produce a valuation result of $88.03 million

DCF	
(1) Present value of free cash flow during the explicit forecast period	$54.69
(2) Terminal value (present value of free cash flow subsequent to the explicit forecast period)	33.33
Total	$88.03
EVA	
(1) Adjusted net worth (starting surplus)	$42.13
(2) Present value of value added amounts during the explicit forecast period	31.81
(3) Present value of continuing value added subsequent to the explicit forecast period	14.08
Total	$88.03

Overview of the Financial Model

The property/casualty insurer financial model for the PSIC valuation performs all of the necessary computations to produce prospective statutory and GAAP financial statements. The major functions of the model are (i) runoff of loss and LAE reserves, (ii) payout of loss and loss adjustment expenses stemming from the earning of the unearned premium reserve, (iii) estimation of the level of future written premium and associated earned premium and application of the loss and expense ratio assumptions, (iv) calculation of investment income, and (v) calculation of federal income tax due.

There are two items of note before discussing the details of PSIC financial model projections. First, the model does not reflect all the changes resulting from the NAIC's codification of statutory accounting principles An example is the recognition of a statutory asset or liability for deferred taxes. Even without these items, the financial model results provide significant insight into the considerations and calculations for valuing a property/casualty insurance company Second, the GAAP balance sheet and income statements are provided for the interested reader. The GAAP results are not discussed in the text because the valuation estimate relies exclusively on amounts computed using statutory accounting.

Exhibit 11 is the Detailed Statutory Balance Sheet for PSIC The "Actual 2001" column shows amounts from PSIC's December 31, 2001 statutory Annual Statement. Balance sheet items are either the sum of amounts from individual lines of business or for PSIC in total. Investment and cash amounts, items (1a) through (1g) and the Total Investments & Cash subtotal, are not segregated by line, neither are capital and surplus

The remaining assets (receivables) and liabilities (payables and loss, LAE, and unearned premium reserves) are the sums of individual line of business amounts. In this example, PSIC wrote and continues to write three lines of business: workers' compensation, auto liability, and general liability, all on a primary basis. Exhibits 18, 19, and 20 show the December 31, 2001 balance sheet amounts and business assumptions for the workers' compensation, auto liability, and general liability books of business, respectively.

The largest single balance sheet item from the line of business data is the net loss and ALAE reserve Sheet 6 for Exhibits 18, 19, and 20 show the loss and LAE reserves as of December 31, 2001 for accident years 2001 and prior for each line of business. Sheet 5 for each line of business shows the payment patterns for the respective 2001 balance sheet reserve amounts

Sheet 4 for Exhibits 18, 19, and 20 shows the other balance sheet items associated with each line of business as of December 31, 2001.

Exhibit 9 is PSIC's Statutory Income Statement. Exhibit 8, Change in Statutory Surplus, uses net income from Exhibit 9 The annual change in statutory surplus equals net income plus change in unrealized capital gains. Net income has three basic components: underwriting income plus investment income less federal income taxes (The PSIC model does not include any "other income" amounts) PSIC's underwriting income equals the sum of individual line of business underwriting income amounts. Investment income and federal income taxes are computed for PSIC in total. Investment income includes investment income on the capital along with the assets generated by line of business.

Sheet 1 for Exhibits 18, 19, and 20 provides the underwriting income by line of business. Sheet 2 provides the calculation notes for the components of the line of business underwriting income. The principal assumptions are:

Net Earned Premium

- Direct written premium ("DWP") annual growth is 4%
- 50% of DWP is earned in year written, 50% in the following year
- Workers compensation and general liability have excess reinsurance; 10% of the DWP is ceded

Net Incurred Loss and LAE

- As shown in Sheet 4 of Exhibits 18, 19, and 20, the selected loss and LAE ratios for each line of business are:

	Direct Loss Ratio	ALAE to Loss Ratio	ULAE to Loss Ratio	Ceded Loss Ratio
Workers' Comp	70.0%	8.0%	8.5%	100.0%
Auto Liability	64.0%	8.5%	7.5%	N/A
General Liability	68.0%	15.0%	8.5%	100.0%

- These gross loss, gross LAE, and ceded ratios are applied to the December 31, 2001 unearned premium reserve and earned premium generated by forecasted written premium.

Total Underwriting Year Expenses

- As shown in Sheet 4 of Exhibits 18, 19, and 20, the underwriting expense ratios for each line of business are (DWP = direct written premium, CWP = ceded written premium):

	Agents' Commission	Premium Tax	Other Underwriting Expenses		Reinsurance Commissions
	(%DWP)	(%DWP)	(%DEP)	(%DWP)	(%CWP)
Workers' Comp	10.0%	3.0%	3.0%	2.25%	0.0%
Auto Liability	15.0%	2.0%	2.25%	3.25%	N/A
General Liability	12.5%	2.0%	4.0%	1.0%	0.0%

Investment income is shown in row (5) of the Statutory Income Statement (Exhibit 9). The sources of investment income are realized capital gains, interest income, and dividends. The annual yield rates (pre-tax) for each asset type are:

Realized Capital Gains

Preferred Stocks	2.5%
Common Stocks	4.0%
Real Estate	4.0%

Interest Income

Taxable Bonds	6.0%
Non-taxable Bonds	4.0%
Cash	3.0%
Real Estate	4.0%
Other	2.0%

Dividends

Preferred Stocks	5.0%
Common Stocks	2 0%

The distribution of invested assets and cash is:

Taxable Bonds	42.0%
Non-taxable Bonds	24.0%
Preferred Stocks	1.0%
Common Stocks	25.0%
Cash	5 0%
Real Estate	1.0%
Other	2.0%
Total	100.0%

Invested assets held at the beginning of a forecasted year will earn a full year of investment income based on the above yield percentages Investment income is also earned on new cash generated by PSIC's insurance operations. The financial model assumes that cash from operations is collected and invested at the mid-point of each forecasted year. The collected cash is invested according the distribution of invested assets and cash shown above. Thus, the distribution is constant for all forecasted years.

Cash flows from operations are shown in Exhibit 13. Premium collections, loss and LAE payments, and underwriting expense payments are modeled for each line of business. Sheet 3 of Exhibits 18, 19, and 20 shows the cash flow from underwriting for each line of business, respectively. In addition to the premium, loss, LAE and underwriting expense assumptions, the line of business underwriting cash flow relies on the following assumptions:

- Loss and LAE payment patterns for each line of business shown in Sheet 5 of Exhibits 18, 19, and 20, respectively. The payment patterns apply to reserves carried as of December 31, 2001 and loss and LAE incurred in 2002 and subsequent accident years

- Lag of 1 month in collection of direct premium.

- Lag of 3 months in paying ceded premium.

- Lag of 1 month in collection of ceded loss recovery.

Federal income tax is the final component for computing net statutory income. The PSIC model followed the 2001 instructions for computing federal income tax for U.S. property/casualty insurance companies.

Total income for valuation equals net statutory income plus unrealized capital gains as shown in Changes in Statutory Surplus, Exhibit 8. Unrealized capital gains are computed as total annual capital gains in equity investments less realized capital gains. The capital gain percentages are:

Preferred Stocks	11.0%
Common Stocks	9.5%

Sensitivity Testing

Table 7 shows the sensitivity of DCF and EVA value estimates to changes in underlying assumptions. Exhibit 21 shows additional detail related to each of these alternative scenarios.

For ease of reference, the assumptions underlying the base case are listed below:

- Starting capital as of December 31, 2001 = $42.13 million
- Surplus-RBC ratio = 2.0
- Workers' compensation loss ratio = 70%
- Auto liability loss ratio = 64%
- General liability loss ratio = 68%
- Average investment yield = 4.26% (weighted average of yields by asset type)
- Premium growth = 3%
- Hurdle rate = 15% for explicit forecast period and subsequent years

Table 7
Sensitivity Testing of Alternative Assumptions

	DCF Model			EVA Model		
	2001-2011	2012 to ∞	Total	2001-2011	2012 to ∞	Total
Base Case	54.7	33.3	88.0	73.9	14 1	88.0
Change in Assumption						
Surplus-RBC ratio = 2.5	43 1	34.7	77.7	67.3	10 4	77.7
Base loss ratios +2%	46 0	30.4	76.4	66.0	10.4	76.4
Base loss ratios −2%	63 3	36.2	99.5	81.8	17 7	99.5
Investment yield +100 basis pts	67.6	39 8	107.5	86.9	20.6	107.5
Investment yield −100 basis pts	41.6	26 8	68.4	60 9	7.5	68.4
Premium growth = 0%	58.1	26.3	84 4	72.5	11.9	84 4
Premium growth = 6%	52 4	37.3	89.8	74.6	15.1	89.8
Hurdle rate +3%	48.3	20.9	69 3	63.2	6.1	69.3
Hurdle rate −3%	62 5	56.4	118 9	87.5	31.4	118.9

Table 8 shows the changes in value implied by the alternative assumptions. **SECTION 2 – Valuation Results: EVA versus DCF** discusses the similarities and differences of the models' structure and results using varying assumptions.

Table 8
Changes from Base Case in Valuation Estimates

	DCF Model			EVA Model		
	2001-2011	2012 to ∞	Total	2001-2011	2012 to ∞	Total
Surplus-RBC ratio = 2.5	(11.6)	1 3	(10.3)	(6.7)	(3 7)	(10.3)
Base loss ratios +2%	(8.7)	(2.9)	(11.7)	(8 0)	(3.7)	(11 7)
Base loss ratios -2%	8.6	2 9	11.5	7.9	3.6	11 5
Investment yield +100 basis pts	12 9	6.5	19.4	12.9	6.5	19.4
Investment yield -100 basis pts	(13.1)	(6 6)	(19.6)	(13 1)	(6.6)	(19 6)
Premium growth = 0%	3.4	(7.0)	(3.6)	(1.4)	(2.2)	(3 6)
Premium growth = 6%	(2.3)	4 0	1.7	0.7	1.1	1 7
Hurdle rate +3%	(6.4)	(12 4)	(18.8)	(10.7)	(8.0)	(18.8)
Hurdle rate -3%	7 8	23.1	30 9	13 6	17 3	30.9

These tables show that company value is very sensitive to changes in the assumptions underlying the valuation. Every sensitivity test alters value by at least 10%, except for the premium growth assumptions Large changes in premium growth assumptions had small impact on value because the underwriting profits of the insurance company are modest. This is apparent in Exhibit 9, which shows the underwriting income contribution to pre-tax operating income for 2001 through 2011.

The hurdle rate for the entire valuation period is also a key assumption. Decreasing the hurdle rate from 15% to 12% for all projection periods increases value by 35%.

An increase in the required surplus (raising the surplus to RBC ratio from 2.0 to 2.5) lowers value. This result is logical in that the higher the capital required, the lower the free cash flows for DCF and the higher the cost of capital for EVA

Value is also very sensitive to changes in the investment yield for the asset portfolio. This result is logical for this company in that over 95% of the pre-tax operating income is related to investment income (as shown in Exhibit 9).

Valuation results will always be sensitive to small changes in loss ratios as shown in Tables 7 and 8. A reduction in loss ratio of 2% for each lines of business results in a increase in value of 13%.

Since the value of any company is a function of the assumptions used, as noted in **SECTION I – Valuation Framework**, a valuation report should clearly identify the source of every assumption. The report should specify whether the assumption was provided by the subject company, derived from historical experience, provided by a potential investor, or developed from other sources. The source of an assumption may be an indication of whether the assumption is conservative, optimistic, or unbiased.

SECTION 5 – Recent Changes and Other Considerations

There are a variety of changes that have occurred over the past 15 years that may affect the valuation of a property/casualty insurer. While many of these changes may not affect valuation methodology, they are relatively new developments that require consideration in the determination of value.

Accounting[34]

- Codification

The starting point for valuation based on EVA and DCF methodologies is the statutory balance sheet. One significant change with respect to the determination of statutory surplus is the recent codification of statutory accounting principles ("SAP").

With the introduction of codified SAP, there are at least two key changes that affect statutory surplus for many companies: (i) the treatment of deferred taxes, and (ii) the requirement to establish a premium deficiency reserve. Both of these changes mitigate the differences between statutory and GAAP accounting.

Codified SAP now requires the accrual of a deferred tax asset ("DTA") or liability ("DTL"). Consider a company that purchases one share of stock on January 1, 2001 for \$100. If the company holds the stock and it appreciates to \$1,000 as of December 31, 2001, the company will be required to accrue a DTL for the unrealized capital gain. (The DTL is calculated as $t \times [1{,}000 - 100]$, where t is the corporate tax rate.) Conversely, the determination of federal taxes using discounted loss reserves results in the accrual of a DTA As a result, a company's statutory surplus is affected by necessary adjustments for DTA's and DTL's

A premium deficiency reserve ("PDR") is required to supplement the unearned premium reserve ("UEPR") when the UEPR is inadequate to fund for future liabilities related to the unearned exposure

Each of these changes resulting from codification affects the starting statutory surplus in a valuation and, as a result, the entity's future earnings. Prior to codification, a shortfall in the UEPR or the value of a DTL or DTA would have been recognized in future earnings as losses are incurred or assets are sold. Codified SAP reflects the associated value immediately on the balance sheet. In computing value prior to codification, the value associated with the PDR, DTA, or DTL would have been recognized on a discounted basis through the present value of future earnings component of the DCF or EVA valuation methods. After codification, value associated with the PDR, DTA or DTL is as recorded in the statutory balance sheet

[34] One might question why accounting changes should affect value. As statutory earnings and statutory capital influence free cash flows (when either capital can be released from a company or additional capital contributions are required), accounting changes that affect statutory income or statutory surplus influence value.

- Fair Value Accounting

Financial assets and liabilities are accounted for in numerous ways under current U.S accounting rules. For property/casualty insurance companies there is GAAP accounting, statutory accounting and tax accounting Each of the various measuring approaches has its advantages and disadvantages. In general, GAAP accounting for property/casualty insurance companies is accounting for a "going concern." It reflects adjustments that make insurance financials comparable to other industries Statutory accounting is a more conservative form of accounting to meet regulatory requirements targeted at protecting policyholders. Tax accounting is the basis of the tax calculation.

Historically, many financial assets were accounted for at cost or amortized cost. These values are readily available and verifiable. Many financial liabilities were at ultimate settlement value, which is a value that in many cases is contractually set and thus readily available and auditable

The adoption of Financial Accounting Standard ("FAS") 115, which requires market value accounting for assets held in a "trading portfolio," led to the discussion of fair value accounting for financial assets and liabilities. With the adoption of FAS 115, several parties raised concerns with requiring assets to be held at market value when the liabilities were not reported at market values. Since then, the Financial Accounting Standards Board has stated a vision of having all financial assets and liabilities reported at fair value, which is considered an economic value.

The "fair value" of an asset or liability could be defined an estimated market value or as the actual market value when a sufficiently active market exists If no sufficiently similar assets or liabilities exist by which to estimate a market value, the estimated market value is based on present value of future cash flows adjusted for risks.

Fair value accounting is most commonly an issue for financial assets or liabilities. Financial assets are generally either cash or contractual rights to receive cash or other financial assets. Financial liabilities are generally obligations to provide financial assets

Fair value accounting may have an important influence in valuing property/casualty insurance companies. If a fair value accounting approach is adopted for statutory accounting, recognition of many flows will be accelerated relative to statutory accounting. As such, the introduction of fair value accounting will change the value estimates derived from the methods described in this paper, with value estimates increasing if accelerated revenues are higher than accelerated expenses and value estimates decreasing when the reserve is true[35].

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[35] The impact on value is relevant whether these accelerated revenues and expenses are recognized in the income statement or solely as a direct adjustment to surplus. As both after-tax operating income and amount of capital affect free cash flows, either change could influence value

For example, any imbedded value associated with investment income on the loss and LAE reserves or profit in the unearned premium reserve would be reflected in fair value accounting at the time the loss or unearned premium reserve is reported. However, fair value accounting, at least initially, may not consider cash flows and associated profits with policy renewals or new business. Therefore, the fair value accounting net worth of an insurance company, initially, may approximate its runoff value.

Regulatory Changes

- Risk-Based Capital Requirements

In 1993, the NAIC adopted RBC standards for property/casualty insurers. These standards are used by regulators to help to identify insurers that require regulatory attention and, as a result, the standards may be viewed as minimum capital requirements. As such, these requirements affect valuation because they can form a key determinant in the amount of capital a company must hold. Further changes in RBC could affect insurance company valuations if there are changes in required capital levels.

- Gramm-Leach-Bliley Act

The Financial Services Modernization Act of 1999 (Gramm-Leach-Bliley Act or "GLBA") enabled closer alignment of insurance companies and other financial institutions such as banks and securities firms. A primary feature of GLBA is that a bank holding company or foreign bank that meets certain eligibility criteria may become a financial holding company ("FHC"). FHC's are authorized to engage in a range of financial activities such as insurance agency and underwriting activities, merchant banking activities, and securities underwriting and dealing.

To date, GLBA has not had a significant impact on the property/casualty insurance industry because there are very few affiliations of insurance companies with other financial institutions. The 1998 merger of Citicorp and Travelers Group to form Citigroup was the first merger between an insurer and a bank since such mergers were prohibited in 1933. (In August 2002, however, Citicorp spun off the property/casualty operations of Travelers to end the affiliation of the banking institution and life insurance operation with the property/casualty insurance operation.) There has been no subsequent merger activity between property/casualty insurers and other financial institutions since the Citicorp merger.

Nonetheless, if a property/casualty insurer were affiliated with an FHC, the affiliation might affect certain assumptions related to the valuation of the insurer. The Federal Reserve Board, which regulates FHC's, is prohibited from directly imposing capital requirements on insurance affiliates, but it does establish capital requirements for FHC's. These FHC capital requirements may have an implicit influence on the capital level of an insurance subsidiary.

Stochastic Analysis of Insurance Company Financial Results

A unique feature of property and casualty insurance is the stochastic nature of claim emergence and settlement. In general, it is difficult to predict the timing of cash flows related to policyholder claims. While almost every line of business has the potential to generate unexpected claim experience, catastrophic insured events are particularly difficult to estimate due to the low frequency and high severity of these events. These events may have a severe and adverse impact on the operating earnings of an insurer and thus should be considered during the financial projection process. There are two broad approaches to modeling future financial projections: scenario testing and stochastic modeling.

Scenario testing is a deterministic approach in which results are projected from a specific set of conditions and assumptions. With this static approach, the user defines a scenario that reflects assumptions about various components of the company. The user is able to define the specific interrelationships of components and evaluate the impact of changes in different factors on the financial projections. This approach produces results that are easy to explain and easy to modify by incorporating one or more alternative assumptions.

Stochastic modeling has become increasingly popular in recent years for the property/casualty industry via dynamic financial analysis ("DFA"). Underlying stochastic models are probability distributions for each of the stochastic variables reflected in the model. Based on the probability distributions and a random number generator, the stochastic model produces a range of outcomes from which probabilities may be determined for the results. Its flexibility and ability to test the impact of a wide range of variables simultaneously make it an appealing approach. With respect to the implementation of stochastic modeling, however, the probability distributions for the stochastic variables and the correlations between components are critical to a meaningful model.

Over the past ten to fifteen years considerable emphasis has been placed on the DFA of insurance company financial results to evaluate capital needs, capital allocation, ceded reinsurance structures, and the risk associated with specific business initiatives. Since valuation formulas include the present value of future earnings, stochastic modeling of insurance financial results would seem like a natural adjunct to valuation.

In practice, valuing an insurance company is often undertaken in a limited timeframe. Valuation is usually based on expected value results for earnings with sensitivity tests related to changes in premium growth rates, changes in loss ratios, changes in hurdle rates, and changes in annual investment yields.

The contribution from stochastic modeling for valuation is that it would provide better definition of "risk" (the distribution of possible outcomes around the expected value) and could be used to derive better estimates of the cost of capital.

Exposure to Natural Catastrophes

As noted by Gorvett, et al.[36], exposure to natural catastrophes has had a very significant impact on the performance of the property/casualty insurance industry worldwide. As a result, the major catastrophic events during the past fifteen years have accelerated the evolution of the modeling of natural catastrophes and also led to a recent proposal to create a pre-funded catastrophe reserve on the statutory balance sheet.

Though the range of sophistication of catastrophe models varies widely, there are three essential elements of most models regardless of whether the model is deterministic or stochastic. First, there must be an estimate of the intensity of the underlying hazard. This estimate is often simulated based on historical information about catastrophes related to the particular hazard. Second, for the underlying hazard, the model requires an estimate of the total damage caused by the hazard. For a given hazard, the damage estimate is primarily dependent on the geographical location of the risk and the value and construction type of the structure affected by the hazard. The final key element is an estimate of the loss to the insurer – this is based directly on the location of policies written and limits provided.

For the purpose of insurer valuation, the primary benefit of catastrophe modeling is related to scenario testing. While it is beneficial to understand the expected average severity of natural catastrophes, catastrophe models are unable to help identify the future timing of these events. As a result, the future earnings stream of an insurer with significant insurance exposure to natural catastrophes is much more difficult to predict.

Due to the immediate and extremely adverse impact catastrophes may have on the balance sheets of property/casualty insurers and reinsurers, there has been a recent NAIC proposal to establish a tax-deferred pre-funded catastrophe reserve. The intent of this proposal is to establish a simple mechanism by which insurers and reinsurers can prudently manage risk created by exposure to natural catastrophes. This mechanism is intended to reduce the uncertainty related to the future earnings stream of insurers with significant exposure to natural catastrophes. The focus of the current proposal is on exposure of property insurance coverages to natural mega-catastrophes (e.g., Hurricane Andrew in 1992) that are expected to occur in the future.

As currently proposed, this "reserve" can be more appropriately viewed as segregated surplus. For the purpose of solvency regulation, the pre-funded nature of this reserve is also expected to come with restrictions on how it may be taken down over time.

This reserve and its funding mechanism will lead to additional considerations related to the determination of starting capital and future earnings for the purpose of a valuation. If the catastrophe reserve is immediately funded out of existing capital and as a liability, the entity's starting capital for the purpose of valuation will be reduced. If, however, the reserve is considered to be segregated surplus, the value of the company will not change. An alternative pre-funding approach is to contribute a percentage of premiums to the

[36] CAS – Foundations of Casualty Actuarial Science, 4th edition, Chapter 10 ("Special Issues")

catastrophe reserve fund This would have no impact on starting capital, but would affect future earnings. The direction of the change, however, is uncertain.

SECTION 6 - Closing

The valuation of a property/casualty insurance company is an important feature of actuarial work. Much of the actuarial literature on valuation focuses on the method referred to throughout this paper as Economic Value Added Other financial service professionals, however, often rely on a discounted cash flow approach to valuation. One of the principal intentions of this paper is to demonstrate that, with a common set of assumptions, the EVA and DCF modeling approaches will produce equivalent values. For both methods, the key factors underlying value are (1) the projection of future income, (2) the required capital, and (3) the hurdle rate. Developing future income estimates, appropriate growth assumptions (and the resultant capital needs), and the appropriate hurdle rate for the entity required sophisticated analysis. Furthermore, there are aspects of valuation, such as the determination of adjustments to the starting capital of the entity, for which experts have varying points of view. Recent changes such as the development of fair value accounting principles will provide further ideas on the valuation of assets and liabilities of a property/casualty insurance company.

We hope that this paper will help actuaries and other financial professionals to explain the valuation process for property/casualty insurance

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APPENDIX – Demonstration of Algebraic Equivalence of EVA and DCF

The general expression for value based on the Discounted Cash Flow (DCF) approach is.

$$\text{(DCF-1)} \quad Value = FC_0 + \sum_{x=1}^{\infty} [OE_x - \Delta C_x] \times (1+h)^{-x}$$

where.

FC_0 = Free cash available at time 0 to be released to shareholders

OE_x = After-tax operating earnings generated in time period x

ΔC_x = Change in required capital over time period x = C_x - C_{x-1},
where C_x = required capital at the end of time period x (this is equivalent to the required capital at the beginning of time period x+1)

h = Hurdle rate (required return on capital)

Equation DCF-1 represents the sum of the free cash available at time 0 and the present value of future free cash flows, where future free cash flows ($OE_x - \Delta C_x$) are defined as after-tax operating earnings less the amount of required capital reinvestment. For ease of illustration, we have made the simplifying assumption that all cash flows occur at the end of the period.

Distributing and separating Equation DCF-1 into two separate sums, we produce.

$$\text{(DCF-2)} \quad Value = FC_0 + \sum_{x=1}^{\infty} OE_x \times (1+h)^{-x} - \sum_{x=1}^{\infty} \Delta C_x \times (1+h)^{-x}$$

If we assume that both operating earnings and capital grow at constant rate g, then:

$$OE_x = OE_{x-1} \times (1+g) = OE_1 \times (1+g)^{x-1}$$

and

$$C_x = C_{x-1} \times (1+g) = C_0 \times (1+g)^x \text{, so}$$

$$\Delta C_x = C_x - C_{x-1} = C_{x-1} \times g = C_0 \times (1+g)^{x-1} \times g$$

Substituting into equation DCF-2, the DCF value becomes:

(DCF-3) $$Value = FC_0 + \sum_{t=1}^{\infty} OE_1 \times (1+g)^{t-1} \times (1+h)^{-t} - \sum_{t=1}^{\infty} C_0 \times g \times (1+g)^{t-1} \times (1+h)^{-t}.$$

By factoring out the constants, this equation is rewritten as:

(DCF-4) $$Value = FC_0 + \frac{OE_1}{(1+h)} \sum_{t=1}^{\infty} \left[\frac{(1+g)}{(1+h)}\right]^{t-1} - \frac{C_0 \times g}{(1+h)} \sum_{t=1}^{\infty} \left[\frac{(1+g)}{(1+h)}\right]^{t-1}.$$

Note that g, the growth rate, will always be less than h, the hurdle rate. As a result, the sum of the infinite geometric series can be solved easily as A ÷ (1 − R), where A is the first term in the series and R is the multiplicative factor used to generate the next term in the series.

The sum converges to $\frac{1}{1-\frac{(1+g)}{(1+h)}}$, which may be rewritten as $\frac{1+h}{h-g}$

When we substitute this into Equation DCF-4, the $(1 + h)$ terms cancel, so the formula for value based on a DCF approach becomes:

(DCF-5) $$Value = FC_0 + \frac{OE_1}{(h-g)} - \frac{C_0 \times g}{(h-g)}$$

This is appropriately viewed as the sum of all free cash flows, or initial capital plus the present value of future earnings, minus the present value of future required capital reinvestments.

The general expression of EVA is:

$$\text{(EVA-1)} \quad Value = SC_0 + \sum_{x=1}^{\infty}\left[OE_x - (h \times C_{x-1})\right] \times (1+h)^{-x}$$

where:

SC_0 = Starting capital, this is equal to the sum of free capital and required capital at time 0 (FC_0 and C_0, respectively, as defined in the DCF discussion)

OE_x, C_x, and h have the same definitions as in the DCF discussion.

This formula represents the required capital at the valuation date (time = 0) plus the present value of future economic profits. Economic profits for time period x are defined as after-tax operating earnings (OE_x) reduced by the cost of capital, which is the product of the hurdle rate and the required capital at the beginning of each period ($h \times C_x$).

Distributing and separating Equation EVA-1 into two separate sums, we produce:

$$\text{(EVA-2)} \quad Value = SC_0 + \sum_{x=1}^{\infty} OE_x \times (1+h)^{-x} - \sum_{x=1}^{\infty}(h \times C_{x-1}) \times (1+h)^{-x}$$

Based on a constant growth rate g for both after-tax operating earnings and capital and the identities defined above in the DCF discussion, the formula for EVA value is restated.

$$\text{(EVA-3)} \quad Value = SC_0 + \sum_{x=1}^{\infty} OE_1 \times (1+g)^{x-1} \times (1+h)^{-x} - \sum_{x=1}^{\infty} h \times C_0 \times (1+g)^{x-1} \times (1+h)^{-x}$$

By factoring out the constants, this may be rewritten as:

$$\text{(EVA-4)} \quad Value = SC_0 + \frac{OE_1}{(1+h)}\sum_{x=1}^{\infty}\left[\frac{(1+g)}{(1+h)}\right]^{x-1} - \frac{(h \times C_0)}{(1+h)}\sum_{x=1}^{\infty}\left[\frac{(1+g)}{(1+h)}\right]^{x-1}$$

Again, we use identities defined in the DCF discussion to simplify Equation EVA-4 to the following:

$$\text{(EVA-5)} \quad Value = SC_0 + \frac{OE_1}{(h-g)} - \frac{h \times C_0}{(h-g)}$$

This can also be expressed as:

(EVA-6) $$Value = SC_0 + \frac{OE_1}{(h-g)} - \frac{(h-g+g)\times C_0}{(h-g)}$$

or

(EVA-7) $$Value = SC_0 + \frac{OE_1}{(h-g)} - \frac{(h-g)\times C_0}{(h-g)} - \frac{g\times C_0}{(h-g)}$$

or

(EVA-8) $$Value = SC_0 + \frac{OE_1}{(h-g)} - C_0 - \frac{g\times C_0}{(h-g)}$$

or

(EVA-9) $$Value = FC_0 + C_0 + \frac{OE_1}{(h-g)} - C_0 - \frac{g\times C_0}{(h-g)}$$

or

(EVA-10) $$Value = FC_0 + \frac{OE_1}{(h-g)} - \frac{g\times C_0}{(h-g)}$$

This is the same result as for the DCF model, as shown in Equation DCF-5.

Basic Valuation Example
Discounted Cash Flow Model

Exhibit 1A

Scenario Assumptions:

Total Earnings: hurdle rate exactly achieved
Annual Growth: 0%

Projected Year	(1) Initial Capital	(2) Required Growth in Capital	(3) Total Required Capital at Beginning of Year	(4) Discount Factor at 15%	(5) Premium Earned During Year	(6) After-tax Investment Income on Capital	(6a) After-tax Investment Income on Original Capital	(6b) After-tax Investment Income on Additional Capital	(7) After-tax Total Income From Insurance Operations	(8) Required Reinv.	(9) Available After-tax Net Income	(10) Indicated Value
1	100	0	100	0.870	200	4	4	0	11	0	15	
2		0	100	0.756	200	4	4	0	11	0	15	
3		0	100	0.658	200	4	4	0	11	0	15	
4		0	100	0.572	200	4	4	0	11	0	15	
5		0	100	0 497	200	4	4	0	11	0	15	
6		0	100	0 432	200	4	4	0	11	0	15	
7		0	100	0.376	200	4	4	0	11	0	15	
8		0	100	0 327	200	4	4	0	11	0	15	
9		0	100	0 284	200	4	4	0	11	0	15	
10		0	100	0 247	200	4	4	0	11	0	15	
<u>Discounted Totals</u>												
(11) Yrs. 1-10	100					20.08	20.08	0.00	55.21	0 00	75 28	75 28
(12) Terminal Value	0					6 59	6 59	0.00	18.13	0.00	24 72	24 72
(13) All Yrs	100					26 67	26 67	0 00	73 33	0 00	100 00	100.00

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3), for year 1, (3) - (1)
(3) = (5) ÷ 2 0, where 2 0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year; subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4%, (6a) = (6) multiplied by the ratio of initial capital to (3), (6b) = (6) - (6a)
(7) = 5 5% of (5), selected so that earnings achieve the hurdle rate exactly; 5.5% = [hurdle rate - investment yield] ÷ premium-to-surplus ratio
(8) = (3) - following year's (3); difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (9)
(11) = Explicit forecast period value
(12) = Terminal value
(13) = (11) + (12) = Value in perpetuity

Basic Valuation Example
Economic Value Added (a) Model ***Exhibit 1B***

Scenario Assumptions:

Total Earnings: hurdle rate exactly achieved
Annual Growth: 0%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)	(10a)	(10b)	(11)	(11a)	(11b)	(12)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Hurdle Required Return on Capital	Hurdle Required Return on Original Capital	Hurdle Required Return on Additional Capital	Total Cost of Capital	Cost of Original Capital	Cost of Additional Capital	Indicated Value
1	100	0	100	0.870	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
2		0	100	0.756	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
3		0	100	0.658	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
4		0	100	0.572	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
5		0	100	0.497	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
6		0	100	0.432	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
7		0	100	0.376	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
8		0	100	0.327	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
9		0	100	0.284	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
10		0	100	0.247	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
<u>Discounted Totals</u>																		
(13) Yrs 1-10	100					20.08	20.08	0.00	55.21	0.00	75.28	(75.28)	(75.28)	0.00	(55.21)	(55.21)	0.00	100.00
(14) Terminal Value	0					6.59	6.59	0.00	18.13	0.00	24.72	(24.72)	(24.72)	0.00	(18.13)	(18.13)	0.00	0.00
(15) All Yrs	100					26.67	26.67	0.00	73.33	0.00	100.00	(100.00)	(100.00)	0.00	(73.33)	(73.33)	0.00	100.00

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3), for year 1, (3) - (1)
(3) = (5) ÷ 2.0, where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4%, (6a) = (6) multiplied by the ratio of initial capital to (3). (6b) = (6) - (6a)
(7) = 5.5% of (5), selected so that earnings achieve the hurdle rate exactly, 5.5% = [hurdle rate - investment yield] ÷ premium-to-surplus ratio
(8) = (3) - following year's (3) difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) × -15% (hurdle rate), (10a) = (10) multiplied by the ratio of initial capital to (3), (10b) = (10) - (10a)
(11) = (6) + (10), (11a) = (6a) + (10a) (11b) = (6b) + (10b)
(12) = (1) + [(6) + (7)] + (10), EVA (a) does not reduce the cost of capital to reflect investment income earned on capital
(13) Forecast Period Value
(14) Terminal Value
(15) = (13) + (14) Value in Perpetuity

Basic Valuation Example
Economic Value Added (b) Model **_Exhibit 1C_**

> *Scenario Assumptions:*
>
> ***Total Earnings: hurdle rate exactly achieved***
> ***Annual Growth: 0%***

Projected Year	(1) Initial Capital	(2) Required Growth in Capital	(3) Total Required Capital at Beginning of Year	(4) Discount Factor at 15%	(5) Premium Earned During Year	(6) After-tax Investment Income on Capital	(6a) After-tax Investment Income on Original Capital	(6b) After-tax Investment Income on Additional Capital	(7) After-tax Total Income From Insurance Operations	(8) Required Reinv	(9) Available After-tax Net Income	(10) Hurdle Required Return on Capital	(10a) Hurdle Required Return on Original Capital	(10b) Hurdle Required Return on Additional Capital	(11) Total Cost of Capital	(11a) Cost of Original Capital	(11b) Cost of Additional Capital	(12) Indicated Value
1	100	0	100	0.870	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
2		0	100	0.756	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
3		0	100	0.658	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
4		0	100	0.572	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
5		0	100	0.497	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
6		0	100	0.432	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
7		0	100	0.376	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
8		0	100	0.327	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
9		0	100	0.284	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
10		0	100	0.247	200	4	4	0	11	0	15	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
<u>Discounted Totals</u>																		
(13) Yrs 1-10	100					20.08	20.08	0.00	55.21	0.00	75.28	(75.28)	(75.28)	0.00	(55.21)	(55.21)	0.00	100.00
(14) Terminal Value	0					6.59	6.59	0.00	18.13	0.00	24.72	(24.72)	(24.72)	0.00	(18.13)	(18.13)	0.00	0.00
(15) All Yrs	100					26.67	26.67	0.00	73.33	0.00	100.00	(100.00)	(100.00)	0.00	(73.33)	(73.33)	0.00	100.00

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3) for year 1, (3) - (1)
(3) = (5) ÷ 2.0 where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4% (6a) = (6) multiplied by the ratio of initial capital to (3) (6b) = (6) - (6a)
(7) = 5.5% of (5) selected so that earnings achieve the hurdle rate exactly; 5.5% = [hurdle rate - investment yield] ÷ premium-to-surplus ratio
(8) = (3) - following year's (3), difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) × -15% (hurdle rate) (10a) = (10) multiplied by the ratio of initial capital to (3) (10b) = (10) - (10a)
(11) = (6) + (10) (11a) = (6a) + (10a), (11b) = (6b) + (10b)
(12) = (1) + (7) + (11), EVA (b) reduces the cost of capital component to reflect investment income earned on capital
(13) Forecast Period Value
(14) Terminal Value
(15) = (13) + (14) Value in Perpetuity

Basic Valuation Example ***Exhibit 2A***

Discounted Cash Flow Model

Scenario Assumptions:

Total Earnings: hurdle rate exactly achieved
Annual Growth: 3%

Projected Year	(1) Initial Capital	(2) Required Growth in Capital	(3) Total Required Capital at Beginning of Year	(4) Discount Factor at 15%	(5) Premium Earned During Year	(6) After-tax Investment Income on Capital	(6a) After-tax Investment Income on Original Capital	(6b) After-tax Investment Income on Additional Capital	(7) After-tax Total Income From Insurance Operations	(8) Required Reinv.	(9) Available After-tax Net Income	(10) Indicated Value
1	100	0	100	0.870	200	4	4	0	11	(3)	12	
2		3	103	0 756	206	4 12	4	0 12	11 33	(3 09)	12 36	
3		3 09	106 09	0.658	212.18	4.24	4	0.24	11.67	(3 18)	12.73	
4		3 18	109 27	0.572	218 55	4.37	4	0 37	12 02	(3 28)	13 11	
5		3 28	112 55	0.497	225.10	4.50	4	0.50	12 38	(3.38)	13.51	
6		3.38	115 93	0.432	231 85	4.64	4	0.64	12 75	(3 48)	13 91	
7		3.48	119 41	0 376	238 81	4.78	4	0.78	13.13	(3.58)	14 33	
8		3.58	122 99	0 327	245 97	4.92	4	0 92	13.53	(3.69)	14 76	
9		3.69	126.68	0.284	253 35	5.07	4	1 07	13 93	(3 80)	15 20	
10		3.80	130 48	0.247	260 95	5 22	4	1.22	14 35	(3.91)	15.66	
<u>Discounted Totals</u>												
(11) Yrs 1-10	100					22.26	20 08	2.19	61.22	(16.70)	66 78	66.78
(12) Terminal Value	0					11.07	6.59	4 48	30 45	(8 30)	33 22	33.22
(13) All Yrs	100					33.33	26.67	6.67	91 67	(25.00)	100.00	100 00

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3); for year 1, (3) - (1)
(3) = (5) / 2, where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, growing by growth rate
(6) = (3) × selected investment yield of 4%; (6a) = (6) multiplied by the ratio of initial capital to (3), (6b) = (6) - (6a)
(7) = 5 5% of (5), selected so that earnings achieve the hurdle rate exactly, 5 5% = [hurdle rate - investment yield] ÷ premium-to-surplus ratio
(8) = (3) - following year's (3), difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (9)
(11) = Explicit forecast period value
(12) = Terminal value
(13) = (11) + (12) = Value in perpetuity

Basic Valuation Example
Economic Value Added (a) Model

Exhibit 2B

Scenario Assumptions.

Total Earnings: hurdle rate exactly achieved
Annual Growth: 3%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)	(10a)	(10b)	(11)	(11a)	(11b)	(12)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Hurdle Required Return on Capital	Hurdle Required Return on Original Capital	Hurdle Required Return on Additional Capital	Total Cost of Capital	Cost of Original Capital	Cost of Additional Capital	Indicated Value
1	100	0	100	0 870	200	4	4	0	11	(3)	12	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0 00	
2		3	103	0 756	206	4 12	4	0 12	11 33	(3 09)	12 36	(15 45)	(15 00)	(0 45)	(11 33)	(11.00)	(0 33)	
3		3 09	106 09	0 658	212 18	4 24	4	0 24	11 67	(3 18)	12 73	(15 91)	(15 00)	(0 91)	(11 67)	(11 00)	(0 67)	
4		3 18	109 27	0 572	218.55	4 37	4	0 37	12 02	(3 28)	13 11	(16 39)	(15 00)	(1 39)	(12 02)	(11 00)	(1 02)	
5		3 28	112 55	0 497	225 10	4 50	4	0 50	12 38	(3 38)	13 51	(16 88)	(15 00)	(1 88)	(12.38)	(11 00)	(1 38)	
6		3 38	115 93	0 432	231 85	4 64	4	0 64	12 75	(3 48)	13 91	(17 39)	(15 00)	(2 39)	(12 75)	(11 00)	(1 75)	
7		3 48	119 41	0.376	238 81	4 78	4	0 78	13 13	(3 58)	14 33	(17 91)	(15 00)	(2 91)	(13 13)	(11 00)	(2 13)	
8		3 58	122 99	0 327	245 97	4 92	4	0 92	13 53	(3 69)	14 76	(18 45)	(15 00)	(3 45)	(13 53)	(11 00)	(2 53)	
9		3 69	126 68	0 284	253 35	5 07	4	1 07	13 93	(3 80)	15 20	(19 00)	(15 00)	(4 00)	(13 93)	(11 00)	(2 93)	
10		3 80	130 48	0 247	260 95	5 22	4	1 22	14 35	(3 91)	15 66	(19 57)	(15 00)	(4 57)	(14 35)	(11 00)	(3 35)	
<u>Discounted Totals</u>																		
(13) Yrs 1-10	100					22 26	20 08	2 19	61 22	(16 70)	66 78	(83 48)	(75 28)	(8 19)	(61 22)	(55 21)	(6 01)	100 00
(14) Terminal Value	0					11 07	6 59	4 48	30 45	(8.30)	33 22	(41 52)	(24 72)	(16 81)	(30 45)	(18 13)	(12 32)	0 00
(15) All Yrs	100					33 33	26 67	6 67	91 67	(25 00)	100 00	(125 00)	(100 00)	(25 00)	(91 67)	(73 33)	(18 33)	100.00

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3). for year 1, (3) - (1)
(3) = (5) ÷ 2 0 where 2 0 represents the target premium-to-surplus ratio
(4) factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4% (6a) = (6) multiplied by the ratio of initial capital to (3). (6b) = (6) - (6a)
(7) = 5 5% of (5) selected so that earnings achieve the hurdle rate exactly 5 5% = [hurdle rate - investment yield] ÷ premium-to-surplus ratio
(8) = (3) - following year's (3). difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) × -15% (hurdle rate). (10a) = (10) multiplied by the ratio of initial capital to (3). (10b) = (10) - (10a)
(11) = (6) + (10) (11a) = (6a) + (10a) (11b) = (6b) + (10b)
(12) = (1) + [(6) + (7)] + (10). EVA (a) does not reduce the cost of capital to reflect investment income earned on capital
(13) Forecast Period Value
(14) Terminal Value

Basic Valuation Example **_Exhibit 2C_**
Economic Value Added (b) Model

Scenario Assumptions
Total Earnings: hurdle rate exactly achieved
Annual Growth. 3%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)	(10a)	(10b)	(11)	(11a)	(11b)	(12)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After-tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Hurdle Required Return on Capital	Hurdle Required Return on Original Capital	Hurdle Required Return on Additional Capital	Total Cost of Capital	Cost of Original Capital	Cost of Additional Capital	Indicated Value
1	100	0	100	0.870	200	4	4	0	11	(3)	12	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
2		3	103	0.756	206	4.12	4	0.12	11.33	(3.09)	12.36	(15.45)	(15.00)	(0.45)	(11.33)	(11.00)	(0.33)	
3		3.09	106.09	0.658	212.18	4.24	4	0.24	11.67	(3.18)	12.73	(15.91)	(15.00)	(0.91)	(11.67)	(11.00)	(0.67)	
4.		3.18	109.27	0.572	218.55	4.37	4	0.37	12.02	(3.28)	13.11	(16.39)	(15.00)	(1.39)	(12.02)	(11.00)	(1.02)	
5		3.28	112.55	0.497	225.10	4.50	4	0.50	12.38	(3.38)	13.51	(16.88)	(15.00)	(1.88)	(12.38)	(11.00)	(1.38)	
6		3.38	115.93	0.432	231.85	4.64	4	0.64	12.75	(3.48)	13.91	(17.39)	(15.00)	(2.39)	(12.75)	(11.00)	(1.75)	
7		3.48	119.41	0.376	238.81	4.78	4	0.78	13.13	(3.58)	14.33	(17.91)	(15.00)	(2.91)	(13.13)	(11.00)	(2.13)	
8		3.58	122.99	0.327	245.97	4.92	4	0.92	13.53	(3.69)	14.76	(18.45)	(15.00)	(3.45)	(13.53)	(11.00)	(2.53)	
9		3.69	126.68	0.284	253.35	5.07	4	1.07	13.93	(3.80)	15.20	(19.00)	(15.00)	(4.00)	(13.93)	(11.00)	(2.93)	
10		3.80	130.48	0.247	260.95	5.22	4	1.22	14.35	(3.91)	15.66	(19.57)	(15.00)	(4.57)	(14.35)	(11.00)	(3.35)	
<u>Discounted Totals</u>																		
(13) Yrs 1-10	100					22.26	20.08	2.19	61.22	(16.70)	66.78	(83.48)	(75.28)	(8.19)	(61.22)	(55.21)	(6.01)	100.00
(14) Terminal Value	0					11.07	6.59	4.48	30.45	(8.30)	33.22	(41.52)	(24.72)	(16.81)	(30.45)	(18.13)	(12.32)	0.00
(15) All Yrs	100					33.33	26.67	6.67	91.67	(25.00)	100.00	(125.00)	(100.00)	(25.00)	(91.67)	(73.33)	(18.33)	100.00

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3), for year 1 (3) - (1)
(3) = (5) ÷ 2.0 where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4%, (6a) = (6) multiplied by the ratio of initial capital to (3); (6b) = (6) - (6a)
(7) = 5.5% of (5) selected so that earnings achieve the hurdle rate exactly, 5.5% = [hurdle rate investment yield] ÷ premium-to-surplus ratio
(8) = (3) - following year's (3), difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) × -15% (hurdle rate), (10a) = (10) multiplied by the ratio of initial capital to (3) (10b) = (10) - (10a)
(11) = (6) + (10) (11a) = (6a) + (10a), (11b) = (6b) + (10b)
(12) = (1) + (7) + (11), EVA (b) reduces the cost of capital component to reflect investment income earned on capital
(13) Forecast Period Value
(14) Terminal Value
(15) = (13) + (14) Value in Perpetuity

Basic Valuation Example
Discounted Cash Flow Model

Exhibit 3A

Scenario Assumptions:

Total Earnings: hurdle rate more than achieved
Annual Growth: 0%

Projected Year	(1) Initial Capital	(2) Required Growth in Capital	(3) Total Required Capital at Beginning of Year	(4) Discount Factor at 15%	(5) Premium Earned During Year	(6) After-tax Investment Income on Capital	(6a) After-tax Investment Income on Original Capital	(6b) After-tax Investment Income on Additional Capital	(7) After-tax Total Income From Insurance Operations	(8) Required Reinv.	(9) Available After-tax Net Income	(10) Indicated Value
1	100	0	100	0.870	200	4	4	0	12	0	16	
2		0	100	0 756	200	4	4	0	12	0	16	
3		0	100	0 658	200	4	4	0	12	0	16	
4		0	100	0 572	200	4	4	0	12	0	16	
5		0	100	0 497	200	4	4	0	12	0	16	
6		0	100	0 432	200	4	4	0	12	0	16	
7		0	100	0 376	200	4	4	0	12	0	16	
8		0	100	0 327	200	4	4	0	12	0	16	
9		0	100	0.284	200	4	4	0	12	0	16	
10		0	100	0 247	200	4	4	0	12	0	16	
<u>Discounted Totals</u>												
(11) Yrs 1-10	100					20.08	20 08	0 00	60.23	0.00	80.30	80 30
(12) Terminal Value	0					6.59	6 59	0 00	19 77	0 00	26.37	26 37
(13) All Yrs	100					26 67	26 67	0 00	80 00	0 00	106 67	106 67

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3), for year 1, (3) - (1)
(3) = (5) ÷ 2.0, where 2 0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4%; (6a) = (6) multiplied by the ratio of initial capital to (3), (6b) = (6) - (6a)
(7) = 6% of (5), selected so that earnings exceed the hurdle rate requirement
(8) = (3) - following year's (3), difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (9)
(11) = Explicit forecast period value
(12) = Terminal value
(13) = (11) + (12) = Value in perpetuity

Basic Valuation Example
Economic Value Added (a) Model

Exhibit 3B

Scenario Assumptions:

Total Earnings: hurdle rate more than achieved
Annual Growth: 0%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)	(10a)	(10b)	(11)	(11a)	(11b)	(12)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After-tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Hurdle Required Return on Capital	Hurdle Required Return on Original Capital	Hurdle Required Return on Additional Capital	Total Cost of Capital	Cost of Original Capital	Cost of Additional Capital	Indicated Value
1	100	0	100	0.870	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
2		0	100	0.756	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
3		0	100	0.658	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
4		0	100	0.572	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
5		0	100	0.497	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
6		0	100	0.432	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
7		0	100	0.376	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
8		0	100	0.327	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
9		0	100	0.284	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
10		0	100	0.247	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
Discounted Totals																		
(13) Yrs 1-10	100					20.08	20.08	0.00	60.23	0.00	80.30	(75.28)	(75.28)	0.00	(55.21)	(55.21)	0.00	105.02
(14) Terminal Value	0					6.59	6.59	0.00	19.77	0.00	26.37	(24.72)	(24.72)	0.00	(18.13)	(18.13)	0.00	1.65
(15) All Yrs	100					26.67	26.67	0.00	80.00	0.00	106.67	(100.00)	(100.00)	0.00	(73.33)	(73.33)	0.00	106.67

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3), for year 1 (3) - (1)
(3) = (5) ÷ 2.0, where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4% (6a) = (6) multiplied by the ratio of initial capital to (3), (6b) = (6) - (6a)
(7) = 6% of (5), selected so that earnings exceed the hurdle rate requirement
(8) = (3) - following year's (3) difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) × -15% (hurdle rate), (10a) = (10) multiplied by the ratio of initial capital to (3), (10b) = (10) - (10a)
(11) = (6) + (10) (11a) = (6a) + (10a), (11b) = (6b) + (10b)
(12) = (1) + [(6) + (7)] + (10). EVA (a) does not reduce the cost of capital to reflect investment income earned on capital
(13) Forecast Period Value
(14) Terminal Value
(15) = (13) + (14) Value in Perpetuity

Basic Valuation Example
Economic Value Added (b) Model

Exhibit 3C

Scenario Assumptions:
Total Earnings: hurdle rate more than achieved
Annual Growth: 0%

Projected Year	(1) Initial Capital	(2) Required Growth in Capital	(3) Total Required Capital at Beginning of Year	(4) Discount Factor at 15%	(5) Premium Earned During Year	(6) After-tax Investment Income on Capital	(6a) After-tax Investment Income on Original Capital	(6b) After-tax Investment Income on Additional Capital	(7) After-tax Total Income From Insurance Operations	(8) Required Reinv	(9) Available After-tax Net Income	(10) Hurdle Required Return on Capital	(10a) Hurdle Required Return on Original Capital	(10b) Hurdle Required Return on Additional Capital	(11) Total Cost of Capital	(11a) Cost of Original Capital	(11b) Cost of Additional Capital	(12) Indicated Value
1	100	0	100	0.870	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
2		0	100	0.756	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
3		0	100	0.658	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
4		0	100	0.572	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
5		0	100	0.497	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
6		0	100	0.432	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
7		0	100	0.376	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
8		0	100	0.327	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
9		0	100	0.284	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
10		0	100	0.247	200	4	4	0	12	0	16	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
<u>Discounted Totals</u>																		
(13) Yrs 1-10	100					20.08	20.08	0.00	60.23	0.00	80.30	(75.28)	(75.28)	0.00	(55.21)	(55.21)	0.00	105.02
(14) Terminal Value	0					6.59	6.59	0.00	19.77	0.00	26.37	(24.72)	(24.72)	0.00	(18.13)	(18.13)	0.00	1.65
(15) All Yrs	100					26.67	26.67	0.00	80.00	0.00	106.67	(100.00)	(100.00)	0.00	(73.33)	(73.33)	0.00	106.67

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3), for year 1, (3) - (1)
(3) = (5) ÷ 2.0, where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year subsequent years increased by the selected growth rate
(6) = (3) * selected investment yield of 4%; (6a) = (6) multiplied by the ratio of initial capital to (3), (6b) = (6) - (6a)
(7) = 6% of (5), selected so that earnings exceed the hurdle rate requirement
(8) = (3) - following year's (3), difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) * -15% (hurdle rate) (10a) = (10) multiplied by the ratio of initial capital to (3), (10b) = (10) - (10a)
(11) = (6) + (10) (11a) = (6a) + (10a), (11b) = (6b) + (10b)
(12) = (1) + (7) + (11) EVA (b) reduces the cost of capital component to reflect investment income earned on capital
(13) Forecast Period Value
(14) Terminal Value
(15) = (13) + (14) Value in Perpetuity

Basic Valuation Example ***Exhibit 4A***
Discounted Cash Flow Model

Scenario Assumptions:

Total Earnings: hurdle rate not achieved
Annual Growth: 0%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After-tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Indicated Value
1	100	0	100	0 870	200	4	4	0	10	0	14	
2		0	100	0.756	200	4	4	0	10	0	14	
3		0	100	0.658	200	4	4	0	10	0	14	
4		0	100	0.572	200	4	4	0	10	0	14	
5		0	100	0 497	200	4	4	0	10	0	14	
6		0	100	0.432	200	4	4	0	10	0	14	
7		0	100	0 376	200	4	4	0	10	0	14	
8		0	100	0 327	200	4	4	0	10	0	14	
9		0	100	0 284	200	4	4	0	10	0	14	
10		0	100	0.247	200	4	4	0	10	0	14	
Discounted Totals												
(11) Yrs. 1-10	100					20.08	20 08	0 00	50 19	0.00	70.26	70 26
(12) Terminal Value	0					6.59	6 59	0 00	16.48	0.00	23 07	23 07
(13) All Yrs.	100					26 67	26 67	0.00	66.67	0 00	93 33	93 33

(1) - assumed
(2) = (3) - previous year's (3); for year 1, (3) - (1)
(3) = (5) ÷ 2 0, where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4%; (6a) = (6) multiplied by the ratio of initial capital to (3), (6b) = (6) - (6a)
(7) = 5% of (5), selected so that earnings are less than the hurdle rate requirement
(8) = (3) - following year's (3), difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (9)
(11) Forecast Period Value
(12) Terminal Value
(13) = (11) + (12) = Value in Perpetuity

Basic Valuation Example
Economic Value Added (a) Model ***Exhibit 4B***

Scenario Assumptions.

Total Earnings: hurdle rate not achieved
Annual Growth: 0%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)	(10a)	(10b)	(11)	(11a)	(11b)	(12)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After-tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Hurdle Required Return on Capital	Hurdle Required Return on Original Capital	Hurdle Required Return on Additional Capital	Total Cost of Capital	Cost of Original Capital	Cost of Additional Capital	Indicated Value
1	100	0	100	0.870	200	4	4	0	10	0	14	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
2		0	100	0.756	200	4	4	0	10	0	14	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
3		0	100	0.658	200	4	4	0	10	0	14	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
4		0	100	0.572	200	4	4	0	10	0	14	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
5		0	100	0.497	200	4	4	0	10	0	14	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
6		0	100	0.432	200	4	4	0	10	0	14	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
7		0	100	0.376	200	4	4	0	10	0	14	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
8		0	100	0.327	200	4	4	0	10	0	14	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
9		0	100	0.284	200	4	4	0	10	0	14	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
10		0	100	0.247	200	4	4	0	10	0	14	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
<u>Discounted Totals</u>																		
(13) Yrs 1-10	100					20.08	20.08	0.00	50.19	0.00	70.26	(75.28)	(75.28)	0.00	(55.21)	(55.21)	0.00	94.98
(14) Terminal Value	0					6.59	6.59	0.00	16.48	0.00	23.07	(24.72)	(24.72)	0.00	(18.13)	(18.13)	0.00	-1.65
(15) All Yrs	100					26.67	26.67	0.00	66.67	0.00	93.33	(100.00)	(100.00)	0.00	(73.33)	(73.33)	0.00	93.33

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3) for year 1 (3) - (1)
(3) = (5) ÷ 2.0 where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4%. (6a) = (6) multiplied by the ratio of initial capital to (3). (6b) = (6) - (6a)
(7) = 5% of (5) selected so that earnings are less than the hurdle rate requirement
(8) = (3) - following year's (3) difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) × -15% (hurdle rate) (10a) = (10) multiplied by the ratio of initial capital to (3). (10b) = (10) - (10a)
(11) = (6) + (10). (11a) = (6a) + (10a). (11b) = (6b) + (10b)
(12) = (1) + (6) + (7) + (10)
(13) Forecast Period Value
(14) Terminal Value
(15) = (13) + (14) Value in Perpetuity

Basic Valuation Example **_Exhibit 4C_**

Economic Value Added (b) Model

Scenario Assumptions:

Total Earnings: hurdle rate not achieved
Annual Growth: 0%

Projected Year	(1) Initial Capital	(2) Required Growth in Capital	(3) Total Required Capital at Beginning of Year	(4) Discount Factor at 15%	(5) Premium Earned During Year	(6) After-tax Investment Income on Capital	(6a) After-tax Investment Income on Original Capital	(6b) After-tax Investment Income on Additional Capital	(7) After-tax Total Income From Insurance Operations	(8) Required Reinv	(9) Available After tax Net Income	(10) Hurdle Required Return on Capital	(10a) Hurdle Required Return on Original Capital	(10b) Hurdle Required Return on Additional Capital	(11) Total Cost of Capital	(11a) Cost of Original Capital	(11b) Cost of Additional Capital	(12) Indicated Value
1	100	0	100	0 870	200	4	4	0	10	0	14	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0 00	
2		0	100	0 756	200	4	4	0	10	0	14	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0 00	
3		0	100	0 658	200	4	4	0	10	0	14	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0 00	
4		0	100	0 572	200	4	4	0	10	0	14	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0 00	
5		0	100	0 497	200	4	4	0	10	0	14	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0 00	
6		0	100	0 432	200	4	4	0	10	0	14	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0 00	
7		0	100	0 376	200	4	4	0	10	0	14	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0 00	
8		0	100	0 327	200	4	4	0	10	0	14	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0 00	
9		0	100	0 284	200	4	4	0	10	0	14	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0 00	
10		0	100	0 247	200	4	4	0	10	0	14	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0.00	
<u>Discounted Totals</u>																		
(13) Yrs 1-10	100					20 08	20 08	0 00	50 19	0 00	70 28	(75 28)	(75 28)	0 00	(55 21)	(55 21)	0 00	94 98
(14) Terminal Value	0					6 59	6 59	0 00	16 48	0 00	23 07	(24 72)	(24 72)	0 00	(18 13)	(18 13)	0 00	-1 65
(15) All Yrs	100					26 67	26 67	0 00	66 67	0 00	93 33	(100 00)	(100 00)	0 00	(73 33)	(73.33)	0 00	93.33

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3), for year 1, (3) - (1)
(3) = (5) ÷ 2 0, where 2 0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, subsequent years increased by the selected growth rate
(6) = (3) * selected investment yield of 4%, (6a) = (6) multiplied by the ratio of initial capital to (3), (6b) = (6) - (6a)
(7) = 5% of (5), selected so that earnings are less than the hurdle rate requirement
(8) = (3) - following year's (3); difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) * -15% (hurdle rate), (10a) = (10) multiplied by the ratio of initial capital to (3), (10b) = (10) (10a)
(11) = (6) + (10), (11a) = (6a) + (10a), (11b) = (6b) + (10b)
(12) = (1) + (7) + (11), EVA (b) reduces the cost of capital component to reflect investment income earned on capital
(13) Forecast Period Value
(14) Terminal Value
(15) = (13) + (14) Value in Perpetuity

Basic Valuation Example ***Exhibit 5A***
Discounted Cash Flow Model

Scenario Assumptions:

Total Earnings: hurdle rate more than achieved
Annual Growth: 3%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After-tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Indicated Value
1	100	0	100	0.870	200	4	4	0	12	(3)	13	
2		3	103	0.756	206	4.12	4	0.12	12.36	(3.09)	13.39	
3		3.09	106.09	0.658	212.18	4.24	4	0.24	12.73	(3.18)	13.79	
4		3.18	109.27	0.572	218.55	4.37	4	0.37	13.11	(3.28)	14.21	
5		3.28	112.55	0.497	225.10	4.50	4	0.50	13.51	(3.38)	14.63	
6		3.38	115.93	0.432	231.85	4.64	4	0.64	13.91	(3.48)	15.07	
7		3.48	119.41	0.376	238.81	4.78	4	0.78	14.33	(3.58)	15.52	
8		3.58	122.99	0.327	245.97	4.92	4	0.92	14.76	(3.69)	15.99	
9		3.69	126.68	0.284	253.35	5.07	4	1.07	15.20	(3.80)	16.47	
10		3.80	130.48	0.247	260.95	5.22	4	1.22	15.66	(3.91)	16.96	
<u>Discounted Totals</u>												
(11) Yrs. 1-10	100					22.26	20.08	2.19	66.78	(16.70)	72.35	72.35
(12) Terminal Value	0					11.07	6.59	4.48	33.22	(8.30)	35.99	35.99
(13) All Yrs	100					33.33	26.67	6.67	100.00	(25.00)	108.33	108.33

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3): for year 1, (3) - (1)
(3) = (5) ÷ 2.0, where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year; subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4%. (6a) = (6) multiplied by the ratio of initial capital to (3); (6b) = (6) - (6a)
(7) = 6% of (5), selected so that earnings exceed the hurdle rate requirement
(8) = (3) - following year's (3), difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (9)
(11) = Explicit forecast period value
(12) = Terminal value
(13) = (11) + (12) = Value in perpetuity

Basic Valuation Example
Economic Value Added (a) Model

Exhibit 5B

Scenario Assumptions.

Total Earnings: hurdle rate more than achieved
Annual Growth: 3%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)	(10a)	(10b)	(11)	(11a)	(11b)	(12)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After-tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Hurdle Required Return on Capital	Hurdle Required Return on Original Capital	Hurdle Required Return on Additional Capital	Total Cost of Capital	Cost of Original Capital	Cost of Additional Capital	Indicated Value
1	100	0	100	0 870	200	4	4	0	12	(3)	13	(15 00)	(15 00)	0 00	(11 00)	(11 00)	0 00	
2		3	103	0 756	206	4 12	4	0 12	12 36	(3 09)	13 39	(15 45)	(15 00)	(0 45)	(11 33)	(11 00)	(0 33)	
3		3 09	106 09	0 658	212.18	4 24	4	0 24	12 73	(3 18)	13 79	(15 91)	(15 00)	(0 91)	(11 67)	(11 00)	(0 67)	
4		3 18	109 27	0 572	218 55	4 37	4	0 37	13 11	(3 28)	14 21	(16 39)	(15 00)	(1 39)	(12 02)	(11 00)	(1 02)	
5		3 28	112 55	0 497	225 10	4 50	4	0 50	13 51	(3 38)	14.63	(16 88)	(15 00)	(1 88)	(12 38)	(11 00)	(1 38)	
6		3 38	115 93	0 432	231 85	4 64	4	0 64	13 91	(3 48)	15 07	(17 39)	(15 00)	(2 39)	(12 75)	(11 00)	(1 75)	
7		3 48	119 41	0 376	238 81	4 78	4	0 78	14 33	(3 58)	15 52	(17 91)	(15 00)	(2 91)	(13 13)	(11 00)	(2 13)	
8		3 58	122.99	0 327	245 97	4 92	4	0 92	14 76	(3 69)	15 99	(18 45)	(15 00)	(3 45)	(13 53)	(11 00)	(2 53)	
9		3 69	126 68	0 284	253 35	5 07	4	1 07	15 20	(3 80)	16 47	(19 00)	(15 00)	(4 00)	(13 93)	(11 00)	(2 93)	
10		3 80	130 48	0 247	260 95	5 22	4	1 22	15 66	(3 91)	16 96	(19 57)	(15 00)	(4 57)	(14 35)	(11 00)	(3 35)	
<u>Discounted Totals</u>																		
(13) Yrs 1-10	100					22 26	20 08	2 19	66 78	(16 70)	72 35	(83 48)	(75 28)	(8 19)	(61 22)	(55 21)	(6 01)	105 57
(14) Terminal Value	0					11 07	6 59	4 48	33 22	(8 30)	35 99	(41 52)	(24 72)	(16 81)	(30 45)	(18 13)	(12 32)	2.77
(15) All Yrs	100					33 33	26 67	6 67	100 00	(25 00)	108 33	(125 00)	(100 00)	(25 00)	(91 67)	(73 33)	(18 33)	108 33

(1) - selected judgmentally for illustration purposes
(2) = (3) previous year's (3), for year 1, (3) - (1)
(3) = (5) ÷ 2 0, where 2 0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4%, (6a) = (6) multiplied by the ratio of initial capital to (3), (6b) = (6) - (6a)
(7) = 6% of (5) selected so that earnings exceed the hurdle rate requirement
(8) = (3) - following year's (3) difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) × -15% (hurdle rate), (10a) = (10) multiplied by the ratio of initial capital to (3), (10b) = (10) - (10a)
(11) = (6) + (10), (11a) = (6a) + (10a), (11b) = (6b) + (10b)
(12) = (1) + [(6) + (7)] + (10). EVA (a) does not reduce the cost of capital to reflect investment income earned on capita
(13) Forecast Period Value
(14) Terminal Value
(15) = (13) + (14) Value in Perpetuity

Basic Valuation Example ***Exhibit 5C***
Economic Value Added (b) Model

Scenario Assumptions:

Total Earnings - hurdle rate more than achieved
Annual Growth. 3%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)	(10a)	(10b)	(11)	(11a)	(11b)	(12)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After-tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Hurdle Required Return on Capital	Hurdle Required Return on Original Capital	Hurdle Required Return on Additional Capital	Total Cost of Capital	Cost of Original Capital	Cost of Additional Capital	Indicated Value
1	100	0	100	0.870	200	4	4	0	12	(3)	13	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
2		3	103	0.756	206	4.12	4	0.12	12.36	(3.09)	13.39	(15.45)	(15.00)	(0.45)	(11.33)	(11.00)	(0.33)	
3		3.09	106.09	0.658	212.18	4.24	4	0.24	12.73	(3.18)	13.79	(15.91)	(15.00)	(0.91)	(11.67)	(11.00)	(0.67)	
4		3.18	109.27	0.572	218.55	4.37	4	0.37	13.11	(3.28)	14.21	(16.39)	(15.00)	(1.39)	(12.02)	(11.00)	(1.02)	
5		3.28	112.55	0.497	225.10	4.50	4	0.50	13.51	(3.38)	14.63	(16.88)	(15.00)	(1.88)	(12.38)	(11.00)	(1.38)	
6		3.38	115.93	0.432	231.85	4.64	4	0.64	13.91	(3.48)	15.07	(17.39)	(15.00)	(2.39)	(12.75)	(11.00)	(1.75)	
7		3.48	119.41	0.376	238.81	4.78	4	0.78	14.33	(3.58)	15.52	(17.91)	(15.00)	(2.91)	(13.13)	(11.00)	(2.13)	
8		3.58	122.99	0.327	245.97	4.92	4	0.92	14.76	(3.69)	15.99	(18.45)	(15.00)	(3.45)	(13.53)	(11.00)	(2.53)	
9		3.69	126.68	0.284	253.35	5.07	4	1.07	15.20	(3.80)	16.47	(19.00)	(15.00)	(4.00)	(13.93)	(11.00)	(2.93)	
10		3.80	130.48	0.247	260.95	5.22	4	1.22	15.66	(3.91)	16.96	(19.57)	(15.00)	(4.57)	(14.35)	(11.00)	(3.35)	
Discounted Totals																		
(13) Yrs 1-10	100					22.26	20.08	2.19	66.78	(16.70)	72.35	(83.48)	(75.28)	(8.19)	(61.22)	(55.21)	(6.01)	105.57
(14) Terminal Value	0					11.07	6.59	4.48	33.22	(8.30)	35.99	(41.52)	(24.72)	(16.81)	(30.45)	(18.13)	(12.32)	2.77
(15) All Yrs	100					33.33	26.67	6.67	100.00	(25.00)	108.33	(125.00)	(100.00)	(25.00)	(91.67)	(73.33)	(18.33)	108.33

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3) for year 1, (3) - (1)
(3) = (5) ÷ 2.0 where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4% (6a) = (6) multiplied by the ratio of initial capital to (3), (6b) = (6) - (6a)
(7) = 6% of (5) selected so that earnings exceed the hurdle rate requirement
(8) = (3) - following year's (3) difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) × -15% (hurdle rate), (10a) = (10) multiplied by the ratio of initial capital to (3), (10b) = (10) - (10a)
(11) = (6) + (10), (11a) = (6a) + (10a) (11b) = (6b) + (10b)
(12) = (1) + (7) + (11) EVA (b) reduces the cost of capital component to reflect investment income earned on capital
(13) Forecast Period Value
(14) Terminal Value
(15) = (13) + (14) Value in Perpetuity

Basic Valuation Example
Discounted Cash Flow Model

Exhibit 6A

Scenario Assumptions:

Total Earnings: hurdle rate not achieved
Annual Growth: 3%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After-tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Indicated Value
1	100	0	100	0 870	200	4	4	0	10	(3)	11	
2		3	103	0.756	206	4 12	4	0.12	10 3	(3.09)	11 33	
3		3.09	106.09	0 658	212 18	4 24	4	0 24	10 61	(3 18)	11.67	
4		3.18	109.27	0.572	218.55	4 37	4	0 37	10 93	(3 28)	12 02	
5		3.28	112 55	0 497	225.10	4.50	4	0.50	11.26	(3 38)	12.38	
6		3 38	115 93	0 432	231 85	4 64	4	0.64	11.59	(3.48)	12.75	
7		3 48	119.41	0.376	238 81	4 78	4	0 78	11 94	(3 58)	13 13	
8		3 58	122 99	0 327	245.97	4.92	4	0.92	12 30	(3 69)	13 53	
9		3.69	126 68	0 284	253.35	5 07	4	1.07	12 67	(3 80)	13 93	
10		3 80	130 48	0.247	260 95	5 22	4	1.22	13.05	(3.91)	14.35	
<u>Discounted Totals</u>												
(11) Yrs. 1-10	100					22.26	20 08	2 19	55 65	(16 70)	61 22	61 22
(12) Terminal Value	0					11 07	6 59	4.48	27 68	(8.30)	30 45	30 45
(13) All Yrs.	100					33 33	26.67	6 67	83 33	(25 00)	91.67	91.67

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3), for year 1, (3) - (1)
(3) = (5) ÷ 2.0, where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4%, (6a) = (6) multiplied by the ratio of initial capital to (3); (6b) = (6) - (6a)
(7) = 5% of (5), selected to be lower than the hurdle rate
(8) = (3) - following year's (3); difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (9)
(11) = Explicit forecast period value
(12) = Terminal value
(13) = (11) + (12) = Value in perpetuity

Basic Valuation Example
Economic Value Added (a) Model

Exhibit 6B

Scenario Assumptions.

Total Earnings: hurdle rate not achieved
Annual Growth. 3%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)	(10a)	(10b)	(11)	(11a)	(11b)	(12)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After-tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Hurdle Required Return on Capital	Hurdle Required Return on Original Capital	Hurdle Required Return on Additional Capital	Total Cost of Capital	Cost of Original Capital	Cost of Additional Capital	Indicated Value
1	100	0	100	0.870	200	4	4	0	10	(3)	11	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
2		3	103	0.756	206	4.12	4	0.12	10.3	(3.09)	11.33	(15.45)	(15.00)	(0.45)	(11.33)	(11.00)	(0.33)	
3		3.09	106.09	0.658	212.18	4.24	4	0.24	10.61	(3.18)	11.67	(15.91)	(15.00)	(0.91)	(11.67)	(11.00)	(0.67)	
4		3.18	109.27	0.572	218.55	4.37	4	0.37	10.93	(3.28)	12.02	(16.39)	(15.00)	(1.39)	(12.02)	(11.00)	(1.02)	
5		3.28	112.55	0.497	225.10	4.50	4	0.50	11.26	(3.38)	12.38	(16.88)	(15.00)	(1.88)	(12.38)	(11.00)	(1.38)	
6		3.38	115.93	0.432	231.85	4.64	4	0.64	11.59	(3.48)	12.75	(17.39)	(15.00)	(2.39)	(12.75)	(11.00)	(1.75)	
7		3.48	119.41	0.376	238.81	4.78	4	0.78	11.94	(3.58)	13.13	(17.91)	(15.00)	(2.91)	(13.13)	(11.00)	(2.13)	
8		3.58	122.99	0.327	245.97	4.92	4	0.92	12.30	(3.69)	13.53	(18.45)	(15.00)	(3.45)	(13.53)	(11.00)	(2.53)	
9		3.69	126.68	0.284	253.35	5.07	4	1.07	12.67	(3.80)	13.93	(19.00)	(15.00)	(4.00)	(13.93)	(11.00)	(2.93)	
10		3.80	130.48	0.247	260.95	5.22	4	1.22	13.05	(3.91)	14.35	(19.57)	(15.00)	(4.57)	(14.35)	(11.00)	(3.35)	
<u>Discounted Totals</u>																		
(13) Yrs 1-10	100					22.26	20.08	2.19	55.65	(16.70)	61.22	(83.48)	(75.28)	(8.19)	(61.22)	(55.21)	(6.01)	94.43
(14) Terminal Value	0					11.07	6.59	4.48	27.68	(8.30)	30.45	(41.52)	(24.72)	(16.81)	(30.45)	(18.13)	(12.32)	-2.77
(15) All Yrs	100					33.33	26.67	6.67	83.33	(25.00)	91.67	(125.00)	(100.00)	(25.00)	(91.67)	(73.33)	(18.33)	91.67

(1) selected judgmentally for illustration purposes
(2) = (3) - previous year's (3); for year 1, (3) - (1)
(3) = (5) ÷ 2.0, where 2.0 represents the target premium-to-surplus ratio
(4) - factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year; subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4%. (6a) = (6) multiplied by the ratio of initial capital to (3). (6b) = (6) - (6a)
(7) = 5% of (5), selected so that earnings are less than the hurdle rate requirement
(8) = (3) - following year's (3), difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) × -15% (hurdle rate). (10a) = (10) multiplied by the ratio of initial capital to (3). (10b) = (10) - (10a)
(11) = (6) + (10). (11a) = (6a) + (10a). (11b) = (6b) + (10b)
(12) = (1) + [(6) + (7)] + (10). EVA (a) does not reduce the cost of capital to reflect investment income earned on capita
(13) Forecast Period Value
(14) Terminal Value
(15) = (13) + (14) Value in Perpetuity

Basic Valuation Example ***Exhibit 6C***
Economic Value Added (b) Model

Scenario Assumptions:

Total Earnings: hurdle rate not achieved
Annual Growth: 3%

	(1)	(2)	(3)	(4)	(5)	(6)	(6a)	(6b)	(7)	(8)	(9)	(10)	(10a)	(10b)	(11)	(11a)	(11b)	(12)
Projected Year	Initial Capital	Required Growth in Capital	Total Required Capital at Beginning of Year	Discount Factor at 15%	Premium Earned During Year	After-tax Investment Income on Capital	After-tax Investment Income on Original Capital	After tax Investment Income on Additional Capital	After-tax Total Income From Insurance Operations	Required Reinv	Available After-tax Net Income	Hurdle Required Return on Capital	Hurdle Required Return on Original Capital	Hurdle Required Return on Additional Capital	Total Cost of Capital	Cost of Original Capital	Cost of Additional Capital	Indicated Value
1	100	0	100	0.870	200	4	4	0	10	(3)	11	(15.00)	(15.00)	0.00	(11.00)	(11.00)	0.00	
2		3	103	0.756	206	4.12	4	0.12	10.3	(3.09)	11.33	(15.45)	(15.00)	(0.45)	(11.33)	(11.00)	(0.33)	
3		3.09	106.09	0.658	212.18	4.24	4	0.24	10.61	(3.18)	11.67	(15.91)	(15.00)	(0.91)	(11.67)	(11.00)	(0.67)	
4		3.18	109.27	0.572	218.55	4.37	4	0.37	10.93	(3.28)	12.02	(16.39)	(15.00)	(1.39)	(12.02)	(11.00)	(1.02)	
5		3.28	112.55	0.497	225.10	4.50	4	0.50	11.26	(3.38)	12.38	(16.88)	(15.00)	(1.88)	(12.38)	(11.00)	(1.38)	
6		3.38	115.93	0.432	231.85	4.64	4	0.64	11.59	(3.48)	12.75	(17.39)	(15.00)	(2.39)	(12.75)	(11.00)	(1.75)	
7		3.48	119.41	0.376	238.81	4.78	4	0.78	11.94	(3.58)	13.13	(17.91)	(15.00)	(2.91)	(13.13)	(11.00)	(2.13)	
8		3.58	122.99	0.327	245.97	4.92	4	0.92	12.30	(3.69)	13.53	(18.45)	(15.00)	(3.45)	(13.53)	(11.00)	(2.53)	
9		3.69	126.68	0.284	253.35	5.07	4	1.07	12.67	(3.80)	13.93	(19.00)	(15.00)	(4.00)	(13.93)	(11.00)	(2.93)	
10		3.80	130.48	0.247	260.95	5.22	4	1.22	13.05	(3.91)	14.35	(19.57)	(15.00)	(4.57)	(14.35)	(11.00)	(3.35)	
Discounted Totals																		
(13) Yrs 1-10	100					22.26	20.08	2.19	55.65	(16.70)	61.22	(83.48)	(75.29)	(8.19)	(61.22)	(55.21)	(6.01)	94.43
(14) Terminal Value	0					11.07	6.59	4.48	27.68	(8.30)	30.45	(41.52)	(24.72)	(16.81)	(30.45)	(18.13)	(12.32)	-2.77
(15) All Yrs	100					33.33	26.67	6.67	83.33	(25.00)	91.67	(125.00)	(100.00)	(25.00)	(91.67)	(73.33)	(18.33)	91.67

(1) - selected judgmentally for illustration purposes
(2) = (3) - previous year's (3) for year 1, (3) - (1)
(3) = (5) ÷ 2.0, where 2.0 represents the target premium-to-surplus ratio
(4) factor to discount from the end of the projected year to the beginning of year 1 at the hurdle rate (15%)
(5) = 200 for first projected year, subsequent years increased by the selected growth rate
(6) = (3) × selected investment yield of 4% (6a) = (6) multiplied by the ratio of initial capital to (3) (6b) = (6) - (6a)
(7) = 5% of (5), selected so that earnings are less than the hurdle rate requirement
(8) = (3) - following year's (3) difference between required capital at the beginning of year and required capital at beginning of following year
(9) = (6) + (7) + (8)
(10) = (3) × -15% (hurdle rate), (10a) = (10) multiplied by the ratio of initial capital to (3), (10b) = (10) - (10a)
(11) = (6) + (10) (11a) = (6a) + (10a) (11b) = (6b) + (10b)
(12) = (1) + (7) + (11). EVA (b) reduces the cost of capital component to reflect investment income earned on capital
(13) Forecast Period Value
(14) Terminal Value
(15) = (13) + (14) Value in Perpetuity

Primary Stock Insurance Company

Valuation Estimates as of December 31, 2001

Exhibit 7
Sheet 1

Establishing Starting Surplus	
(1) Booked Statutory Surplus @ 12/31/01	45 000
(2) Selected Starting Surplus for Future Income Projections (based on 12/31/01 RBC level)	42 131
(3) Income Recognized @ 12/31/01	2 869

Monitoring and Selecting Surplus	As of 12/31 of										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(4) Selected Surplus	$ 42 131	$ 46 506	$ 51 328	$ 55 362	$ 58 890	$ 62 150	$ 65 288	$ 68 388	$ 71 500	$ 74 652	$ 77 864
(5) Indicated Risk Based Capital Company Action Level	21 069	23 253	25 664	27 681	29 445	31 075	32 644	34 194	35 750	37 326	38 932
(6) Surplus to RBC Ratio	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%
(7) Surplus Growth		4 375	9 197	13 231	16 759	20 019	23 157	26 257	29 369	32 521	35 733
(8) Net Written Premium to Surplus Ratio	1 99	1 88	1 77	1 71	1 67	1 64	1 63	1 62	1 61	1 60	1 60
(9) Loss and LAE Reserves to Surplus Ratio	2 54	2 71	2 74	2 75	2 75	2 75	2 74	2 74	2 73	2 73	2 72

Estimated Future Income	During:									
	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(10) Total Company Net Income	10 445	12 054	13 187	14 153	14 977	15 752	16 507	17 248	17 978	18 713

Hurdle Rate	@ 12/31/										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(11) Selected Hurdle Rate		15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%
(12) Present Value Factor for 12/31/01 (assuming year end income recognition)	1 000	0 870	0 756	0 658	0 572	0 497	0 432	0 376	0 327	0 284	0 247

Expected Annual Perpetual Growth Rate of Capital & Income After 2011	2 0%

Application of DCF Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(13) Total Company Net Income	2 869	10 445	12 054	13 187	14 153	14 977	15 752	16 507	17 248	17 978	18 713			
(13pv) PV of Total Company Net Income	2 869	9 083	9 115	8 671	8 092	7 446	6 810	6 206	5 638	5 110	4 626	73 665	36 293	109 957
(14) Reinvestment Cost to Grow Company		4 375	4 822	4 034	3 528	3 260	3 138	3 100	3 112	3 152	3 212			
(14pv) PV of Reinv. Cost to Grow Company		3 804	3 646	2 652	2 017	1 621	1 357	1 165	1 017	896	794	18 970	2 961	21 931
(15) Free Cash Flow	2 869	6 070	7 232	9 153	10 625	11 717	12 614	13 407	14 136	14 824	15 501			
(15pv) PV of Free Cash Flow	2 869	5 278	5 468	6 018	6 075	5 825	5 453	5 040	4 621	4 214	3 832	54 694	33 332	88 026

Application of EVA Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(16) Statutory Surplus @ 12/31/01	42 131													
(17) Total Company Net Income	2 869	10 445	12 054	13 187	14 153	14 977	15 752	16 507	17 248	17 978	18 713			
(17pv) PV of Total Company Net Income	2 869	9 083	9 115	8 671	8 092	7 446	6 810	6 206	5 638	5 110	4 626	73 665	36 293	109 957
(18) Cost of Capital		6 320	6 978	7 699	8 304	8 834	9 323	9 783	10 258	10 725	11 198			
(18pv) PV of Cost of Capital		5 495	5 275	5 062	4 748	4 392	4 030	3 682	3 353	3 049	2 768	41 854	22 208	64 062
(19) Cost of Starting Capital		6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320			
(19pv) PV of Cost of Starting Capital		5 495	4 779	4 155	3 613	3 142	2 732	2 376	2 066	1 796	1 562	31,717	10 414	42 131
(20) Cost of Growth Capital			658	1 380	1 985	2 514	3 003	3 474	3 939	4 405	4 878			
(20pv) PV of Cost of Growth Capital			496	907	1 135	1 250	1 298	1 306	1 288	1 252	1 206	10 137	11 794	21 931
(21) Excess Returns in Year	2 869	4 125	5 076	5 488	5 849	6 144	6 430	6 714	6 990	7 251	7 515			
(21pv) PV of Excess Returns in Year	2 869	3 587	3 840	3 608	3 344	3 054	2 780	2 524	2 285	2 061	1 858	31 810	14 085	45 895
(22) Total Indicated Value														88 026

Primary Stock Insurance Company

Calculation Notes for Valuation Estimates as of December 31, 2001

(1) from Primary Stock Insurance Company's 12/31/01 booked balance sheet
(2) selected starting surplus for future income projections based on the selected 200.0% surplus to indicated RBC ratio
(3) = (1) − (2)

(4) selected surplus based on the selected 200.0% surplus to indicated RBC ratio
(5) calculated RBC at the Company Action Level - based on Exhibit 12 for 2002 and subsequent
(6) selection of 200.0% surplus to RBC ratio for determining required surplus at each year end
(7) cumulative increase in (4) from starting surplus (4) − (2)
(8) NPW for all lines ÷ (4)
(9) net loss and LAE reserves for all lines ÷ (4)

(10) from Exhibit 8, line (11)

(11) is selected hurdle rate of 15.0% used for determining Cost of Capital in EVA method and the present value of future earnings and value added
(12) = 1.000 at 12/31/01, for future years = (1.0 + 15.0%) raised to (2001 − year) exponent

(13) = (10)
(13pv) = (13) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(13)_{2011}$ × (1 + Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(14) annual change in (4)
(14pv) = (14) × (12) for each year. Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = $(4)_{2011}$ × Growth Rate ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(15) = (13) − (14)
(15pv) = (13pv) − (14pv)

(16) = (2)

(17) = (10)
(17pv) = (17) × (12) for each year; Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(17)_{2011}$ × (1 + Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(18) = (19) + (20)
(18pv) = (19pv) + (20pv)
(19) = (2) × (11)
(19pv) = (19) × (12) for each year. Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = (16) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞
(20) = $(7)_{\text{prior year}}$ × (11)
(20pv) = (20) × (12) for each year. Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = [$(4)_{2011}$ × Hurdle Rate ÷ (Hurdle Rate − Growth Rate) − (16)] × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(21) = (17) − (18)
(21pv) = (17pv) − (18pv)

(22) = (16) + $(21pv)_{\text{All Years}}$

Primary Stock Insurance Company Exhibit 8

Changes In Statutory Surplus	**Actual 2001**	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**
(1) Net Income	$8.531	$6 933	$8.111	$8.909	$9.592	$10.160	$10 697	$11.218	$11 726	$12.218	$12.715
(2) Changes in Unrealized Cap Gains	0	3 512	3 943	4 278	4.563	4.816	5 055	5.288	5.522	5.758	5 998
(3) Changes in Non-admitted Assets	0	0	0	0	0	0	0	0	0	0	0
(4) Capital Paid In	0	0	0	0	0	0	0	0	0	0	0
(5) Increase in Surplus Notes	0	0	0	0	0	0	0	0	0	0	0
(6) Principal Repayment	0	0	0	0	0	0	0	0	0	0	0
(7) Other Surplus Adjustments	0	0	0	0	0	0	0	0	0	0	0
(8) Contributions to Meet P/S Target		0	0	0	0	0	0	0	0	0	0
(9) Contributions to Meet R/S Target		0	0	0	0	0	0	0	0	0	0
(10) Change in Statutory Reserve	0	0	0	0	(2)	1	0	1	0	0	0
(11) Subtotal: Company Income	8.531	10 445	12.054	13.187	14.153	14.977	15.752	16.507	17.248	17.976	18.713
(12) Dividends to Stockholders	0	(6 070)	(7.232)	(9.153)	(10.625)	(11.717)	(12.614)	(13.407)	(14.136)	(14.824)	(15 501)
(12) Total Surplus Adjustments	$8.531	$4.375	$4 822	$4 034	$3 528	$3.260	$3.138	$3 100	$3.112	$3.152	$3 212

Calculation Notes:

(1) = After-tax net income from the statutory income statement (Exhibit 9)

(3), (4), (5), (6) (7) are set to $0

(11) = sum of lines (1) through (10)

(12) is the maximum dividend that satisfies required surplus based on risk-based capital multiple of 2.

(13) = (11) + (12)

Primary Stock Insurance Company Exhibit 9

Statutory Income Statement

	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(1) Net Earned Premium	$82,385	$85,680	$89,108	$92,671	$96,379	$100,234	$104 244	$108 412	$112 749	$117,259	$121,952
(2) Net Incurred Losses & LAE	63,988	66,547	69 208	71,974	74,856	77 850	80,963	84 200	87 571	91,074	94 719
(3) Underwriting Expenses	18,022	18,743	19,492	20,271	21 083	21 927	22,803	23 716	24 664	25 650	26,676
(4) Underwriting Income	$375	$390	$408	$426	$440	$457	$478	$496	$514	$535	$557
(5) Investment Income	11,000	10 410	11 685	12 683	13,525	14 277	14 984	15 677	16,368	17,068	17,780
(6) Other Income	0	0	0	0	0	0	0	0	0	0	0
(7) Policyholder Dividends	0	0	0	0	0	0	0	0	0	0	0
(8) Pre-Tax Operating Income	$11,375	$10 800	$12 093	$13,109	$13,965	$14,734	$15,462	$16 173	$16,882	$17,603	$18 337
(9) Federal Income Tax	2,844	3 867	3,982	4 200	4,373	4 574	4,765	4,955	5,156	5,385	5,622
(10) Net Income	$8,531	$6 933	$8 111	$8 909	$9,592	$10,160	$10 697	$11 218	$11 726	$12,218	$12,715

Calculation Notes:

(1) = sum of net earned premium for all lines of business

(2) = sum of net incurred losses and LAE for all lines of business

(3) = sum of underwriting expenses for all lines of business

(4) = (1) - (2) - (3)

(6) = set to $0 for the company

(7) = set to $0 for the company

(8) = (4) + (5) + (6) - (7)

(10) = (8) - (9)

Primary Stock Insurance Company Exhibit 10

Summary Statutory Balance Sheet	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(1) Invested Assets	$185,216	$213 090	$233,984	$251,262	$266 218	$279 994	$293,292	$306 473	$319 754	$333 230	$346 972
(2) Other Assets	7,500	6,828	7,127	7,431	7,744	8,065	8,397	8,741	9,099	9,470	9 854
(3) Total Assets	$192,716	$219 918	$241,111	$258 693	$273,962	$288,059	$301,689	$315 214	$328,853	$342,700	$356,826
(4) Net Loss & LAE Reserves	$107,085	$126 175	$140,678	$152,257	$161 965	$170,680	$178,971	$187,110	$195,258	$203 474	$211 810
(5) Net Unearned Premium Reserve	42 000	43 680	45,426	47,244	49 134	51,100	53 143	55,269	57,479	59,778	62 169
(6) Other Liabilities	1,500	3,557	3,679	3 830	3 973	4 129	4,287	4,447	4,616	4,796	4,983
(7) Total Liabilities	$150,585	$173 412	$189,783	$203,331	$215 072	$225 909	$236,401	$246 826	$257,353	$268,048	$278,962
(8) Statutory Surplus	$42,131	$46,506	$51 328	$55 362	$58,890	$62,150	$65,288	$68 388	$71,500	$74 652	$77,864
(9) Total Liabilities & Surplus	$192 716	$219 918	$241,111	$258,693	$273 962	$288 059	$301 689	$315 214	$328,853	$342 700	$356 826

Calculation Notes:

(1) Exhibit 11, Line (1)
(2) = (Premium Receivable + Receivables from Reinsurers + Other Assets) from Exhibit 11
(3) = (1) + (2)

(4) Exhibit 11, Line (8)
(5) Exhibit 11, Line (9)
(6) = [(10) + (11) + (12) + (13) + (14)] from Exhibit 11
(7) = (4) + (5) + (6)

(8) = (3) − (7)
(9) = (7) + (8)

Primary Stock Insurance Company

Exhibit 11
Sheet 1

Detailed Statutory Balance Sheet	Actual 2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(1a) Taxable Bonds	$77,791	$89,496	$98,273	$105,529	$111.812	$117.596	$123,182	$128.718	$134 295	$139,956	$145.728
(1b) Non-taxable Bonds	44,452	51.142	56,156	60.303	63,892	67 199	70 390	73,554	76 741	79,975	83,273
(1c) Stocks - Preferred	1,852	2.131	2,340	2.513	2.662	2.800	2,933	3,065	3.198	3 332	3.470
(1d) Stocks - Common	46,304	53 273	58,496	62.816	66,555	69,999	73.323	76 618	79,939	83,308	86,743
(1e) Cash	9,261	10,655	11,699	12,563	13,311	14,000	14 665	15,324	15,988	16,662	17.349
(1f) Real Estate	1.852	2,131	2.340	2.513	2.662	2,800	2.933	3.065	3,198	3.332	3.470
(1g) Other Income Producing Assets	3.704	4.262	4,680	5.025	5.324	5.600	5.866	6,129	6.395	6,665	6,939
(1) Total Investments & Cash	$185,216	$213,090	$233,984	$251,262	$266,218	$279,994	$293,292	$306.473	$319 754	$333,230	$346.972
(2) Premium Receivable	$7.500	$6.500	$6.760	$7,030	$7,310	$7,602	$7.907	$8,222	$8 552	$8,894	$9.249
(3) Receivables from Reinsurers	0	328	. 367	401	434	463	490	519	547	576	605
(4) Other Assets	0	0	0	0	0	0	0	0	0	0	0
(5) Total Assets	$192,716	$219,918	$241,111	$258,693	$273 962	$288.059	$301,689	$315,214	$328.853	$342,700	$356,826
(6) Net Loss Reserve	$79,567	$98,763	$113,298	$124,727	$134,151	$142,471	$150.267	$157,814	$165,275	$172,717	$180,202
(7) Net LAE Reserve	27,518	27,412	27,380	27 530	27,812	28.208	28,703	29,296	29.983	30,757	31,608
(8) Net Loss & LAE Reserve	107,085	126.175	140,678	152,257	161,965	170,680	178,971	187.110	195.258	203.474	211.810
(9) Net Unearned Premium Reserve	42,000	43,680	45,426	47,244	49.134	51,100	53,143	55,269	57,479	59,778	62 169
(10) Expenses Payable	1,000	1,030	1,061	1,093	1,126	1,162	1,198	1 236	1.275	1,316	1.359
(11) Income Taxes Payable	0	967	996	1,050	1,093	1,143	1.191	1 238	1.288	1,345	1,404
(12) Dividends Declared and Unpaid											
(12a) Policyholders	0	0	0	0	0	0	0	0	0	0	0
(12b) Stockholders	0	0	0	0	0	0	0	0	0	0	0
(13) Balances due Reinsurers	500	1,560	1,622	1,687	1,754	1.824	1,898	1,973	2,053	2 135	2,220
(14) Other Liabilities	0	0	0	0	0	0	0	0	0	0	0
(15) Total Liabilities	$150,585	$173,412	$189,783	$203,331	$215,072	$225,909	$236.401	$246.826	$257,353	$268 048	$278.962
(16) Capital	$40.000	$40,000	$40,000	$40.000	$40,000	$40.000	$40,000	$40,000	$40,000	$40.000	$40,000
(17) Surplus Notes	0	0	0	0	0	0	0	0	0	0	0
(18) Unassigned Funds	2.131	6,506	11,328	15,362	18,890	22,150	25.288	28.388	31,500	34,652	37,864
(19) Policyholder Surplus	$42.131	$46.506	$51,328	$55,362	$58,890	$62.150	$65,288	$68,388	$71.500	$74,652	$77,864
(20) Total Liabilities and Surplus	$192.716	$219,918	$241.111	$258,693	$273,962	$288,059	$301.689	$315.214	$328,853	$342,700	$356.826

Calculation Notes for Detailed Statutory Balance Sheet

(1a) = 42.0% of (1) Total Investments & Cash
(1b) = 24.0% of (1) Total Investments & Cash
(1c) = 1.0% of (1) Total Investments & Cash
(1d) = 25.0% of (1) Total Investments & Cash
(1e) = 5.0% of (1) Total Investments & Cash
(1f) = 1.0% of (1) Total Investments & Cash
(1g) = 2.0% of (1) Total Investments & Cash

(1) 2001 value is input, subsequent values = prior year value + net cash in* + changes in unrealized capital gains**

(2) 2001 value is input, subsequent values = prior year value + direct written premium for all lines- direct premium collected for all lines
(3) 2001 value is input, subsequent values = prior year value + ceded loss & LAE paid- ceded loss & LAE received
(4) value set to $0 for all years
(5) = (1) + (2) + (3) + (4)

(6) =
(7) =
(8) = (6) + (7)
(9) 2001 value is input, subsequent values = prior year value + net written premium for all lines- net earned premium for all lines
(10) 2001 value is input, subsequent values = prior year value + (agents' commissions + other underwriting expenses + premium taxes) for all lines- underwriting expenses paid*
(11) 2001 value is input, subsequent values = prior year value + federal income tax- federal income tax paid*
(12) values set to $0 for all years
(13) 2001 value is input, subsequent values = prior year value + (ceded written premium- premium ceded + reinsurance commission - reinsurance commission paid) for all lines
(14) value set to $0 for all years
(15) = (8) + (9) +(10) + (11) + (12a) + (12b) + (13) + (14)

(16) 2001 value is input, subsequent values = prior year value + capital paid in**
(17) 2001 value is input, subsequent values = prior year value + Increase in Surplus Notes**- Principal Repayment**
(18) = (19) - (16) - (17)
(19) = (5) - (15)
(20) = (15) + (19)

* Value is from Exhibit 13
** Value is from Exhibit 8

Exhibit 12

Primary Stock Insurance Company

Risk-Based Capital

INVESTED ASSET RISK	Factor	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Bonds	0 3%	$140 638	$154.429	$165 832	$175.704	$184 795	$193.572	$202.272	$211 036	$219.931	$229.001
Common Stocks	15 0%	53 273	58.496	62 816	66 555	69.999	73.323	76 618	79.939	83 308	86.743
Preferred Stocks	2.3%	2.131	2 340	2 513	2 662	2 800	2.933	3 065	3 198	3 332	3.470
Cash	0 3%	10 655	11.699	12.563	13 311	14.000	14.665	15.324	15.988	16 662	17.349
Real Estate	10 0%	2.131	2.340	2.513	2.662	2.800	2.933	3.065	3.198	3.332	3.470
Short-Term Investments	0 3%	4 262	4.680	5.025	5.324	5 600	5.866	6 129	6.395	6.665	6.939
Fixed-Income RBC		$487	$512	$550	$583	$613	$642	$671	$700	$730	$760
Equity-Asset RBC		$8.253	$9.062	$9.731	$10 311	$10.844	$11.359	$11.870	$12 384	$12.906	$13 438
CREDIT RISK											
Reinsurance Ceded (excl US affiliates pools)	10 0%	$29.359	$32 092	$34.694	$37 198	$39 653	$42.090	$44 523	$46.959	$49 410	$51.880
All Other Receivables	1 0%	0	0	0	0	0	0	0	0	0	0
Credit RBC		$2 936	$3.209	$3.469	$3 720	$3 965	$4.209	$4.452	$4.696	$4.941	$5.188
PREMIUM RISK											
Total of By-Line RBC		$5.111	$5.537	$5 987	$6 465	$6 972	$7 510	$8.078	$8.681	$9.318	$9 994
Premium Concentration Factor		0 357	0 357	0 357	0 357	0 357	0 357	0 357	0 357	0 357	0 357
Premium RBC		$4 125	$4.469	$4 832	$5 218	$5.627	$6 061	$6 520	$7.006	$7 521	$8.066
RESERVE RISK											
Total of By-Line RBC		$23.909	$26 565	$28.728	$30 561	$32.218	$33.795	$35.341	$36 886	$38.444	$40.025
Reserve Concentration Factor		0 430	0 418	0 412	0 410	0 410	0 410	0 410	0 410	0 410	0 410
Reserve RBC		$19 821	$21.927	$23 660	$25 152	$26.515	$27.813	$29.086	$30.357	$31 639	$32.941
GROWTH RISK											
Three-Year Average Growth		4 0%	4 0%	4 0%	4 0%	4 0%	4 0%	4 0%	4 0%	4 0%	4 0%
Net Written Premium	22 5%	$87 360	$90.854	$94.489	$98 269	$102.200	$106 287	$110 537	$114 960	$119.558	$124.343
Net Loss & LAE Reserves	45 0%	126.175	140.678	152.257	161 963	170 679	178 970	187.110	195.258	203.474	211 810
Growth RBC		$0	$0	$0	$0	$0	$0	$0	$0	$0	$0
INDICATED RBC (Upper end of Company Action Level)		$23 253	$25 664	$27 681	$29 445	$31 075	$32.644	$34.194	$35 750	$37.326	$38 932
ADJUSTED SURPLUS		$46.506	$51 328	$55 362	$58.890	$62.150	$65.288	$68 388	$71 500	$74.652	$77.864
Ratio to Authorized Control Level		400 00%	400 00%	400 00%	400 00%	400 00%	400 00%	400 00%	400 00%	400 00%	400 00%

Primary Stock Insurance Company Exhibit 13 Sheet 1

Cash Flow from Operations	Actual 2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(1) Direct Premium Collected	$90 962	$94 600	$97 084	$100 968	$105 008	$109 208	$113 574	$118 118	$122 841	$127 756	$132 869
(2) Premium Ceded	4 981	5 180	6 428	6 684	6 952	7 230	7 518	7 820	8 132	8 458	8 796
(3) Net Premium Collected	85 981	89 420	90 656	94 284	98 056	101 978	106 056	110 298	114 709	119 298	124 073
(4) Reinsurance Commissions Paid	0	0	0	0	0	0	0	0	0	0	0
(5) Interest & Dividends	9 400	8 344	9 366	10 165	10 840	11 443	12,010	12 565	13 119	13 680	14 251
(6) Realized Capital Gains	1 600	2 066	2 319	2 518	2 685	2 834	2 974	3,112	3 249	3 388	3 529
(7) Capital Received	0	0	0	0	0	0	0	0	0	0	0
(8) Capital Contributions	0	0	0	0	0	0	0	0	0	0	0
(9) Increase in Surplus Notes	0	0	0	0	0	0	0	0	0	0	0
(10) Other Income	0	0	0	0	0	0	0	0	0	0	0
(11) Total Collected	$96 981	$99 830	$102 341	$106 967	$111 581	$116 255	$121 040	$125 975	$131,077	$136 366	$141 853
(12) Gross Losses Paid	$38 806	$40 358	$47 610	$53 360	$58 086	$61 983	$65 395	$68 639	$71 837	$75 083	$78 397
(13) Loss Recoveries Received	3 404	3 540	4 195	4 526	4 838	5 127	5 410	5 695	5 986	6 281	6 586
(14) Net Losses Paid	35 402	36 818	43 415	48 834	53 248	56 856	59 985	62 944	65 851	68 802	71 811
(15) Gross LAE Paid	10 603	11 027	11 496	11 849	12 269	12 707	13 169	13 652	14 158	14 692	15 256
(16) LAE Recoveries Received	0	60	167	254	334	400	455	507	558	607	655
(17) Net LAE Paid	10 603	10 967	11 329	11 595	11 935	12 307	12 714	13 145	13 600	14 085	14 601
(18) Non-admitted Assets Purchased		0	0	0	0	0	0	0	0	0	0
(19) Total Underwriting Expenses Paid	17,893	18 713	19 461	20 239	21 050	21 891	22,767	23 678	24 625	25 609	26 633
(20) Federal Income Tax Paid	2 133	2 900	3 953	4 146	4 330	4 524	4 717	4 908	5 106	5 328	5 563
(21) Stockholder Dividends Paid	0	6 070	7 232	9 153	10 625	11 717	12 614	13 407	14 136	14 824	15 501
(22) Policyholder Dividends Paid	0	0	0	0	0	0	0	0	0	0	0
(23) Principal Repayment	0	0	0	0	0	0	0	0	0	0	0
(24) Interest Expense	0	0	0	0	0	0	0	0	0	0	0
(25) Total Paid	$66 131	$75 468	$85 390	$93 967	$101 188	$107 295	$112 797	$118 082	$123 318	$128 648	$134 109
(26) Net Cash from Operations	$30 850	$24 362	$16 951	$13 000	$10 393	$8 960	$8 243	$7 893	$7 759	$7 718	$7 744
NET CHANGE IN CASH	$30 850	$1 394	$1 044	$884	$748	$689	$665	$659	$664	$674	$687

Calculation Notes for Cash Flow from Operations

(1) = Direct Premium Collected for all lines

(2) = Premium Ceded for all lines

(3) = (1) − (2)

(4) = Reinsurance Commissions Paid for all lines

(7) value set to \$0 for all years

(8) value set to \$0 for all years

(9) value set to \$0 for all years

(10) value set to \$0 for all years

(11) = (3) + (4) + (5) − (6) + (7) + (8) + (9) + (10)

(12) = Gross Losses Paid for all lines

(13) = Loss Recoveries Received for all lines

(14) = (12) − (13)

(15) = Gross ALAE Paid for all lines + ULAE Paid for all lines

(16) = ALAE Recoveries Received for all lines

(17) = (15) − (16)

(18) value set to \$0 for all years

(19) Underwriting Expense Paid for all lines

(20) = 75% × Federal Income Tax + prior year Federal Income Tax Payable

(21) is the maximum dividend that satisfies required surplus based on risk-based capital multiple of 2.0

(22) value set to \$0 for all years

(25) = (14) + (17) + (18) + (19) + (20) − (21) + (22) + (23) − (24)

(26) = (11) − (25)

Primary Stock Insurance Company

Exhibit 14

GAAP Income Statement	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(1) Net Earned Premium	$82,385	$85 680	$89 108	$92 671	$96,379	$100 234	$104 244	$108,412	$112,749	$117,259	$121 952
(2) Net Incurred Losses & LAE	63 986	66,547	69 208	71,974	74 856	77 850	80 963	84,200	87 571	91 074	94,719
(3) Net Underwriting Expenses	17 693	18 401	19 214	19 983	20,782	21 615	22 480	23,377	24 311	25,286	26 296
(4) Underwriting Income	$704	$732	$686	$714	$741	$769	$801	$835	$867	$899	$937
(5) Investment Income	$11 000	$10 410	$11 685	$12 683	$13,525	$14,277	$14 984	$15 677	$16,368	$17,068	$17 780
(6) Other income	0	0	0	0	0	0	0	0	0	0	0
(7) Policyholder Dividends Incurred	0	0	0	0	0	0	0	0	0	0	0
(8) Pre-Tax Operating Income	$11 704	$11,142	$12,371	$13,397	$14 266	$15,046	$15 785	$16,512	$17 235	$17 967	$18,717
(9) Federal Income Tax	2 926	3 101	3,452	3 736	3 977	4,193	4 399	4,601	4,802	5,006	5 215
(10) Net Income	$8 778	$8 041	$8,919	$9 661	$10 289	$10 853	$11,386	$11,911	$12 433	$12,961	$13,502

Calculation Notes:

(1) = sum of net earned premium for all lines of business

(2) = sum of net incurred losses and LAE for all lines of business

(3) = sum of GAAP underwriting expenses for all lines of business

(4) = (1) - (2) - (3)

(6) = set to $0 for the company

(7) = set to $0 for the company

(8) = (4) + (5) + (6) - (7)

(10) = (8) - (9)

Primary Stock Insurance Company

Exhibit 15

Summary GAAP Balance Sheet	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(1) Invested Assets	$185 216	$213 090	$233,984	$251,262	$266 218	$279 994	$293,292	$306,473	$319,754	$333 230	$346,972
(2) Ceded Loss & LAE Reserves	26,134	29 030	31 725	34,293	36 763	39,190	41,600	44 004	46 412	48,834	51 275
(3) Ceded Unearned Premium Reserves	3 000	3,120	3 246	3 375	3,510	3,650	3,796	3 947	4,105	4,269	4,440
(4) Other Assets	14 100	13,770	14 347	14,939	15 553	16,186	16 841	17,524	18,235	18 970	19,734
(5) Total Assets	$228 450	$259 010	$283 302	$303 869	$322 044	$339,020	$355 529	$371,948	$388 506	$405 303	$422 421
(6) Gross Loss & LAE Reserves	$133 219	$155 205	$172,403	$186 550	$198 726	$209 869	$220 570	$231,114	$241,670	$252 308	$263,085
(7) Gross Unearned Premium Reserves	45,000	46,800	48 672	50 619	52,644	54,750	56 939	59 216	61,584	64,047	66 609
(8) Other Liabilities	1,500	2,791	2,383	2,070	1,817	1,592	1,384	1,190	1,005	806	586
(9) Total Liabilities	$179 719	$204,796	$223,458	$239 239	$253,187	$266,211	$278 893	$291,520	$304,259	$317 161	$330 280
(10) Total Capital & Surplus	48 731	54,214	59,844	64 630	68,857	72,809	76 636	80 428	84,247	88,142	92,141
(11) Total Liabilities & Surplus	$228 450	$259 010	$283 302	$303,869	$322,044	$339,020	$355,529	$371,948	$388,506	$405 303	$422,421

Calculation Notes.

(1) Exhibit 16, Line (1)
(2) Exhibit 16 Line (2) + Line (3)
(3) Exhibit 16 Line (4)
(4) = (Premium Receivable + Receivables from Reinsurers + Other Assets) from Exhibit 16
(5) = (1) + (2) + (3) + (4)

(6) Exhibit 16, Line (10) + Line (11)
(7) Exhibit 16 Line (12)
(8) = [(10) + (11) + (12) + (13) + (14)] from Exhibit 16
(9) = (6) + (7) + (8)

(10) = (5) - (9)
(11) = (9) + (10)

Primary Stock Insurance Company

Exhibit 16
Sheet 1

Detailed GAAP Balance Sheet	Actual 2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(1) Investments & Cash	$185,216	$213,090	$233,984	$251 262	$266,218	$279,994	$293,292	$306,473	$319,754	$333 230	$346,972
(2) Ceded Unpaid Losses	26,134	28,392	30 531	32 598	34 618	36,628	38,641	40 666	42 710	44,780	46,878
(3) Ceded Unpaid LAE	0	638	1 194	1,695	2 145	2,562	2,959	3 338	3 702	4 054	4 397
(4) Ceded Unearned Premium Reserves	3,000	3 120	3,246	3,375	3 510	3,650	3,796	3,947	4 105	4,269	4 440
(5) Premiums Receivable	7,500	6,500	6,760	7,030	7 310	7,602	7,907	8 222	8,552	8,894	9,249
(6) Deferred Acquisition Costs	6,600	6,942	7 220	7,508	7 809	8,121	8 444	8 783	9,136	9,500	9 880
(7) Receivables from Reinsurers	0	328	367	401	434	463	490	519	547	576	605
(8) Other Assets	0	0	0	0	0	0	0	0	0	0	0
(9) Total Assets	$228 450	$259,010	$283,302	$303,869	$322 044	$339,020	$355 529	$371 948	$388 506	$405,303	$422 421
(10) Gross Unpaid Losses	$105,701	$127,155	$143 829	$157 325	$168 769	$179 099	$188,908	$198 480	$207,985	$217,497	$227,080
(11) Gross Unpaid LAE	27,518	28 050	28,574	29,225	29 957	30 770	31,662	32 634	33 685	34 811	36,005
(12) Gross Unearned Premium Reserve	45,000	46,800	48 672	50 619	52 644	54,750	56,939	59,216	61 584	64 047	66 609
(13) Premium Deficiency Reserve	0	0	0	0	0	0	0	0	0	0	0
(14) Expenses Payable	1,000	1 030	1 061	1,093	1 126	1,162	1,198	1 236	1 275	1,316	1,359
(15) Balances Due Reinsurers	500	1 580	1,622	1 687	1 754	1 824	1 898	1,973	2 053	2 135	2,220
(16) Dividends Payable											
(16a) Policyholders	0	0	0	0	0	0	0	0	0	0	0
(16b) Stockholders	0	0	0	0	0	0	0	0	0	0	0
(17) Federal Income Taxes Payable											
(17a) Current	0	967	996	1,050	1,093	1,143	1 191	1,238	1,288	1,345	1,404
(17b) Deferred	0	(766)	(1 296)	(1,760)	(2 156)	(2 537)	(2,903)	(3 257)	(3,611)	(3,990)	(4 397)
(18) Surplus Notes	0	0	0	0	0	0	0	0	0	0	0
(19) Accrued Interest	0	0	0	0	0	0	0	0	0	0	0
(20) Other Liabilities	0	0	0	0	0	0	0	0	0	0	0
(21) Total Liabilities	$179,719	$204 796	$223 458	$239 239	$253 187	$266,211	$278 893	$291 520	$304,259	$317,161	$330 280
(22) Capital	$40 000	$40,000	$40 000	$40,000	$40 000	$40 000	$40 000	$40,000	$40 000	$40,000	$40 000
(23) Unrealized Capital Gains	0	3,512	7,455	11,733	16 296	21,112	26,167	31 455	36 977	42 735	48,733
(24) Retained Earnings	8,731	10,702	12 389	12,897	12,561	11,697	10 469	8 973	7 270	5 407	3,408
(25) Total Capital & Surplus	$48,731	$54,214	$59,844	$64 630	$68,857	$72,809	$76,636	$80 428	$84 247	$88,142	$92,141
(26) Total Liabilities & Surplus	$228 450	$259,010	$283 302	$303 869	$322 044	$339,020	$355 529	$371,948	$388,506	$405,303	$422 421

Primary Stock Insurance Company

Exhibit 16
Sheet 2

Calculation Notes for Detailed GAAP Balance Sheet

(1) shown in detail on Exhibit 11
(4) 2001 value is input. subsequent values = prior year value + (GWP - NWP + GEP - NEP) for all lines
(5) 2001 value is input. subsequent values = prior year value + DWP for all lines - Direct Premium Collected for all lines
(6) 2001 value is input. subsequent values = prior year value + [underwriting expenses (statutory) - underwriting expenses (GAAP) + reinsurance commission (GAAP)] for all lines
(7) 2001 value is input. subsequent values = prior year value + ceded loss & LAE paid - ceded loss & LAE received
(8) value set to $0 for all years
(9) = (1) + (2) + (3) + (4) + (5) + (6) + (7) + (8)

(12) 2001 value is input, subsequent values = prior year value + GWP for all lines - GEP for all lines
(14) 2001 value is input, subsequent values = prior year value + (agents' commissions + other underwriting expenses + premium taxes) for all lines - Underwriting Expenses Paid*
(15) 2001 value is input, subsequent values = prior year value + (ceded written premium - premium ceded + reinsurance commission - reinsurance commission paid) for all lines
(16), (16a), (16b) values set to $0 for all years
(17a) 2001 value is input. subsequent values = prior year value + Federal Income Tax - Federal Income Tax Paid*
(18) 2001 value is input, subsequent values = prior year value + Increase in Surplus Notes** - Principal Repayment**
(19) unpaid principal and interest associated with (18)
(20) value set to $0 for all years
(21) = (10) + (11) + (12) + (13) + (14) + (15) + (16a) + (16b) + (17a) + (17b) + (18) + (19) + (20)

(22) 2001 value is input, subsequent values = prior year value + capital paid in**
(24) = (25) - (22) - (23)
(25) = (9) - (21)

(26) = (21) + (25)

* Value is from Exhibit 13
** Value is from Exhibit 8

Primary Stock Insurance Company Exhibit 17

Changes in GAAP Net Worth	Actual 2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(1) Net Income	$8,778	$8,041	$8 919	$9 661	10,289	$10 853	$11,386	$11 911	$12 433	$12,961	$13,502
(2) Stockholder Dividends Incurred	0	6 070	7,232	9,153	10,625	11,717	12,614	13 407	14,136	14 824	15,501
(3) Other Surplus Adjustments	0	0	0	0	0	0	0	0	0	0	0
(4) Change in Retained Earnings	$8 778	$1 971	$1 687	$508	($336)	($864)	($1,228)	($1 496)	($1 703)	($1,863)	($1,999)
(5) Capital Contributions	0	0	0	0	0	0	0	0	0	0	0
(6) Change in Unrealized Capital Gains	0	3 512	3 943	4 278	4,563	4 816	5,055	5,288	5 522	5,758	5,998
(7) Change in Net Worth	$8,778	$5 483	$5,630	$4 786	$4,227	$3 952	$3 827	$3,792	$3 819	$3 895	$3 999

Calculation Notes:

(1) = GAAP Net Income
(3) value set to $0 for all years
(4) = (1) - (2) + (3)

(5) value set to $0 for all years
(7) = (4) + (5) + (6)

Primary Stock Insurance Company

Exhibit 18
Sheet 1

Workers Compensation

	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Statutory Underwriting Income for Line of Business										
(A) Net Earned Premium	$27,540	$28,642	$29,787	$30,979	$32,218	$33,507	$34,847	$36,241	$37,690	$39,199
(B) Net Incurred Loss and LAE	21,650	22,516	23,415	24,354	25,328	26,340	27,393	28,490	29,629	30,815
(C) Total Underwriting Expenses	5,676	5,903	6,139	6,385	6,640	6,906	7,182	7,470	7,768	8,079
(D) Underwriting Income	$214	$223	$233	$240	$250	$261	$272	$281	$293	$305
Modeled Amounts										
(1) Direct Written Premium	$31,200	$32,448	$33,748	$35,096	$36,500	$37,960	$39,478	$41,057	$42,699	$44,408
(2) Direct Earned Premium	30,600	31,824	33,097	34,421	35,798	37,230	38,719	40,268	41,878	43,554
(3) Ceded Written Premium	3,120	3,245	3,375	3,510	3,650	3,796	3,948	4,106	4,270	4,441
(4) Ceded Earned Premium	3,060	3,182	3,310	3,442	3,580	3,723	3,872	4,027	4,188	4,355
(5) Net Written Premium	28,080	29,203	30,371	31,586	32,850	34,164	35,530	36,951	38,429	39,967
(6) Net Earned Premium	27,540	28,642	29,787	30,979	32,218	33,507	34,847	36,241	37,690	39,199
(7) Direct Incurred Losses	21,420	22,277	23,168	24,095	25,059	26,061	27,103	28,188	29,315	30,488
(8) Ceded Incurred Losses	3,060	3,182	3,310	3,442	3,580	3,723	3,872	4,027	4,188	4,355
(9) Net Incurred Losses	18,360	19,095	19,858	20,653	21,479	22,338	23,231	24,161	25,127	26,133
(10) Direct Incurred ALAE	1,714	1,782	1,853	1,928	2,005	2,085	2,168	2,255	2,345	2,439
(11) Ceded ALAE	245	255	265	275	286	298	310	322	335	348
(12) Net Incurred ALAE	1,469	1,527	1,588	1,653	1,719	1,787	1,858	1,933	2,010	2,091
(13) Gross Incurred ULAE	1,821	1,894	1,969	2,048	2,130	2,215	2,304	2,396	2,492	2,591
(14) Net Incurred Loss & LAE	21,650	22,516	23,415	24,354	25,328	26,340	27,393	28,490	29,629	30,815
(15) Agents' Commissions	3,120	3,245	3,375	3,510	3,650	3,796	3,948	4,106	4,270	4,441
(16) Other Underwriting Expenses	1,620	1,685	1,752	1,822	1,895	1,971	2,050	2,132	2,217	2,306
(17) Premium Taxes	936	973	1,012	1,053	1,095	1,139	1,184	1,232	1,281	1,332
(18) Reinsurance Commissions	0	0	0	0	0	0	0	0	0	0
(19) Total Underwriting Expenses	5,676	5,903	6,139	6,385	6,640	6,906	7,182	7,470	7,768	8,079
Modeled GAAP Amounts										
(20) Gross Reserves	$62,716	$66,999	$70,848	$74,575	$78,303	$82,088	$85,937	$89,864	$93,879	$97,978
(21) Ceded Reserves	17,172	18,539	19,870	21,175	22,486	23,811	25,156	26,520	27,902	29,303
(22) Agents' Commissions	3,060	3,182	3,310	3,442	3,580	3,723	3,872	4,027	4,188	4,355
(23) Underwriting Expenses	1,620	1,685	1,752	1,822	1,895	1,971	2,050	2,132	2,217	2,306
(24) Premium Tax	873	955	993	1,033	1,074	1,117	1,162	1,208	1,256	1,307
(25) Reinsurance Commissions	0	0	0	0	0	0	0	0	0	0
(26) Total Underwriting Expenses	5,553	5,822	6,055	6,297	6,549	6,811	7,084	7,367	7,661	7,968

Primary Stock Insurance Company Exhibit 18
Sheet 2

Workers Compensation

Calculation Notes Statutory Underwriting Income, Modeled Amounts, and Modeled GAAP Amounts

(A) = (6)   (B) = (14)   (C) = (19)   (D) = (A) - (B) - (C)

(1) = 2001 DWP × (annual growth rate)$^{(\text{year} - 2001)}$
(2) = (prior year UEPR) + (earned% × DWP)
(3) = excess ceded % × DWP
(4) = excess ceded % × DEP
(5) = (1) - (3)
(6) = (2) - (4)

(7) = GEP × expected loss ratio
(8) = (4) × ceded loss ratio
(9) = (7) - (8)
(10) = (7) × Gross ALAE to loss %
(11) = [(8) ÷ (7)] × (10)
(12) = (10) - (11)
(13) = (7) × Gross ULAE to loss %
(14) = (9) + (12) + (13)

(15) = DWP × Agents' commission %
(16) = Amount of fixed underwriting + (% of DEP × DEP) + (% of DWP × DWP)
(17) = DWP × Premium tax %
(18) = Reinsurance commission % × (3)
(19) = (15) + (16) + (17) + (18)

(22) = Business earned in 1st year % × DWP × Agents' commission % + 2001 GAAP Deferred Commission *in 2002*
  = Agents' commission % × DEP *in other years*
(23) = (16) × [1 − Deferrable %] + (16) × Business earned in 1st year % × Deferrable % + 2001 GAAP Deferred U/W Expense *in 2002*
  = (16) × [1−Deferrable %] + Deferrable % × [(16) × Business earned in 1st year % + (1−Business earned in 1st year %) × prior yr (16)] *in other years*
(24) = Business earned in 1st year % × DWP × Premium Tax % + 2001 GAAP Deferred Premium Tax *in 2002*
  = Premium Tax % × DEP *in other years*
(25) = (4) × Reinsurance Commission %
(26) = (22) + (23) + (24) + (25)

Primary Stock Insurance Company

Exhibit 18
Sheet 3

Workers Compensation

	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Underwriting Cash Flow For Lines of Business										
(A) Total Collected	$28 510	$29 130	$30 296	$31 508	$32 768	$34 079	$35 442	$36,859	$38,333	$39 868
(B) Net Loss and LAE Payments	18 114	19 617	20 911	21,945	22 924	23 892	24,901	25 939	27 010	28 131
(C) Underwriting Expense Paid	5,666	5,893	6,128	6,374	6,628	6,894	7,169	7,457	7,754	8,065
(D) Cash Flow from Underwriting	$4,730	$3 620	$3 257	$3 189	$3,216	$3,293	$3,372	$3 463	$3 569	$3,672
Modeled Amounts										
(1) Gross Premium Collected	$31 100	$32 344	$33 638	$34,984	$36 383	$37 838	$39,352	$40 925	$42 562	$44 266
(2) Premium Ceded	2,590	3 214	3,342	3 476	3 615	3 759	3,910	4 066	4 229	4 398
(3) Net Premium Collected	28 510	29 130	30 296	31 508	32 768	34 079	35,442	36 859	38 333	39,868
(4) Reinsurance Commissions	0	0	0	0	0	0	0	0	0	0
(5) Total Collected	28 510	29,130	30,296	31 508	32 768	34 079	35 442	36 859	38 333	39,868
(6) Gross Losses Paid	15 990	17 834	19 277	20,405	21 421	22,411	23,430	24 477	25 554	26 677
(7) Loss Recoveries Received	1,686	1 992	2 143	2,284	2 413	2 540	2 670	2 806	2 947	3,095
(8) Net Loss Paid	14,304	15 842	17 134	18 121	19 008	19 871	20 760	21 671	22 607	23 582
(9) Gross ALAE Paid	1 279	1,427	1 542	1 632	1 714	1,793	1 874	1 958	2 044	2,134
(10) ALAE Recoveries Received	18	61	87	114	130	144	155	167	180	194
(11) Net ALAE Paid	1 261	1,366	1,455	1 518	1,584	1 649	1 719	1,791	1,864	1 940
(12) ULAE Paid	2 549	2 409	2 322	2 306	2,332	2,372	2 422	2 477	2,539	2 609
(13) Net Loss & LAE Payments	18,114	19 617	20 911	21 945	22 924	23 892	24 901	25,939	27,010	28,131
(14) Agents' Commissions	3,110	3 235	3 364	3 499	3 638	3 784	3 935	4 093	4 256	4 427
(15) Other Underwriting Expenses	1 620	1 685	1,752	1 822	1,895	1 971	2 050	2,132	2,217	2 306
(16) Premium Taxes Paid	936	973	1,012	1 053	1 095	1 139	1,184	1 232	1 281	1 332
(17) Underwriting Expense Paid	5,666	5,893	6,128	6,374	6,628	6,894	7,169	7,457	7,754	8,065

Calculation Notes.

(A) = (5)   (B) = (13)   (C) = (17)   (D) = (A) − (B) − (C)

(1) = DWP × (1 − Monthly Premium Collection Lag / 12) + 2001 Direct Premium Uncollected *in 2002*

= DWP × (1 − Monthly Premium Collection Lag / 12) + prior year DWP × Monthly Premium Collection Lag / 12 *in other years*

(2) = Ceded WP × (1 − Monthly Ceding Premium Lag / 12) + 2001 Ceded Unearned Premium *in 2002*

= Ceded WP × (1 − Monthly Ceding Premium Lag / 12) + prior year Ceded WP × Premium Collection Lag / 12 *in other years*

(3) = (1) − (2)

(4) = Reinsurance commission % × (2)

(5) = (3) + (4)

(8) = (6) − (7)

(11) = (9) − (10)

(13) = (8) + (11) + (12)

(14) = Agents' commission from cash flow × (1 − Monthly Premium Collection Lag / 12) + 2001 Unpaid Agents Commission *In 2002*

= Agents' commission* × (1 − Monthly Premium Collection Lag / 12) + Prior year Agents commission* × Monthly Premium Collection Lag / 12 *in other yea*

(15) = Fixed underwriting$ + (% of DEP × DEP) + (% of DWP × DWP)

(16) = DWP × Premium tax %

(17) = (14) + (15) + (16)

Financial Modeling Assumptions

Workers Compensation

Modeling Assumptions for Current and Future Business

Amounts as of 12/31/01		Assumptions for Future Business	
Gross Unearned Premium	$15 000	Gross Written Premium in 2002	$31 200
Net Unearned Premium	$13,500	Annual Growth Rate of Business	4 00%
Ceded Unearned Premium	1 500		
Gross Premium Uncollected	$2 500	Percent of Business Earned in 1st Year	50 00%
Unpaid Agent s Commission	250	Premium Collection Lag (in months)	1
Ceded Premium Not Yet Remitted	250		
Ceded Paid Losses Not Yet Coll	0	Expected Loss Ratio	70 00%
Ceded Paid ALAE Not Yet Coll	0	Gross ALAE as a % of Loss	6 00%
Reinsurance Comm Not Yet Coll	0	Gross ULAE as a % of Loss	6 50%
GAAP Deferred Commission	1 500		
GAAP Deferred U/W Expense	0	Agents' Commission as % of DWP	10 00%
GAAP Deferred Premium Tax	405	Premium Tax as % of DWP	3 00%
GAAP Deferred Reins Commission	0	Other Underwriting Expenses	
		Fixed	$0
		Variable (% of DEP)	3 00%
		Variable (% of DWP)	2 25%
		% Deferrable	0 00%
		Reinsurance (per occ Excess)	
		Percent of premium ceded	10 00%
		Lag in Ceding Premium (in months)	3
		Ceded Loss Ratio	100 00%
		Ceded Loss Collection Lag (in months)	1

Loss and LAE reserves are carried at nominal (undiscounted) value for both SAP and GAAP

'Reserve strengthening' adjustments are not needed ultimate loss and LAE amounts do not deteriorate or improve over time

Financial Modeling Assumptions

Workers Compensation

Loss Payment and Discounting

Payment Patterns for Loss and LAE

Accident Year +	Gross	Ceded	IRS
0	21.00%	8.00%	25.00%
1	30.00%	18.00%	33.00%
2	14.00%	9.00%	16.00%
3	10.00%	9.75%	12.00%
4	4.00%	4.25%	4.00%
5	3.00%	3.25%	2.00%
6	2.00%	2.25%	1.50%
7	2.00%	2.50%	0.75%
8	1.75%	2.50%	0.75%
9	1.50%	2.50%	0.75%
10	1.50%	2.75%	0.75%
11	1.25%	2.50%	0.75%
12	1.00%	2.25%	0.75%
13	1.00%	2.50%	0.50%
14	0.75%	2.00%	0.50%
15	0.75%	2.25%	1.00%
16	0.50%	1.75%	
17	0.50%	2.00%	
18	0.50%	2.25%	
19	0.25%	1.25%	
20	0.25%	1.50%	
21	0.25%	1.50%	
22	0.25%	1.50%	
23	0.25%	1.50%	
24	0.25%	1.50%	
25	0.25%	1.50%	
26	0.25%	1.50%	
27	0.25%	1.50%	
28	0.25%	1.50%	
29	0.25%	1.50%	
30	0.25%	1.50%	
Total	100.00%	100.00%	100.00%

Interest Rate for Discounted Tax Reserves

Accident Year	Rate
1982	7.20%
1983	7.20%
1984	7.20%
1985	7.20%
1986	7.20%
1987	7.20%
1988	7.77%
1989	8.16%
1990	8.37%
1991	7.00%
1992	7.00%
1993	7.00%
1994	7.00%
1995	7.00%
1996	7.00%
1997	7.00%
1998	7.00%
1999	7.00%
2000	7.00%
2001	7.00%
2002	7.00%
2003	7.00%
2004	7.00%
2005	7.00%
2006	7.00%
2007	7.00%
2008	7.00%
2009	7.00%
2010	7.00%
2011	7.00%

Financial Modeling Assumptions

Workers Compensation

Prior Years' Information

Accident Year	Estimates of Ultimate Loss and LAE @ 12/31/01: Gross Loss	Ceded Loss	Net Loss	Gross ALAE	Ceded ALAE	Net ALAE	ULAE	Paid Loss and LAE @ 12/31/01: Gross Loss	Ceded Loss	Net Loss	Gross ALAE	Ceded ALAE	Net ALAE	ULAE
1982	8 971	1,282	7,689	718	0	718	763	8 746	1,089	7 857	700	0	700	740
1983	9 330	1,333	7,997	746	0	746	793	9 073	1,113	7 960	726	0	726	761
1984	9 703	1,386	8,317	776	0	776	825	9 412	1 140	8 272	753	0	753	783
1985	10 091	1,442	8,649	807	0	807	858	9,738	1,153	8 584	779	0	779	806
1986	10 494	1,499	8 995	840	0	840	892	10,075	1,169	8 905	806	0	806	830
1987	10 914	1,559	9,355	873	0	873	928	10,423	1,189	9 234	834	0	834	853
1988	11,351	1,622	9 729	908	0	908	965	10,755	1,200	9 555	860	0	860	868
1989	11,805	1,686	10 118	944	0	944	1 003	11 097	1,214	9 882	888	0	888	873
1990	12,277	1,754	10 523	982	0	982	1,044	11,418	1,219	10 199	913	0	913	868
1991	12,768	1 824	10 944	1 021	0	1 021	1,085	11,747	1,227	10 520	940	0	940	857
1992	13,279	1 897	11 382	1 062	0	1 062	1,129	12,051	1 228	10 822	964	0	964	835
1993	13,810	1 973	11 837	1 105	0	1 105	1,174	12,325	1 223	11,102	986	0	986	810
1994	14,362	2 052	12,311	1 149	0	1 149	1,221	12,603	1 221	11,382	1 008	0	1,008	769
1995	14,937	2 134	12,803	1,195	0	1 195	1,270	12,846	1 216	11,629	1 028	0	1,028	736
1996	15,534	2 219	13,315	1,243	0	1 243	1,320	13 049	1 209	11,839	1,044	0	1,044	713
1997	16 156	2 308	13,848	1,292	0	1 292	1,373	13 248	1 206	12,042	1,060	0	1,060	687
1998	16 802	2,400	14,402	1 344	0	1 344	1,428	13 274	1 176	12,097	1,062	0	1 062	657
1999	17 474	2 496	14,978	1,398	0	1 398	1 485	13 106	1,117	11,988	1,048	0	1 048	624
2000	18 173	2,596	15,577	1,454	0	1 454	1 545	11 813	909	10 904	945	0	945	587
2001	18,900	2 700	16,200	1,512	0	1 512	1 607	3,969	216	3 753	318	0	318	546

Accident Year	Loss and LAE Reserves @ 12/31/01: Gross Loss	Ceded Loss	Net Loss	Gross ALAE	Ceded ALAE	Net ALAE	ULAE	Net Earned Premium
1982	224	192	32	18	0	18	23	12,815
1983	257	220	37	21	0	21	32	13,328
1984	291	246	45	23	0	23	41	13,861
1985	353	288	65	28	0	28	51	14,416
1986	420	330	90	34	0	34	62	14,992
1987	491	370	121	39	0	39	74	15 592
1988	596	422	174	48	0	48	96	16 216
1989	708	472	236	57	0	57	130	16,864
1990	859	535	324	69	0	69	177	17,539
1991	1 021	597	424	82	0	82	228	18 240
1992	1 228	669	560	98	0	98	293	18,970
1993	1 485	750	735	119	0	119	384	19 729
1994	1 759	831	928	141	0	141	452	20 518
1995	2 091	918	1 174	167	0	167	533	21 338
1996	2 486	1 010	1 476	199	0	199	607	22 192
1997	2 908	1 102	1,806	233	0	233	687	23 080
1998	3 528	1 224	2,304	282	0	282	771	24 003
1999	4 369	1 379	2,989	349	0	349	861	24 963
2000	6 361	1 688	4,673	509	0	509	958	25 962
2001	14,931	2,484	12 447	1 194	0	1 194	1,060	27 000
Total	46 367	15 726	30,640	3 709	0	3 709	7 503	

Primary Stock Insurance Company

Exhibit 19
Sheet 1

Auto Liability

	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Statutory Underwriting Income for Line of Business										
(A) Net Earned Premium	$30 600	$31,824	$33 097	$34,421	$35 798	$37 230	$38 719	$40 268	$41,879	$43 554
(B) Net Incurred Loss and LAE	22 718	23,626	24 571	25 553	26,576	27 639	28 745	29 896	31 091	32 335
(C) Total Underwriting Expenses	7,007	7,287	7,578	7,881	8,197	8,524	8 866	9,220	9 589	9 972
(D) Underwriting Income	$875	$911	$948	$987	$1,025	$1,067	$1,108	$1,152	$1,199	$1 247
Modeled Amounts										
(1) Direct Written Premium	$31 200	$32 448	$33 746	$35 096	$36,500	$37,960	$39,478	$41 057	$42 700	$44 408
(2) Direct Earned Premium	30 600	31 824	33 097	34,421	35,798	37 230	38 719	40 268	41,879	43 554
(3) Ceded Written Premium	0	0	0	0	0	0	0	0	0	0
(4) Ceded Earned Premium	0	0	0	0	0	0	0	0	0	0
(5) Net Written Premium	31 200	32 448	33,746	35 096	36,500	37 960	39 478	41 057	42,700	44 408
(6) Net Earned Premium	30 600	31 824	33,097	34 421	35 798	37 230	38 719	40,268	41 879	43 554
(7) Direct Incurred Losses	19,584	20 367	21 182	22,029	22 911	23 827	24 780	25 772	26 803	27,875
(8) Ceded Incurred Losses	0	0	0	0	0	0	0	0	0	0
(9) Net Incurred Losses	19 584	20 367	21,182	22 029	22,911	23,827	24 780	25 772	26 803	27,875
(10) Direct Incurred ALAE	1 665	1 731	1 800	1 872	1 947	2 025	2 106	2 191	2 278	2 369
(11) Ceded ALAE	0	0	0	0	0	0	0	0	0	0
(12) Net Incurred ALAE	1 665	1,731	1 800	1 872	1,947	2 025	2 106	2 191	2 278	2 369
(13) Gross Incurred ULAE	1 469	1 528	1,589	1,652	1 718	1 787	1,859	1,933	2 010	2,091
(14) Net Incurred Loss & LAE	22 718	23,626	24,571	25 553	26,576	27 639	28 745	29 896	31,091	32 335
(15) Agents' Commissions	4 680	4,867	5 062	5,264	5 475	5 694	5 922	6,159	6 405	6,661
(16) Other Underwriting Expenses	1 703	1,771	1,841	1 915	1 992	2 071	2,154	2,240	2 330	2,423
(17) Premium Taxes	624	649	675	702	730	759	790	821	854	888
(18) Reinsurance Commissions	0	0	0	0	0	0	0	0	0	0
(19) Total Underwriting Expenses	7,007	7 287	7,578	7,881	8,197	8 524	8 866	9,220	9 589	9 972
Modeled GAAP Amounts										
(20) Gross Reserves	$26 347	$33,470	$38 521	$42,146	$44,953	$47 388	$49,672	$51,904	$54 083	$56 246
(21) Ceded Reserves	0	0	0	0	0	0	0	0	0	0
(22) Agents Commissions	4 590	4 774	4 985	5 163	5 370	5 585	5 808	6 040	6 282	6,533
(23) Underwriting Expenses	1 703	1,771	1 841	1,915	1,992	2 071	2 154	2 240	2,330	2,423
(24) Premium Tax	612	638	662	688	716	745	774	805	838	871
(25) Reinsurance Commissions	0	0	0	0	0	0	0	0	0	0
(26) Total Underwriting Expenses	6,905	7,181	7,468	7,766	8,078	8,401	8,736	9,085	9 450	9,827

Primary Stock Insurance Company — Exhibit 19, Sheet 2

Auto Liability

Calculation Notes Statutory Underwriting Income, Modeled Amounts, and Modeled GAAP Amounts

(A) = (6)   (B) = (14)   (C) = (19)   (D) = (A) − (B) − (C)

(1) = 2001 DWP × (annual growth rate)$^{(\text{year} - 2001)}$
(2) = (prior year UEPR) + (earned% × DWP)
(3) = excess ceded % × DWP
(4) = excess ceded % × DEP
(5) = (1) − (3)
(6) = (2) − (4)

(7) = GEP × expected loss ratio
(8) = (4) × ceded loss ratio
(9) = (7) − (8)
(10) = (7) × Gross ALAE to loss %
(11) = [(8) ÷ (7)] × (10)
(12) = (10) − (11)
(13) = (7) × Gross ULAE to loss %
(14) = (9) + (12) + (13)

(15) = DWP × Agents' commission %
(16) = Fixed underwriting$ + (% of DEP × DEP) + (% of DWP × DWP)
(17) = DWP × Premium tax %
(18) = Reinsurance commission % × (3)
(19) = (15) + (16) + (17) − (18)

(22) = Business earned in 1st year % × DWP × Agents' commission % + 2001 GAAP Deferred Commission *in 2002*
  = Agents' commission % × DEP *in other years*
(23) = (16) × [1 − Deferrable %] + (16) × Business earned in 1st year % × Deferrable % + 2001 GAAP Deferred U/W Expense *in 2002*
  = (16) × [1−Deferrable %] + Deferrable % × [(16) × Business earned in 1st year % + (1−Business earned in 1st year %) × prior yr (16)] *in other years*
(24) = Business earned in 1st year % × DWP × Premium Tax % + 2001 GAAP Deferred Premium Tax *in 2002*
  = Premium Tax % × DEP *in other years*
(25) = (4) × Reinsurance Commission %
(26) = (22) + (23) + (24) + (25)

Primary Stock Insurance Company Exhibit 19
Sheet 3

Auto Liability

	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Underwriting Cash Flow For Lines of Business										
(A) Total Collected	$32 400	$32 396	$33 692	$35 040	$36 442	$37,899	$39 415	$40 991	$42 632	$44 337
(B) Net Loss and LAE Payments	12 526	16 503	19 519	21,928	23 770	25 205	26 460	27 664	28 913	30 172
(C) Underwriting Expense Paid	7 062	7 279	7 570	7 873	8 188	8 515	8,857	9,210	9 579	9 961
(D) Cash Flow from Underwriting	$12 812	$8 614	$8 603	$5,239	$4,484	$4,179	$4 098	$4,117	$4 140	$4,204
Modeled Amounts										
(1) Gross Premium Collected	$32,400	$32,396	$33 692	$35 040	$36,442	$37,899	$39,415	$40,991	$42 632	$44 337
(2) Premium Ceded	0	0	0	0	0	0	0	0	0	0
(3) Net Premium Collected	32,400	32,396	33 692	35 040	36 442	37,899	39 415	40 991	42,632	44,337
(4) Reinsurance Commissions	0	0	0	0	0	0	0	0	0	0
(5) Total Collected	32 400	32 396	33,692	35 040	36,442	37 899	39,415	40,991	42 632	44 337
(6) Gross Losses Paid	10 006	13,674	16,472	18 684	20,352	21 629	22 730	23 777	24,861	25,951
(7) Loss Recoveries Received	0	0	0	0	0	0	0	0	0	0
(8) Net Loss Paid	10 006	13 674	16 472	18 684	20 352	21 629	22 730	23,777	24 861	25 951
(9) Gross ALAE Paid	851	1,162	1 400	1 588	1,730	1 838	1 932	2 021	2,113	2 206
(10) ALAE Recoveries Received	0	. 0	0	0	0	0	0	0	0	0
(11) Net ALAE Paid	851	1 162	1 400	1 588	1 730	1,838	1 932	2 021	2 113	2 206
(12) ULAE Paid	1,689	1 667	1 647	1 656	1 688	1,738	1 798	1 866	1,939	2,015
(13) Net Loss & LAE Payments	12 526	16 503	19 519	21 928	23 770	25 205	26 460	27 664	28,913	30 172
(14) Agents' Commissions	4 735	4,859	5 054	5 256	5 466	5 685	5 913	6 149	6,395	6 650
(15) Other Underwriting Expenses	1 703	1 771	1 841	1 915	1 992	2 071	2 154	2,240	2 330	2 423
(16) Premium Taxes Paid	624	649	675	702	730	759	790	821	854	888
(17) Underwriting Expense Paid	7,062	7,279	7 570	7 873	8,188	8 515	8 857	9,210	9,579	9 961

Calculation Notes:

(A) = (5)   (B) = (13)   (C) = (17)   (D) = (A) − (B) − (C)

(1) = DWP × (1 − Monthly Premium Collection Lag / 12) + 2001 Direct Premium Uncollected *in 2002*

= DWP × (1 − Monthly Premium Collection Lag / 12) + prior year DWP × Monthly Premium Collection Lag / 12 *in other years*

(2) = Ceded WP × (1 − Monthly Ceding Premium Lag / 12) + 2001 Ceded Unearned Premium *in 2002*

= Ceded WP × (1 − Monthly Ceding Premium Lag / 12) + prior year Ceded WP × Premium Collection Lag / 12 *in other years*

(3) = (1) − (2)

(4) = Reinsurance commission % × (2)

(5) = (3) + (4)

(8) = (6) − (7)

(11) = (9) − (10)

(13) = (8) + (11) + (12)

(14) = Agents' commission from cash flow × (1 − Monthly Premium Collection Lag / 12) + 2001 Unpaid Agents Commission *in 2002*

= Agents' commission* × (1 − Monthly Premium Collection Lag / 12) + Prior year Agents commission* × Monthly Premium Collection Lag / 12 *in other yea*

(15) = Fixed underwriting$ + (% of DEP × DEP) + (% of DWP × DWP)

(16) = DWP × Premium tax %

(17) = (14) + (15) + (16)

Exhibit 19
Sheet 4

Financial Modeling Assumptions

Auto Liability

Modeling Assumptions for Current and Future Business

Amounts as of 12/31/01		Assumptions for Future Business	
Gross Unearned Premium	$15 000	Gross Written Premium in 2002	$31,200
Net Unearned Premium	$15,000	Annual Growth Rate of Business	4 00%
Ceded Unearned Premium	0		
Gross Premium Uncollected	$2 500	Percent of Business Earned in 1st Year	50 00%
Unpaid Agent's Commission	250	Premium Collection Lag (in months)	0 5
Ceded Premium Not Yet Remitted	0		
Ceded Paid Losses Not Yet Coll	0	Expected Loss Ratio	64 00%
Ceded Paid ALAE Not Yet Coll	0	Gross ALAE as a % of Loss	8 50%
Reinsurance Comm Not Yet Coll.	0	Gross ULAE as a % of Loss	7 50%
GAAP Deferred Commission	2,250		
GAAP Deferred U/W Expense	0	Agents' Commission as % of DWP	15 00%
GAAP Deferred Premium Tax	300	Premium Tax as % of DWP	2 00%
GAAP Deferred Reins Commission	0	Other Underwriting Expenses	
		Fixed	$0
		Variable (% of DEP)	2 25%
		Variable (% of DWP)	3 25%
		Reinsurance (per occ Excess)	
		Percent of premium ceded	0 00%
		Lag in Ceding Premium (in months)	0
		Ceded Loss Ratio	#DIV/0!
		Ceded Loss Collection Lag (in months)	1

Loss and LAE reserves are carried at nominal (undiscounted) value for both SAP and GAAP

'Reserve strengthening' adjustments are not needed ultimate loss and LAE amounts do not deteriorate or improve over time

Exhibit 19
Sheet 5

Financial Modeling Assumptions

Auto Liability

Loss Payment and Discounting

Payment Patterns for Loss and LAE

Accident Year •	Gross	Ceded	IRS
0	26 00%	22 00%	30 00%
1	26.00%	23 00%	29 00%
2	18 00%	16 00%	19 00%
3	13 00%	18 00%	10 00%
4	8 00%	10 00%	6 00%
5	4 00%	4.00%	3 00%
6	2 00%	3 00%	1 00%
7	1 00%	2 00%	1 00%
8	1.00%	1 00%	0 50%
9	1 00%	1 00%	0 50%
10	0 00%	0 00%	0 00%
11	0 00%	0 00%	0 00%
12	0 00%	0 00%	0 00%
13	0 00%	0 00%	0 00%
14	0 00%	0.00%	0 00%
15	0 00%	0.00%	0 00%
16	0 00%	0 00%	
17	0 00%	0 00%	
18	0 00%	0 00%	
19	0 00%	0 00%	
20	0 00%	0 00%	
21	0.00%	0 00%	
22	0 00%	0 00%	
23	0 00%	0 00%	
24	0 00%	0 00%	
25	0 00%	0 00%	
26	0.00%	0 00%	
27	0 00%	0.00%	
28	0 00%	0 00%	
29	0 00%	0 00%	
30	0 00%	0 00%	
Total	100 00%	100 00%	100 00%

Interest Rate for Discounted Tax Reserves

Accident Year	Rate
1982	7 20%
1983	7 20%
1984	7 20%
1985	7 20%
1986	7 20%
1987	7 20%
1988	7 77%
1989	8 16%
1990	8 37%
1991	7 00%
1992	7 00%
1993	7 00%
1994	7 00%
1995	7 00%
1996	7 00%
1997	7 00%
1998	7 00%
1999	7.00%
2000	7 00%
2001	7 00%
2002	7.00%
2003	7.00%
2004	7 00%
2005	7 00%
2006	7 00%
2007	7 00%
2008	7 00%
2009	7 00%
2010	7 00%
2011	7 00%

Financial Modeling Assumptions

Auto Liability

Prior Years' Information

	Estimates of Ultimate Loss and LAE @ 12/31/01							Paid Loss and LAE @ 12/31/01						
Accident Year	Gross Loss	Ceded Loss	Net Loss	Gross ALAE	Ceded ALAE	Net ALAE	ULAE	Gross Loss	Ceded Loss	Net Loss	Gross ALAE	Ceded ALAE	Net ALAE	ULAE
1982	4 557	0	4 557	387	0	387	342	4,557	0	4 557	387	0	387	342
1983	4 739	0	4 739	403	0	403	355	4,739	0	4 739	403	0	403	355
1984	4 928	0	4 928	419	0	419	370	4 928	0	4 928	419	0	419	370
1985	5 126	0	5,126	436	0	436	384	5 126	0	5 126	436	0	436	384
1986	5 331	0	5,331	453	0	453	400	5 331	0	5 331	453	0	453	400
1987	5,544	0	5,544	471	0	471	416	5 544	0	5,544	471	0	471	416
1988	5,766	0	5,766	490	0	490	432	5 766	0	5 766	490	0	490	432
1989	5,996	0	5 996	510	0	510	450	5,996	0	5 996	510	0	510	450
1990	6 236	0	6 236	530	0	530	468	6,236	0	6 236	530	0	530	468
1991	6 485	0	6 485	551	0	551	486	6,485	0	6 485	551	0	551	486
1992	6 745	0	6 745	573	0	573	506	6,745	0	6 745	573	0	573	506
1993	7 015	0	7 015	596	0	596	526	7 015	0	7,015	596	0	596	500
1994	7,295	0	7 295	620	0	620	547	7 222	0	7 222	614	0	614	492
1995	7,587	0	7,587	645	0	645	569	7 435	0	7,435	632	0	632	484
1996	7,891	0	7,891	671	0	671	592	7 654	0	7,654	651	0	651	473
1997	8,206	0	8,206	698	0	698	615	7 796	0	7 796	663	0	663	462
1998	8,534	0	8,534	725	0	725	640	7 766	0	7 766	660	0	660	416
1999	8 876	0	8 876	754	0	754	666	7,387	0	7 387	626	0	626	366
2000	9 231	0	9 231	785	0	785	692	6,462	0	6 462	549	0	549	277
2001	9 600	0	9 600	816	0	816	720	2 496	0	2 496	212	0	212	72

	Loss and LAE Reserves @ 12/31/01							
Accident Year	Gross Loss	Ceded Loss	Net Loss	Gross ALAE	Ceded ALAE	Net ALAE	ULAE	Net Earned Premium
1982	0	0	0	0	0	0	0	7 120
1983	0	0	0	0	0	0	0	7 404
1984	0	0	0	0	0	0	0	7 701
1985	0	0	0	0	0	0	0	8 009
1986	0	0	0	0	0	0	0	8 329
1987	0	0	0	0	0	0	0	8 662
1988	0	0	0	0	0	0	0	9,009
1989	0	0	0	0	0	0	0	9,369
1990	0	0	0	0	0	0	0	9,744
1991	0	0	0	0	0	0	0	10 133
1992	0	0	0	0	0	0	0	10,539
1993	0	0	0	0	0	0	26	10,960
1994	73	0	73	6	0	6	55	11,399
1995	152	0	152	13	0	13	85	11 855
1996	237	0	237	20	0	20	118	12 329
1997	410	0	410	35	0	35	154	12 822
1998	768	0	768	65	0	65	224	13 335
1999	1,509	0	1,509	128	0	128	300	13,868
2000	2,769	0	2,769	235	0	235	415	14 423
2001	7,104	0	7,104	604	0	604	648	15 000
Total	13 022	0	13 022	1 107	0	1 107	2,026	

Primary Stock Insurance Company

Exhibit 20
Sheet 1

General Liability

	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Statutory Underwriting Income for Line of Business										
(A) Net Earned Premium	$27,540	$28 642	$29 787	$30 979	$32 218	$33 507	$34 846	$36,240	$37 690	$39 199
(B) Net Incurred Loss and LAE	22 179	23 066	23,988	24 949	25 946	26 984	28 062	29 185	30 354	31,569
(C) Total Underwriting Expenses	6,060	6 302	6 554	6 817	7 090	7 373	7 668	7 974	8 293	8 625
(D) Underwriting Income	($699)	($726)	($755)	($787)	($818)	($850)	($884)	($919)	($957)	($995)
Modeled Amounts										
(1) Direct Written Premium	$31,200	$32 448	$33,746	$35 096	$36 500	$37 959	$39 477	$41 057	$42 699	$44 408
(2) Direct Earned Premium	30 600	31 824	33,097	34 421	35 798	37 230	38 718	40 267	41 878	43,554
(3) Ceded Written Premium	3 120	3 245	3 375	3 510	3 650	3 796	3,948	4,106	4 270	4,441
(4) Ceded Earned Premium	3 060	3 182	3 310	3 442	3 580	3 723	3 872	4 027	4,188	4 355
(5) Net Written Premium	28,080	29 203	30 371	31 586	32 850	34,163	35,529	36 951	38 429	39 967
(6) Net Earned Premium	27,540	28 642	29 787	30 979	32 218	33,507	34 846	36 240	37,690	39 199
(7) Direct Incurred Losses	20 808	21,640	22 506	23,406	24 343	25 316	26 328	27 382	28,477	29 617
(8) Ceded Incurred Losses	3,060	3 182	3 310	3 442	3 580	3 723	3 872	4 027	4 188	4 355
(9) Net Incurred Losses	17,748	18 458	19 196	19 964	20 763	21,593	22 456	23 355	24 289	25 262
(10) Direct Incurred ALAE	3 121	3 246	3 376	3 511	3 651	3 797	3 949	4 107	4 272	4 443
(11) Ceded ALAE	459	477	497	516	537	558	581	604	628	653
(12) Net Incurred ALAE	2 662	2 769	2 879	2 995	3 114	3 239	3 368	3 503	3 644	3 790
(13) Gross Incurred ULAE	1,769	1 839	1 913	1,990	2,069	2 152	2 238	2 327	2 421	2 517
(14) Net Incurred Loss & LAE	22 179	23 066	23 988	24 949	25 946	26 984	28,062	29 185	30 354	31,569
(15) Agents Commissions	3,900	4 056	4 218	4,387	4 563	4 745	4,935	5,132	5 337	5,551
(16) Other Underwriting Expenses	1 536	1 597	1 661	1 728	1,797	1 869	1 943	2 021	2,102	2 186
(17) Premium Taxes	624	649	675	702	730	759	790	821	854	888
(18) Reinsurance Commissions	0	0	0	0	0	0	0	0	0	0
(19) Total Underwriting Expenses	6 060	6 302	6 554	6 817	7 090	7,373	7,668	7 974	8,293	8,625
Modeled GAAP Amounts										
(20) Gross Reserves	$66,142	$71 933	$77,179	$82 003	$86 613	$91 095	$95 505	$99,902	$104 346	$108 862
(21) Ceded Reserves	11 858	13,186	14,423	15 589	16 704	17 789	18 848	19 893	20 932	21,972
(22) Agents Commissions	3 825	3 978	4 137	4 303	4 475	4 654	4 840	5 033	5,235	5 444
(23) Underwriting Expenses	1 536	1 597	1,661	1 728	1 797	1 869	1,943	2 021	2.102	2 186
(24) Premium Tax	582	636	662	688	716	745	774	805	838	871
(25) Reinsurance Commissions	0	0	0	0	0	0	0	0	0	0
(26) Total Underwriting Expenses	5,943	6 211	6 460	6 719	6 988	7 268	7 557	7 859	8,175	8 501

Calculation Notes Statutory Underwriting Income, Modeled Amounts, and Modeled GAAP Amounts

(A) = (6)   (B) = (14)   (C) = (19)   (D) = (A) − (B) − (C)

(1) = 2001 DWP × (annual growth rate)$^{(\text{year} - 2001)}$
(2) = (prior year UEPR) − (earned% × DWP)
(3) = excess ceded % × DWP
(4) = excess ceded % × DEP
(5) = (1) − (3)
(6) = (2) − (4)

(7) = GEP × expected loss ratio
(8) = (4) × ceded loss ratio
(9) = (7) − (8)
(10) = (7) × Gross ALAE to loss %
(11) = [(8) ÷ (7)] × (10)
(12) = (10) − (11)
(13) = (7) × Gross ULAE to loss %
(14) = (9) + (12) + (13)

(15) = DWP × Agents commission %
(16) = Fixed underwriting$ + (% of DEP × DEP) + (% of DWP × DWP)
(17) = DWP × Premium tax %
(18) = Reinsurance commission % × (3)
(19) = (15) + (16) + (17) + (18)

(22) = Business earned in 1st year % × DWP × Agents commission % + 2001 GAAP Deferred Commission *in 2002*
  = Agents commission % × DEP *in other years*
(23) = (16) × [1 − Deferrable %] + (16) × Business earned in 1st year % × Deferrable % + 2001 GAAP Deferred U/W Expense *in 2002*
  = (16) × [1−Deferrable %] + Deferrable % × [(16) × Business earned in 1st year % + (1−Business earned in 1st year %) × prior yr (16)] *in other years*
(24) = Business earned in 1st year % × DWP × Premium Tax % + 2001 GAAP Deferred Premium Tax *in 2002*
  = Premium Tax % × DEP *in other years*
(25) = (4) × Reinsurance Commission %
(26) = (22) + (23) + (24) + (25)

Primary Stock Insurance Company — Exhibit 20, Sheet 3

General Liability

	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Underwriting Cash Flow For Lines of Business										
(A) Total Collected	$28,510	$29 130	$30 296	$31 508	$32 768	$34 078	$35 441	$36 859	$38 333	$39 868
(B) Net Loss and LAE Payments	17,145	18 624	19 999	21 310	22 469	23,602	24 728	25,848	26 964	28 109
(C) Underwriting Expense Paid	5 985	6,289	6 541	6,803	7,075	7 358	7 652	7 958	8 276	8 607
(D) Cash Flow from Underwriting	$5 380	$4 217	$3 756	$3 395	$3 224	$3,118	$3 061	$3 053	$3 093	$3 152
Modeled Amounts										
(1) Gross Premium Collected	$31 100	$32 344	$33 638	$34 984	$36 383	$37 837	$39 351	$40 925	$42 562	$44 266
(2) Premium Ceded	2,590	3 214	3 342	3 476	3 615	3 759	3,910	4 066	4 229	4,398
(3) Net Premium Collected	28 510	29 130	30,296	31,508	32 768	34 078	35 441	36 859	38 333	39,868
(4) Reinsurance Commissions	0	0	0	0	0	0	0	0	0	0
(5) Total Collected	28 510	29 130	30,296	31,508	32 768	34,078	35 441	36 859	38 333	39 868
(6) Gross Losses Paid	14 362	16,102	17 611	18 997	20,210	21 355	22,479	23,583	24 668	25,769
(7) Loss Recoveries Received	1 854	2 203	2 383	2 554	2 714	2 870	3,025	3 180	3 334	3,491
(8) Net Loss Paid	12,508	13 899	15 228	16 443	17 498	18 485	19 454	20 403	21 334	22 278
(9) Gross ALAE Paid	2,154	2 415	2 642	2 850	3,031	3,203	3 372	3,537	3 700	3 865
(10) ALAE Recoveries Received	42	106	167	220	270	311	352	391	427	461
(11) Net ALAE Paid	2 112	2,309	2 475	2 630	2 761	2 892	3 020	3 146	3 273	3,404
(12) ULAE Paid	2 525	2,416	2 296	2,237	2 212	2,225	2 254	2 299	2,357	2,427
(13) Net Loss & LAE Payments	17 145	18 624	19,999	21,310	22 469	23 602	24 728	25 848	26 964	28 109
(14) Agents Commissions	3 825	4 043	4 205	4 373	4 548	4,730	4 919	5,116	5 320	5 533
(15) Other Underwriting Expenses	1 536	1,597	1 661	1 728	1 797	1 869	1,943	2 021	2 102	2,186
(16) Premium Taxes Paid	624	649	675	702	730	759	790	821	854	888
(17) Underwriting Expense Paid	5,985	6 289	6 541	6 803	7,075	7 358	7 652	7 958	8 276	8,607

Calculation Notes:

(A) = (5)   (B) = (13)   (C) = (17)   (D) = (A) − (B) − (C)

(1) = DWP × (1 − Monthly Premium Collection Lag / 12) + 2001 Direct Premium Uncollected *in 2002*

= DWP × (1 − Monthly Premium Collection Lag / 12) + prior year DWP × Monthly Premium Collection Lag / 12 *in other years*

(2) = Ceded WP × (1 − Monthly Ceding Premium Lag / 12) + 2001 Ceded Unearned Premium *in 2002*

= Ceded WP × (1 − Monthly Ceding Premium Lag / 12) + prior year Ceded WP × Premium Collection Lag / 12 *in other years*

(3) = (1) − (2)

(4) = Reinsurance commission % × (2)

(5) = (3) + (4)

(8) = (6) − (7)

(11) = (9) − (10)

(13) = (8) + (11) + (12)

(14) = Agents' commission from cash flow × (1 − Monthly Premium Collection Lag / 12) + 2001 Unpaid Agents Commission *in 2002*

= Agents' commission* × (1 − Monthly Premium Collection Lag / 12) + Prior year Agents' commission* × Monthly Premium Collection Lag / 12 *in other years*

(15) = Fixed underwriting$ + (% of DEP × DEP) + (% of DWP × DWP)

(16) = DWP × Premium tax %

(17) = (14) + (15) + (16)

Exhibit 20
Sheet 4

Financial Modeling Assumptions

General Liability

Modeling Assumptions for Current and Future Business

Amounts as of 12/31/01		Assumptions for Future Business	
Gross Unearned Premium	$15 000	Gross Written Premium in 2002	$31 200
Net Unearned Premium	$13 500	Annual Growth Rate of Business	4 00%
Ceded Unearned Premium	1,500		
Gross Premium Uncollected	$2,500	Percent of Business Earned in 1st Year	50 00%
Unpaid Agent's Commission	250	Premium Collection Lag (in months)	1
Ceded Premium Not Yet Remitted	250		
Ceded Paid Losses Not Yet Coll.	0	Expected Loss Ratio	68 00%
Ceded Paid ALAE Not Yet Coll	0	Gross ALAE as a % of Loss	15 00%
Reinsurance Comm Not Yet Coll	0	Gross ULAE as a % of Loss	8 50%
GAAP Deferred Commission	1 875		
GAAP Deferred U/W Expense	0	Agents Commission as % of DWP	12 50%
GAAP Deferred Premium Tax	270	Premium Tax as % of DWP	2 00%
GAAP Deferred Reins Commission	0	Other Underwriting Expenses	
		Fixed	$0
		Variable (% of DEP)	4 00%
		Variable (% of DWP)	1.00%
		Reinsurance (per occ Excess)	
		Percent of premium ceded	10.00%
		Lag in Ceding Premium (in months)	3
		Ceded Loss Ratio	100 00%
		Ceded Loss Collection Lag (in months)	1

Loss and LAE reserves are carried at nominal (undiscounted) value for both SAP and GAAP

"Reserve strengthening" adjustments are not needed, ultimate loss and LAE amounts do not deteriorate or improve over time

Financial Modeling Assumptions

General Liability

Loss Payment and Discounting

Payment Patterns for Loss and LAE

Accident Year +	Gross	Ceded	IRS
0	15 00%	10 00%	17 00%
1	19.00%	14 00%	21 00%
2	17 00%	12 00%	19 00%
3	12.00%	10 00%	11 00%
4	10 00%	9 00%	9 00%
5	6 00%	6 25%	5 70%
6	5 00%	6 25%	4 00%
7	4 00%	5 25%	3 50%
8	3 00%	4 50%	2 50%
9	2 00%	3 50%	2 00%
10	1 75%	3 50%	1 75%
11	1 50%	3 50%	1 50%
12	1 25%	3 50%	1 00%
13	1 00%	3 00%	0 50%
14	0 75%	2 50%	0 50%
15	0 50%	2 00%	0 00%
16	0 25%	1 25%	
17	0 00%	0 00%	
18	0 00%	0 00%	
19	0 00%	0 00%	
20	0 00%	0 00%	
21	0 00%	0 00%	
22	0 00%	0 00%	
23	0 00%	0.00%	
24	0 00%	0.00%	
25	0 00%	0 00%	
26	0 00%	0 00%	
27	0 00%	0 00%	
28	0 00%	0 00%	
29	0 00%	0 00%	
30	0 00%	0 00%	
Total	100 00%	100 00%	99 95%

Interest Rate for Discounted Tax Reserves

Accident Year	Rate
1982	7 20%
1983	7 20%
1984	7 20%
1985	7 20%
1986	7 20%
1987	7 20%
1988	7 77%
1989	8 16%
1990	8 37%
1991	7 00%
1992	7 00%
1993	7 00%
1994	7 00%
1995	7 00%
1996	7 00%
1997	7 00%
1998	7 00%
1999	7 00%
2000	7 00%
2001	7 00%
2002	7 00%
2003	7.00%
2004	7.00%
2005	7 00%
2006	7 00%
2007	7 00%
2008	7 00%
2009	7 00%
2010	7 00%
2011	7 00%

Financial Modeling Assumptions

General Liability

Prior Years' Information

	Estimates of Ultimate Loss and LAE @ 12/31/01							Paid Loss and LAE @ 12/31/01						
Accident Year	Gross Loss	Ceded Loss	Net Loss	Gross ALAE	Ceded ALAE	Net ALAE	ULAE	Gross Loss	Ceded Loss	Net Loss	Gross ALAE	Ceded ALAE	Net ALAE	ULAE
1982	8 714	1 282	7,433	1 307	0	1 307	741	8,714	1 282	7 433	1,307	0	1 307	733
1983	9,063	1 333	7,730	1,359	0	1 359	770	9 063	1 333	7 730	1,359	0	1 359	755
1984	9,426	1 386	8 039	1 414	0	1 414	801	9 426	1 386	8 039	1 414	0	1,414	777
1985	9,803	1 442	8 361	1 470	0	1 470	833	9 803	1 442	8 361	1 470	0	1 470	800
1986	10,195	1 499	8 695	1 529	0	1 529	867	10 195	1 499	8 695	1 529	0	1,529	823
1987	10 602	1,559	9 043	1 590	0	1,590	901	10,576	1 540	9 036	1 586	0	1,586	847
1988	11 027	1,622	9 405	1 654	0	1,654	937	10,944	1 569	9 375	1 642	0	1 642	862
1989	11 468	1,686	9,781	1,720	0	1 720	975	11,296	1,589	9 706	1 694	0	1 694	887
1990	11,926	1,754	10,172	1 789	0	1 789	1 014	11,626	1,600	10 028	1,744	0	1 744	912
1991	12 403	1 824	10 579	1,861	0	1 861	1 054	11,938	1 601	10 338	1 791	0	1 791	928
1992	12 899	1 897	11 003	1,935	0	1 935	1 098	12 222	1 598	10 624	1,833	0	1 833	943
1993	13 415	1 973	11 443	2 012	0	2 012	1 140	12 476	1 593	10 883	1,871	0	1,871	958
1994	13 952	2 052	11,900	2 093	0	2,093	1 186	12 696	1 585	11 111	1 904	0	1,904	949
1995	14 510	2 134	12 376	2 177	0	2,177	1 233	12 769	1,552	11 217	1 915	0	1,915	925
1996	15 091	2 219	12,871	2 264	0	2,264	1 283	12,676	1 498	11 178	1 901	0	1 901	898
1997	15 694	2 308	13 386	2 354	0	2 354	1,334	12 398	1,414	10,965	1 860	0	1 860	800
1998	16 322	2 400	13 922	2 448	0	2 448	1,387	11 915	1,320	10,595	1 787	0	1 787	694
1999	16 975	2 496	14 479	2 546	0	2 546	1,443	10 694	1 148	9 546	1 604	0	1,604	577
2000	17 654	2 596	15 058	2 648	0	2 648	1 501	9,003	935	8 069	1 351	0	1,351	450
2001	18 360	2,700	15,660	2 754	0	2 754	1,561	2,754	270	2 484	413	0	413	312

	Loss and LAE Reserves @ 12/31/01							
Accident Year	Gross Loss	Ceded Loss	Net Loss	Gross ALAE	Ceded ALAE	Net ALAE	ULAE	Net Earned Premium
1982	0	0	0	0	0	0	7	12 815
1983	0	0	0	0	0	0	15	13,328
1984	0	0	0	0	0	0	24	13,861
1985	0	0	0	0	0	0	33	14,416
1986	0	0	0	0	0	0	43	14 992
1987	27	19	7	4	0	4	54	15 592
1988	83	53	30	12	0	12	75	16 216
1989	172	97	75	26	0	26	88	16 864
1990	298	153	145	45	0	45	101	17 539
1991	465	223	242	70	0	70	127	18 240
1992	677	299	378	102	0	102	154	18 970
1993	939	380	559	141	0	141	182	19 729
1994	1 256	467	789	188	0	188	237	20,518
1995	1 741	581	1,160	261	0	261	308	21,338
1996	2,414	721	1 693	362	0	362	385	22,192
1997	3 296	894	2 401	494	0	494	534	23 080
1998	4 407	1 080	3 327	661	0	661	694	24 003
1999	6 281	1 348	4 933	942	0	942	866	24 963
2000	8 650	1 662	6 989	1 298	0	1,298	1 050	25 962
2001	15 606	2 430	13 176	2 341	0	2,341	1 248	27 000
Total	46 312	10 408	35 904	6,947	0	6 947	6 226	

Primary Stock Insurance Company

Exhibit 21
Sheet 1

Valuation Estimates as of December 31, 2001 Surplus to RBC Ratio of 2.5

Establishing Starting Surplus	
(1) Booked Statutory Surplus @ 12/31/01	45,000
(2) Selected Starting Surplus for Future Income Projections (based on 12/31/01 RBC level)	52,671
(3) Income Recognized @ 12/31/01	(7,671)

Monitoring and Selecting Surplus	As of 12/31 of										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(4) Selected Surplus	$ 52,671	$ 58,560	$ 64,630	$ 69,708	$ 74,145	$ 78,251	$ 82,200	$ 86,100	$ 90,019	$ 93,983	$ 98,025
(5) Indicated Risk Based Capital Company Action Level	21,068	23,424	25,852	27,883	29,658	31,300	32,880	34,440	36,007	37,593	39,210
(6) Surplus to RBC Ratio	250.0%	250.0%	250.0%	250.0%	250.0%	250.0%	250.0%	250.0%	250.0%	250.0%	250.0%
(7) Surplus Growth		5,888	11,959	17,038	21,474	25,580	29,529	33,429	37,347	41,311	45,354
(8) Net Written Premium to Surplus Ratio	1.59	1.49	1.41	1.36	1.33	1.31	1.29	1.28	1.28	1.27	1.27
(9) Loss and LAE Reserves to Surplus Ratio	2.03	2.15	2.18	2.18	2.18	2.18	2.18	2.17	2.17	2.17	2.16

Estimated Future Income	During									
	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(10) Total Company Net Income	11,074	12,761	13,958	14,978	15,852	16,672	17,467	18,236	19,009	19,791

Hurdle Rate	@ 12/31										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(11) Selected Hurdle Rate		15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%
(12) Present Value Factor for 12/31/01 (assuming year end income recognition)	1.000	0.870	0.756	0.658	0.572	0.497	0.432	0.376	0.327	0.284	0.247

Expected Annual Perpetual Growth Rate of Capital & Income After 2011	2.0%

Application of DCF Method	@ 12/31	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(13) Total Company Net Income	(7,671)	11,074	12,761	13,958	14,978	15,852	16,672	17,467	18,236	19,009	19,791			
(13pv) PV of Total Company Net Income	(7,671)	9,630	9,649	9,178	8,564	7,881	7,208	6,566	5,961	5,404	4,892	67,261	38,384	105,645
(14) Reinvestment Cost to Grow Company		5,889	6,070	5,078	4,438	4,106	3,949	3,900	3,919	3,964	4,043			
(14pv) PV of Reinv. Cost to Grow Company		5,121	4,590	3,339	2,537	2,041	1,707	1,466	1,281	1,127	999	24,208	3,728	27,936
(15) Free Cash Flow	(7,671)	5,185	6,691	8,881	10,541	11,746	12,723	13,567	14,318	15,045	15,749			
(15pv) PV of Free Cash Flow	(7,671)	4,509	5,058	5,839	6,027	5,840	5,501	5,100	4,680	4,277	3,893	43,053	34,656	77,709

Application of EVA Method	@ 12/31	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(16) Statutory Surplus @ 12/31/01	52,671													
(17) Total Company Net Income	(7,671)	11,074	12,761	13,958	14,978	15,852	16,672	17,467	18,236	19,009	19,791			
(17pv) PV of Total Company Net Income	(7,671)	9,630	9,649	9,178	8,564	7,881	7,208	6,566	5,961	5,404	4,892	67,261	38,384	105,645
(18) Cost of Capital		7,901	8,784	9,695	10,456	11,122	11,738	12,330	12,915	13,503	14,097			
(18pv) PV of Cost of Capital		6,870	6,642	6,374	5,978	5,529	5,075	4,635	4,222	3,838	3,485	52,649	27,958	80,607
(19) Cost of Starting Capital		7,901	7,901	7,901	7,901	7,901	7,901	7,901	7,901	7,901	7,901			
(19pv) PV of Cost of Starting Capital		6,870	5,974	5,195	4,517	3,928	3,416	2,970	2,583	2,246	1,953	39,652	13,020	52,671
(20) Cost of Growth Capital			883	1,794	2,555	3,221	3,837	4,429	5,014	5,602	6,197			
(20pv) PV of Cost of Growth Capital			668	1,179	1,461	1,601	1,659	1,665	1,639	1,592	1,532	12,997	14,938	27,936
(21) Excess Returns in Year	(7,671)	3,173	3,977	4,264	4,522	4,730	4,934	5,137	5,321	5,506	5,694			
(21pv) PV of Excess Returns in Year	(7,671)	2,759	3,007	2,803	2,585	2,352	2,133	1,931	1,739	1,565	1,407	14,612	10,426	25,038
(22) Total Indicated Value														77,709

Primary Stock Insurance Company

Calculation Notes for Valuation Estimates as of December 31, 2001

(1) from Primary Stock Insurance Company's 12/31/01 booked balance sheet

(2) selected starting surplus for future income projections based on the selected 250.0% surplus to indicated RBC ratio

(3) = (1) − (2)

(4) selected surplus based on the selected 250.0% surplus to indicated RBC ratio

(5) calculated RBC at the Company Action Level - based on Exhibit 12 for 2002 and subsequent

(6) selection of 250.0% surplus to RBC ratio for determining required surplus at each year end

(7) cumulative increase in (4) from staring surplus (4) − (2)

(8) NPW for all lines ÷ (4)

(9) net loss and LAE reserves for all lines ÷ (4)

(10) from Exhibit 8, line (11)

(11) is selected hurdle rate of 15.0% used for determining Cost of Capital in EVA method and the present value of future earnings and value added

(12) = 1.000 at 12/31/01 for future years = (1.0 + 15.0%) raised to (2001 − year) exponent

(13) = (10)

(13pv) = (13) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(13)_{2011}$ × (1 + Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(14) annual change in (4)

(14pv) = (14) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(4)_{2011}$ × Growth Rate ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$.

All Years = Total '01 to '11 + Total '12 to ∞

(15) = (13) − (14)

(15pv) = (13pv) − (14pv)

(16) = (2)

(17) = (10)

(17pv) = (17) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(17)_{2011}$ × (1 + Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(18) = (19) + (20)

(18pv) = (19pv) + (20pv)

(19) = (2) × (11)

(19pv) = (19) × (12) for each year. Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = (16) × $(12)_{2011}$.

All Years = Total '01 to '11 + Total '12 to ∞

(20) = $(7)_{\text{prior year}}$ × (11)

(20pv) = (20) × (12) for each year; Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = [$(4)_{2011}$ × Hurdle Rate ÷ (Hurdle Rate − Growth Rate) − (16)] × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(21) = (17) − (18)

(21pv) = (17pv) − (18pv)

(22) = (16) + $(21pv)_{\text{All years}}$

Primary Stock Insurance Company

Exhibit 21
Sheet 3

Valuation Estimates as of December 31, 2001 Using Base Loss Ratios +2%

Establishing Starting Surplus	
(1) Booked Statutory Surplus @ 12/31/01	45 000
(2) Selected Starting Surplus for Future Income Projections (based on 12/31/01 RBC level)	42 131
(3) Income Recognized @ 12.31/01	2 869

Monitoring and Selecting Surplus	As of 12/31 of										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(4) Selected Surplus	$ 42,131	$ 47 388	$ 52 612	$ 56 976	$ 60 758	$ 64 242	$ 67 578	$ 70 860	$ 74 148	$ 77 470	$ 80 871
(5) Indicated Risk Based Capital Company Action Level	21 069	23 694	26 306	28 488	30 379	32 121	33 789	35 430	37 074	38 735	40 435
(6) Surplus to RBC Ratio	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%
(7) Surplus Growth		5 257	10 481	14 845	18 627	22 111	25 447	28 729	32 017	35 338	38 740
(8) Net Written Premium to Surplus Ratio	1 69	1 84	1 73	1 66	1 62	1 59	1 57	1 56	1 55	1 54	1 54
(9) Loss and LAE Reserves to Surplus Ratio	2 64	2 70	2 73	2 74	2 74	2 74	2 73	2 73	2 72	2 72	2 71

Estimated Future Income	During									
	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(10) Total Company Net Income	9 008	10 692	11 864	12 842	13 649	14 404	15 131	15 843	16 549	17 259

Hurdle Rate	@ 12/31/										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(11) Selected Hurdle Rate		15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%
(12) Present Value Factor for 12/31/01 (assuming year end income recognition)	1 000	0 870	0 756	0 658	0 572	0 497	0 432	0 376	0 327	0 284	0 247

Expected Annual Perpetual Growth Rate of Capital & Income After 2011	2 0%

Application of DCF Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	12 to ∞	Years
(13) Total Company Net Income	2 869	9 008	10 692	11 864	12 842	13 649	14 404	15 131	15 843	16 549	17 259			
(13pv) PV of Total Company Net Income	2 869	7 833	8 085	7 801	7 342	6 786	6 227	5 688	5 179	4 704	4 266	66 781	33 473	100 254
(14) Reinvestment Cost to Grow Company		5 257	5 224	4 364	3 782	3 484	3 336	3 282	3 288	3 322	3 401			
(14pv) PV of Reinv. Cost to Grow Company		4 571	3 950	2 869	2 162	1 732	1 442	1 234	1 075	944	841	20 821	3 076	23 897
(15) Free Cash Flow	2 869	3 751	5 468	7 500	9 060	10 165	11 068	11 849	12 555	13 227	13 858			
(15pv) PV of Free Cash Flow	2 869	3 262	4 135	4 931	5 180	5 054	4 785	4 454	4 104	3 760	3 425	45 960	30 398	76 357

Application of EVA Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	12 to ∞	Years
(16) Statutory Surplus @ 12 31/01	42 131													
(17) Total Company Net Income	2 869	9 008	10 692	11 864	12 842	13 649	14 404	15 131	15 843	16 549	17 259			
(17pv) PV of Total Company Net Income	2 869	7 833	8 085	7 801	7,342	6 786	6 227	5 688	5 179	4 704	4 266	66 781	33 473	100 254
(18) Cost of Capital		6 320	7 108	7 892	8 546	9 114	9 636	10 137	10 629	11 122	11 621			
(18pv) PV of Cost of Capital		5 495	5 375	5 189	4 886	4 531	4 166	3 811	3 475	3 162	2 872	42 962	23 065	66 028
(19) Cost of Starting Capital		6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320			
(19pv) PV of Cost of Starting Capital		5 495	4 776	4 155	3 613	3 142	2 732	2 376	2 066	1 796	1 562	31 717	10 414	42 131
(20) Cost of Growth Capital			789	1 572	2 227	2 794	3 317	3 817	4 309	4 803	5 301			
(20pv) PV of Cost of Growth Capital			596	1 034	1 273	1 389	1 434	1 435	1 409	1 365	1 310	11 245	12 651	23 897
(21) Excess Returns in Year	2 869	2 688	3 584	3 972	4 296	4 535	4 768	4 994	5 214	5 427	5 639			
(21pv) PV of Excess Returns in Year	2 869	2 338	2 710	2 612	2 456	2 255	2 061	1 878	1 704	1 543	1 394	23 819	10 407	34 226
(22) Total Indicated Value														76 357

Primary Stock Insurance Company

Calculation Notes for Valuation Estimates as of December 31, 2001

(1) from Primary Stock Insurance Company's 12/31/01 booked balance sheet
(2) selected starting surplus for future income projections based on the selected 200.0% surplus to indicated RBC ratio
(3) = (1) − (2)

(4) selected surplus based on the selected 200.0% surplus to indicated RBC ratio
(5) calculated RBC at the Company Action Level - based on Exhibit 12 for 2002 and subsequent
(6) selection of 200.0% surplus to RBC ratio for determining required surplus at each year end
(7) cumulative increase in (4) from starting surplus (4) − (2)
(8) NPW for all lines ÷ (4)
(9) net loss and LAE reserves for all lines ÷ (4)

(10) from Exhibit 8 line (11)

(11) is selected hurdle rate of 15.0% used for determining Cost of Capital in EVA method and the present value of future earnings and value added
(12) = 1.000 at 12/31/01; for future years = (1.0 + 15.0%) raised to (2001 − year) exponent

(13) = (10)
(13pv) = (13) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(13)_{2011}$ × (1+ Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(14) annual change in (4)
(14pv) = (14) × (12) for each year; Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = $(4)_{2011}$ × Growth Rate ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(15) = (13) − (14)
(15pv) = (13pv) − (14pv)

(16) = (2)

(17) = (10)
(17pv) = (17) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(17)_{2011}$ × (1+ Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(18) = (19) + (20)
(18pv) = (19pv) + (20pv)
(19) = (2) × (11)
(19pv) = (19) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = (16) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞
(20) = $(7)_{\text{prior year}}$ × (11)
(20pv) = (20) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = [$(4)_{2011}$ × Hurdle Rate ÷ (Hurdle Rate − Growth Rate) − (16)] × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(21) = (17) − (18)
(21pv) = (17pv) − (18pv)

(22) = (16) + $(21pv)_{\text{All Years}}$

Primary Stock Insurance Company Exhibit 21
Sheet 5

Valuation Estimates as of December 31, 2001 Using Base Loss Ratios -2%

Establishing Starting Surplus	
(1) Booked Statutory Surplus @ 12/31/01	45 000
(2) Selected Starting Surplus for Future Income Projections (based on 12/31/01 RBC level)	42,131
(3) Income Recognized @ 12/31/01	2 869

Monitoring and Selecting Surplus	As of 12/31 of										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(4) Selected Surplus	$ 42 131	$ 45 690	$ 50 093	$ 53 810	$ 57 078	$ 60 104	$ 63 044	$ 65 958	$ 68 894	$ 71 880	$ 74 946
(5) Indicated Risk Based Capital Company Action Level	21 069	22 845	25 046	26 905	28 539	30 052	31 522	32 979	34 447	35 940	37 473
(6) Surplus to RBC Ratio	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%	200 0%
(7) Surplus Growth		3 559	7 962	11 679	14 947	17 973	20 913	23 827	26 763	29 749	32 815
(8) Net Written Premium to Surplus Ratio	1 99	1 91	1 81	1 76	1 72	1 70	1 69	1 68	1 67	1 66	1 66
(9) Loss and LAE Reserves to Surplus Ratio	2 64	2 72	2 75	2 76	2 76	2 75	2 75	2 74	2 74	2 73	2 73

Estimated Future Income	During									
	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(10) Total Company Net Income	11 879	13 415	14 515	15 470	16 288	17 067	17 829	18 590	19 354	20 148

Hurdle Rate	@ 12/31/										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(11) Selected Hurdle Rate		15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%	15 0%
(12) Present Value Factor for 12/31/01 (assuming year end income recognition)	1 000	0 870	0 756	0 658	0 572	0 497	0 432	0 376	0 327	0 284	0 247

Expected Annual Perpetual Growth Rate of Capital & Income After 2011	2 0%

Application of DCF Method	@ 12/31/	During:										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(13) Total Company Net Income	2 869	11 879	13 415	14 515	15 470	16 288	17 067	17 829	18 590	19 354	20 148			
(13pv) PV of Total Company Net Income	2 869	10 330	10 144	9 544	8 845	8 098	7 379	6 703	6 077	5 502	4 980	80 469	39 076	119 545
(14) Reinvestment Cost to Grow Company		3 559	4 403	3 717	3 268	3 026	2 940	2 914	2 936	2 986	3 066			
(14pv) PV of Reinv. Cost to Grow Company		3 095	3 329	2 444	1 868	1 504	1,271	1 095	960	849	758	17,174	2 850	20 024
(15) Free Cash Flow	2 869	8 320	9 012	10 798	12,202	13 262	14 127	14 915	15 654	16 368	17 082			
(15pv) PV of Free Cash Flow	2 869	7 235	6 814	7,100	6 977	6 594	6 107	5 607	5,117	4 653	4 222	63 295	36 226	99 521

Application of EVA Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(16) Statutory Surplus @ 12/31/01	42 131													
(17) Total Company Net Income	2 869	11 879	13 415	14 515	15 470	16 288	17 067	17 829	18 590	19 354	20 148			
(17pv) PV of Total Company Net Income	2 869	10 330	10 144	9 544	8 845	8 098	7 379	6 703	6 077	5 502	4 980	80 469	39 076	119 545
(18) Cost of Capital		6 320	6 854	7 514	8 072	8 562	9 016	9 457	9 894	10 334	10 782			
(18pv) PV of Cost of Capital		5 495	5 182	4 941	4 615	4 257	3 898	3 555	3 234	2 938	2 665	40 780	21 376	62 155
(19) Cost of Starting Capital		6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320			
(19pv) PV of Cost of Starting Capital		5 495	4 779	4 155	3 613	3 142	2 732	2 376	2 066	1 796	1 562	31 717	10 414	42 131
(20) Cost of Growth Capital			534	1 194	1 752	2 242	2 696	3 137	3 574	4 014	4 462			
(20pv) PV of Cost of Growth Capital			404	785	1 002	1 115	1 166	1 179	1 168	1 141	1,103	9 063	10 961	20 024
(21) Excess Returns in Year	2 869	5 559	6 562	7 001	7 399	7 726	8 051	8 372	8 696	9 020	9 366			
(21pv) PV of Excess Returns in Year	2 869	4 834	4 961	4 603	4 230	3 841	3 481	3 147	2 843	2 564	2 315	39 690	17 700	57 390
(22) Total Indicated Value														99 521

Primary Stock Insurance Company

Calculation Notes for Valuation Estimates as of December 31, 2001

(1) from Primary Stock Insurance Company's 12/31/01 booked balance sheet

(2) selected starting surplus for future income projections based on the selected 200.0% surplus to indicated RBC ratio

(3) = (1) − (2)

(4) selected surplus based on the selected 200.0% surplus to indicated RBC ratio

(5) calculated RBC at the Company Action Level - based on Exhibit 12 for 2002 and subsequent

(6) selection of 200.0% surplus to RBC ratio for determining required surplus at each year end

(7) cumulative increase in (4) from starting surplus, (4) − (2)

(8) NPW for all lines ÷ (4)

(9) net loss and LAE reserves for all lines ÷ (4)

(10) from Exhibit 8, line (11)

(11) is selected hurdle rate of 15.0% used for determining Cost of Capital in EVA method and the present value of future earnings and value added

(12) = 1.000 at 12/31/01; for future years = (1.0 + 15.0%) raised to (2001 − year) exponent

(13) = (10)

(13pv) = (13) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(13)_{2011}$ × (1+ Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(14) annual change in (4)

(14pv) = (14) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(4)_{2011}$ × Growth Rate ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(15) = (13) − (14)

(15pv) = (13pv) − (14pv)

(16) = (2)

(17) = (10)

(17pv) = (17) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(17)_{2011}$ × (1+ Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(18) = (19) + (20)

(18pv) = (19pv) + (20pv)

(19) = (2) × (11)

(19pv) = (19) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = (16) × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(20) = $(7)_{\text{prior year}}$ × (11)

(20pv) = (20) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = [$(4)_{2011}$ × Hurdle Rate ÷ (Hurdle Rate − Growth Rate) − (16)] × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(21) = (17) − (18)

(21pv) = (17pv) − (18pv)

(22) = (16) + $(21pv)_{\text{All Years}}$

Exhibit 21
Sheet 7

Primary Stock Insurance Company

Valuation Estimates as of December 31, 2001 Using Base Investment Yields +100bp

Establishing Starting Surplus	
(1) Booked Statutory Surplus @ 12/31/01	45,000
(2) Selected Starting Surplus for Future Income Projections (based on 12/31/01 RBC level)	42,131
(3) Income Recognized @ 12/31/01	2,869

Monitoring and Selecting Surplus	As of 12/31 of										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(4) Selected Surplus	$ 42,131	$ 48,510	$ 51,332	$ 55,366	$ 58,892	$ 62,154	$ 65,295	$ 68,394	$ 71,506	$ 74,658	$ 77,870
(5) Indicated Risk Based Capital Company Action Level	21,066	23,255	25,666	27,683	29,446	31,077	32,647	34,197	35,753	37,329	38,935
(6) Surplus to RBC Ratio	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%
(7) Surplus Growth		4,379	9,201	13,235	16,761	20,023	23,164	26,263	29,375	32,527	35,739
(8) Net Written Premium to Surplus Ratio	1.89	1.88	1.77	1.71	1.67	1.64	1.63	1.62	1.61	1.60	1.60
(9) Loss and LAE Reserves to Surplus Ratio	2.54	2.71	2.74	2.75	2.75	2.75	2.74	2.74	2.73	2.73	2.72

Estimated Future Income	During									
	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(10) Total Company Net Income	12,440	14,289	15,615	16,740	17,707	18,601	19,468	20,324	21,184	22,056

Hurdle Rate	@ 12/31/										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(11) Selected Hurdle Rate		15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%
(12) Present Value Factor for 12/31/01 (assuming year end income recognition)	1.000	0.870	0.756	0.658	0.572	0.497	0.432	0.376	0.327	0.284	0.247

Expected Annual Perpetual Growth Rate of Capital & Income After 2011	2.0%

Application of DCF Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(13) Total Company Net Income	2,869	12,440	14,289	15,615	16,740	17,707	18,601	19,468	20,324	21,184	22,056			
(13pv) PV of Total Company Net Income	2,869	10,817	10,805	10,267	9,571	8,804	8,042	7,319	6,644	6,022	5,452	88,611	42,776	129,387
(14) Reinvestment Cost to Grow Company		4,379	4,822	4,034	3,526	3,262	3,141	3,099	3,112	3,152	3,212			
(14pv) PV of Reinv. Cost to Grow Company		3,808	3,646	2,652	2,016	1,622	1,358	1,165	1,017	896	794	18,974	2,961	21,936
(15) Free Cash Flow	2,869	8,061	9,467	11,581	13,214	14,445	15,460	16,369	17,212	18,032	18,844			
(15pv) PV of Free Cash Flow	2,869	7,010	7,158	7,615	7,555	7,182	6,684	6,154	5,627	5,125	4,658	67,638	39,815	107,452

Application of EVA Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(16) Statutory Surplus @ 12/31/01	42,131													
(17) Total Company Net Income	2,869	12,440	14,289	15,615	16,740	17,707	18,601	19,468	20,324	21,184	22,056			
(17pv) PV of Total Company Net Income	2,869	10,817	10,805	10,267	9,571	8,804	8,042	7,319	6,644	6,022	5,452	88,611	42,776	129,387
(18) Cost of Capital		6,320	6,977	7,700	8,305	8,834	9,323	9,794	10,259	10,726	11,196			
(18pv) PV of Cost of Capital		5,495	5,275	5,063	4,748	4,392	4,031	3,682	3,354	3,049	2,768	41,857	22,210	64,067
(19) Cost of Starting Capital		6,320	6,320	6,320	6,320	6,320	6,320	6,320	6,320	6,320	6,320			
(19pv) PV of Cost of Starting Capital		5,495	4,779	4,155	3,613	3,142	2,732	2,376	2,066	1,796	1,562	31,717	10,414	42,131
(20) Cost of Growth Capital			657	1,380	1,985	2,514	3,003	3,475	3,939	4,406	4,879			
(20pv) PV of Cost of Growth Capital			497	907	1,135	1,250	1,298	1,306	1,288	1,253	1,206	10,140	11,795	21,936
(21) Excess Returns in Year	2,869	6,120	7,313	7,915	8,435	8,873	9,278	9,674	10,065	10,458	10,857			
(21pv) PV of Excess Returns in Year	2,869	5,322	5,529	5,204	4,823	4,412	4,011	3,637	3,290	2,973	2,684	44,754	20,567	65,321
(22) Total Indicated Value														107,452

Primary Stock Insurance Company

Calculation Notes for Valuation Estimates as of December 31, 2001

(1) from Primary Stock Insurance Company's 12/31/01 booked balance sheet
(2) selected starting surplus for future income projections based on the selected 200.0% surplus to indicated RBC ratio
(3) = (1) − (2)

(4) selected surplus based on the selected 200.0% surplus to indicated RBC ratio
(5) calculated RBC at the Company Action Level - based on Exhibit 12 for 2002 and subsequent
(6) selection of 200.0% surplus to RBC ratio for determining required surplus at each year end
(7) cumulative increase in (4) from starting surplus. (4) − (2)
(8) NPW for all lines ÷ (4)
(9) net loss and LAE reserves for all lines ÷ (4)

(10) from Exhibit 8 line (11)

(11) is selected hurdle rate of 15.0% used for determining Cost of Capital in EVA method and the present value of future earnings and value added
(12) = 1.000 at 12/31/01. for future years = (1.0 + 15.0%) raised to (2001 − year) exponent

(13) = (10)
(13pv) = (13) × (12) for each year. Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = $(13)_{2011}$ × (1 + Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(14) annual change in (4)
(14pv) = (14) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(4)_{2011}$ × Growth Rate ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(15) = (13) − (14)
(15pv) = (13pv) − (14pv)

(16) = (2)

(17) = (10)
(17pv) = (17) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(17)_{2011}$ × (1 + Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(18) = (19) + (20)
(18pv) = (19pv) + (20pv)
(19) = (2) × (11)
(19pv) = (19) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = (16) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞
(20) = $(7)_{\text{prior year}}$ × (11)
(20pv) = (20) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = [$(4)_{2011}$ × Hurdle Rate ÷ (Hurdle Rate − Growth Rate) − (16)] × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(21) = (17) − (18)
(21pv) = (17pv) − (18pv)

(22) = (16) + $(21pv)_{\text{All Years}}$

Primary Stock Insurance Company

Exhibit 21
Sheet 9

Valuation Estimates as of December 31, 2001 Using Base Investment Yields -100bp

Establishing Starting Surplus	
(1) Booked Statutory Surplus @ 12/31/01	45,000
(2) Selected Starting Surplus for Future Income Projections (based on 12/31/01 RBC level)	42,131
(3) Income Recognized @ 12/31/01	2,869

Monitoring and Selecting Surplus	As of 12/31 of:										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(4) Selected Surplus	$ 42,131	$ 46,504	$ 51,326	$ 55,358	$ 58,886	$ 62,146	$ 65,284	$ 68,384	$ 71,496	$ 74,646	$ 77,858
(5) Indicated Risk Based Capital Company Action Level	21,069	23,252	25,663	27,679	29,443	31,073	32,642	34,192	35,748	37,323	38,929
(6) Surplus to RBC Ratio	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%
(7) Surplus Growth		4,373	9,195	13,227	16,755	20,015	23,153	26,253	29,365	32,515	35,727
(8) Net Written Premium to Surplus Ratio	1.99	1.88	1.77	1.71	1.67	1.64	1.63	1.62	1.61	1.60	1.60
(9) Loss and LAE Reserves to Surplus Ratio	2.64	2.71	2.74	2.75	2.75	2.75	2.74	2.74	2.73	2.73	2.72

Estimated Future Income	During									
	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(10) Total Company Net Income	8,440	9,807	10,749	11,553	12,232	12,872	13,492	14,101	14,712	15,331

Hurdle Rate	@ 12/31/										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(11) Selected Hurdle Rate		15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%
(12) Present Value Factor for 12/31/01 (assuming year end income recognition)	1.000	0.870	0.756	0.658	0.572	0.497	0.432	0.376	0.327	0.284	0.247

Expected Annual Perpetual Growth Rate of Capital & Income After 2011	2.0%

Application of DCF Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(13) Total Company Net Income	2,869	8,440	9,807	10,749	11,553	12,232	12,872	13,492	14,101	14,712	15,331			
(13pv) PV of Total Company Net Income	2,869	7,339	7,416	7,068	6,605	6,081	5,565	5,072	4,610	4,182	3,790	60,597	29,734	90,330
(14) Reinvestment Cost to Grow Company		4,373	4,822	4,032	3,528	3,260	3,138	3,100	3,112	3,150	3,212			
(14pv) PV of Reinv. Cost to Grow Company		3,803	3,646	2,651	2,017	1,621	1,357	1,165	1,017	895	794	18,966	2,961	21,927
(15) Free Cash Flow	2,869	4,067	4,985	6,717	8,025	8,972	9,734	10,392	10,989	11,562	12,119			
(15pv) PV of Free Cash Flow	2,869	3,537	3,769	4,417	4,588	4,461	4,208	3,907	3,592	3,287	2,996	41,630	26,773	68,403

Application of EVA Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(16) Statutory Surplus @ 12/31/01	42,131													
(17) Total Company Net Income	2,869	8,440	9,807	10,749	11,553	12,232	12,872	13,492	14,101	14,712	15,331			
(17pv) PV of Total Company Net Income	2,869	7,339	7,416	7,068	6,605	6,081	5,565	5,072	4,610	4,182	3,790	60,597	29,734	90,330
(18) Cost of Capital		6,320	6,976	7,699	8,304	8,833	9,322	9,793	10,258	10,724	11,197			
(18pv) PV of Cost of Capital		5,495	5,275	5,062	4,748	4,392	4,030	3,681	3,353	3,049	2,768	41,852	22,206	64,058
(19) Cost of Starting Capital		6,320	6,320	6,320	6,320	6,320	6,320	6,320	6,320	6,320	6,320			
(19pv) PV of Cost of Starting Capital		5,495	4,779	4,155	3,613	3,142	2,732	2,376	2,066	1,796	1,562	31,717	10,414	42,131
(20) Cost of Growth Capital			656	1,379	1,984	2,513	3,002	3,473	3,938	4,405	4,877			
(20pv) PV of Cost of Growth Capital			496	907	1,134	1,250	1,298	1,306	1,287	1,252	1,206	10,135	11,792	21,927
(21) Excess Returns in Year	2,869	2,120	2,831	3,050	3,249	3,399	3,550	3,699	3,843	3,988	4,134			
(21pv) PV of Excess Returns in Year	2,869	1,844	2,141	2,005	1,858	1,690	1,535	1,391	1,256	1,134	1,022	18,744	7,528	26,272
(22) Total Indicated Value														68,403

Primary Stock Insurance Company

Calculation Notes for Valuation Estimates as of December 31, 2001

(1) from Primary Stock Insurance Company's 12/31/01 booked balance sheet
(2) selected starting surplus for future income projections based on the selected 200.0% surplus to indicated RBC ratio
(3) = (1) − (2)

(4) selected surplus based on the selected 200.0% surplus to indicated RBC ratio
(5) calculated RBC at the Company Action Level - based on Exhibit 12 for 2002 and subsequent
(6) selection of 200.0% surplus to RBC ratio for determining required surplus at each year end
(7) cumulative increase in (4) from starting surplus, (4) − (2)
(8) NPW for all lines ÷ (4)
(9) net loss and LAE reserves for all lines ÷ (4)

(10) from Exhibit 8, line (11)

(11) is selected hurdle rate of 15.0% used for determining Cost of Capital in EVA method and the present value of future earnings and value added
(12) = 1.000 at 12/31/01, for future years = (1.0 + 15.0%) raised to (2001 − year) exponent

(13) = (10)
(13pv) = (13) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(13)_{2011}$ × (1 + Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(14) annual change in (4)
(14pv) = (14) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(4)_{2011}$ × Growth Rate ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(15) = (13) − (14)
(15pv) = (13pv) − (14pv)

(16) = (2)

(17) = (10)
(17pv) = (17) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(17)_{2011}$ × (1 + Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(18) = (19) + (20)
(18pv) = (19pv) + (20pv)
(19) = (2) × (11)
(19pv) = (19) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = (16) × $(12)_{2011}$;
All Years = Total '01 to '11 + Total '12 to ∞
(20) = $(7)_{\text{prior year}}$ × (11)
(20pv) = (20) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = [$(4)_{2011}$ × Hurdle Rate ÷ (Hurdle Rate − Growth Rate) − (16)] × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(21) = (17) − (18)
(21pv) = (17pv) − (18pv)

(22) = (16) + $(21pv)_{\text{all years}}$

Primary Stock Insurance Company

Exhibit 21
Sheet 11

Valuation Estimates as of December 31, 2001 Using Base Hurdle Rate +3%

Establishing Starting Surplus	
(1) Booked Statutory Surplus @ 12/31/01	45,000
(2) Selected Starting Surplus for Future Income Projections (based on 12/31/01 RBC level)	42,131
(3) Income Recognized @ 12/31/01	2,869

Monitoring and Selecting Surplus	As of 12/31 of										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(4) Selected Surplus	$ 42,131	$ 46,506	$ 51,328	$ 55,362	$ 58,890	$ 62,150	$ 65,288	$ 68,388	$ 71,500	$ 74,652	$ 77,864
(5) Indicated Risk Based Capital Company Action Level	21,069	23,253	25,664	27,681	29,445	31,075	32,644	34,194	35,750	37,326	38,932
(6) Surplus to RBC Ratio	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%
(7) Surplus Growth		4,375	9,197	13,231	16,759	20,019	23,157	26,257	29,369	32,521	35,733
(8) Net Written Premium to Surplus Ratio	1.99	1.88	1.77	1.71	1.67	1.64	1.63	1.62	1.61	1.60	1.60
(9) Loss and LAE Reserves to Surplus Ratio	2.54	2.71	2.74	2.75	2.75	2.75	2.74	2.74	2.73	2.73	2.72

Estimated Future Income	During									
	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(10) Total Company Net Income	10,445	12,054	13,187	14,153	14,977	15,762	16,507	17,248	17,978	18,713

Hurdle Rate	@ 12/31										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(11) Selected Hurdle Rate		18.0%	18.0%	18.0%	18.0%	18.0%	18.0%	18.0%	18.0%	18.0%	18.0%
(12) Present Value Factor for 12/31/01 (assuming year end income recognition)	1.000	0.847	0.718	0.609	0.516	0.437	0.370	0.314	0.266	0.225	0.191

Expected Annual Perpetual Growth Rate of Capital & Income After 2011	2.0%

Application of DCF Method	@ 12/31	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(13) Total Company Net Income	2,869	10,445	12,054	13,187	14,153	14,977	15,752	16,507	17,248	17,676	18,713			
(13pv) PV of Total Company Net Income	2,869	8,852	8,657	8,026	7,300	6,547	5,835	5,182	4,589	4,053	3,575	65,484	22,793	88,277
(14) Reinvestment Cost to Grow Company		4,375	4,822	4,034	3,528	3,260	3,138	3,100	3,112	3,152	3,212			
(14pv) PV of Reinv. Cost to Grow Company		3,708	3,463	2,455	1,820	1,425	1,162	973	828	711	614	17,158	1,660	18,818
(15) Free Cash Flow	2,869	6,070	7,232	9,153	10,625	11,717	12,614	13,407	14,136	14,824	15,501			
(15pv) PV of Free Cash Flow	2,869	5,144	5,194	5,571	5,480	5,122	4,673	4,209	3,761	3,342	2,962	48,326	20,933	69,259

Application of EVA Method	@ 12/31	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(16) Statutory Surplus @ 12/31/01	42,131													
(17) Total Company Net Income	2,869	10,445	12,054	13,187	14,153	14,977	15,752	16,507	17,248	17,676	18,713			
(17pv) PV of Total Company Net Income	2,869	8,852	8,657	8,026	7,300	6,547	5,835	5,182	4,589	4,053	3,575	65,484	22,793	88,277
(18) Cost of Capital		7,584	8,371	9,239	9,865	10,600	11,187	11,752	12,310	12,870	13,437			
(18pv) PV of Cost of Capital		6,427	6,012	5,623	5,140	4,633	4,144	3,689	3,275	2,902	2,567	44,412	16,737	61,149
(19) Cost of Starting Capital		7,584	7,584	7,584	7,584	7,584	7,584	7,584	7,584	7,584	7,584			
(19pv) PV of Cost of Starting Capital		6,427	5,446	4,616	3,912	3,315	2,809	2,381	2,018	1,710	1,449	34,081	8,050	42,131
(20) Cost of Growth Capital			787	1,655	2,382	3,017	3,603	4,168	4,726	5,286	5,854			
(20pv) PV of Cost of Growth Capital			566	1,008	1,228	1,318	1,335	1,309	1,257	1,192	1,118	10,331	8,687	19,018
(21) Excess Returns in Year	2,869	2,861	3,683	3,946	4,166	4,377	4,565	4,755	4,938	5,106	5,276			
(21pv) PV of Excess Returns in Year	2,869	2,425	2,645	2,403	2,160	1,913	1,691	1,493	1,314	1,151	1,008	21,072	6,056	27,128
(22) Total Indicated Value														69,259

Primary Stock Insurance Company

Calculation Notes for Valuation Estimates as of December 31, 2001

(1) from Primary Stock Insurance Company's 12/31/01 booked balance sheet
(2) selected starting surplus for future income projections based on the selected 200.0% surplus to indicated RBC ratio
(3) = (1) − (2)

(4) selected surplus based on the selected 200.0% surplus to indicated RBC ratio
(5) calculated RBC at the Company Action Level - based on Exhibit 12 for 2002 and subsequent
(6) selection of 200.0% surplus to RBC ratio for determining required surplus at each year end
(7) cumulative increase in (4) from starting surplus, (4) − (2)
(8) NPW for all lines ÷ (4)
(9) net loss and LAE reserves for all lines ÷ (4)

(10) from Exhibit 8, line (11)

(11) is selected hurdle rate of 18.0% used for determining Cost of Capital in EVA method and the present value of future earnings and value added
(12) = 1.000 at 12/31/01, for future years = (1.0 + 18.0%) raised to (2001 − year) exponent

(13) = (10)
(13pv) = (13) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(13)_{2011}$ × (1+ Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$.
All Years = Total '01 to '11 + Total '12 to ∞

(14) annual change in (4)
(14pv) = (14) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(4)_{2011}$ × Growth Rate ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$.
All Years = Total '01 to '11 + Total '12 to ∞

(15) = (13) − (14)
(15pv) = (13pv) − (14pv)

(16) = (2)

(17) = (10)
(17pv) = (17) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(17)_{2011}$ × (1+ Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(18) = (19) + (20)
(18pv) = (19pv) + (20pv)
(19) = (2) × (11)
(19pv) = (19) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = (16) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞
(20) = $(7)_{\text{final year}}$ × (11)
(20pv) = (20) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = [$(4)_{2011}$ × Hurdle Rate ÷ (Hurdle Rate − Growth Rate) − (16)] × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(21) = (17) − (18)
(21pv) = (17pv) − (18pv)
(22) = (16) + $(21pv)_{\text{All Years}}$

Exhibit 21
Sheet 13

Primary Stock Insurance Company

Valuation Estimates as of December 31, 2001 Using Base Hurdle Rate -3%

Establishing Starting Surplus	
(1) Booked Statutory Surplus @ 12/31/01	45 000
(2) Selected Starting Surplus for Future Income Projections (based on 12/31/01 RBC level)	42 131
(3) Income Recognized @ 12/31/01	2 869

Monitoring and Selecting Surplus	As of 12/31 of										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(4) Selected Surplus	$ 42 131	$ 46 506	$ 51 328	$ 55 362	$ 58 890	$ 62 150	$ 65 288	$ 68 388	$ 71 500	$ 74 652	$ 77 864
(5) Indicated Risk Based Capital Company Action Level	21 066	23 253	25 664	27 681	29 445	31 075	32 644	34 194	35 750	37 326	38 932
(6) Surplus to RBC Ratio	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%
(7) Surplus Growth		4 375	9 197	13 231	16 759	20 019	23 157	26 257	29 369	32 521	35 733
(8) Net Written Premium to Surplus Ratio	1.99	1.88	1.77	1.71	1.67	1.64	1.63	1.62	1.61	1.60	1.60
(9) Loss and LAE Reserves to Surplus Ratio	2.54	2.71	2.74	2.75	2.76	2.75	2.74	2.74	2.73	2.73	2.72

Estimated Future Income	During									
	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(10) Total Company Net Income	10 445	12 054	13 187	14 153	14 977	15 752	16 507	17 248	17 978	18 713

Hurdle Rate	@ 12/31/										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(11) Selected Hurdle Rate		12.0%	12.0%	12.0%	12.0%	12.0%	12.0%	12.0%	12.0%	12.0%	12.0%
(12) Present Value Factor for 12/31/01 (assuming year end income recognition)	1.000	0.893	0.797	0.712	0.636	0.567	0.507	0.452	0.404	0.361	0.322

Expected Annual Perpetual Growth Rate of Capital & Income After 2011	2.0%

Application of DCF Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(13) Total Company Net Income	2 869	10 445	12 054	13 187	14 153	14 977	15 752	16 507	17 248	17 978	18 713			
(13pv) PV of Total Company Net Income	2 869	9 326	9 609	9 388	8 994	8 498	7 980	7 467	6 966	6 482	6 025	83 604	61 456	145 060
(14) Reinvestment Cost to Grow Company		4 375	4 822	4 034	3 528	3 260	3 138	3 100	3 112	3 152	3 212			
(14pv) PV of Reinv. Cost to Grow Company		3 906	3 844	2 871	2 242	1 850	1 590	1 402	1 257	1 137	1 034	21 133	5 014	26 147
(15) Free Cash Flow	2 869	6 070	7 232	9 153	10 625	11 717	12 614	13 407	14 136	14 824	15 501			
(15pv) PV of Free Cash Flow	2 869	5 420	5 765	6 515	6 752	6 648	6 391	6 065	5 709	5 348	4 991	62 471	56 442	118 913

Application of EVA Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(16) Statutory Surplus @ 12/31/01	42 131													
(17) Total Company Net Income	2 869	10 445	12 054	13 187	14 153	14 977	15 752	16 507	17 248	17 978	18 713			
(17pv) PV of Total Company Net Income	2 869	9 326	9 609	9 388	8 994	8 498	7 980	7 467	6 966	6 482	6 025	83 604	61 456	145 060
(18) Cost of Capital		5 056	5 581	6 159	6 643	7 067	7 458	7 835	8 207	8 580	8 958			
(18pv) PV of Cost of Capital		4 514	4 449	4 384	4 222	4 010	3 778	3 544	3 314	3 094	2 884	38 194	30 084	68 278
(19) Cost of Starting Capital		5 056	5 056	5 056	5 056	5 056	5 056	5 056	5 056	5 056	5 056			
(19pv) PV of Cost of Starting Capital		4 514	4 030	3 599	3 213	2 869	2 561	2 287	2 042	1 823	1 628	28 566	13 565	42 131
(20) Cost of Growth Capital			525	1 104	1 588	2 011	2 402	2 779	3 151	3 524	3 903			
(20pv) PV of Cost of Growth Capital			419	785	1 009	1 141	1 217	1 257	1 273	1 271	1 257	9 628	16 519	26 147
(21) Excess Returns in Year	2 869	5 389	6 473	7 028	7 510	7 910	8 294	8 672	9 041	9 398	9 755			
(21pv) PV of Excess Returns in Year	2 869	4 812	5 160	5 002	4 772	4 488	4 202	3 923	3 652	3 388	3 141	45 410	31 372	76 782
(22) Total Indicated Value														118 913

Primary Stock Insurance Company

Calculation Notes for Valuation Estimates as of December 31, 2001

(1) from Primary Stock Insurance Company's 12/31/01 booked balance sheet
(2) selected starting surplus for future income projections based on the selected 200.0% surplus to indicated RBC ratio
(3) = (1) − (2)

(4) selected surplus based on the selected 200.0% surplus to indicated RBC ratio
(5) calculated RBC at the Company Action Level - based on Exhibit 12 for 2002 and subsequent
(6) selection of 200.0% surplus to RBC ratio for determining required surplus at each year end
(7) cumulative increase in (4) from starting surplus (4) − (2)
(8) NPW for all lines ÷ (4)
(9) net loss and LAE reserves for all lines ÷ (4)

(10) from Exhibit 8, line (11)

(11) is selected hurdle rate of 12.0% used for determining Cost of Capital in EVA method and the present value of future earnings and value added
(12) = 1.000 at 12/31/01; for future years = (1.0 + 12.0%) raised to (2001 − year) exponent

(13) = (10)
(13pv) = (13) × (12) for each year; Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = $(13)_{2011}$ × (1+ Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(14) annual change in (4)
(14pv) = (14) × (12) for each year; Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = $(4)_{2011}$ × Growth Rate ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$.

All Years = Total '01 to '11 + Total '12 to ∞

(15) = (13) − (14)
(15pv) = (13pv) − (14pv)

(16) = (2)

(17) = (10)
(17pv) = (17) × (12) for each year; Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = $(17)_{2011}$ × (1+ Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(18) = (19) + (20)
(18pv) = (19pv) + (20pv)
(19) = (2) × (11)
(19pv) = (19) × (12) for each year; Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = (16) × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(20) = $(7)_{\text{end year}}$ × (11)
(20pv) = (20) × (12) for each year; Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = [$(4)_{2011}$ × Hurdle Rate ÷ (Hurdle Rate − Growth Rate) − (16)] × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(21) = (17) − (18)
(21pv) = (17pv) − (18pv)

(22) = (16) + $(21pv)_{\text{All Years}}$

Primary Stock Insurance Company Exhibit 21
Sheet 15

Valuation Estimates as of December 31, 2001 No Growth in Explicit Forecast Period

Establishing Starting Surplus	
(1) Booked Statutory Surplus @ 12/31/01	45 000
(2) Selected Starting Surplus for Future Income Projections (based on 12/31/01 RBC level)	42 131
(3) Income Recognized @ 12/31/01	2 869

Monitoring and Selecting Surplus	As of 12/31 of										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(4) Selected Surplus	$ 42 131	$ 46 096	$ 49 934	$ 52 574	$ 54 362	$ 55 628	$ 56 550	$ 57 240	$ 57 760	$ 58 154	$ 58 444
(5) Indicated Risk Based Capital Company Action Level	21 069	23 048	24 967	26 287	27 181	27 814	28 275	28 620	28 880	29 077	29 222
(6) Surplus to RBC Ratio	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%
(7) Surplus Growth		3 965	7 803	10 443	12 231	13 497	14 419	15 109	15 629	16 023	16 313
(8) Net Written Premium to Surplus Ratio	2.00	1.82	1.68	1.60	1.55	1.51	1.49	1.47	1.45	1.44	1.44
(9) Loss and LAE Reserves to Surplus Ratio	2.54	2.72	2.74	2.75	2.75	2.75	2.74	2.74	2.73	2.73	2.72

Estimated Future Income	During									
	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(10) Total Company Net Income	10 718	12 174	13 025	13 611	13 977	14 240	14 431	14 568	14 665	14 730

Hurdle Rate	@ 12/31/										
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(11) Selected Hurdle Rate		15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%
(12) Present Value Factor for 12/31/01 (assuming year end income recognition)	1.000	0.870	0.756	0.658	0.572	0.497	0.432	0.376	0.327	0.284	0.247

Expected Annual Perpetual Growth Rate of Capital & Income After 2011	2.0%

Application of DCF Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(13) Total Company Net Income	2 869	10 718	12 174	13 025	13 611	13 977	14 240	14 431	14 568	14 665	14 730			
(13pv) PV of Total Company Net Income	2 869	9 320	9 205	8 564	7 782	6 949	6 156	5 425	4 762	4 169	3 641	68 843	28 568	97 411
(14) Reinvestment Cost to Grow Company		3 965	3 838	2 640	1 788	1 266	922	690	520	394	290			
(14pv) PV of Reinv. Cost to Grow Company		3 448	2 902	1 738	1 022	629	399	259	170	112	72	10 749	2 223	12 972
(15) Free Cash Flow	2 869	6 753	8 336	10 385	11 823	12 711	13 318	13 741	14 048	14 271	14 440			
(15pv) PV of Free Cash Flow	2 869	5 872	6 303	6 826	6 760	6 320	5 758	5 166	4 592	4 057	3 569	58 094	26 346	84 440

Application of EVA Method	@ 12/31/	During										Total	Total	All
	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	'01 to '11	'12 to ∞	Years
(16) Statutory Surplus @ 12/31/01	42 131													
(17) Total Company Net Income	2 869	10 718	12 174	13 025	13 611	13 977	14 240	14 431	14 568	14 665	14 730			
(17pv) PV of Total Company Net Income	2 869	9 320	9 205	8 564	7 782	6 949	6 156	5 425	4 762	4 169	3 641	68 843	28 568	97 411
(18) Cost of Capital		6 320	6 914	7 490	7 886	8 154	8 344	8 483	8 586	8 664	8 723			
(18pv) PV of Cost of Capital		5 495	5 228	4 925	4 509	4 054	3 607	3 189	2 807	2 463	2 156	38 434	16 669	55 103
(19) Cost of Starting Capital		6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320			
(19pv) PV of Cost of Starting Capital		5 495	4 779	4 155	3 615	3 142	2 732	2 376	2 066	1 796	1 562	31 717	10 414	42 131
(20) Cost of Growth Capital			595	1,170	1 566	1 835	2 025	2 163	2 266	2 344	2 403			
(20pv) PV of Cost of Growth Capital			450	770	896	912	875	813	741	666	594	6 717	6 255	12 972
(21) Excess Returns in Year	2 869	4 398	5 260	5 535	5 725	5 823	5 896	5 949	5 982	6 001	6 007			
(21pv) PV of Excess Returns in Year	2 869	3 825	3 977	3 639	3 275	2 895	2 549	2 238	1 956	1 706	1 485	30 409	11 899	42 309
(22) Total Indicated Value														84 440

Primary Stock Insurance Company

Calculation Notes for Valuation Estimates as of December 31, 2001

(1) from Primary Stock Insurance Company's 12/31/01 booked balance sheet

(2) selected starting surplus for future income projections based on the selected 200.0% surplus to indicated RBC ratio

(3) = (1) − (2)

(4) selected surplus based on the selected 200.0% surplus to indicated RBC ratio

(5) calculated RBC at the Company Action Level - based on Exhibit 12 for 2002 and subsequent

(6) selection of 200.0% surplus to RBC ratio for determining required surplus at each year end

(7) cumulative increase in (4) from starting surplus (4) − (2)

(8) NPW for all lines ÷ (4)

(9) net loss and LAE reserves for all lines ÷ (4)

(10) from Exhibit 8, line (11)

(11) is selected hurdle rate of 15.0% used for determining Cost of Capital in EVA method and the present value of future earnings and value added

(12) = 1.000 at 12/31/01; for future years = (1.0 + 15.0%) raised to (2001 − year) exponent

(13) = (10)

(13pv) = (13) × (12) for each year. Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = $(13)_{2011}$ × (1 + Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(14) annual change in (4)

(14pv) = (14) × (12) for each year. Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = $(4)_{2011}$ × Growth Rate ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$;

All Years = Total '01 to '11 + Total '12 to ∞

(15) = (13) − (14)

(15pv) = (13pv) − (14pv)

(16) = (2)

(17) = (10)

(17pv) = (17) × (12) for each year. Total '01 to '11 is the total of the estimates by year. Total '12 to ∞ = $(17)_{2011}$ × (1 + Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(18) = (19) + (20)

(18pv) = (19pv) + (20pv)

(19) = (2) × (11)

(19pv) = (19) × (12) for each year. Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = (16) × $(12)_{2011}$.

All Years = Total '01 to '11 + Total '12 to ∞

(20) = $(7)_{prior\ year}$ × (11)

(20pv) = (20) × (12) for each year. Total '01 to '11 is the total of the estimates by year; Total '12 to ∞ = [$(4)_{2011}$ × Hurdle Rate ÷ (Hurdle Rate − Growth Rate) − (16)] × $(12)_{2011}$

All Years = Total '01 to '11 + Total '12 to ∞

(21) = (17) − (18)

(21pv) = (17pv) − (18pv)

(22) = (16) + $(21pv)_{All\ Years}$

Primary Stock Insurance Company Exhibit 21
Sheet 17

Valuation Estimates as of December 31, 2001 6% Premium Growth in Explicit Forecast Period

Establishing Starting Surplus	
(1) Booked Statutory Surplus @ 12/31/01	45 000
(2) Selected Starting Surplus for Future Income Projections (based on 12/31/01 RBC level)	42 131
(3) Income Recognized @ 12/31/01	2 869

Monitoring and Selecting Surplus	As of 12/31 of: 2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(4) Selected Surplus	$ 42,131	$ 46 692	$ 52 006	$ 56 786	$ 61 266	$ 65 656	$ 70 110	$ 74 730	$ 79 520	$ 84 538	$ 89 806
(5) Indicated Risk Based Capital Company Action Level	21 066	23 346	26 003	28 393	30 633	32 828	35 055	37 365	39 760	42 269	44 903
(6) Surplus to RBC Ratio	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%	200.0%
(7) Surplus Growth		4 561	9 875	14 655	19 135	23 525	27 979	32 599	37 389	42 407	47 675
(8) Net Written Premium to Surplus Ratio	1.99	1.91	1.81	1.76	1.73	1.71	1.70	1.69	1.68	1.68	1.68
(9) Loss and LAE Reserves to Surplus Ratio	2.64	2.71	2.74	2.75	2.75	2.75	2.74	2.74	2.73	2.73	2.72

Estimated Future Income	During 2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(10) Total Company Net Income	10 311	11 984	13 256	14 417	15 482	16 531	17 588	18 674	19 803	21 018

Hurdle Rate	@ 12/31/ 2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
(11) Selected Hurdle Rate		15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%	15.0%
(12) Present Value Factor for 12/31/01 (assuming year end income recognition)	1.000	0.870	0.756	0.658	0.572	0.497	0.432	0.376	0.327	0.284	0.247

Expected Annual Perpetual Growth Rate of Capital & Income After 2011	2.0%

Application of DCF Method	@ 12/31/ 2001	During 2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	Total '01 to '11	Total '12 to ∞	All Years
(13) Total Company Net Income	2 869	10 311	11 984	13 256	14 417	15 482	16 531	17 588	18 674	19 803	21 018			
(13pv) PV of Total Company Net Income	2 869	8 966	9 062	8 716	8 243	7 697	7 147	6 612	6 105	5 629	5 195	76 241	40 763	117 004
(14) Reinvestment Cost to Grow Company		4 561	5 314	4 780	4 480	4 390	4 454	4 620	4 790	5 018	5 268			
(14pv) PV of Reinv. Cost to Grow Company		3 966	4 018	3 143	2 561	2 183	1 926	1 737	1 566	1 426	1 302	23 828	3 415	27,243
(15) Free Cash Flow	2 869	5 750	6 670	8 476	9 937	11 092	12 077	12 968	13 884	14 785	15 750			
(15pv) PV of Free Cash Flow	2 869	5 000	5 043	5 573	5 682	5 515	5 221	4 875	4 539	4 203	3 893	52 413	37 348	89 761

Application of EVA Method	@ 12/31/ 2001	During 2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	Total '01 to '11	Total '12 to ∞	All Years
(16) Statutory Surplus @ 12/31/01	42 131													
(17) Total Company Net Income	2 869	10 311	11 984	13 256	14 417	15 482	16 531	17 588	18 674	19 803	21 018			
(17pv) PV of Total Company Net Income	2 869	8 966	9 062	8 716	8 243	7 697	7 147	6 612	6 105	5 629	5 195	76 241	40 763	117 004
(18) Cost of Capital		6 320	7 034	7 801	8 518	9 190	9 848	10 517	11 210	11 928	12 681			
(18pv) PV of Cost of Capital		5 495	5 296	5 129	4 870	4 569	4 258	3 954	3 664	3 391	3 134	43 760	25 614	69 374
(19) Cost of Starting Capital		6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320	6 320			
(19pv) PV of Cost of Starting Capital		5 495	4 779	4 155	3 613	3 142	2 732	2 376	2 066	1 796	1 562	31 717	10 414	42 131
(20) Cost of Growth Capital			684	1 481	2 198	2 870	3 529	4 197	4 890	5 608	6 361			
(20pv) PV of Cost of Growth Capital			517	974	1 257	1 427	1 526	1 578	1 598	1 594	1 572	12 044	15 200	27 243
(21) Excess Returns in Year	2 869	3 991	4 950	5 455	5 899	6 292	6 683	7 072	7 465	7 875	8 337			
(21pv) PV of Excess Returns in Year	2 869	3 471	3 766	3 587	3 373	3 128	2 889	2 658	2 440	2 239	2 061	32 480	15 149	47 630
(22) Total Indicated Value														89 761

Primary Stock Insurance Company

Calculation Notes for Valuation Estimates as of December 31, 2001

(1) from Primary Stock Insurance Company's 12/31/01 booked balance sheet
(2) selected starting surplus for future income projections based on the selected 200.0% surplus to indicated RBC ratio
(3) = (1) − (2)

(4) selected surplus based on the selected 200.0% surplus to indicated RBC ratio
(5) calculated RBC at the Company Action Level - based on Exhibit 12 for 2002 and subsequent
(6) selection of 200.0% surplus to RBC ratio for determining required surplus at each year end
(7) cumulative increase in (4) from starting surplus (4) − (2)
(8) NPW for all lines ÷ (4)
(9) net loss and LAE reserves for all lines ÷ (4)

(10) from Exhibit 8 line (11)

(11) is selected hurdle rate of 15.0% used for determining Cost of Capital in EVA method and the present value of future earnings and value added
(12) = 1.000 at 12/31/01 for future years = (1.0 + 15.0%) raised to (2001 − year) exponent

(13) = (10)
(13pv) = (13) × (12) for each year Total '01 to '11 is the total of the estimates by year Total '12 to ∞ = $(13)_{2011}$ × (1+ Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(14) annual change in (4)
(14pv) = (14) × (12) for each year Total '01 to '11 is the total of the estimates by year, Total '12 to ∞ = $(4)_{2011}$ × Growth Rate ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(15) = (13) − (14)
(15pv) = (13pv) − (14pv)

(16) = (2)

(17) = (10)
(17pv) = (17) × (12) for each year Total '01 to '11 is the total of the estimates by year Total '12 to ∞ = $(17)_{2011}$ × (1+ Growth Rate) ÷ (Hurdle Rate − Growth Rate) × $(12)_{2011}$;
All Years = Total '01 to '11 + Total '12 to ∞

(18) = (19) + (20)
(18pv) = (19pv) + (20pv)
(19) = (2) × (11)
(19pv) = (19) × (12) for each year; Total '01 to '11 is the total of the estimates by year Total '12 to ∞ = (16) × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞
(20) = $(7)_{\text{prior year}}$ × (11)
(20pv) = (20) × (12) for each year Total '01 to '11 is the total of the estimates by year Total '12 to ∞ = [$(4)_{2011}$ × Hurdle Rate ÷ (Hurdle Rate − Growth Rate) − (16)] × $(12)_{2011}$
All Years = Total '01 to '11 + Total '12 to ∞

(21) = (17) − (18)
(21pv) = (17pv) − (18pv)

(22) = (16) + $(21pv)_{\text{all years}}$

Acquisition Valuation of P&C Insurance Companies

Jaroslav Danhel and Petr Sosik

Acquisition Valuation of P&C Insurance Companies

Research team: Jaroslav Danhel, Petr Sosik
(University of Economics in Prague)

Abstract

This paper presents a general approach and specific aspects of the valuation of P/C Insurers. It combines corporate finance, the economics of P/C Insurers, and actuarial versus financial views. Although the primary purpose of the paper is to investigate the acquisition valuation of P/C Insurers, its conclusions are applicable to other areas as well.

We discuss strategic aspects such as the purpose of valuation, motivation for acquisitions, status quo valuation, valuation of synergy, valuation of control, valuation of embedded (real) options, and so forth.

We introduce the main valuation methods and their applications to the P/C Insurance Industry. We develop the application of the EVA-based valuation approach. We examine the accounting versus the economic approach, the determination of Net Asset Value, Cost of Capital, cash flow projection, scenario testing versus stochastic analysis, the inputs of cash flow modeling, sensitivity analysis, the valuation of embedded options, and so on A special focus will be limitations of the valuation, including critical analysis of key assumptions.

The appendix includes a case study of the acquisition of a P/C insurer from the Central / Eastern European region (CEE). Practical aspects of experience with the CEE are presented.

1. Introduction

The assessment of the value of P/C Insurance Companies represents one of the traditional tasks of actuaries. The subjects interested in this issue range from investors, through company management to regulatory bodies and rating agencies. The particular interests of any of those parties determine the respective valuation objective. The valuation might be performed due to M&A purpose, internally as a base for an adequate risk and financial management or as a financial assessment executed by regulators and rating agencies.

This paper presents a general approach and specific aspects of the valuation of P/C Insurers. It combines corporate finance, the economics of P/C Insurers, and actuarial versus financial views. The paper balances the theoretical and practical aspects. Although the primary purpose of the paper is to investigate the acquisition valuation of P/C Insurers, its conclusions are applicable to other areas as well.

We are not restricted to a precise and complex valuation model with all possible actuarial and financial inputs. The success of a valuation model is not determined by its complexity. Valuation should be based primarily on the investor's point of view. Logic and a manager's intuition play a substantial role. Decomposition of an acquisition price may make it more acceptable. The application of other valuation methods is beneficial to confirm acquisition price.

The first section contains key features of the economics of P/C Insurers with respect to relevance and implications for valuation. In the next section, we distinguish between strategic and actuarial/financial aspects. Here, we discuss strategic aspects such as the purpose of valuation, motivation for acquisitions, status quo valuation, valuation of synergy, valuation of control, valuation of embedded (real) options, and so forth. Thirdly, we describe valuation methods of Corporate Finance and its application to P/C Insurance

The core of the paper, Section 5.2, shifts to the actuarial and financial aspects of the valuation process. We examine the application of an EVA-based valuation to P/C Insurers, the accounting versus the economic approach, the determination of Net Asset Value, Cost of Capital, cash flow projection, scenario testing versus stochastic analysis, the inputs of cash flow modeling, sensitivity analysis, the valuation of embedded options, and so on.

A special focus will be limitations of the valuation, including critical analysis of key assumptions. Identification of key value drivers in the actuarial and financial assumptions and a detailed sensitivity analysis are necessary. Scenario testing is used in cash flow projections because of a possible information deficit of parameters.

The appendix includes a case study of the acquisition of a P/C insurer from the Central / Eastern European region (CEE). Practical aspects of experience with the CEE are presented.

2. Economics of P/C Insurance and Consequences for Valuation

In this section, we recall key theoretical fundamentals for the basis of P/C insurance. We highlight significant principles for the basis of the P/C valuation.

The common features of financial institutions may be seen in the mix of asset and risk transformation. We make the following hypothesis based on the specific nature of risk and asset transformation. The financial institution's mix of business and its position in the economy should be clearly identified and reflected in the valuation.

Does this hypothesis mean that the valuation of P/C insurers and financial institutions are based on different principles than those applied to non-financial institutions? Insurers, as financial intermediaries, play a substantial role in market economies. There is no substantial difference between P/C insurance companies and non-financial firms with respect to ownership. The majority of P/C insurers operate as joint-stock companies. This fact determines the objective from the investor's point of view. According to the traditional microeconomic approach, both P/C insurers and other non-financial firms run their business with the objective of maximizing shareholder value.

This starting assumption implies that the P/C valuation should be based on general valuation principles developed in corporate finance[1]. This requirement is in line with the investor's point of view. We discuss the implication in Section 4. The next step takes the specifics of the insurer into consideration, with respect to their role in the economy and the nature of the insurance business. We do this in order to correctly apply valuation principles.

We identify specifics of the P/C insurance industry with substantial consequences for valuation.

1) **The stochastic nature of the insurance process**
   Key stochastic variables include number of claims, claim amounts, claims occurrence, and payoff patterns. The valuation model should take this uncertainty into account. For actuarial applications, stochastic analysis or deterministic scenario testing and sensitivity analysis may be used.

2) **The long-term nature of the insurance business**
   This aspect is closely related to the previous one. The time horizon of cash outflow to settle incurred claim events can range from months to decades. Actuaries cannot rely exceptionally on accounting statements, which by definition represent a short-term and retrospective point of view. The valuation of P/C insurers should take a prospective and long-term view.

3) **The specific structure of the insurer's assets and liabilities**

[1] The valuation principles are the same whatever company is concerned See Damodaran A. The dark side of valuation: valuing old tech, new tech. and new economy companies: Prentice Hall . 2001. p 454 "Three fundamentals determine the value of a business a firm's capacity to generate cash flow from existing investments. the expected growth in these cash flows over time. and the uncertainty whether or not cash flow will be generated in the first place"

The specific structure of insurance assets and liabilities with respect to maturity, degree of risk, uncertainty, and liquidity are of key importance for both NAV determination and cash flow projection

4) **Market imperfections**
Market imperfections lead to an understanding of the information asymmetry between the insurer and its clients, the existence of moral hazard, adverse selection, and the negative consequences of bankruptcy. All of these items justify the existence of state regulation.

5) **State regulation**
The statutory solvency requirements must be fully reflected in the valuation process. An example is seen in adjustments to NAV determination.

6) **Rating Agencies**
Appraisals and reports performed by rating agencies have a substantial influence on how investors and the general public view the company. The agencies are a source of information for the valuation process, mainly in appreciating the adequacy of the acquisition price.

7) **Dependence on the legal environment**
Besides the economic aspects of risk transfer, any insurance contract involves legal aspects as well. The long-term nature of insurance is substantially affected by certain long-term liabilities such as products liability and environmental claims. Future judicial decisions should be considered in the stress testing the valuation model.

8) **Dependence on macroeconomic development**
The greater the time delay inherent in the insurance process, the more sensitive the company's results on key macroeconomic variables such as inflation, interest rates, GDP growth, stock market developments, and so on. Sensitivity testing is used to model various scenarios of future economic development. Stress testing models extreme cases.

To summarize, the long-term nature of insurance, its stochastic character, information asymmetry, the close connection to macroeconomic developments, and dependency on state regulation introduce a substantial amount of uncertainty and complexity to the valuation. We stress that a long-term prospective approach must balance the inputs from accounting statements, which represent a short-term view. An economic approach balances the actuarial and financial tools. Neglecting any of these points in the valuation may lead to misleading assumptions with a substantial impact on decision making.

3. The Acquisition Valuation Process of a P/C Insurer

3.1. Strategic Issues

In this section, we briefly describe the theory of the strategic aspects of the acquisition valuation.
Certain strategic issues should be addressed before starting the valuation modeling. Factors such as motivation, expectation, restrictions, and psychological fears are more qualitative than quantitative. They are difficult to quantify but are at least implicitly considered in the model. The valuation process consists of two interconnected parts:

1. **Strategic**
It is the task of a manager to analyze the motives of an acquisition, possible synergy and diversification, control issues, improvement of operational efficiencies by managerial know-how via restructuring of an acquired firm, and the consideration of other strategic options. In this respect, actuaries and financial analysts are dependant on subjective managerial input.

2. **Actuarial / financial**
The valuation task is delegated to actuaries and financial analysts once the necessary strategic issues are analyzed. The correct valuation model is based on the managerial assumptions and inputs of the strategy.

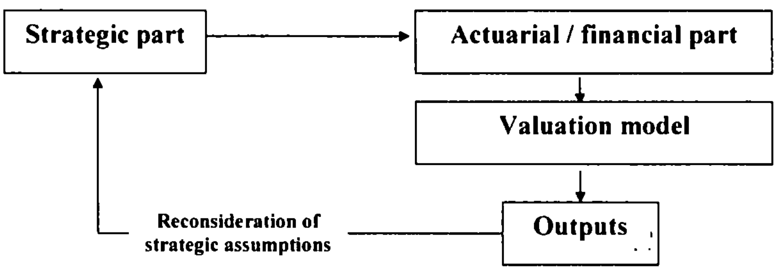

Figure 1: Valuation process of P/C Insurers

The strategic issues below are a simplified summary of ways in which actuaries and financial analysts depend on decision-makers.

- Motives for an acquisition
- Expected short, middle, and long-term impact on growth (sales, cross-selling) and profitability (economies of scale)
- Risks connected with the acquisition
- Other expectations and aspects of the acquisition

3.2. Motivation for an Acquisition

We distinguish the following basic motives for acquisition of a P/C Insurer, based on Damoradan's classification[2].

1. **Excess Capital of Acquiring Firms**
   The positive developments of the stock markets at the end of the 1990's boosted the capital of many insurance companies and led to capital in excess of the economically needed levels. Managers, trying to find adequate investment opportunities, have launched a wave of M&A activities. If excess capital were the only motivation, the acquisition would be very risky as the inputs may be significantly overestimated. Excess capital as a motivation is primarily determined by the demand-side, which may automatically push the acquisition price to inadequate levels.

2. **Undervaluation of Target Firms**
   The presumption here is the ability of the acquiring company to recognize that firms are undervalued by financial markets. Such ability suggests access to better information than is available to other investors in the market. We note that acceptance of this motive denies the validity of the efficient market hypothesis This acquisition motive suggests a speculative investment, making a profit from the disparity in purchase price and sales price rather than a strategy.

3. **Synergy**
   Synergy is defined as the positive value-added by combining two firms. Many managers consider synergy as the primary motive for M&A activities. Although it is a popular justification, many studies have shown that synergy effects often overestimate the valuation. We should carefully analyze the extent to which synergy effects are adequate as inputs to the model Traditionally, positive synergy effects are distinguished by economies of scale (lower relative costs) and growth synergy (higher growth) in the following areas:
   - Distribution (cross-selling opportunities, E-business, tied agency networks)
   - Operations (the company's infrastructure, IT-infrastructure, managerial know-how)
   - Underwriting and claims settlement (expertise, good reputation)
   - Asset management (know-how)

4. **Diversification**
   Diversification[3] reduces the volatility of the company's earnings. Together with synergy effects, diversification is often mentioned. as a leading motive for acquisition. The quantification of diversification can be very questionable. Nevertheless, we identify areas where diversification benefits may be found:

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[2] See Damoradan A Investment Valuation Tools and Techniques for Determining the Value of Any Asset, John Wiley and Sons, Inc., 2002

[3] Traditionally, there is a discussion in the financial theory and practice as to whether the diversification effects on the companies' level are negligible or not According to the financial theory based on the hypothesis of efficient markets without any transaction costs, the diversification reduces only the company-specific risks, which can be diversified by investors creating diversified investments portfolios Based on this, it would mean that the contributions resulting from the diversification should not be rewarded and therefore valued on the company's level in the acquisition process.

- Extension of an existing product offer
- Sound portfolio structure (e.g. property vs. liability products, reducing exposure in auto business)
- Extending the base of existing clients
- Creation of larger, homogenous portfolios
- Cross-industry acquisition (other financial institutions such as life insurers or asset management firms)
- Territory diversification (cross-border transactions)

5. **Effects of Control**
The effects of control include positive value-added from the restructuring of poorly managed firms. If there are easily identifiable operational deficiencies in the target company which can be improved in the short-term or middle-term future, they should be considered in the valuation.

6. **Managerial Self-Interest**
Besides the economic factors, there are psychological aspects stemming from the manager's incentive to increase personal power.

7. **Tax Considerations**
Tax considerations may be seen as a special case of synergy if the combined firm pays less tax than the separate firms would pay. As an example, a profitable firm may acquire a firm with tax deductible losses.

8. **Increased Market Power**
Market power depends on the concentration and competitiveness of the insurance market. The higher the market power, the higher the possibility of influencing market developments such as price.

9. **Regulation**
The acquisition may allow the combined firm to better meet statutory solvency requirements. This may be the case when a well-capitalized company acquires a weaker company.

10. **Embedded Options**
The initial acquisition investment may include the option of future investments. The initial investment is the necessary condition for exercising the option. Examples include expansion, entering new markets, or the sale of new products. Section 4.6 addresses real (embedded) options in more detail.

A clear message should be that the methodology and results of valuation modeling depend critically on an analysis of acquisition motives. In practice, there is a combination of motives. The analysis of motivation may capture other strategic issues. Excluding an analysis of motivation from the overall valuation analysis may result in misleading valuation assumptions

3.3. Decomposition of the Valuation Process

We may analyze the motives of an acquisition once they are identified. Appropriate conclusions are made regarding the quantification of strategic valuation inputs and their incorporation into the model. Appropriate questions may be:

- When to consider the effects of synergy
- How to evaluate synergy
- Risks inherent in synergy valuation
- Who should pay for the positive added-value of synergy
- What portion of the total acquisition price is the synergy premium
- What are the assumptions underlying synergy premium

Similar questions may be asked for diversification, control effects, and real (embedded) options.

Damodaran's classification (see (7)) suggests the following decomposition of the valuation process to make the price determination transparent.

The value of a company $V = V_{SQ} + CP + SP + EO$

1. **Status Quo Valuation (V_{SQ})**

The target company is valued according to current financial, actuarial, and business inputs as well as managerial know-how. In other words, we suppose there is no change in the company's operations. This first step provides a base from which the control and synergy premium is estimated. The variable V_{SQ} is the value of a company based on Status Quo Valuation. If the investment is speculative and the motive is undervaluation, V_{SQ} is the maximum price to be paid.

2. **Valuation of Control Premium (CP)**

The value of control premium is the difference between the value of an optimally managed firm and the value resulting from Status Quo Valuation:

Control Premium (CP) = Value of an optimally managed firm – V_{SQ}

CP results from the right of the acquiring firm to take necessary steps in restructuring to improve the target company's operational efficiency. If the acquisition motive is control, the acquiring company should be willing to pay the value of control premium.

3. **Valuation of Synergy Premium (SP)**

SP represents the positive added-value from combining two firms and includes diversification premium. Theoretically, synergy premium (SP) is calculated as:

Synergy Premium (SP) = Value of the combined firms – Value of the target firm – Value of the Acquiring Firm

SP is based on the presumption that the value of the combined firms is greater than the sum of the values of the acquiring firm and target firm operating independently:

$V(A+B) > V(A) + V(B)$

The acquiring company's flexibility in reaching the desired positive synergy effects determines the willingness to pay the synergy premium. The acquiring company is less willing to pay the premium if it sees many possible targets affording the required synergy effect. In other words, the more flexibility you have, the less you need to pay. Note that it is also possible to reach the desired synergy effect by internal (organic) growth. The acquiring firm is not restricted to the acquisition of established entities.

4. Valuation of Embedded Options (EO)

The value of options to expand initial investments via new markets or new products, to postpone expansion, or to abandon projects should be taken into consideration. Traditional discounted cash flow models do not consider the value of options which are often embedded in investments. Section 4.6 discusses the valuation of real (embedded) options in further detail.

In general, there are two possible ways to consider the value of control / synergy / diversification effects and real (embedded) options in the valuation process. The easier way is an implicit inclusion of underwriting, financial, operational, and other business inputs. The second way is to exclude these effects and model them separately as outlined above. An explicit treatment enables us to better understand the impact of particular valuation assumptions and to analyze the adequacy of control / synergy / diversification weights on acquisition price. We may view explicit treatment as a more transparent and a safer way to review assumptions. However, we must be aware of certain difficulties. The decomposition approach may incorporate some inputs more than once. Its usefulness depends critically on the analyst's ability to explicitly define inputs for each step of the proposed valuation process, without creating uncertainty by implementing speculative inputs.

3.4. Summary

Any valuation is, to a certain extent, a subjective task. Strategic inputs, which describe soft terms such as expectations, are the subjective factors. For this reason, the valuation study should emphasize all key strategic assumptions and their corresponding implications.

The following are key outputs of the strategic process which must be provided to actuaries and financial analysts.

- Expected growth rates for each product line
- Improvements to operations (cost cuttings)
- Necessary investments for the operations
- Cross-selling opportunities
- Synergy / diversification effects
- Growth synergy
- Economies of scale
- Embedded options in the acquisition

Strategic inputs are critically influenced by the particular time period over which the valuation occurs. We strongly recommend not considering the inputs as fixed because of this time dependence. The preparation of several scenarios or sensitivity analysis of key strategic inputs provides feedback to the decision makers.

4. Valuation Methods and Applications to the P/C Insurance Industry

4.1. Introduction

In this section, we introduce the main theoretical approaches to the valuation of companies. The valuation methods are very well known, see Damodaran (7). We discuss underlying principles in order to apply them correctly. We discuss specifics in the application to P/C Insurance, including advantages and disadvantages of each method. An overview is justified in that an analyst does not rely solely on one method when valuing a company. Other approaches are taken into consideration to confirm the range of possible outcomes and to check the correctness of the valuation assumptions.

The value of a company is defined as the difference between the value of its assets and the value of its liabilities. Two basic questions arise.

1. Identification of the terms of the assets and liabilities
2. Assigning values to particular assets and liabilities

The identification task is to recognize all assets and liabilities. Assets represent future economic benefits resulting in cash inflows, while liabilities represent future economic burdens resulting in cash outflows. A portfolio of assets ranges from tangibles, such as buildings and equipment to intangibles, such as goodwill, strategy, business opportunities, and employees. Questions to be asked are:

- Which assets should be included in the valuation?
- How should intangible assets, such as the firm's ability to generate future profits (goodwill) be reflected?
- Does flexibility in decision-making and other business opportunities (real options) constitute an asset? If so, which conditions must be fulfilled prior to inclusion?
- When should a liability be recognized?
- How should a potential risk (liability) be recognized?
- Which leading principles should be followed in the identification of assets and liabilities?
- How should a conservative approach balance a probabilistic approach?

One sees that the basic framework of the identification task is crucial. Substantial uncertainty can arise from the identification process itself. The second task then consists in assigning appropriate values to all assets and liabilities

There are many approaches to both tasks The various approaches, the valuation methods, are determined by underlying principles in identifying and valuing assets and liabilities. We follow corporate financial theory to distinguish the following basic valuation methods.

- Book value approach
- Stock market approach
- Relative valuation
- Discounted cash flow approach (DCF)
- Option-pricing theory

To create a general framework, we recognize certain criteria by which we classify the valuation methods.
Any valuation approach applies these classification criteria to various extents.

1. Prospective vs. retrospective approach
   Some valuation methods, such as DCF, are based exclusively on the prospective valuation. A retrospective valuation, such as the book value approach, follows primarily from past events.

2. The source and character of the inputs
   Either the objective or the subjective character of the valuation predominates. The former utilizes publicly available data from accounting statements (book value approach) or the stock markets (stock market approach) On the other hand, cash flow projection (DCF) is based predominantly on analysts' subjective assumptions about uncertain future.

3. Accounting vs. an economic approach
   This aspect is closely related to the source and character of the inputs. The valuation methods and applications vary to the extent that they follow the accounting or the economic approach

4. Underlying theory
   Another distinctive criterion is the analyst's degree of dependence on a particular financial theory. We may follow a theory in its strict form, such as the efficient market hypothesis or stock market approach. We may also include rules of thumb such as P/E ratios. There is always a particular financial theory underlying a valuation model. How rigorously it is applied depends on the analyst.

In the following sections, we introduce the principles of methodologies, modified forms of which are widely used in practice. The intention is not to provide an extensive coverage of the field. We emphasize the applicability and limitations of each method and related issues to be taken into consideration We focus on the application to P/C Insurance.

4.2. Book Value Approach

The book value approach is the most straightforward of the methods. Accounting statements are analyzed and adjustments are made to better reflect the market environment. The value of the company is derived by deducting the value of the liabilities from the value of the assets

Value = Value of assets – Value of Liabilities

The issue is how to value the particular items of assets and liabilities. It is advisable to make relevant market revaluations since an exclusive reliance on accounting prices does not give an adequate picture. In concrete terms with respect to P/C Insurers, it means paying special attention to the following items. We present more detail in the discussion of accounting principles vs. economic approach of Section 5.2 1

- Financial investments (book vs. market values)
- Goodwill (value of future business)

- Treatment of deferred acquisition costs (DAC)
- Exclusion of assets which have no connection to future business
- Receivables from reinsurance and direct insurance
- Claims reserves (reserve adequacy, reserve discounting)
- Unearned premium reserve (premium deficiencies)
- Treatment of equalization and catastrophe reserves
- Tax considerations (taxes, deferred taxes)
- Other market adjustments (cleaning of the balance sheet)

The pros and cons of the book value approach with respect to P/C Insurance are summarized as follows:

(+) Simplicity, clarity, transparency
(+) Few assumptions as to future uncertainty are needed
(-) Primarily based on accounting assumptions
(-) A retrospective approach contradicts the long-term nature of the P/C insurance business and the investors' point of view
(-) Accounting prices may not reflect a current market environment
(-) It may not consider the value of future profits and other intangible assets

We find the largest disadvantage of the book value approach, with respect to the valuation of P/C insurers, to be the focus on accounting statements as the primary source of information. The book value approach is a static and retrospective approach. It contradicts the long-term nature of the P/C insurance business and the investors' point of view. Market adjustments to relevant insurance assets and liabilities are a possible solution to this drawback. Another disadvantage is that values of certain intangible assets, such as the value of future business, business opportunities, and strategic options are not considered. There is a risk that these assets may not be captured properly.

Despite these objections, the book value approach is widely utilized in practice and occupies a more or less important place in any acquisition study. We note that the principles of the approach are a basis for the determination of NAV and are the first component of an EVA-based valuation methodology.

4.3. Stock Market Approach (Efficient Market Hypothesis)

According to the stock market approach, the value of a company is determined by the price at which its shares are being traded.

Value = Number of shares * share price

The number of shares is immediately available if we neglect special cases such as stock market programs for managers or employees Problems can occur with the appropriate selection of share price. Should we utilize the current price or an average? What should the time horizon be for an average price? Analysts often prefer an average price over several months. There is no particular rule. It depends on the particular situation.

The application of a stock market approach is the simplest approach. It has a very strong theoretical background, the efficient market hypothesis. It implicitly assumes that the markets are efficient and that market price represents an unbiased estimate of true value. The pioneering work in this field was performed by E.F. Fama (15) The theory is based on many restrictive assumptions distinguishing strong, semi-strong, and weak forms of market efficiency but has far-reaching consequences for valuation. In an efficient market, the expected return from any investment will be consistent in the long-term with the associated risk of that investment. The extent to which we accept the validity of an efficient market hypothesis will influence several steps of our valuation (e.g determination of terminal value and projected ROE vs. CoC).

The strict underlying assumptions of the hypothesis suggest there are risks inherent in the application of the method. Firstly, we must carefully analyze to what extent the assumptions are fulfilled Are the markets really efficient? If not, to what extent can we rely on the information provided by a stock exchange? We must consider how share prices are determined and which factors influence the price-determination process. Several psychological effects lead to over-valued or under-valued shares.

The analysis of adequate share price represents a substantial part of investors' decision making Recent developments in the stock markets show that there is a tendency to substantially overestimate the valuation inputs at "good times" and underestimate the inputs at "bad times" However, the phenomenon of over- / under-valuation is not a concern limited to the application of a stock market approach to valuation. If investors' expectations are too optimistic or pessimistic, it will probably influence the assumptions of other valuation methods as well. Nevertheless, what matters is that ex-ante analysis of share price adequacy be a key objective when utilizing the stock market approach.

The pros and cons of the stock market approach are summarized as follows:

(+) It is the simplest approach
(+) Objectivity in the application of publicly available inputs
(+) No valuation assumptions are needed, other than validity of the efficient market hypothesis
(-) It relies exclusively on the efficient market hypothesis
(-) The strong assumptions of the efficient market hypothesis are not fulfilled in practice
(-) The share price may not reflect the company's long-term perspective
(-) Uncertainty as to how share prices are determined
(-) Related issues of under- / over-valuation of share price

The share price alone is rarely an adequate basis for valuation. There are many risks resulting from the very strong assumptions of the theory. However, the share price is a significant price indicator in any acquisition valuation process and is usually the outgoing (minimum) price for negotiations, above which the acquisition price is determined.

4.4. Relative Valuation

Although we try to develop a sophisticated valuation model, in reality the prices of most assets are determined in the market by a comparison to prices of similar assets. In a quick valuation, most analysts will probably utilize the principles of relative valuation. In relative valuation, the value of Company A is derived from the value of a comparable Company B or a set of comparable firms, a peer group. The relative valuation methodology utilizes standard variables such as earnings, book value, profits, and sales We define a relative measure or multiple to be the ratio of value or price to a standardized variable.

(Value of Company A / standardized variable of Company A) = (Value of Company B / standardized variable of Company B) = Relative Measure (Multiple)

4.4.1. General Overview of Relative Measures

In fact, there is an unlimited range of possible relative measures. The only condition for a relative measure is that it be economically relevant and justifiable for the particular case. The following measures predominate.

1. **Earnings multiples**
   The value of an asset is related to the cash flow it generates. An example is the price-earnings ratio (P/E), which expresses the share price (P) to current or expected earnings per share (EPS).
   $$P/E = \frac{P}{EPS}$$

2. **Book Value multiples**
   An example is the price-book value ratio (P/BV) obtained by dividing the share price (P) by the book value of equity per share (BV).
   $$P/BV = \frac{P}{BV}.$$

3. **Revenue multiples**
   The share price or value of a company may be related to revenue or sales as a measure of business volume. Often utilized is the price-sales ratio (P/S), where market value per share is divided by revenues per share (S).
   $$P/S = \frac{P}{S}.$$

4.4.2. P/C Insurance Industry Specific Relative Measures

All three of the above categories are seen in the P/C insurance industry. Sales ratios express the value of a company as a multiple of gross written premium or net written premium. P/E ratios and book value ratios are also used.

Note that P/E ratios and book value ratios are determined from accounting quantities such as earnings and book value of equity. Accounting principles serve as a first approximation of company value. Premium, as the denominator of sales or revenue ratios, is not as dependent on accounting rules. Many practitioners prefer sales ratios.

We demonstrate the application of relative valuation by means of a decomposition which relates value to premium This procedure reveals the implicit assumptions underlying a given relative measure. For simplicity, denote premium by P, not to be confused with the same abbreviation for price. We express company value (V) as related to premium (P). We do not distinguish between gross and net premium.

Consider a group of peer companies with ratio V/P The task at hand consists of two parts:

- Determine a V/P ratio for the valuated company based on a peer group analysis
- Analyze the underlying assumptions

Decomposition of the V/P ratio into relevant driving factors makes this a feasible task In the first step, we suppose that the value V, as determined by the (V/P) ratio, corresponds to the sum of discounted values of future cash flows or profits. In other words, we utilize the DCF approach. If we replace cash flows by profits, assume that all profits are to be distributed, and suppose an infinite time horizon with stable profits, the value is given in perpetuity by the form:

$$V = Profit / CoC \tag{4.1}$$

where, CoC = cost of capital
Profit = expected profit at time t = 1.

For the sake of simplicity, the profit variable is defined as follows:

$$Profit = P - L - C + IR \tag{4.2}$$

where: P = premium
L = claims
C = costs
IR = investment result.

$$V = Profit / CoC = (P - L - C + IR) / CoC. \tag{4.3}$$

Substituting equation (4.3) into the V/P ratio, we obtain:

$$V/P = \frac{1}{CoC} * \frac{Profit}{P} = \frac{1}{CoC} * \frac{P - L - C + IR}{P} = \frac{1}{CoC} * (1 - L/P - C/P + IR/P). \tag{4.4}$$

The sum (L/P + C/P) is the combined ratio. We further define the investment result (IR) as the product of investment yield (IY) and the state of the investment portfolio (I).

$$IR = I * R. \tag{4.5}$$

The investment portfolio (I) as a percentage of premium (P) is known by the term asset leverage (AL).

$$AL = I / P. \tag{4.6}$$

Substituting the relationships (4.5) and (4.6) into equation (4.4), we obtain:

$$V/P = \frac{1}{CoC} * [(1 - \text{combined ratio}) + AL*IY). \tag{4.7}$$

How do we interpret this equation? It is derived from the equilibrium relationship between relative measure (V/P) and the value determined by the DCF approach. We thereby express the V/P ratio in relation to:

- CoC (a measure of investment risk)
- Combined ratio (a result of the underwriting)
- Investment yield (a result of investment)
- Asset leverage (a measure of time delay in the insurance process, a function of product mix)

Based on this ratio decomposition, we better understand the assumptions underlying a given relative measure. The decomposition also elucidates structural differences among companies including portfolio structure, combined ratio, and average time delay of insurance processes. Complicating factors include growth in profits and retained profits but the principles in a more complicated analysis would be the same.

We illustrate this type of ratio decomposition with some real figures. We extend equation (4.1) by a parameter representing growth rate of profit. If profit is assumed to grow at a flat rate of g% over an infinite time horizon, the DCF approach yields a value (V) given by:

$$V = \frac{Profit}{CoC - g} \text{ for g = growth rate of profit.} \tag{4.8}$$

Equation (4.7) is modified accordingly:

$$\frac{V}{P} = \frac{1}{CoC - g} * [(1 - \text{combined ratio}) + AL*IY). \tag{4.9}$$

A rule of thumb widely used by practitioners is that the value of an insurance company moves in the range of 1:3 times annual premium. Based on equation (4.9), we explore this rule of thumb with respect to changing combined ratio and growth rate of profit. The other parameters remain fixed.

V/P ratio in dependence on combined ratio and growth rate of profits

V/P = [1 / (CoC – g)] * [(1 – combined ratio) + AL*IY), where:

V/P	Value (V) as related to premium (P)
CoC	Cost of Capital
G	Growth rate of profits
AL	Asset leverage (I / P)
IY	Investment yield

Assumptions:		**Parameters:**
CoC	9%	Combined rat.
IY	5%	Growth (g)
AL	2,0	

		Combined ratio										
		95%	96%	97%	98%	99%	100%	101%	102%	103%	104%	105%
Growth (g)	5%	3,75	3,50	3,25	**3,00**	2,75	2,50	2,25	2,00	1,75	1,50	1.25
	4%	**3,00**	2,80	2,60	2,40	2.20	2.00	1,80	1,60	1,40	1,20	1,00

3%	2,50	2,33	2,17	2,00	1,83	1,67	1,50	1,33	1,17	1,00	0,83
2%	2,14	2,00	1,86	1,71	1,57	1,43	1,29	1,14	1,00	0,86	0,71
1%	1,88	1,75	1,63	1,50	1,38	1,25	1,13	1,00	0,88	0,75	0,63
0%	1,67	1,56	1,44	1,33	1,22	1,11	1,00	0,89	0,78	0,67	0,56

Under the assumptions of this simplified model, the V/P ratio of 3 corresponds to either a 5% growth rate and 98% combined ratio or a 4% growth rate and 95% combined ratio Although very simple, this type of scenario analysis provides very strong conclusions concerning the implicit valuation assumptions.

What matters in a relative measure is the set of assumptions. The assumptions are the same in the DCF approach. all cash flow components of the profit, growth in profits as a cash flow, and risk. The decomposition expresses the assumptions in a transparent, explicit form.

The application of relative valuation is a simple but good rule of thumb for the appreciation of value adequacy, enabling us to restrict the range of possible outcomes. However, there are dangers. What are the main risks of the method? The method of relating one firm's value to that of a comparable firm by means of one financial parameter is simplistic. This assumption is made in retail industries with relative ease. An application of the assumption to P/C Insurers omits key structural differences.

- Product mix
- Risk profile
- Company size
- Differences in distribution channels, target audience, and organizational infrastructure
- Differences in life cycles

Additionally, the method automatically assumes that the value of the other company is "correct". For these reasons, we advise a decomposition of the relative measure to get an explicit set of assumptions concerning profitability, growth in profitability, and risk.

The advantages and disadvantages of relative valuation are summarized as follows:

(+) Quick and quite simple calculation
(+) Restricted number of explicit valuation assumptions
(-) Hidden valuation assumptions
(-) Possibly difficulties in finding comparable firms
(-) Inherent assumptions regarding the "correct" value of comparable firms
(-) Appropriateness (economic relevance) of the relative measure for the determination of value

As an exclusive measure for the P/C Industry, the relative measure approach is simplifying and thus very dangerous. It omits structural differences. We therefore strongly emphasize the analysis of hidden valuation assumptions. Its simplicity allows the use of the measure as an additional method supporting the basic and more sophisticated valuation model.

4.5. Discounted Cash Flow Approach (DCF)

The various modifications of the DCF approach serve as a basis for the majority of valuation models. The leading principle of the theory is the rule of present value. The value of any asset is determined by the present value of the expected future cash flow.

The basic valuation equation of the DCF approach is written as follows:

$$\text{Value} = \sum_{t=1}^{T} \frac{CF_t}{(1+r)^t} \tag{4.10}$$

where: T = time horizon over which there is cash flow on the asset
CF$_t$ = cash flow in time period t
r = discount rate reflecting the risk of the cash flow.

DCF models are classified into two main types of model. The first approach values the shareholder's equity The second branch values the debtor's equity as well. The difference in the two models lies in the relevant cash flows and in the discount rate applied to those cash flows. In the first model (1), the discount rate is the cost of equity, the rate of return required by shareholders. In the second model (2), the value of the firm is obtained by discounting at the cost of debt weighted on the average cost of equity.

The first model, a dividend discount model (DDM), assumes dividends to be the only relevant cash flow. A strict application of the DDM is too restrictive since many firms do not pay adequate dividends. Free cash flow to equity is a broader definition, see Damodaran (7). We consider the specific asset-liability structure of financial service firms and choose the first approach for P/C Insurance companies. We value the cash flow to equity by discounting at a selected cost of equity, the cost of capital (CoC).

$$\text{Value of equity} = \sum_{t=1}^{T} \frac{CF\_to\_equity_t}{(1+CoC)^t}. \tag{4.11}$$

A discussion of key inputs is deferred:

- Cash flow projection (Section 5.2.3)
- Discount rate CoC (Section 5.2.2)
- Comparison of the DCF and EVA approaches (Section 5.2 1)

The advantages and disadvantages of the Discounted Cash Flow approach are summarized as follows

(+) Prospective valuation of future profit
(+) A full theoretical justification: "The value of any asset is determined by the present value of expected future cash flows."
(+) The basis for major valuation models in practice
(-) Several assumptions estimated for a long time horizon (cash flow determination, CoC)
(-) Sensitivity to inputs
(-) Single scenario approach with no variability in future cash flows

Traditionally, there has been broad agreement in the financial theory that the value of a firm is determined by the present value of expected cash flows and thus DCF serves as a basis for any valuation model However, several theoretical objections have recently been raised concerning an exclusive application of the DCF model. DCF does not capture the variability of cash flows, which plays a very substantial role in the valuation of strategic issues Embedded options (real options) include the flexibility to expand projects, to postpone additional expansion, or to abandon projects.

4.6. Option Pricing Theory (OPT)

Thesis: "Firms sometimes invest in projects because the investments allow them either to make further investments or to enter other markets in the future. In such cases, we can view the initial projects as yielding options allowing the firm to invest in other projects, and they should be willing to pay a price for such options. Put another way, a firm may accept a negative net present value (NPV) on the initial project because of the possibility of high positive NPV on future projects." (Damodaran A.: Investment Valuation: Tools and Techniques for Determining the Value of Any Asset; John Wiley and Sons, Inc.; 2002, p. 796)

The 1990s witnessed a full acceptance of this thesis. The cash flows of certain assets are contingent upon future events. Such assets are referred to as real options and are characterized by two basic aspects of the option.

- The value of the first asset is derived from the value of a second asset
- The cash flow of the asset is contingent on the occurrence or non-occurrence of an event

Traditional DCF models underestimate the value of real options. Thus, option pricing theory has become a necessary tool to reflect these specific cases in the valuation. We present the fundamental principles of option pricing theory. The pioneering work into option valuation is connected with the papers written by Black-Scholes (1) and Merton (26).

Definition: Options provide the holder with the right to buy (call option) or sell (put option) a specified quantity of an underlying asset at a fixed price (strike price: exercise price) at (European option) or also before (American option) the expiration date of the option. Since it is a right and not an obligation, the holder can choose not to exercise the right and allow the option to expire.

The value of an option depends on the following factors

1. The relationship between strike price and the current value of the underlying asset. The higher the strike price, as compared to the current value of the underlying asset, the higher the option price.
2. The variance in the value of the underlying asset. The higher the variance in the value of the underlying asset, the higher the price of the option
3. Time to expiration. The greater the time to expiration, the more valuable the option.
4. Risk-free interest rate. The rate is connected to opportunity costs over the lifetime of an option.

The value of an option in financial theory is determined by the well known Black-Scholes-Merton formula. We present the binomial option pricing model, a discrete counterpart to the continuous Black-Scholes-Merton model. In a binomial tree, the option price is associated with the upward or downward movement in stock price. The simplest case is the one-step binomial tree where the stock price moves either up or down into one of two positions and the option price takes on at most one of two associated values[4]

The binomial model provides insight to the logic of option pricing theory in a case study of the motor third party liability (M-TPL) market of the Czech Republic.

BOX: Case Study - Application of Option Pricing Theory

There was demonopolization of the Motor third party liability (M-TPL) market in the Czech Republic 3 years ago It can be understood as a special case of market share acquisition.

First, we assume that according to the analysts' calculations this kind of acquisition is connected with the negative **NPV of -50 mio**. The management has to make the decision as to whether the company should invest in this project or not.

If the decision were based only on the rule of the positive net present value, the company would not make this investment. On the other hand, executive management argues with the thesis that the acquisition is connected with the unique opportunity of further expansion in the future. This aspect was not considered in the NPV calculation. Managers build on the assumptions that the Czech insurance market is underdeveloped and a substantial growth in all branches can be expected In their views, the acquired M-TPL clients' base represents the cross selling opportunities If that is the case, and the company does not make the investment, it gives up the right (option) of a good outgoing position in the expected future expansion. In other words, **acquisition of M-TPL market share implicitly includes the option to expand** as well.

What is the value of this embedded option to expand?

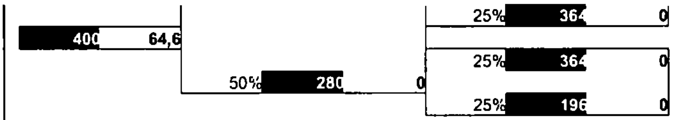
Let us assume that the costs connected with the additional expansion in the future would be 500 mio (strike price) and at the moment the current NPV is estimated at 400 mio (current value of an asset, t=0). The option will be exercised only if the NPV at time of expiration (t=2) exceeds the costs of expansion (strike price).

NPV of entry into M-TPL market	**-50**
Option to expend:	
Costs of expansion - Call strike price	500
Borrowing interest rate	10%

Furthermore, we assume the following binomial process (the expected development of NPV over the next two years).

t=0		t=1			t=2		
Current NPV	Call price	Probability	Current NPV	Call price	Probability	Current NPV	Call price
					25%	676	176
		50%	520	106,7			

4 For more on binomial trees we refer to Hull J C Options, Futures, & Other Derivatives, Prentice Hall, 1999, Chapter 9



Explanation: The value of the call option is arrived at by applying arbitrage theory. We can replicate the cash flow from the option as a combination of borrowing and purchase of underlying asset (current NPV).

Value of the call = Current NPV * Option delta - Borrowing needed to replace portfolio,

where: Option delta = $(C_U - C_D) / (NPV_U - NPV_D)$
Borrowing needed to replace portfolio = Option delta * $NPV_D / (1+i)$
NPV_U = the value of NPV if current NPV goes UP
NPV_D = the value of NPV if current NPV goes DOWN
C_U = call price of option if current NPV is NPV_U
C_D = call price of option if current NPV is NPV_D

Running the calculation backwards. At the expiration time t=2 the option value (call price) is given by the positive difference between current NPV and strike price. Going back to the present, we can calculate the option value at the time t=1, based on the above equations. Looking at the lower branch of the binomial process, the value is obviously 0 (current NPV at the time t=1 can go from 280 to 364 or 196 vs. strike price = 500). In this case, the option would not be exercised. The option value for the upper branch at the time t=1 (current NPV of 520 can move on to either 676 or 364) is given by the above equations.

Option delta = 176 / (676-364); Borrowing = Option delta * 364 / (1+10%); Value of the call = 106,7

Similarly, we can calculate the value of expansion option at the time t=0.

Conclusion:	
NPV of entry into M-TPL market	**-50**
Value of option to expand	**64,6**
NPV of entry into M-TPL market with option to expand	**14,6**

The company should enter into the M-TPL market although the value according to the NPV calculation is negative. That is because of the acquisition of the option to expand. The value of this option is estimated to be higher than negative NPV from entry into M-TPL market.

It is important to keep in mind that we meet with real (embedded) options in daily life. In fact, options are present anywhere we have a certain amount of flexibility at our disposal in decision-making.

DCF-based models assume a passive treatment of assets and liabilities but managers have many opportunities to change a pre-defined course in reaction to current developments. Real options are then crucially important and their values can be substantial. Real options quantify the value of strategic aspects in a very sophisticated way. Managers who have familiarized themselves with option pricing theory justify their investments by the value of embedded options. Therefore, we must apply the valuations very carefully. It means correctly identifying any real option to be considered in the valuation. As an example, if the option is freely available to all market participants, it should not represent an option to be considered in the valuatión.

The advantages and disadvantages of the OPT approach are summarized as follows:

(+) Overcomes the drawback of DCF by reflecting the variability of future cash flows
(+) Full theoretical justification in main-stream financial theory
(+) The application is becoming a standard tool in specific areas
(+/-) A very sophisticated model
(-) Requires many valuation assumptions, including variability parameters
(-) Sets a high requirement on analysts' and decision makers' capabilities
(-) Very sensitive to inputs
(-) Is easily manipulated and misused

The theoretical concept of real options was initially used in the valuation of start-up companies and fast growth sectors such as new economy and biotechnology sectors. OPT is also recognized in the value of a company as a call option. Stockholders act as holders of an option on the company's assets with a strike price at the level of the company's liabilities.

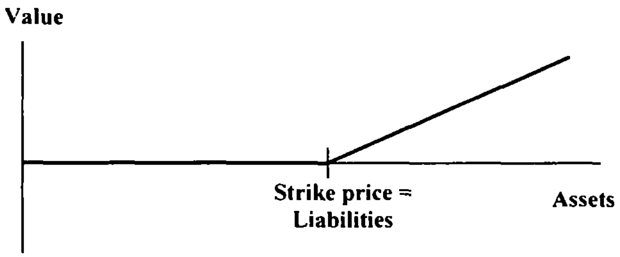


Figure 2: Value as a call option on company's assets

In the P/C Industry, it potentially changes the way of thinking of an investment. The crucial contribution of real option valuation is the recognition of the flexibility in decision-making. Real options may be equated with future opportunities and dangers. Option pricing theory offers an opportunity to embrace this aspect of the valuation process.

4.7. Conclusion

We have briefly presented the main theoretical methodologies for the valuation of a firm with special emphasis on an application to the P/C Insurance industry. We introduced the models in their strict form, stressing underlying assumptions. We discussed how the models are to be applied, as well as the limitations and inherent risks.

Analysts use a wide range of valuation methods in practice, derived by modifications and combinations of the basic models. The models differ in underlying theory, basic assumptions, complexity, and outcome. We cannot say which valuation model is best. It always depends on the specific case. What matters is the precise application of the selected model with respect to its underlying assumptions. Although we usually rely on one basic valuation methodology, which is a combination of several methods in the valuation model, it is advisable to not restrict oneself to only one method. We strongly recommend the application of methods other than the basic valuation method to uncover inconsistencies in the inputs. We should also confront the results of whatever methods we choose. Do the results meet our expectations? Are the results reasonable?

In the next section, we present the Economic Value Added (EVA) approach as a modification to basic valuation methods. EVA is widely used and plays an important role in practice. It follows primarily from the principles of DCF. Firstly, we discount the excess of future profits net of the costs of holding capital. Secondly, we consider the current state of invested capital, the NAV determination, where the principles of a book value approach are recognized. EVA methodology has become a very popular tool in financial management for profitability measurement and valuation. The task is to justify our thesis that this approach serves as a good basis for the valuation modeling of P/C Insurers.

5. EVA as a Basis for the Valuation of P/C Insurers

In the previous section, we created a framework of basic valuation methods as developed in corporate finance This framework provides coherence to the Economic Value Added methodology we present here. We explore the EVA-based valuation as a methodology built upon traditional DCF models. An EVA valuation approach extends the DCF approach by a further consideration of investor's needs. EVA is a measure of surplus value created by an investment. It is defined as profit adjusted by the cost of holding capital.

In this Section, we firstly examine the theoretical background of EVA. We then develop, step by step, a valuation methodology for P/C Insurers.

5.1. Theoretical Background

The EVA methodology was created at the beginning of the 1990's by the consultancy Stern, Stewart & Co. Stewart defined EVA as "operating profit less the cost of all capital employed to produce those earnings"[5]. We note that the concept of economic profit brings nothing new to economic theory[6]. EVA methodology is widely used in financial management for the measurement of profitability and the valuation of a company.

We can summarize the main thesis of the EVA approach as follows:
"The EVA valuation approach meets one of the most important requirements consisting of the preference of the investor's point of view instead of that of the company."

5.1.1. The EVA-based Valuation

We derive the basic equations of the EVA valuation approach. In its simplest form, EVA is defined as Profit after Tax (PaT) earned on invested capital and adjusted by the costs of holding capital, reflecting investors' opportunity costs. The cost of holding capital is defined as the product of invested capital and the required return on invested capital.

$$\mathbf{EVA_t = PaT_t - CoC * Invested\ Capital_{t-1}} \qquad \mathbf{(5.1)}$$

where: EVA_t = EVA in year t
PaT_t = Profit after Tax in year t
Invested $Capital_{t-1}$ = capital provided by investors at the end of the previous year (= at the beginning of the current year)
CoC = Cost of Capital (equity).

We need three basic inputs to calculate EVA:

1. After tax profit generated on invested capital
2. The rate of CoC (discussed further in the text)
3. Invested capital

Either the book value of equity or Net Asset Value (NAV), following the economic approach to valuation, can be used to define invested capital. The economic approach to valuation

[5] Stewart, G.B. The Quest for Value Harper Collins 1991. New York.
[6] A basic course of microeconomics covers the topic of economic profit, taking into account expensed costs and opportunity costs (e.g. costs of capital)

better reflects investors' requiremenets and the market environment, we operate exclusively with NAV.

$$\mathbf{EVA_t = PaT_t - CoC * NAV_{t-1}.} \tag{5.2}$$

Positive EVA implies a company's ability to generate profits above a level required by investors for a given level of risk. The company, therefore, brings additional (added) value to investors. Negative EVA is interpreted as a negative message to investors, expressing that profits are not comparable to other investments with the same risk or opportunity cost.

Dividing equation (5.2) by the NAV yields a reformulation of the basic equation in relative terms:

$$EVA / NAV = PaT / NAV - CoC * NAV / NAV. \tag{5.3}$$

PaT / NAV represents return on invested capital, which we denote for simplicity as ROE[7]:

$$EVA / NAV = ROE - CoC \tag{5.4}$$

$$\mathbf{EVA = (ROE - CoC) * NAV} \tag{5.5}$$

where. ROE = return on invested capital (NAV).

Equation (5.5) is to be understood as follows. The company generates positive added value if the return on invested capital (ROE) exceeds the cost of capital (CoC). It implies that EVA can be increased either through higher operating efficiency under the same level of risk, increasing ROE, or by reaching the same profit by lowering the risk to decrease CoC.

Next, Market Value Added (MVA) is defined as the present value of future EVAs:

$$MVA = \sum_{t=1}^{T} \frac{EVA_t}{(1+CoC)^t} + \frac{EVA_{T+1}}{CoC*(1+CoC)^T} \tag{5.6}$$

$$MVA = \sum_{t=1}^{T} \frac{PaT_t - CoC * NAV_{t-1}}{(1+CoC)^t} + \frac{PaT_{T+1} - CoC * NAV_T}{CoC*(1+CoC)^T} \tag{5.7}$$

where: EVA_t = EVA in year t
T = No. of years over which EVA is explicitly estimated (from the period T+1 calculated as perpetuity)
CoC = Cost of Capital (equity)
NAV_{t-1} = (market) value of invested capital at the end of previous year
PaT_t = Profit after Tax in year t.

Based on the EVA methodology, the value of a company (V) is defined as the sum of invested capital (NAV) and the present value of future EVAs (MVA):

$$\mathbf{V = NAV_0 + MVA} \tag{5.8}$$

[7] Throughout this text, we understand the term ROE to mean return on invested capital, represented by NAV according to the economic approach

where: NAV_0 = (market) value of invested capital as of appraisal date.

$$V = NAV_0 + \sum_{t=1}^{T} \frac{EVA_t}{(1+CoC)^t} + \frac{EVA_{T+1}}{CoC*(1+CoC)^T}. \tag{5.9}$$

$$V = NAV_0 + \sum_{t=1}^{T} \frac{PaT_t - CoC*NAV_{t-1}}{(1+CoC)^t} + \frac{PaT_{T+1} - CoC*NAV_T}{CoC*(1+CoC)^T} \tag{5.10}$$

In other words, the value of a company is determined on one hand by the current state of invested capital (NAV), as the difference between the (market) values of assets and liabilities, and on the other hand by the discounted excesses of future profits above the level of yield on alternative investments (CoC). By this equation, the valuation task is divided into two separate steps:

- Determination of NAV,
- Projection of future cash flow including the determination of discount rate (CoC), consisting of explicit cash flow modeling and determination of terminal value (TV).

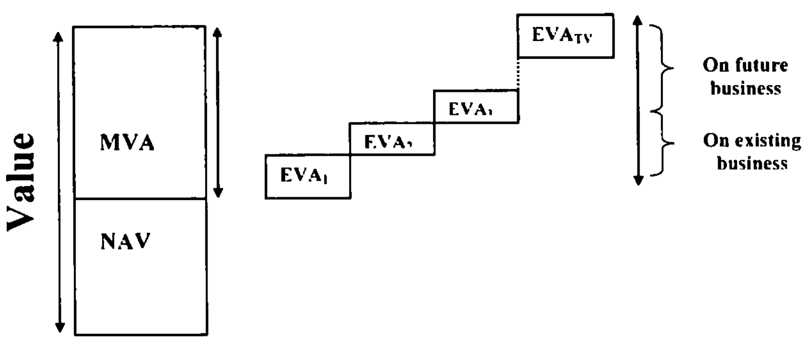


Figure 3: Main features of EVA-based valuation[8]

EVA vs. DCF approach

We have already several times mentioned, that EVA-based valuation approach is deduced from DCF. What is the interrelationship between DCF and EVA-based valuation approach? For simplicity, let us assume the infinite horizon (present value of future profits calculated as perpetuity) So, the values of a company are defined as follows.

1) According to EVA

By the EVA approach, assuming an infinite horizon with stable profits, the value of the company (V) is derived from equation (5.10).

[8] The values of EVAs are depicted already at their discounted values

$$\mathrm{V} = \mathrm{NAV}_0 + \frac{EVA_1}{CoC} = \mathrm{NAV}_0 + \frac{PaT_1 - CoC * NAV_0}{CoC} = \frac{CoC * NAV_0 + PaT_1 - CoC * NAV_0}{CoC}$$

$$= \frac{PaT_1}{CoC}. \tag{5.11}$$

2) According to DCF

Substituting cash-flow in the basic DCF equation (4.9) by after tax profit distributed to shareholders and assuming an infinite horizon with stable profits, we obtain a perpetuity.

$$\mathrm{V} = \frac{PaT_1}{CoC}. \tag{5.12}$$

Conclusion: (5.11.) = (5.12.)

We have just proven for infinite horizon[9] with stable profits, there is no difference between EVA-based valuation and DCF.

Generally, we suppose in the theoretical models the distribution of profits to shareholders. However, in the valuation models we very often assume that profits are retained in the company to finance additional growth. Therefore, we must always very precisely keep the same treatment of distributed / retained profits across the whole valuation[10]. Let us illustrate both extreme cases, either full distribution or full retention of profits, with regard to DCF and EVA on the following example.

Comparison between EVA based valuation and DCF

Inputs:

NAV_0	100	$ROE_{1..10}$	12%
CoC	10%	$ROE_{>=11}$	10%

	Distributed profits: $NAV_t = NAV_{t-1}$					Retained profits: $NAV_t = NAV_{t-1} + PaT_t$				
				DCF approach	EVA based approach				DCF approach	EVA based approach
T	NAV_t	PaT	EVA	Discounted PaT	Discounted EVA	NAV_t	PaT	EVA	Discounted PaT	Discounted EVA
0	100,0					100,0				
1	100,0	12,0	2,0	10,9	1,8	112,0	12,0	2,0		1,8
2	100,0	12,0	2,0	9,9	1,7	125,4	13,4	2,2		1,9
3	100,0	12,0	2,0	9,0	1,5	140,5	15,1	2,5		1,9
4	100,0	12,0	2,0	8,2	1,4	157,4	16,9	2,8		1,9
5	100,0	12,0	2,0	7,5	1,2	176,2	18,9	3,1		2,0

[9] We would come to the same conclusion, if we used for the first several years explicit modeling and for the rest the calculation of terminal value as perpetuity (this can be proven in the same way)

[10] As a warning remark, however obvious, the authors met the practical valuation applications, which considered retained profits both as a component of discounted cash flow and the item increasing NAV and thus terminal value.

6	100,0	12,0	2,0	6,8	1,1	197,4	21,1	3,5		2,0
7	100,0	12,0	2,0	6,2	1,0	221,1	23,7	3,9		2,0
8	100,0	12,0	2,0	5,6	0,9	247,6	26,5	4,4		2,1
9	100,0	12,0	2,0	5,1	0,8	277,3	29,7	5,0		2,1
10	100.0	12,0	2,0	4,6	0,8	310,6	33,3	5,5		2,1
Sum (t = 1, ..., 10)				73,7	12,3				-	19,7
11	100.0	10.0	-	3,5	-	341,6	31,1	-	10,9	-
Terminal Value				38,6	-				119,7	-
NAV_0				-	100,0				-	100,0
Value				**112,3**	**112,3**				**119,7**	**119,7**

Remarks: In the case of retained profits, the value of invested capital (NAV) at the end of a period is given by the sum of the value of invested capital at the beginning of that period plus retained profits. Since we are discounting only the cash flow to shareholders, this case implies that the value according to DCF is given only by the terminal value. The crucial vulnerability of DCF can be seen in the weight, which is given by the terminal value. The higher portion of retained profits, the higher share of terminal value on the total value in the DCF applications. Although from the theoretical standpoint both EVA and DCF are derived from the same background and therefore should bring the same results, we find EVA-based valuation as better reflecting the practical needs (see chapter 5.1.2).

Remark: Comparison of Miccolis concept with EVA (see Miccolis (27))
The Miccolis concept concerning valuation of P/C Insurance Companies based on the term of economic (respectively actuarial) value is developed from the same fundamentals as EVA based valuation.

Economic value = Current net worth (1) + some adjustments (2) + discounted value of future earnings (3) – costs of capital (4) = NAV (1+2) + MVA (3+4)

Miccolis offers the same approach. The key contribution of EVA is a full acceptance and incorporation into current Corporate Finance Theory.

5.1.2. Main Advantages of the EVA-based Valuation Approach

In the following paragraphs, we explain why EVA - when correctly applied – can represent a good theoretical tool for valuation of P/C insurers and can offer some advantages as compared with DCF. One can ask, what are the unique aspects in the application of EVA-based valuation for P/C Insurers The most probable answer, that there is nothing special, can be at the first glance surprising. But when looking at this issue in a more detail, this feature is becoming the biggest advantage of EVA applications. In fact, the clarity and understandable interpretation makes EVA a very useful tool for valuation in insurance industry as well, building upon the traditional discounted cash flow models. Generally, we can identify the following key arguments for the application of EVA-based valuation in the P/C Insurance industry:

1. Consistent with Shareholder Value Management. An emphasis is placed on the investors' needs. The EVA approach enables us to clearly identify the investors' requirements and expectations with regard to risk-reward trade-off. It makes the valuation transparent.

2. Consistent with a traditional accounting approach. Furhermore, the importance of CoC, the discount rate, is seen more easily in this approach.

3. Consistent with current corporate finance theory. The EVA methodology is thoroughly developed and fully integrated into current corporate financial theory. It is consistent with a firm's accounting statements.

4. Consistent with an actuarial approach. EVA as a tool for the measurement of profitability and for valuation purposes is easily incorporated into actuarial DFA models.

5. The economic approach is easily incorporated. The valuation analysis is derived from accounting statements and is thus easily transformed into an economic point of view for NAV determination and consecutive cash flow projection.

6 Standardization and general acceptance. The simplicity of the leading principles and comprehensibility has contributed to the general acceptance of the theory. What would be the contribution of a theoretical approach if nobody understood it and therefore did not trust its results?

7. The decomposition of an EVA valuation allows a clear understanding of the components of the acquisition price. Adequacy of the acquisition price can easily be seen from the two components NAV and discounted future profits. The NAV, the difference between assets and liabilities, is the current state of invested capital. Future expectations of profitability are calculated as the discounted excesses of future profits.

8. Treatment of terminal value. The determination of terminal value in a DCF application is problematic The treatment of terminal value in the presented EVA-based valuation approach is based on the assumption that in the long term infinite horizon, the company's ability to beat the market in reaching a higher return on invested capital (ROE) than the average corresponding to CoC is restricted (see 5.2.3.2). This implies that the terminal value in an EVA application should be set to zero, under the assumption that ROE = CoC in the long term In comparison to the DCF approach, EVA represents a safer and more controllable tool, ensuring no overestimation of future late profits and terminal value.

9. A clear link between the profitability and the valuation of a company EVA provides a clear connection between a performance measure for a given time period, the flow, and the value of the company at a particular point in time, a state.

5.2. The Application of an EVA-based Valuation to P/C Insurers

We have established a comprehensive theoretical framework for EVA-based valuations, provided a comparison to the DCF approach, and described key contributions of this approach to the valuation of P/C insurers. We now shift our attention to creating procedures for an application to the P/C Insurance industry. Appendix C provides an illustrative case study of an acquisition of a P/C Insurer from the CEE region, highlighting specific considerations.

We start with the determination of NAV from the basic equation (5.9) of the framework we have developed.

$$V = NAV_0 + \sum_{t=1}^{T} \frac{EVA_t}{(1+CoC)^t} + \frac{EVA_{T+1}}{CoC*(1+CoC)^T}.$$

We compare the accounting and economic approaches to a valuation. An economic approach is preferable to an accounting approach since it better corresponds to the character of the P/C insurance business and the investors' point of view. We then consider the issue of cash flow projection including the determination of the appropriate discount rate CoC. Figure 3 of Appendix A illustrates the main features of an EVA-based valuation of P/C Insurers.

5.2.1. Determination of Net Asset Value (NAV)

The value of equity, reported in accounting statements and following accounting standards, generally results from the application of valuation principles to insurance assets and liabilities. There are other factors that potential investors should take into account. Net Asset Value (NAV) captures both the accounting term of equity and factors not captured by statutory accounting.

NAV = (Market) Value of Assets – (Market) Value of Liabilities

The value of NAV is the difference between assets and liabilities and depends on how particular items are valued. Accounting statements are the primary information source for the determination of NAV. However, NAV is an economic approach to valuation, including factors such as market values, best estimate adjustments, current market environment, and other factors to correctly reflect the investors' point of view. NAV includes the objective of the decision-makers.

Generally, the determination of NAV consists of several steps:

- Valuation of insurance assets
  = investments; other assets
  - from statutory accounting to economic approach
- Valuation of insurance liabilities
  = technical reserves
  - from statutory accounting to economic approach
- Other factors to be taken into consideration – e.g. solvency and other operational deficiencies

To illustrate the interrelationships between individual steps and resulting implications with respect to the determination of NAV, see also the following figure representing the logic structure of this chapter.

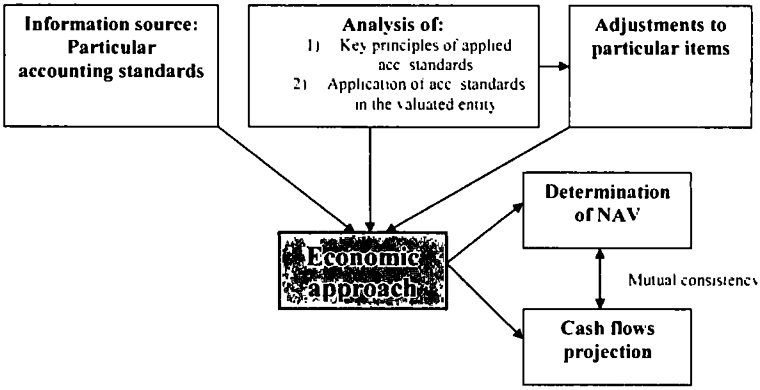


Figure 4: Primary implications of our approach

Accounting Standards and the Economic Approach

Knowledge of accounting standards is not sufficient for our purposes. We must go into further detail and explore how the accounting principles were applied and interpreted in the company's books Many accounting experts are convinced of the exactness of accounting information and say there is only one "true and fair" picture of how to report the financial condition of a firm. However, there is always a certain amount of uncertainty in this respect. It occurs mainly in the accounting of financial services firms where there is a substantial time delay inherent in the business and room for differing interpretations (e.g. reserve adequacy). We also admit the danger of creative accounting, which after the recent accounting scandals seems to be possible anywhere, including countries with long and established traditional accounting systems.

NAV determination considers not only proven accounting principles but also the way in which they were applied and interpreted in a particular company.

The second part of this statement cannot be underestimated in the valuation process. We are of the opinion that a certain amount of skepticism towards information provided by accounting statements makes sense and can be beneficial for the valuation process. As discussed in Section 2, this rule of a little skepticism is becoming important in the P/C Insurance business.

Differences between accounting standards arise primarily from their respective objectives. As an example, consider the differences between SAP and U.S.-GAAP. SAP serves as a basis for state supervision and focuses primarily on surplus adequacy, the company's ability to meet obligations to policyholders. Therefore, the balance sheet is a major concern. GAAP, on the other hand, is based on accrual accounting and provides information concerning the components of a company's earnings. There are accounting standards whose objectives are a basis for tax calculations, and so on.

The objectives of an accounting standard further determine its underlying principles. In this respect, one criterion for classification can be the extent to which best estimate practice (market point of view) vs. the prudence of accounting principles (conservatism) are used. Accounting standards can also vary from the weights that are given either to accrual (matching concept) or cash flow principles. Furthermore, the valuation at historical costs vs. market (fair) values shows the inconsistency between accounting standards. We can also find accounting standards with some specific instruments and tools for insurance business, which can be contradictory to other accounting principles (e g. equalization reserves).
With regard to unification in the field of P/C Insurance accounting, there are currently two leading accounting standards - US-GAAP in North America (but also for Europeans companies which are traded on the U.S. stock markets) and IAS in Europe (currently for insurance business under reconstruction).
There is agreement that the trend with respect to valuation of insurance assets is in the direction of application of (fair) market prices. On the other hand, concerning the valuation of technical reserves, the standards follow the principle of conservatism and do not allow the discounting of reserves. Furthermore, unforeseen losses / profits are recognized immediately (e.g. premium deficiencies). The booking of equalization reserves is generally not allowed

The Economic Approach

One of the theses of our approach emphasizes the preference of the economic approach, being in line with the investors' point of view. This hypothesis was also supported in the analysis of specifics of the P/C Insurance business and their impacts for valuation, where we have intuitively accepted the necessity of implementing an economic approach to valuation, as better reflecting the specifics of P/C Insurance business. We are really convinced that the long term nature of the insurance business, uncertainty, and dependence on the legal and economic environments imply that the valuation methodology is an application of the economic approach. In our view, accounting standards cannot capture all the factors.
But we have not defined this term yet. At the first glance, everyone has a certain idea of what under an economic approach is to be understood. But there is no unified definition of this term in the economic practice. **The economic approach should generally extend the accounting information using the analyst's best estimate adjustments and other factors that need to be taken into consideration to correctly reflect the investors' point of view.**

Based on that, we could define the economic approach according to the following principles:
1. Long-term, prospective approach
2. Market valuation, best estimate practice
3. The rule of present value (time value of money)
4. More priority given to cash-flow rather than accrual accounting
5. More focus on the balance sheet instead of P&L
6. All known factors must be considered
7. All known uncertainty must be considered
8. Partly subjective character highlighting the analyst's role

Theoretically, we could make the following very strong statement. A clear application of the economic approach, and an inclusion of all current factors in the NAV determination lead to the valuation of a company determined only by NAV. In this case, NAV would implicitly include the firm's ability to generate profits corresponding to the risk of an investment, above a "normal" level. Goodwill is an example. The second component of our valuation equation, the discounted excess of future profits (MVA) from cash-flow projections, would correspond exactly to the level of cost of capital, with an implication that MVA is zero by definition.

Of course, precise applications of an economic approach in the valuation models are only a theoretical issue However, understanding this extreme case is important for the application of a particular economic approach in practical situations. We must further state the necessity of consistency between the NAV determination and cash-flow projections in the economic approach (see Figure 4). This consistency must be fundamental to valuation modeling with decisive practical implications.

To clarify the entire complex relationship of valuation by accounting principles vs. the economic approach, we show detail from several balance sheet items as well as other factors considered in the NAV determination.

I. Investments	**I. Liabilities**
1) Fixed income	**1) Claims reserves**
- available for sale	**2) Unearned premium reserves**
- held to maturity	**3) Equalization and catastrophe reserves**
- trading	**4) Other Liabilities**
2) Equity securities	
- available for sale	**II. Shareholder's equity (NAV)**
- trading	
3) Short term investments	
4) Mortgages and other loans	
5) Investment real estate	
II. Other assets	
- thereof: DAC	
- thereof: other deferred expenses	

Figure 5: Simplified balance sheet of P/C Insurance Company

The analyses of balance sheet items as shown on the figure above will be explored with respect to:

1. Country accounting standards (CAS) represented by Czech accounting standards as the base information source
2. US-GAAP (playing more and more important role in Europe as well)
3. Economic approach

Our goal is not to provide readers with a comprehensive description of precise accounting treatment. On the other hand, the presented overview will be aimed at some selected specifics and their impacts and consequences for valuation in order to record the most problematic issues and to keep the complexity of the paper. Whereby the strongest emphasis will be placed on the application of the economic approach.

Issues to be addressed are as follows:

- **Investments (book vs. market values)**
- **Treatment of DAC**
- **Other assets (w/o DAC) – e.g. deferrals**
- **Receivables from reinsurance and direct insurance**

- **Claims reserves (reserve adequacy, reserve discounting)**
- **Unearned premium reserves (premium deficiencies)**
- **Equalization and catastrophe reserves**
- **Solvency requirements (statutory, RBC)**
- **Other operational deficiencies**
- **Treatment of goodwill (elimination)**
- **Tax considerations (taxes, deferred taxes)**
- **Other market adjustments (cleaning of balance sheet)**

Investments
The treatment of investments across different accounting standards can vary in many respects. First, the classification by investment classes and the subsequent accounting valuation is of a big concern. Furthermore, the definitions of book and market values, the issue concerning recognition of changes in market values in P&L and balance sheet and so on represent the areas in which analysts should be interested. The differences in investment valuation can be very substantial. For instance, under US-GAAP, most bonds and equities are carried in a balance sheet in market values, except for "held to maturity" which is carried in amortized costs. But there are still certain items in an investment portfolio, such as real estate, which are valued at historical costs. On the other hand, under Czech accounting standards (until 2001) were the unrealized losses recognized both in P&L and balance sheet immediately after they occurred, while unrealized gains (hidden reserves) were forbidden to be considered either in P&L or in balance sheet (the principle of prudence). However, the worldwide trend moves on to unification in the direction of US-GAAP, representing a more market-orientated approach.
Therefore, the clear identification of valuation principles utilized in the accounting approach to investments is a critical assumption for the further treatment of investments in the economic approach.
But not only the used accounting standards should serve as an outgoing base for investments valuation according to economic principles. There are other factors such as market liquidity, information asymmetry on the market, the availability of credit ratings or market efficiency, which should be taken into consideration, as well. It is clear that all these points are of less importance in the developed markets, which works efficiently. But when valuing an investment portfolio of an insurer from a developing economy, it can be a crucial issue and a point of many struggles between negotiating parties. Here, we can find the cases that despite of the availability of market prices we cannot use since they do not reflect the reality due to e.g low market liquidity.
Generally, we can say that the less liquidity and efficiency in the market, the higher space (and probably necessity) for analyst's adjustments above book and "quasi-market" prices.
To sum up, the valuation of investments according to an economic approach should follow as much as possible market prices (where available), whereby other factors need to be taken into consideration as well.
The above statement has been related to investment portfolio of an insurer covering technical reserves. In addition, an insurer can hold **strategic investments** in subsidiaries, where the valuation differences between accounting standards can be completely different. The best way is to exclude these investments and to value them separately. In this case, the role of analysts when valuing according to economic approach is even more important.

Deferred acquisition costs (DAC)
Generally, the accounting standards allow to defer the policy acquisition costs, following the accrual principle of matching between premium income (as earned premium) and the corresponding expenses (matching concept). The range of policy acquisition costs however,

which are supposed to be deferred according to earned premium income, can vary substantially. Under SAP for instance, the policy acquisition costs are included into P&L as they are incurred (no DAC). On the other hand, under US-GAAP both commissions for renewals and new business and internal acquisition expenses are to be deferred in proportion to earned premium. The Czech accounting standards represent in this respect a compromise; here only commissions from new business are allowed for deferrals.
Concerning the application of economic approach, we do not see any problem with deferrals of policy acquisition costs, under the assumption that there is a clear link between DAC and future business. Furthermore, if the time horizon for amortization of DAC is restricted to one year, it should not pose a problem for the valuation model to keep the consistent treatment with cash flow projection as well.

Other assets (w/o DAC)
We also recommend very careful analysis of other assets (w/o DAC), which can include some very doubtful items. Generally, the majority of other assets are carried in historical costs. First, we would suggest dividing this part of balance sheet into those items, which are connected to the insurance business, and the remaining items, not directly influencing the insurance business The next criterion for classification of other assets should be, whether they really result in future economic benefits. Whatever kind of other assets not directly connected with insurance business, we propose to value them according to the economic approach as conservatively as possible, mainly when the effects on future business are negligible. The valuation at liquidating prices we find as the most feasible solution in this respect
Furthermore, we advise paying close attention to all deferrals (excluding DAC) and other similar items. They can be treated under different accounting standards in a different way completely. The detailed analysis of these items, which can be shown in the balance sheet either explicitly or are hidden under tangible property (and amortized), should not be also underestimated. The clear connection to future business (profits) would be the decisive criterion. We are convinced, that an analyst cannot go too far wrong by following the principle that all doubtful assets are to be charged directly against NAV.

In summary, we see the following main rules, which are to be applied by the transmission of other assets into economic approach:
- Clear connection to insurance business
- Clear connection to future profits
- Elimination of any accounting playing with deferrals
- Preference of cash-flow to accrual accounting
- Controversial items are to be excluded from NAV

Receivables from reinsurance and direct insurance
All receivables either from reinsurance or direct insurance should be according to economic approach reclassified with respect to probability of getting money back. The appropriate revaluations should be charged directly against NAV. In the case of receivables from reinsurance, we can use the rating as a measure of default probability.

Claims reserves
Concerning valuation of claims reserves, we must explore two issues: **discounting** and reserves adequacy. Generally, the accounting standards require claims reserves to be estimated at their ultimate amounts, in which claims are expected to be settled. There can be some exceptions, where reserve discounting is allowed – e.g. under US-GAAP mainly for

claims with fixed and determinable payments (e.g. pensions from worker's compensation). However, the majority of claims reserves are valued without reflecting time value of money. The reason for non-discounting is that most claims reserves are estimates and the amount and timing of the payment cannot be determined with certainty (principal of conservatism). However, non-discounting means that expenditure is not matched with corresponding income. While all claims provisions must be provided when the premium is earned, the investment income, which may be used to pay the claims, is not recognized until a later accounting period. This leads to different results (and equity development) in both cases.
Let us illustrate the issue of profit recognition on the following figure. It is based on the assumption that claims reserves were set correctly (no run-off result) You can see that in the case of non-discounting the profits are recognized later on.

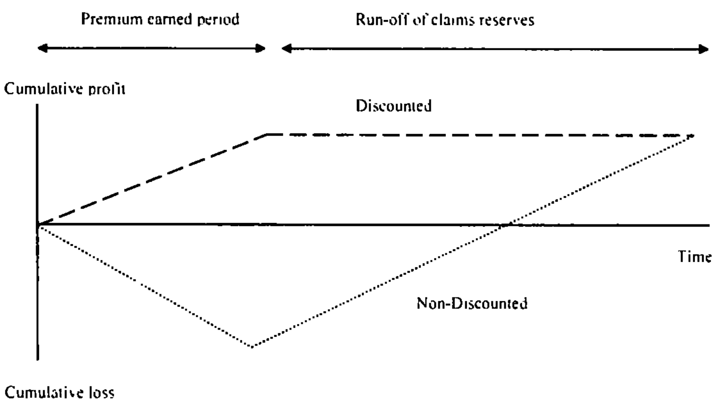


Figure 6: Profit recognition – discounting vs. non-discounting of claims reserves

Based on the definition of economic approach to valuation, it is obvious that reserve discounting is fully consistent with this concept. However, we can meet many possible ways in practice how to cope with this issue in the valuation modeling. Besides the theoretical correctness, we must always take into considerations other practical aspects as well. Nevertheless, what always matters is the mutual consistency of the selected treatment.
First, we can keep claims reserves exclusively at discounted values, what represents the most sophisticated solution requiring very precise and consistent treatment across the whole valuation model. In this case, we must be aware of the fact that any change in the assumptions regarding future interest rates or inflation would immediately impact, besides cash flow projection, the value of NAV as well. The other way is to keep reserve discounting on a separate account, enabling to balance both transparency and economical correctness. Last but not least, if we decide not to consider reserve discounting in the valuation model, then we must keep in mind that all changes in variables effective from the future must be reflected later on in the cash flow projection.

Reserve adequacy is the other actuarial issue. Although many accounting standards are derived from assumptions of a best estimate (neither overestimation nor underestimation) valuation practice, the principle of prudence is applied more widely than the best estimate

approach in many European countries. Whatever accounting standard is applied, a detailed analysis of reserve adequacy is required.
Following the economic approach, we must consider all reserve redundancies or deficiencies, directly impacting NAV. It means, that according to the economic approach the valuation of claims reserves should follow best estimate practice, ensuring the correct states of the outgoing balance sheet as of the date of appraisal.

Unearned premium reserves
Concerning premium deficiencies (when the earned premium from business in force is not sufficient to cover expected claims and expenses), some accounting standards (US-GAAP, IAS) require creation of premium deficiency reserves immediately after they are recognized. On the other hand, there are many other accounting standards (e.g. Czech accounting principles), which do not address this issue so precisely. The application of premium deficiency reserve means the direct recognition of expected future losses in the balance sheet against the decrease of NAV. That is fully consistent with the economic approach.

Equalization and catastrophe reserves
This item represents one of the most controversial points, on which different accounting bodies have not completely agreed yet. In many European countries, the insurers are still obliged (or allowed – depends on interpretation) to create equalization and other similar reserves in order to smooth the fluctuations in claims development.
But according to both leading accounting standards (US-GAAP, IAS), this type of reserve does not represent a certain liability and therefore is not booked as such. On the other hand, both standards, supported by the state regulation (solvency, RBC applications), argue that catastrophic risks are to be implicitly included in the required level of capital. In an economic approach to valuation, these items are not recognized as liabilities, consistent with US-GAAP and IAS. We must reclassify them as NAV in cases where they are booked as liabilities.

Solvency requirements
At the beginning, it is worth distinguishing between statutory solvency required by the state supervision and the required risk capital (risk solvency), resulting from the risk profile of an insurer.
First, as far as the statutory solvency requirement is concerned, it is for sure that any deficit in this respect must be fully considered in the determination of NAV. The required statutory solvency is very often lower than that one corresponding to the risk profile of an insurer, whose level should support continuing business under the defined probability of failure over a certain period. Once we determined the level of required risk capital, the negative gap as compared with current available capital must be fully reflected in the determination of NAV. We propose to deduct the whole capital deficiency from NAV of a target company in order to determine the acquisition price. The explanation for it can be found in the argument, that the total costs of an acquisition consist not only of the paid acquisition price but also the additional capital injections, which are necessary to cover undercapitalisation. In order to insure future profits (ongoing concern), the target company must be adequately capitally equipped. In other words, the acquiring company must provide the target company with additional capital to generate future profits from an acquisition. When we deduct the deficit in solvency from NAV (as a component of acquisition price) and consequently suppose capital increase (in fact another component of acquisition price), we are setting the consistent outgoing level of capital for cash flow projection. All the relationships concerning solvency requirements and their impacts on valuation are illustrated on the following figure.

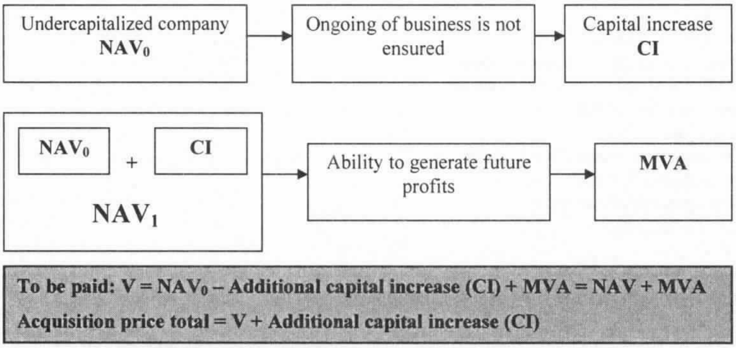


Figure 7: Solvency requirement considerations

Other operational deficiencies
If there are some deficiencies in the operational part of the target company, which in fact represent liabilities requiring future investments, and are not shown in the accounting statement, they should be reflected in the NAV determination according to the economic approach as well. The ongoing concern should be the criterion for recognition of operational deficiencies (the company's insufficient IT-infrastructure).

Elimination of goodwill
We are of the opinion that goodwill should be eliminated from the assets, when determining NAV according to the economic approach. Goodwill, in its theoretical sense, represents the ability of a company given by its staff, market position and operational and sales infrastructure to generate future profits. Of course, this ability matters in the valuation. But in our concept it is considered on another place - in the cash flow projection. If we kept the value of goodwill in NAV, we would count it twice (once under NAV and for the second time in cash flow projection).

Other market adjustments (cleaning of the balance sheet)
Under this item we understand all other analyst's adjustments, which are in line with the applied economic approach to valuation. They can result either from "creative" or else doubtful accounting, with the aim of cleaning the balance sheet as of the date of appraisal. For instance in the CEE region, the authors met with several cases of distrustful accounting, creating an artificial picture about business volume and so on. Therefore, it can sometimes prove very difficult for outsiders to become fully aware of the real economic sense hidden in the information provided by accounting.

Tax considerations
All above adjustments should be also considered with respect to their relevant tax impacts.

Summary of NAV determination (from accounting to the economic approach to valuation)
Equity (from statutory accounting)
+/- market adjustments to investment portfolio covering technical reserves
+/- market adjustments to strategic investments

- elimination of goodwill
+/- treatment of DAC
+/- market adjustments to other assets
+/- reserves adequacy of claims reserves
+/- reserves discounting
+/- premium deficiencies
+ elimination of equalization reserves
- solvency deficiencies
- other operating deficiencies
+/- tax considerations
NAV

5.2.2. Determination of Cost of Capital (CoC)

One of the most important inputs to cash flow projection is the appropriate rate at which future cash flows are discounted (Cost of Capital – CoC). Given the specific capital structure of P/C insurance companies (since premium is received in advance, there is no need for debt financing), the determination of CoC involves only the quantification of the cost of equity capital. CoC is to be interpreted as the rate required by investors to make an investment in the firm's equity. Investors' expectations with respect to risk and return are reflected in this input. From the managers' point of view, on the other hand, it represents the minimal return to be reached. Although the setting of the discount rate is always partly arbitrary in the valuation modeling, the majority of models are derived from the principles of the Capital Asset Pricing Model (CAPM). We introduce this basic concept of financial theory. After that, we address some issues specific to P/C insurance companies.

5.2.2.1. The CAPM

The Capital Asset Pricing Model (CAPM) is derived from the assumption that we can distinguish between firm-specific diversifiable risk and systematic (market) risk, which affects all investments in the market and cannot be diversified. The investor is rewarded only for systematic (market) risk, since firm-specific risk can be avoided through diversification.
The theoretical foundations for the CAMP were established by the paper written by H. Markowitz (24). Here, Markowitz presented the theoretical concept of portfolio diversification and thus gave birth to the modern portfolio theory. The CAPM itself is connected with three names of W Sharpe (30), J. Lintner (23) and J. Treynor. They extended the Markowitz mean-variance model by introducing the beta factor (the risk premium as a function of beta).

According to the CAPM, the expected (required) return on security (investment) R_i is given by the sum of the risk free rate and the risk premium.

R_i = Risk free rate + Risk premium **(5.13)**

The risk premium depends on the systematic risk (= market risk which cannot be eliminated through diversification). It is the contribution of the security (investment) to the overall market risk, measured by the factor beta. It can be written as:

$$R_i = R_f + \beta * [E(R_m) - R_f] \qquad (5.14)$$

where. R_i = the required return on the security (investment)

R_f = Risk free rate

$[E(R_m) - R_f]$ = Expected market risk premium

$E(R_m)$ = Expected return on market portfolio

β = Beta of the security (investment), defined as the portion of the total market variance which is explained by the security (investment).

Replacing the required return on the security (investment) R_i by the rate of CoC, we can write:

$$\mathbf{CoC = R_f + \beta * [E(R_m) - R_f].} \tag{5.15}$$

We need three inputs to CAMP to determine the rate of CoC:

- **The Risk Free Rate**
  The risk free rate is the yield on a risk free asset. An asset is risk free if its expected return can be determined with certainty. The implication is that there is no default and reinvestment risk. Therefore, the risk free rate should be optimally calculated from government zero-coupon bond. Concerning the maturity of such a bond, both models use the short term yield on T-Bills and long-term yields on government bonds We recommend that the maturity (duration) of the risk free asset should correspond to the duration of the cash flow of the investment. An acquisition valuation utilizes the yield to maturity of a long-term (e.g. 10-15 year) government zero coupon bond. If there is no zero coupon bond with the above characteristics available in the market, the payoff pattern can be decomposed as a series of zero coupon bonds.

- **The Risk Premium**
  The risk premium is the additional rate required by investors to invest in the market portfolio (in the original version of the CAPM it is understood to be overall wealth in the economy). It measures what investors, on average, demand as an extra return for investing in the market portfolio relative to the risk free rate. In practice, we usually estimate the risk premium by considering the historical performance of stock market indexes as compared to the yields on risk free assets. Since both the risk free rate and the yield on market portfolio implicitly include the effect of changing inflation rate, the historical risk premium is already net of inflation effects. We use a historical average over a long time horizon in the calculation The geometric mean seems to be more appropriate than the arithmetic mean. Generally, the risk premium is assumed to be in the range of 6%-8%[11].

- **Beta**
  Beta measures the risk that the investment adds to the market portfolio. Beta of an asset is defined as the covariance of the asset (R_i) with the market portfolio (R_m) divided by the variance of the market portfolio.

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[11] For instance, in the case of determination of CoC for a P C Insurer from the developing countries (e g. CEE region) we can face the problem of no available transparent historical data for the calculation of market risk premium and insurance betas Here, we can either use one of the modified approaches for developing economies or the information provided by rating agencies (risk premium resulting from country sovereign rating) or our own analyst's estimation.

$$\beta = \mathbf{cov}\ (R_m, R_i) / \mathbf{var}\ (R_m) = \mathbf{corr}\ (R_m, R_i) * \sigma (R_i) / \sigma (R_m). \qquad (5.16)$$

It tells us what portion of the total market variance is explained by the respective asset. It is clear that the higher the correlation between the respective asset and the market portfolio[12] and the higher the standard deviation of that asset, the higher the amount of systematic risk which is inherent in that investment as compared with the overall market.
In the CAPM, the risk premium for an investment is captured by the beta factor. Beta is usually estimated by regression of historical data.

In summary, when.

- $\beta > 1$: the asset is characterized by higher fluctuation than the market portfolio, is thus more risky, and the investors require a higher return than on the overall market portfolio.
- $\beta < 1$: the asset is characterized by lower fluctuation than the market portfolio, is thus less risky, and the investors require a lower return than on the overall market portfolio.
- $\beta = 1$: the asset is characterized by the same fluctuation as the market portfolio, and is as risky as the market portfolio

Criticism of the CAPM

Recently, several objections have emerged to the standard version of the CAPM. They concern both the practical evidence and its theoretical foundations. There are several studies in the financial literature denying the empirical validity of the CAPM (e.g. market anomalies such as size effect, January effect, etc [13]). Concerning the theoretical foundations, academics argue that the static (single-period) CAPM does not fully address the issues. This criticism resulted in Merton's intertemporal CAPM[14] and the consumption CAPM of Breedon[15].

An alternative theory to the CAPM is represented by the Arbitrage Pricing Model (APT). The theoretical foundations of the APT were established in the paper of S. Ross (28). Like the CAPM, there are two sources of risks: firm-specific (diversifiable) and market (systematic, undiversifiable). The expected risk premium is affected by undiversifiable risk. While there is only one source of market risk captured in the market portfolio in the CAPM, the total risk premium under the APT consists of multiple risk premiums, each one relating to a specific market risk exposure.

Despite all the objections, the key contribution of the Capital Asset Pricing Model (CAPM) is that it provides an insight to the relationship between required return and risk. It recognizes that only market (systematic, undiversifiable) risk matters. The distinction between diversifiable and undiversifiable risk, as the basic underlying assumption of the CAPM, has a timeless validity.

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[12] It implies the smaller diversification effect by adding the asset to the market portfolio.
[13] For instance, see the work of Fama E F., French K R Size and Book-to-Market Factors in Earnings and Returns. Journal of Finance 50. 1995 They showed that stocks of small companies and those with a high book-to-market ratio reach above average returns
[14] Merton R C An Intertemporal Capital Asset Pricing Model; Econometrica. Vol 41. No 5. September 1973.
[15] Breeden D T An Intertemporal Capital Asset Pricing Model with Stochastic Consumption and Investment Opportunities. Journal of Financial Economics. September 1979

Currently, the CAPM represents the standard and most widely used model for measuring market risk in practice. Because of its elegance and simplicity, it is also widely accepted among practitioners in the P/C Insurance industry.

5.2.2.2. Applications to the P/C Insurance Industry – Selected Issues

The following questions are a big concern to actuaries

- What should the required rate of CoC be for the P/C insurance industry?
- What is the riskiness of the P/C insurance companies as compared with other corporations?

There are no unique answers to these questions. On the one hand, we find arguments justifying a higher rate of CoC, assuming that P/C insurance companies are more risky because of their long tailed business, catastrophic risks, and lower transparency to investors. On the other hand, the analysis of historical betas refutes these arguments. This issue is not completely solved either in theory or in practice. However, we believe there is no systematic reason to treat the determination of CoC in a completely different way than in other industries. This statement is in line with our thesis that the higher or lower riskiness of the P/C insurance industry should be considered primarily in the appropriate amount of risk-adjusted capital. The implications are as follows:

Risk profile ⇒ Risk adjusted capital ⇒ Required rate of return (CoC).

With regard to the application of the CAPM in the P/C insurance industry[16], we briefly address the following issues, which are often discussed among the practitioners:

- **Insurance Betas**
  We generally assume that the beta of the valuated company is the same as the industry beta. Nevertheless, we must keep in mind that such a simplification neglects the differences in the risk profile both on the investment side (asset risks) and the underwriting side (underwriting risks) Generally, the betas for the P/C insurance industry are estimated to be less than 1.

- **The LOB Specific Discount Rate**
  In some valuation models, we meet with LOB specific discount rates, reflecting differing risks by LOB. Recalling the implication of the previous paragraph, Risk profile ⇒ Risk adjusted capital ⇒ Required rate of return (CoC), we prefer to apply the same rate of CoC across the entire firm. First, we consider only the systematic risk of the particular LOB in the amount of risk-adjusted capital. Thus, the riskier LOB implies a higher amount of required capital and a higher expected (required) profit margin. Then, the risk-adjusted profitability is compared to the benchmark of the CoC for all LOB This procedure is fully consistent with the EVA-based valuation approach. Here, the item costs of holding capital, as a product of CoC and risk-adjusted capital[17], implicitly include the riskiness of a particular LOB.

Conclusion

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[16] For further discussions we are refering to Felblum S., Thandi N. Financial Pricing Models for Property-Casualty Insurance Products. The Target Return on Capital. CAS Paper. 2003

[17] As a remark we assume that the invested capital is allocated to LOB on a risk-adjusted basis.

The rate of CoC is the critical input in any DCF (EVA) based valuation It represents the discount rate of future cash flows. In an EVA-based valuation, the alternative costs of holding capital are reflected in this input. Although its determination follows a particular theoretical concept, usually CAPM, the resulting rate is always arbitrary and is influenced by subjective factors as well. Many questions specific to the business of insurance arise in an application to the P/C insurance industry. For this reason, the rate of Cost of Capital and its individual components should be analyzed very carefully and tested by sensitivity analysis. Small changes in this parameter substantially impact the range of possible outcomes.

5.2.3. Cash Flow Projections

Cash flow projections together with CoC determine the second component of an EVA-based valuation approach termed Market Value Added (MVA). MVA is defined as the sum of projected profits net of costs of holding capital at discounted values. MVA can be a substantial part of the acquisition price value. The calculation consists of explicit cash flow modeling and determination of terminal value, which is discussed later in this paper.

$$V = NAV_0 + MVA = NAV_0 + \sum_{t=1}^{T} \frac{EVA_t}{(1+CoC)^t} + \frac{EVA_{T+1}}{CoC*(1+CoC)^T}$$

We will focus on the most important aspects of the cash flow projections of P/C Insurers The projection of cash flows sets a higher modeling standard than, as an example, the determination of NAV, which is influenced mainly by the accounting methodology and the scope of the particular economic approach

The analysis of cash flow projections starts with two basic assumptions:

1. Ongoing concern. The assumption that the entity will run its business as an ongoing concern has a basis in the purpose of the acquisition valuation.
2. Consistency The cash flow projections should be consistent with the NAV determination with respect to the valuation principles, e.g. economic approach vs. accounting standards.

For an ongoing concern, the future cash flows can be divided into:

- Run-off of existing business
- Future business

What are the crucial issues with respect to run-off business? Recall how NAV was derived and all the factors included in the economic approach The runoff of existing business is substantially influenced by the extent to which the economic approach is utilized in the NAV determination For instance, if we utilize a clear economic approach including the discounting of reserves, then the runoff of existing business, except for the unearned premium reserve[18], would already be fully considered in NAV. However, as we have said before, the application of a clear economic approach does not always fit the needs of practical valuation modeling. For that reason, we are precise in following the exact appraisal principles of NAV determination.

[18] Depends on interpretations Under certain circumstances, run-off from unearned premium reserve can be understood only as an application of accrual principle in accounting (matching) So, the economic sense behind that is slightly different from the run-off of claims reserves.

The run-off of existing business consists of the following two items:

- Run-off of unearned premium reserve. Unearned premium reserve (UEP) represents the deferral of written premium, according to pro-rata temporis. The release of UEP is related to the incurred losses in that the incurred loss is written as a claims ratio to earned premium in the cash flow calculation. The calculation includes operational expenses and the release of DAC. Investment income is included since it is generated from assets covering both unearned premium reserve and claims reserves. The projection of UEP run-off should be in line with all premium deficiencies / redundancies in the NAV determination. In other words, UEP run-off can be seen as a matching concept in the accounting

- Run-off of claims reserves. The discounting of reserves is a key parameter in the run-off of claims reserves[19]. The value of the run-off equals zero if the discounted reserves are best estimate values since realized investment income is offset by the amortization of the discount. If the claims reserves are undiscounted, the run-off consists of the investment income that the assets supporting the undiscounted claims reserve yield. We assume the reserves are set up correctly.

The tail of the run-off of existing business is of crucial importance for cash flow projection. The long-term nature of the insurance business necessitates that the analyst examine the impacts of variables such as inflation, claims inflation, or interest rates on the run-off of the reserve.

We now explore the various aspects of cash flow projection taken into consideration when creating the appropriate valuation model.

5.2.3.1. Scenario Testing vs. Stochastic Analysis

The first question to address is whether to calculate cash-flow projections based on scenario testing, stochastic analysis, or a combination of both. There are many studies discussing the advantages and disadvantages of both approaches with respect to the objectives of the applications (e g. for details see Feldblum (17), (23)). We briefly describe the relevance of the issue to valuations.

Scenario testing represents the deterministic approach to modeling, where static set of input variables is used. The sets of input assumptions are determined by an analyst as reasonable scenarios of the future development. Therefore, the mutual consistency between inputs is of a great concern. We can use a deterministic model to answer: "What happens, if ...?" questions. Furthermore, the extensions by sensitivity and stress testing are of a big contribution.
The authors of (23) summarized the main advantage of deterministic scenarios as follows: "One advantage of deterministic scenarios is that they can be tailored to reflect management's judgment and develop a consistent, plausible expectation about the future. Therefore, it is important that the economic variables describing the scenario be consistent with each other and with the underwriting and other variables, as well."

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[19] Besides the ex-post adequacy of claims reserves resulting from the stochastic character of the insurance process

Stochastic analysis, on the other hand, uses variables that are selected randomly from probability distributions. It enables you to quantify some function of the variables, such as probability distribution of profit, NAV and so on. When using stochastic analysis, we must pay close attention to the correct set of interrelations between variables.

What conclusions are to be taken concerning scenario testing and stochastic analysis in respect to the valuation purpose?
It is clear that stochastic analysis can provide some pieces of information (e.g. probability distributions), which we cannot obtain from scenario testing On the other hand, the necessity of correct and consistent inputs is much higher in stochastic analysis. That is the key assumption If it is fulfilled, the application of stochastic analysis for valuation of P/C Insurers can be very useful. Otherwise, when this assumption is not fulfilled, the results from stochastic analysis could lead to some misleading conclusions.

In our point of view, there are always some limitations concerning the access to correct data and their correct interpretation in the acquisition valuation process (**possible information deficit of an analyst**). Based on this thesis, we would prefer to base the valuation modeling on the scenario testing approach with the maximum attention paid to the sensitivity analysis of key actuarial and financial parameters. We find this way as a safer one, eliminating the risk of "garbage in, garbage out". In other words, it is no black box for an analyst. Furthermore, the acquisition valuation is more sensitive to and more influenced by strategic inputs The inputs are, by definition, of a psychological nature and the contributions of stochastic analysis could be eliminated to a certain extent.

Other factors influencing the selection of a correct valuation model include the availability of data, the transparency and reliability of data, historical time series, transparency of definitions used, and so on.
In summary, the consistency of inputs as well as the setting of correct relationships between them belongs to the highest principles, which are subordinated to others. When creating a valuation model (either deterministic or stochastic), there must be always considered the harmony between the quality of available data and the requirements for complexity of a valuation model as the highest priority

5.2.3.2. EVA Time Horizon and Terminal Value

The selection of the time horizon over which EVA is explicitly estimated should take into account, on one hand, the long term character of insurance process, where the payout pattern of claims reserves is of a crucial importance and the analyst's ability to set correct assumptions for the far future, on the other hand.
Generally, in the insurance industry the rule is that the explicit cash flow modeling is reasonable up to 15-20 years. Above this level, all assumptions are becoming too speculative, so the determination of terminal value comes into question. However, within this 15-year level of explicit modeling we would also recommend identification of the initial phase over which some aggressive assumptions (e.g. growth rates, synergy effects) are acceptable. We are of the opinion, that this initial phase should not exceed 5 years, as the period over which the assumptions can be estimated with the highest correctness. Then, the inputs during the second phase should be assumed in a more conservative manner.
In this respect, we recommend establishing a controlling mechanism, such as the positive gap between ROE and CoC. This gap should decrease towards the end of the modeled horizon.

As already mentioned, the EVA based valuation approach offers a very elegant treatment of the terminal value.

We assume that after a long time period, e.g. 15 years, there is no reason that the return on equity should exceed the cost of capital[20]. In other words, the company will operate at a level of profitability equal to CoC. Under this assumption, the terminal value equals zero.

ROE = CoC ⇒ EVA = 0 ⇒ TERMINAL VALUE = 0 **(5.17)**

Terminal value is often overestimated in the acquisition price when the valuation follows a traditional DCF approach. Following (5.17), we can be sure of no overestimation of the terminal value in an EVA based approach.

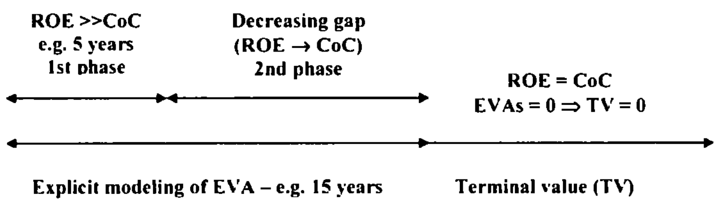

Figure 8: Periods of cash flow modeling (an example)

5.2.3.3. Inputs to Cash Flow Modeling

Once we decide on the valuation model and the time horizon of the model, we shift to the identification of inputs to the cash flow projection. Inputs are categorized by the general economic environment, legal and political stability, industry specific factors and developments in the insurance market, and company specific factors. Figure 1 in Appendix A illustrates this point.

Each level of this hierarchy requires different data sources and also different treatment in respect to impacts and consequences for valuation. This classification (hierarchy) also corresponds to the scope to which the inputs are controllable and influenceable by the acquiring company in the middle, respectively long term. While the first three classes of factors are to be understood as externally determined and therefore uninfluenceable (by industry-specific factors under the assumption of complete competition in the insurance market), the last group of inputs – company-specific – can be in the middle term to a certain extent controllable and manageable by the management of the company.

1. Factors of macroeconomic development

Generally, the insurance industry belongs to the most exposed sectors to macroeconomic development. The variables such as inflation, risk free rate, term structure of interest rates, stock market index, growth of GDP and so on can substantially influence the characteristics of both existing and future business. Taking into account the long-term character of the

[20] This assumption results from the application of efficient market hypothesis (see chapter 4.3 for details)

insurance business, even small changes in any of those parameters matter. For that reason, it is necessary to implement into the valuation model the relationships between economic variables and underwriting and investments parameters. The valuation model should at least embrace the following interrelations. There is usually a close correlation between inflation and risk free rate. Risk free rate determines through the yield structure of interest rates the investment income and through Cost of Capital the discount rate. On the other hand, inflation correlates to claims inflation, as a parameter impacting on the ultimate value of claims reserves. Furthermore, stock market index is in the long term determined by nominal growth of GDP, consisting of its inflationary and real component Next, the investment yield on stocks influences the rate required by investors to make an investment – Cost of Capital and so on.
Putting together all key interrelations, we should be prepared for answering questions: "What happens, if ...". For that reason, a detailed sensitivity analysis is needed, under the assumption that the established relations between key variables are also a subject of uncertainty and should be tested on sensitivity as well. Moreover, stress testing on some extreme developments (long recession, deflation, inflationary shocks, etc.) could identify some too large risk exposures.

2. Factors of external legal and political development

The perceived stability in respect to legal and political development significantly influences how the country is appreciated among the potential investors. Among others, this aspect determines the country risk premium (e.g. according to rating), as a component of Cost of Capital. With regard to valuation, we recommend testing the impacts of changed risk premium on the present value of future cash flow. Furthermore, it is advisable in the case of the developing economies to consider the possibility of extreme events (e.g. political instability), implying the application of stress testing.

3. Factors of insurance market development

For the cash flow projection industry-specific inputs are very important as well. Here, we point out the expected growth of total insurance market (insurance penetration - total premium as % of GDP), setting restrictions and limitations for applied growth rates. Furthermore, the prediction of insurance market structure could provide us with some necessary inputs when projecting growth rates on the level per lines of business. In some models, there are also incorporated the characteristics concerning underwriting cycles (hard vs. soft market). Finally, we should also take into account less quantifiable factors such as trends in the state regulation, integration of financial services and so on

4. Factors of valuated entity

The first task regarding company-specific factors consists of the identification of the inputs to be considered in the valuation model. The relevant classification could follow the two-dimensional basic hierarchy. First, the insurance process and its particular components represent one criterion for classification. According to that, we could distinguish the following areas of company-specific inputs:

- Analysis of premium assumptions ⇒ premium module
- Analysis of expenses assumptions ⇒ expenses module
- Analysis of claims assumptions ⇒ claims module

- Analysis of investments assumptions ⇒ investments module

The next criterion for classification results from the subject setting the assumptions. Here, we must explicitly treat the **strategic inputs** (to be delivered by management) as compared with the **underwriting / financial inputs** (responsibility of actuaries and financial analysts). Strategic inputs, as discussed in Section 3, have connections mainly to premium and expenses (respectively claims) modules:

- Growth rates above market level
- Growth rates vs. claims ratio assumptions
- Synergy/diversification effects
- Growth synergy
- Economy of scale
- Embedded options

Every valuation model should start with the **premium module**, where the assumptions concerning growth rates of premium per lines of business would be the covering output. The growth rate of the premium should be in the next step broken down into the growth rate of number of new business, the growth rate of average premium of new business, the average growth rate of premium in the portfolio (valorization rate) and the cancellation rate (portfolio outflow). Furthermore, the assumptions about payment pattern are beneficial. The premium module should be also linked to industry-specific factors (growth of market premium etc.).
With the premium module are very closely connected the parameters of **expenses**. Usually, the expenses are modeled separately for commissions, other acquisition expenses (both are modeled as variable ratios related to written or earned premium) and operational expenses, which need to be divided into fixed and variable parts. Just in the fixed part of operational expenses we can identify together with the applied growth rate of premium the effects of economy of scale.
The **claims module** includes besides the claims ratio of accident / calendar years also the modeling of run-off of claims reserves and the assumption about payout pattern. It is advisable to link the ultimate value of reserves to claims inflation (see connections to macroeconomic factors).
Finally, the **investment module** should be completely connected to the macroeconomic module. It includes CoC calculation, projection of investment yields per investment classes, portfolio structure and reinvestment rules.

The sources for company-specific inputs are predominantly:

- Analysis of historical performance (development, trends, etc.)
- Risk portfolio analysis (underwriting, investment and operational risks)
- Analyst's expectations

Implementation of cash flow model

Once we have identified key factors influencing cash flow projection, we must put them together and build up a valuation model. Whereby the setting of relationships and interdependencies poses the most significant requirement. Here, the feasibility of selected dependencies is of a crucial importance. Typically, the whole model of a P/C Insurer consists of the consecutive dependencies creating a modular construction, where the first variable influences the second one which is connected to the third one and so on (for example, see Figure 2 in the Appendix A). Next, by setting the relationships between variables, it is worth keeping in mind their implied consequences on the total result. On the one hand, there are

factors whose impacts on the total result are due to dependencies and relationships partly compensated and thus reduced by the opposite change in other factors. On the other hand, a change in one parameter can cause the consecutive changes in other parameters affecting the result predominantly in the same direction (dependencies even multiply the initial effect). In order to be sure that the model works as we intended, it can be a good logical exercise to anticipate all the effects resulting from an initial change in one parameter. In our point of view, an analyst should always be able to intuitively foresee the impacts of any change in underlying assumptions on total result (we are speaking about deterministic model). Otherwise, the valuation model, despite its deterministic character, is becoming a very difficult controllable "black box" with the implications on its credibility.
Consequently, we also recommend setting up some checking mechanisms on the level of respective modules and dependencies between them. For instance, they can be in the form of difference ratios, such as the difference between projected premium growth rate and market growth rate (check of adequacy of applied growth rates), the gap between ROE and CoC or the difference between premium growth and growth rate of expenses (effect of economy of scale).

Finally, we must establish several output sheets in the valuation model, serving for the presentation purpose:

- P&L statement
- Balance sheet
- Key financial indicators (solvency and capital requirements, NAV, internal rate of return, etc.)
- Overview of main valuation assumptions etc.

5.2.3.4. Scenario Testing and Sensitivity Analysis

At the beginning of this chapter, it is worth remembering that we suppose the exclusive application of deterministic modeling. Based on the completed valuation model with all the variables and interrelations, we start constructing various scenarios
The main advantage of scenario testing consists in the possibility of reflecting several analysts' respectively management's judgments concerning future development. That is under the assumption that the variables are in each scenario consistent with each other (macroeconomic, underwriting, investments variables etc.) Generally, the selection of scenarios should consider the environment in which the insurer operates as a whole. It should reflect the reasonable expectation about future development The analysts must ensure the feasibility of assumed interactions between variables. For example, an increase in interest rates would not be probably in line with the decrease in discount rate (CoC) and so on. Due to the character of deterministic modeling (partly subjective determination of scenarios), there is a risk of either too favorable or adverse set of assumptions For that reason, different scenarios are to be prepared, ranging from optimistic to pessimistic alternatives. On the other hand, in order to keep the valuation study sufficiently effective and manageable, it is practical to work with the limited number of scenarios (e.g. maximally 5 scenarios)

First, we start with a **base-case scenario** as an outgoing base for further alternative scenarios and sensitivity testing. The base-case scenario consists of the most probable valuation inputs concerning expectations about future development, with the primary focus on the mutual consistency between the variables. The inputs to base-case scenario are determined according to best estimate practice. In the next step, it is recommended to develop also the **worst-case scenario** as the combination of several adverse but still reasonable assumptions, in order to explicitly present the extreme range of possible outcomes and to warn against potential dangers of an acquisition. In addition to base case scenario, we continue building up additional (3, 4) **alternative scenarios**, consisting of alternative selections of one or more particular variables and their particular values. We are of the opinion that the application of alternative scenarios comes into question mainly in the case of analysis of various strategic inputs. For example, we can explore the effects of changed growth rates, synergy effects, claims development or CoC. The reason behind that is to draw the attention of the decision makers to the impacts of some crucial (e.g. strategic) assumptions, pointing out the uncertainty inherent in the valuation.

While different alternative scenarios should embrace several simple modifications of base-case scenario reflecting the analyst's judgments about the potential changes in assumptions (mainly strategic inputs), the contribution of **sensitivity analysis** offers the possibility of going further into detail, without loosing the necessary transparency and clarity with regard to presentation purpose.
Not only analysts, but also decision makers, should be interested in the issues what happens if something other happens In order to answer this kind of questions: "What happens, if ...?", we must adjust the model for applications of sensitivity testing, which is usually linked to base-case scenario.

We see the main contributions of sensitivity analysis for the valuation of P/C Insurance companies:

- To emphasize that the outcomes of the valuation models are dependent on the particular set of assumptions; if they are changed, the results are also different; there is either "no correct" or "no wrong" outcome. The importance of selected valuation assumptions is to be always pointed out.
- To highlight the uncertainty inherent in the valuation.
- Consistent with the stochastic and long term character of insurance business.
- The possibility of identifying key value drivers, as the variables mostly affecting the results.

We should test the sensitivity on at least the following areas of inputs and their respective components:

- Premium growth rates (new business, avg. premium, cancellation rate)
- Claims development (accident claims ratio, run-off result, payout pattern, claims inflation)
- Expense ratios (commissions, other acquisition expenses, fixed vs. variable part of operational expenses, economy of scale)
- Discount rates (CoC, risk free rate, risk premium)
- Investments yields (yield structure of interest rates, risk free rate, inflation rate, stock market index)

Sometimes it cannot be satisfying to test the variables under the fixed interdependencies. It can be also very beneficial to test the sensitivity of a variable, while isolating the related effects on other variables (other variables are kept constant = no interdependencies). Based on this, we can eliminate the combined effects from the model and explore separately the change in one parameter, without affecting others On the one hand, we can test the change in inflation rate including the corresponding effects on risk free rate, interest rates, investment income, CoC, claims inflation and ultimate values of claims reserves and so on. On the other hand, we can exclude all the relationships and test the change in inflation only in relation to claims inflation affecting ultimate values of claims reserves.

As already mentioned, one of the conclusions from sensitivity analysis should be the identification of **key value drivers** (NAV, CoC, growth rates, synergy, diversification and control effects and so on), which substantially impact the determination of final value.

Finally, we can implement into the valuation study the appendix to sensitivity analysis - **stress testing** – consisting of selecting several extremes, but still reasonable, assumptions about future development and the potential impacts on the insurer

5.2.4. Decomposition of the Acquisition Price

After we have prepared several scenarios and tested key variables for sensitivity, comes the question whether the results are reasonable and plausible. We should start analyzing the acquisition price with the following questions.

- Do we understand the outcomes of the valuation model?
- Are they in line with our expectations?
- How can we appreciate that the resulted outcomes (acquisition price) are adequate?

One of the possible solutions can be to break down the acquisition price into the particular components and then all the components appreciate separately step by step. It enables better understanding of the sources that generate the acquisition price (value).

The primary decomposition of acquisition price is already defined by the basic equation of the EVA-based valuation approach.

$$V = NAV_0 + MVA = NAV_0 + \sum_{t=1}^{T} \frac{EVA_t}{(1+CoC)^t} + \frac{EVA_{T+1}}{CoC*(1+CoC)^T}$$

According to that, we distinguish between:

- NAV, as the difference between (market) values of assets and liabilities. It represents the state of invested capital as of the appraisal date.
- Discounted values of future profits net of costs of holding capital (MVA), consisting of the period of explicit modeling and the terminal value.

Already this basic decomposition can reveal some key connections. It is certain that the first component – NAV – is the safer one with respect to the adequacy of the acquisition price. MVA, on the other hand, is created to a large extent by expectations about the future development. It is worth emphasizing, that there are no recommendations what should be the correct portion of value of "positive expectations about the future" – MVA. However, we should analyze the adequacy of MVA as related to the total value (V = NAV + MVA) very carefully. The higher the share of MVA, the more aggressive the acquisition seems to be.

Nevertheless, we can do a small logical contemplation Let us suppose the case of distributed profits and infinite horizon with the same assumptions concerning ROE and CoC over the whole infinite. We are interested in the relationship between ROE and CoC and the corresponding impact on the proportions between NAV and MVA on the total value (V).

Going out from equation (5.7), we can easily prove that:

$$MVA = NAV * (ROE - CoC) / CoC. \quad (5.18)$$

Substituting the term NAV from (5.18) into (5.8) yields:

$$\mathbf{MVA / V = (ROE - CoC) / ROE.} \quad \mathbf{(5.19)}$$

Following the equation (5.19), the share of MVA on the total VALUE (V) is given by the excess of ROE above CoC as related to ROE. For given CoC = 10%, we can illustrate this dependency on the following graph.

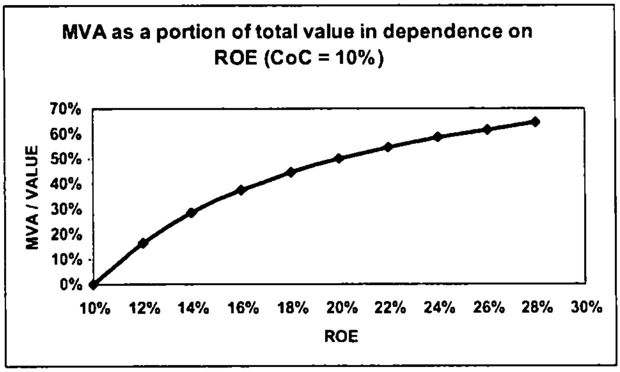

Figure 9: Dependency between the portion of MVA on the total value and ROE (CoC = 10%)

You can see that the share of MVA on the total value (NAV + MVA) exceeds 50% if the projected ROE is higher than 20%.
This example clearly illustrates the connection between the high share of MVA on the total value and the corresponding positive expectations about the future, as embodied in the projected ROE ratio. It is up to analysts to appreciate whether the portions of MVA are still realistic or not. See also Appendix C illustrating the value in dependence on ROE and CoC for 15-year and infinite horizon.

In the next step, we can further break down the value of MVA according to the valuation process as described in Section 3.3:

- Status quo valuation
- Valuation of synergy / diversification effects
- Valuation of effects of control
- Valuation of embedded options

It means to explicitly model all the assumptions concerning synergy / diversification and control effects and to follow all the consecutive steps (for details see chapter three). Then, we could decompose MVA into:

- Value based on status quo valuation
- Synergy / diversification acquisition premium
- Control acquisition premium
- Value of embedded options

There is always a danger that a high acquisition price is justified by high weights of synergy / diversification and control acquisition premium and value of embedded options. Thus, such a decomposition should provide detailed insight into this issue in order to avoid either too positive expectations or any double counting of key items (e.g high growth rates are considered both in the status quo valuation and in the overestimation of synergy effects). For that reason, a careful discussion about the structure of acquisition price is necessary.

5.2.5. Application of other Valuation Methods

Although the valuation modeling is usually based on the DCF (EVA) approach, the final acquisition price is in practice always compared with benchmarks given by either relative valuation or stock market approach. For that reason, a simple application of both is beneficial. It could provide us with some important information about the adequacy of acquisition price, as well. For details see Section 4.

5.2.6. Presentation of Outcomes

The last section of the valuation process is to be devoted to the presentation of outcomes. The analysts should not in any case underestimate this point. Without trying to present here the comprehensive issue, the set of information provided to decision makers with respect to valuation process should include the following parts:

- Summary of key assumptions on which valuation was based
- Risks inherent in valuation
- Presentation of selected scenarios (in the form of simplified P&L and overview of key financial indicators)
- Presentation of conclusions
- Recommendations to Board of Directors

6. Summary and Final Considerations

6.1. Summary

In the presented paper, we tried to capture the whole issue of acquisition valuation of P/C Insurance companies as comprehensive as possible. The paper was aimed at analyzing the acquisition valuation of P/C Insurers from different perspectives, whereby some aspects were only roughly suggested, without going too much into detail. We did not wish to present the next in a long line of other already existing actuarial models of the P/C insurance business. On the contrary, the presented study can be complementary to those models. It should provide actuaries with different insights into this topic from various perspectives, which are not so often discussed in the actuarial profession. The paper should serve as a theoretical background, combining both the knowledge of corporate finance and economics of P/C Insurance. Moreover, we tried to balance theoretical and practical aspects, whereby sometimes we only outlined the practical implications and consequences and let on the reader, if interested, to explore a particular problem more deeply. Our approach was primarily derived from the financial perspective, on which the actuarial models should build up.

In the first part, we discussed the specifics of the P/C Insurance business with the aim to define, besides other generally accepted financial principles to valuation, the basic playground for the valuation of P/C Insurance companies Subsequently, we explored the strategic part of the acquisition process, highlighting the importance of synergy, diversification and control effects. Here, we emphasized the appropriate conclusions and consequences for valuation modeling. Then, the short description of basic valuation methods, as applied in corporate finance, was conducted with the conclusion to base our valuation approach on the EVA principles. After comprehensive theoretical introduction, we dealt with two components of EVA-based valuation: determination of Net Asset Value (NAV) and cash flow projection (MVA). Here, the connections to either applied accounting methodology vs economic approach to valuation were discussed, with the focus on consistency between NAV determination and cash flow projection. Finally, we explored different aspects of cash flow modeling. This paper concludes with some final considerations.

6.2. Final Thoughts about Limitations of Valuation

It is for sure, that there is no unique valuation approach for P/C Insurers. There is no "the only correct" approach. There is no "completely wrong" approach. Nevertheless, we could summarize several principles to be fulfilled, whatever valuation approach (model) is concerned.

1. Reflection of investors' point of view
2. Reflection of specifics of P/C insurance business
3. Reflection of strategic aspects
4. Preference of economic approach
5. Conservatism concerning future development
6. The consistency of a model matters
7. Clarity and transparency of a model is subordinated to its complexity
8. To emphasize the assumptions on which a particular result is based
9. There will be always a large piece of uncertainty
10. To keep "the big picture"
11. Reflection of outsiders' point of view

12. No model can foresee future
13. Logic and managerial intuition will always play an important role

Furthermore, we must be aware that every valuation process runs in the real time under the current external environment. It can substantially influence the expectations, as one of the key determinants of valuation inputs (time-dependency of valuation). Therefore, the appreciation of the same fact can vary at different periods completely. Valuation will be always partly subjective and will bring different outcomes depending on concrete personalities of analysts and decision-makers. With regard to the determination of final acquisition price, we are pretty sure that just the inputs determined by expectations and other strategic aspects are more important than any other (actuarial) assumptions. In addition, to avoid any misunderstanding and misleading interpretations, every valuation should strongly emphasize its underlying assumptions, which can be very changeable over time. We are convinced, that this changing environment is becoming more and more important in the areas, which has been found up to now as quite deterministic and predictable.

To sum up, we must always keep in mind the uncertainty (undeterminability) of external environment concerning future development. This fact, on the one hand, gives reasons for the existence of insurance industry as a risk transformer, but on the other hand implies the uncertainty inherent in the valuation. Whatever the detailed valuation model, we cannot by definition embrace the whole complexity of external world. However, there are still remaining some principles that are timeless.

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Glossary of Key Terms

Actuarial / financial part of valuation process: Following the strategic part, it includes analysis of actuarial, financial and investment valuation inputs and their appropriate reflection in the valuation model.
Book value approach: The value of a company is arrived at by analyzing accounting statements, whereby different adjustments of particular items can be made to better reflect the market environment
Control effects: They are given by additional positive value from restructuring of poorly managed firms.
Cost of capital (CoC): Given the specific capital structure of P/C insurance companies, the determination of CoC involves only the quantification of the cost of equity capital. CoC is to be interpreted as the rate required by investors to make an investment in the firm's equity. At this rate future cash flow is discounted and thus investors' expectations concerning risk vs. reward tradeoff are reflected.
Discounted cash flow approach (DCF): The leading principle of DCF is the rule of present value. The value of any asset is determined by the present value of the expected future cash flow.
Diversification effects: There are given by the reduction of volatility.
Economic approach to valuation: The long-term, prospective approach to valuation, which reflects both the current market environment and investors' point of view. The economic approach should generally extend the accounting information using the analyst's best estimate adjustments and other factors that need to be taken into consideration to correctly reflect the investors' point of view.
Economic Value Added (EVA): EVA is defined as Profit after Tax (PaT) earned on invested capital and adjusted by the costs of holding capital, reflecting investors' opportunity costs.
Embedded option: See Real option
Equity: The book value of equity, as reported in the accounting statements, results from the application of valuation principles to insurance assets and liabilities according to particular accounting standards (see Valuation according to accounting principles).
EVA based valuation: Derived from the principles of DCF, the value of a company is determined by the sum of invested capital (NAV), as the difference between the (market) values of assets and liabilities, and by the discounted excesses of future profits (MVA)
Investors' point of view: The valuation based on investors' point of view primarily goes out from the thesis that the companies are running their businesses with the objective of maximizing shareholder value from the long-term point of view.
Market Value Added (MVA): MVA is defined as the sum of discounted future EVAs
Net Asset Value (NAV): In comparison with the accounting term of equity, the determination of NAV, as the difference between (market) values of assets and liabilities, points out current market environment as well as other factors, which need to be taken into consideration to correctly reflect investors' point of view. NAV is determined by the economic approach (see Valuation according to economic approach).
Option Pricing Theory: If the investment embodies a strategic option such as flexibility to expend a project, to postpone additional expansion or to abandon a project, the value of such an option (see Real option) should be deduced from Option-pricing theory.
Real option: Traditional DCF -based valuation methodologies may fail in including of some strategic aspects, which are embedded in the investments, such as flexibility to expend a project, to postpone additional expansion or to abandon a project. Since the underlying assets are represented by real investments (business opportunities), we speak about real options

(embedded options). Because of the similarity to financial options, their valuation follows Option Pricing Theory.

Relative valuation: It goes out from the principle that the value of a company is derived from the value of a comparable company. It utilizes standardized variables such as earnings, book value, profits, and sales

Stock market approach: Following efficient market hypothesis, the value of a company is determined by the price at which its shares are being publicly traded

Strategic part of valuation process: It includes analysis of motives behind an acquisition, its possible synergy, diversification, and control effects. Usually, managers are supposed to provide strategic inputs.

Synergy effects: They represent the additional positive value from combining two firms. It causes the whole to be greater than the sum of the parts: $V(A + B) > V(A) + V(B)$.

Valuation according to accounting principles: The primary emphasis is placed on the information provided by accounting statements (compare with Economic approach to valuation)

Value decomposition: The decomposition of the valuation process into the consecutive steps makes the price determination transparent (status quo, control and synergy premium, value of embedded options)

Appendix A: Figures

Figure 1: Inputs to Cash Flow Modeling

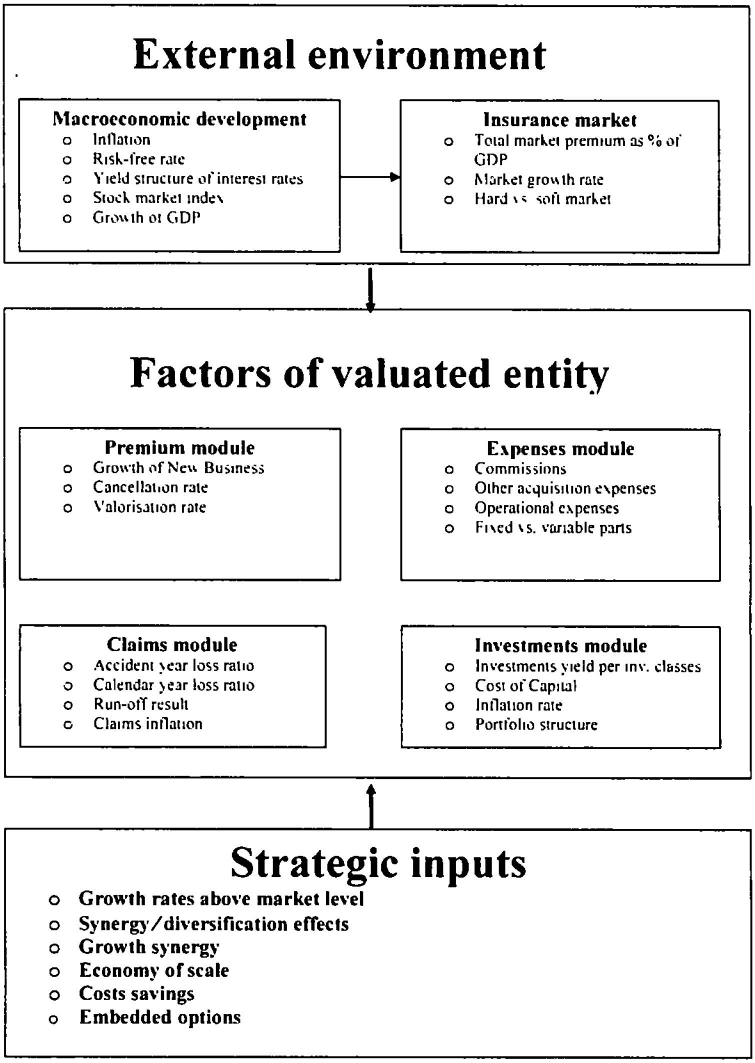


Figure 2: An Example of Interdependencies between Variables

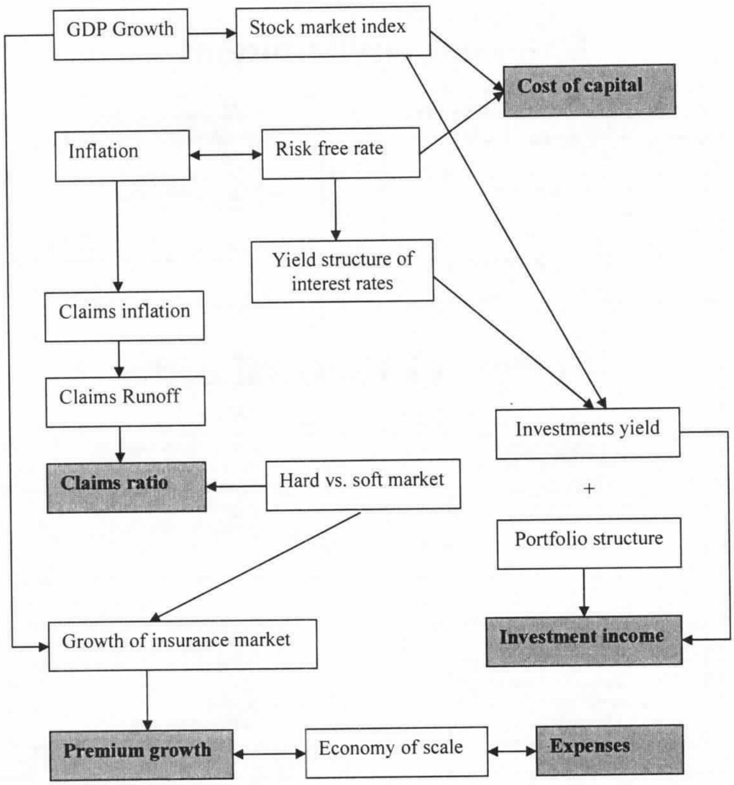
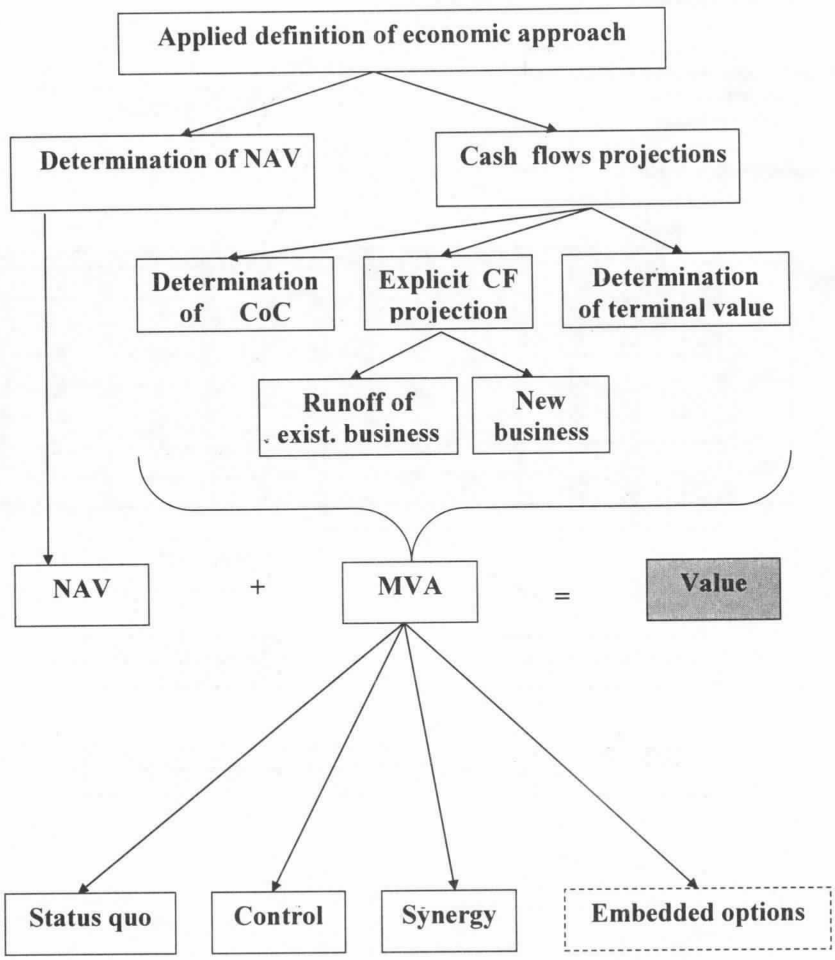


Figure 3: EVA-based Valuation Approach – Value Decomposition



Appendix B:

Value in Dependence on ROE and CoC

Value in dependence on ROE and CoC

- Infinite horizon

- 15-year horizon

Assumption: $NAV_0 = 100$

CoC		ROE 8%	9%	10%	11%	12%	13%	14%	15%	16%	17%	18%	19%	20%
8%	Infinite	100	113	125	138	150	163	175	188	200	213	225	238	250
	15 years	100	109	117	126	134	143	151	160	168	177	186	194	203
9%	Infinite	89	100	111	122	133	144	156	167	178	189	200	211	222
	15 years	92	100	108	116	124	132	140	148	156	164	173	181	189
10%	Infinite	80	90	100	110	120	130	140	150	160	170	180	190	200
	15 years	85	92	100	108	115	123	130	138	146	153	161	168	176
11%	Infinite	73	82	91	100	109	118	127	136	145	155	164	173	182
	15 years	78	86	93	100	107	114	122	129	136	143	150	158	165
12%	Infinite	67	75	83	92	100	108	117	125	133	142	150	158	167
	15 years	73	80	86	93	100	107	114	120	127	134	141	148	154
13%	Infinite	62	69	77	85	92	100	108	115	123	131	138	146	154
	15 years	68	74	81	87	94	100	106	113	119	126	132	139	145
14%	Infinite	57	64	71	79	86	93	100	107	114	121	129	136	143
	15 years	63	69	75	82	88	94	100	106	112	118	125	131	137
15%	Infinite	53	60	67	73	80	87	93	100	107	113	120	127	133
	15 years	59	65	71	77	82	88	94	100	106	112	118	123	129

Appendix C:
Case study: Acquisition of P/C Insurer from the CEE Region

In the appendix, we extend the paper by a discussion of the acquisition valuation process of a P/C Insurer from the CEE region. It is demonstrated on an illustrative case study. The organization is as follows. In the first section, we identify several specifics to be emphasized by analysts, which the authors find as the most relevant. We draw the appropriate consequences for the given valuation task. Next, we follow the procedure from the chapter 5.2 and illustrate the presented valuation approach on a case study of the acquisition valuation of P/C Insurer from the CEE region.

1. Specifics of the Valuation Process

Although the basic valuation principles are the same whatever insurance company is concerned, there are certainly some particular aspects to be taken into consideration when speaking about emerging markets. Here, we go out from the thesis that a precise identification and analysis of all such specifics is of a crucial importance (prerequisite) for the correct application of the valuation procedure as introduced throughout the paper. Generally, we find the following specifics to be taken into account in respect to the valuation procedure in emerging markets, as represented by the CEE region.

1) Economic environment
First, the changing economic environment is of a key importance. The emerging markets are very often characterized by higher inflation, higher currency fluctuations, instability of balance of payments, budget deficits etc. Any instability must be reflected in the mainly country risk premium as a part of Cost of Capital[21].

2) Legal environment
The legal environment is also developing very quickly in these countries. It concerns of both commercial (e.g. commercial code) and insurance law (e.g. state regulation, new definitions of insurance contract). It is just the stability of the legal and institutional framework (e.g. the level of enforceability of the law) that substantially affects the inflow of foreign investments into emerging markets. Thus, the legal and institutional environment plays an important role in the acquisition strategies.

3) Political risk
This aspect is closely connected to the previous one. It is obvious that the amount of political uncertainty also affects the setting of an appropriate discount rate (CoC).

Remark: With regard to the CEE region we must point out the continuing approaching of the local economies to the EU level (expected access in 2004) in all above-mentioned aspects (economical, legal and political stability)

4) Capital markets
The standard allocation and pricing functions of capital markets is still being established. Capital markets are characterized by several inefficiencies: information asymmetry, lower market liquidity, higher transaction costs, and so on. In addition, the role of stock markets is

[21] As a special case, we can mention the valuation process in the highly inflationary economy

substantially underdeveloped. All these factors affect both the investment imporunities for insurance companies and the valuation of an investments portfolio.

5) Underdeveloped insurance markets
The insurance markets are underdeveloped in the most emerging economies, offering good perspectives for high growth in the future (e.g. aggressive acquisition policy). This aspect represents very often the leading acquisition motive. The product mix is usually characterized by less amount of liability coverage as compared with property lines (predominantly car hull). We must also consider the level of market competitiveness. There can be several market imperfections, whereby information asymmetry plays a substantial role The market concentration can be also quite large with a few "big market players"

6) Regulation of insurance markets
The state regulation is characterized by ongoing standardization (proficiency of the regulatory staff, regulation standards etc.)

7) Efficiency of P/C Insurance companies
Generally, the operational part of P/C Insurers is distinguished by less efficiency That is the reason, why the acquisition is usually connected with several necessary restructuring steps (costs cuttings, reduction of staff etc.). Consequently, the combined ratio consists of higher portion of expense ratio at the expense of claims ratio than it is usual in the developed markets. Therefore, it can be a crucial issue for the cash flow projection to correctly estimate the speed of the standardization process (decreasing expense ratio + increasing claims ratio).

8) Accounting standards
Usually, the credibility of information provided by accounting is supposed to be lower than in the developed economies. The analysts can very often meet with very creative bookkeeping what makes their task more difficult.

9) Quality of data for cash flow modeling
The analysts can also face with problems concerning the access to correct and reliable data for the cash flow projection. The historical time series, if available, are very often spoiled due to all the changes in the external environment. Therefore, the information sources must focus rather on an ex-ante (expectation-based) approach than on the analysis of historical data.

In summary, the acquisition valuation process in emerging markets can be characterized by higher uncertainty in all significant variables (external environment, insurance market, company's specifics). On the other hand, there are large growth perspectives and other strategic opportunities.

The overview of all relevant factors including the corresponding consequences for valuation procedure is recapitulated on the following table (Figure 9). Some of them will be directly considered in the presented case study

	Factor	Consequence for valuation
1.	**External environment**	
1.1	Higher fluctuation of key macroeconomic variables	CoC determination:.eventually reflection of inflationary environment

1.2.	Institutional and legal framework	CoC determination
1.3.	Continuing approaching to the EU	Increasing economical and legal stability ⇒ standardization of respective inputs
2.	**Insurance market**	
2.1.	Underdeveloped ins. market	Higher growth rates to be expected
2.2.	State regulation (increasing requirements)	Uncertainty concerning strengthening (changing) rules
2.3	Competitiveness (higher concentration)	Reflection of the market power
3.	**Insurance companies**	
3.1.	Lower operational efficiency	E.g. cost cuttings
3.2.	"Ratios standardization"	Higher costs ratio vs. relatively low claims ratio; to correctly reflect convergence to more standard levels
3.3.	Product mix (dominance of property products)	Taking into account large risk exposures against one product (e.g. car hull)
3.4.	Creditworthiness of accounting	NAV determination
	- hidden liabilities	Analysis of reserves adequacy
	- "artificial" business volume	Profitability analysis of volume-driven business
	- creative bookkeeping	Cleaning of balance sheet
3.5	Reliability and quality of internal data	CF modeling must be based rather than on historical experience on the analysts' expectations

Figure 9: Overview of relevant specifics

2. Case study

Now, we explore the case study of an acquisition of a P/C Insurer from the CEE region[1]. In the first part, we begin with the strategic part where we define the outgoing strategic assumptions. They create a general framework for the consecutive valuation modeling (actuarial / financial part).
The key emphasis is placed on the proper application of key assumptions, the correct reflection all the specifics and the appropriate interpretation of results. We concentrate only on several selected issues (economic adjustments by NAV determination, base case scenario, sensitivity analysis etc.), whereby some aspects will be omitted or assumed as given. By no means, it represents a comprehensive valuation study.

2.1. Strategic Part

Let us assume that a foreign insurance company is interested in entry into an insurance market from the CEE region through the acquisition of already established company. The management of the acquiring company specifies the following set of strategic inputs, which corresponds to the long-term expectations. The inputs provided by management can be divided into the following areas.

1. Economic environment

[1] We do not specify any particular country. However, there can be identified several links to Prague.

Because of the general convergence towards the EU economic environment, we expect a stable economic environment in the long term (GDP growth above the EU level, price stability, etc.).

2. Insurance market

The growth opportunities of the insurance market are the leading motives behind the acquisition. According to the management, we can go out from the assumption that the insurance penetration (market premium as % of GDP) should reach the current level of the EU of 3% in the long term.

3. Company specific inputs

Growth: The strategic target is to increase the current market share of 8,5% (e.o. 2001) to 15% in 15 years. More concretely, we expect the high-growth period during the first 5 years. After that, we suppose the decreasing positive gap between the company and market growth rates. It leads to the market share stabilization in the long run.

Product mix: Slightly increasing share of liability products is assumed

Operational costs: The substantial improvements in the operational part of the business are expected in the middle term (costs savings, higher operational efficiency).

Economy of scale: Both the growth above the market level and the increasing operational efficiency will have positive impacts on the economy of scale. Whereby, the strategic target is to push down the current expense ratio of 33% to the desired level of 25% in the long term.

Claims development: The hardening market competition leads to the increasing claims ratio in the long run up to 75-77%.

In addition, management addresses the following issues:

1) What are the lower and upper boundaries for the acquisition value (serving as a base for the negotiations)?
2) What are the key value drivers?
3) What are the main uncertainties and risks inherent in the acquisition?
4) What happens if the strategic assumptions about market and company growth will not be accomplished?
5) How would look like the worst-case scenario?

2.2. Valuation Modeling (Actuarial / Financial Part)

Based on the set of strategic assumptions, the actuaries and financial analysts must construct the appropriate valuation model. That will be based on the EVA-based valuation approach in the scope as discussed in the paper. We assume the following technical restrictions:

1) Task: to determine the value as of the end of 2001
2) Deterministic modeling
3) We model the product mix as one portfolio (product)
4) We do not consider any reinsurance (gross = net)
5) Explicitly projected period of 15 years (2002 – 2016)

6) Currency: in Mio of local currency
7) We concentrate exclusively on the economic approach to valuation (for simplicity no reserves discounting)

We deal with the following valuation steps in order to provide management with the sufficient support for the negotiations.

1) Construction of the base case scenario
2) Determination of NAV
3) Cash flow projection
4) Identification of key value drivers (sensitivity analysis)
5) Applications of other valuation methods (relative valuation)
6) Construction of the worst-case scenario
7) Summary

2.2.1. Base Case Scenario

Under the base case scenario we understand the combination of the most probable valuation inputs concerning expectations about the future development. It serves as an outgoing base for sensitivity testing. Following equation (5.9), we begin with the determination of NAV. After that, we discuss cash flow projection.

2.2.1.1. Determination of NAV

Let us assume the opening balance sheet according to CAS as of the end of 2001, as illustrated on Figure 10. Our task is to make the appropriate adjustments[22] to get the amount of NAV according to the economic approach.

	Balance sheet	2001		
		CAS	Adjustments	Economic approach
	Assets			
1	**Intangible (ex Goodwill)**	50	0	50
2	**Goodwill**	150	-150	0
3	**Investments**	6 213	93	6 306
3.1	Real estate	246	-57	189
3.2	Investment in aff Enterprises	0	0	0
3.3	Investments held to maturity	3 494	100	3 594
3.4	Investments available for sales	1 842	50	1 892
3.5	Investments tradable	631	0	631
3.6	Others	0	0	0
4	**Receivables**	692	-250	442
4.1	on direct insurance	650	-250	400
4.2	on reinsurance business	2	0	2
4.3	Others	40	0	40

[22] The here presented overview can be understood as a representative sample of adjustments the analysts can face with when valuing a P/C Insurance company from the CEE region. In no case, it covers a comprehensive listing

5	**DAC**	**398**	**-100**	**298**
6	**Other assets**	**146**	**-50**	**96**
7	**Deferred tax assets**	0	204	204
	Total	**7 648**	**-253**	**7 395**

	Liabilities			
1	**Net asset value (Equity)**	**1 877**	**-453**	**1 424**
1.1	Paid in Capital	1 000	0	1 000
1.2	Retained earnings	730	-453	277
1.3	Unappropriated profit/accumulated losses	0	0	0
1.4	Profit / loss of the current year	147	0	147
2+3+4	**Liabilities**	**5 771**	**200**	**5 971**
2	**Technical provisions**	**5 103**	**200**	**5 303**
2 1	Unearned premiums reserves	2 189	100	2 289
2.2	Claim reserves	2 714	300	3 014
2 3	Equalisation reserve	200	-200	0
2.4	Bonus Reserve	0	0	0
2.5	Other underwriting fund and provisions	0	0	0
3	**Other provisions/liability**	**668**	**0**	**668**
4	**Deferred tax liabilities**	**0**	**0**	**0**
	Total	**7 648**	**-253**	**7 395**

Figure 10: Determination of NAV - from statutory accounting to economic approach

Explanation to adjustments

Assets:

Row 2: Elimination of goodwill (goodwill will be considered as a part of MVA within cash flow projection).

Row 3.1: Real estate was overestimated according to statutory accounting (best estimate).

Row 3.3: Since fixed income was carried at purchase values net of unrealized losses, we proceed the revaluation following market prices (hidden reserves).

Row 3.4: Since fixed income was carried at purchase values net of unrealized losses, we proceed the revaluation following market prices (hidden reserves)

Row 4.1: The company holds a large amount of outstanding receivables from direct insurance. Whereby the analysts are convinced (based on the credit risk analysis) that the created accounting adjustments to receivables are substantially underestimated. This adjustment decreases net amount of receivables.

Row 5: The company capitalized some items of marketing expenses in the past. Since no substantial impact on future business has been proved, we exclude these items from DAC and charge them directly against NAV.

Row 6: This adjustment corresponds to the balance sheet cleaning (deferrals etc.).

Liabilities:

Row 2.1: Reflection of premium deficiencies. There are still many unprofitable policies in the company portfolio as a result of the former strategy pushing the business volume at the expense of profitability.

Row 2.2: Best estimate of claims reserves (for the sake of simplicity no reserves discounting). The analysis of claims reserves adequacy has revealed some deficiencies.

Row 2.3: Elimination of equalization reserve (according to economic approach does not represent a particular liability).

Tax effects:
Row 7: Tax impacts from all above listed adjustments (statutory tax rate of 31% is applied).

NAV change:
Row 1.4: If we sum all the adjustments including the tax impacts, then we get the amount of change in NAV.

Conclusion:
Following the economic approach, we determined the amount of **NAV at 1 424 Mio** as of the end of 2001. What is important, this amount is sufficiently above statutory solvency requirements (above 30% of written premium as compared with statutory requirement of approx. 22%) and in line with risk capital concept (additional assumption). Therefore, no capital injection is necessary. Although we were very conservative concerning some items (e.g. deferrals, cleaning of balance sheet, premium deficiencies), there is still some amount of uncertainty left:

- The appropriateness of adjustments to receivables (no data for a reliable analysis available)
- Claims reserves adequacy (too short time series in order to proceed a more reliable analysis of claims reserves adequacy)

If we considered these uncertainties, than the worst case scenario would drop the estimated NAV by additional cca. 300 mio.

2.2.1.2. Cash Flow Modeling

Based on the set of strategic assumptions, we can summarize the following inputs to cash flow modeling, which determine the base case scenario.

A. Macroeconomic factors
Since the given economy has reached the stability in all relevant economic variables recently (inflation rate, interest rates and so on), we do not expect any dramatic movements in this respect. That is the reason, why we suppose key macroeconomic variables to be stable in the long term, as follows:
Real growth of GDP 3,50%,
Inflation rate 3,00%,
Risk free rate 4,00%.

B. Industry-specific factors
The projected growth of the insurance penetration (market premium as a % of GDP) is supposed to reach the current level of the EU of 3% in the long term (in 15 years). Whereby the spread over time is assumed to be linear. In addition, we expect the continuing trends towards the higher industry efficiency (decreasing expense ratios, increasing claims ratios).

C. Company-specific inputs
Analyzing all the company-specific factors, we determine the following trends in key variables.

C.1. Premium module

Key Inputs	2002	2003	2004	2005	2006
Growth rate of Nr of New business	12,0%	20,0%	16,0%	12,8%	10,2%

Commentary: Starting with 2003, we expect a substantial increase of Nr of New business with a declining tendency. From 2007, we suppose stable growth of 7,5%

Key outputs					
Growth rate of premium	9,5%	11,5%	13,2%	13,8%	13,4%
Market share	8,8%	9,0%	9,4%	9,9%	10,4%

Commentary: From 2007, we expect the decreasing growth rate (=decreasing gap between market and company growth rate). The market share is developing accordingly

C.2. Expenses module

Key inputs	2002	2003	2004	2005	2006
Commission rate as % of premium	13,0%	12,7%	12,4%	12,0%	11,7%
Other acquisition expenses as % of premium	7,0%	6,8%	6,7%	6,5%	6,3%
Improvements in operational efficiency (costs savings in Mio)	-14	-32	-32	-9	-5

Operational expenses - variable part (in 2001. 252 mio) grow in line with premium

Operational expenses - fixed part (in 2001: 342 mio) grow in line with inflation rate + 1,5%

Commentary: Higher transparency in commission schemes should push down the overall acquisition costs ratio to 18% in the long run. Furthermore, we suppose substantial costs savings in overheads (as compared with the outgoing base of 2001) due to improved operational efficiency during the first 5 years.

Key output					
Expense ratio	33,3%	32,0%	30,6%	29,6%	28,6%

Commentary: The projected expense ratio is expected to dramatically decrease at the beginning of the projected period because of both costs savings and the effects of economy of scale. The development aims at reaching the strategic target of 25% in the long term.

C.3. Claims module

	2002	2003	2004	2005	2006
Accident year claims ratio	70,0%	70,0%	70,0%	70,0%	70,0%

Commentary: Since the insurance market is not still efficient enough, we find projected claims ratio of 70% by 2006 as conservative estimation. From 2007, we expect gradual increase up to 77% in the long term. Furthermore, we assume the shifts in payoff pattern (longer time delay) because of increasing portion of liability coverage

C.4. Investments module

Key inputs	2002	2003	2004	2005	2006
Risk free rate	4,7%	4,8%	4,5%	4,0%	4,0%
Beta	100,0%	100,0%	100,0%	100,0%	100,0%

Market risk premium	5,5%	5,5%	5,5%	5,5%	5,5%
CoC	**10,2%**	**10,3%**	**10,0%**	**9,5%**	**9,5%**
Investment yield	6,8%	6,9%	6,6%	6,1%	6,1%

Commentary: Concerning the determination of CoC, its first component - risk free rate is determined by YTM on government bond with 10y maturity. The risk free rate includes country risk. Since there is no reliable market information (from stock market) for determination of risk premium, we must rely on the analysts' assumptions (derived from standard markets). The total investment yield is determined as the weighted average of portfolio structure and the corresponding yields on different asset classes (derived from risk free rate + appropriate risk and term premiums). From 2007, we keep the same assumptions as in 2006.

Figure 11: Overview of company-specific inputs

2.2.1.3. Valuation Outputs

Incorporating all these inputs into the valuation model, we get the P&L statement over the projected period (see Figure 12). You can see the profitability development at the bottom of the table, as measured by the gap between ROE and CoC. At the first glance, the development seems to be reasonable. The increasing profitability over the first 5 years corresponds to both successful restructuring steps (cost cuttings etc.) and effects of economy of scale. The rest of explicitly projected period is affected be the tightening insurance market (increasing claims ratio), pushing down ROE slightly above the rate of CoC in the long term. That is in line with the management's assumption that the company cannot beat the market continuously in the long run (efficient market hypothesis). Following this thesis, the determination of terminal value in the base case scenario go out from the assumption ROE = CoC, what implies terminal value to be zero.

P&L Statement	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Premiums written	5 011	5 586	6 338	7 199	8 165	9 233	10 415	11 724	13 177	14 789	16 581	18 573	20 789	23 256	26 000
Change in unearned premium reserves	277	285	370	435	483	534	591	655	720	806	896	996	1 106	1 233	1 373
Premiums earned	4 734	5 299	5 968	6 764	7 682	8 699	9 824	11 069	12 457	13 983	15 685	17 577	19 681	22 022	24 626
Claims paid accident year	1 991	2 176	2 419	2 715	3 047	3 444	3 881	4 363	4 894	5 481	6 130	6 848	7 636	8 513	9 480
Claims reserves accident year	1 365	1 530	1 752	2 020	2 330	2 706	3 133	3 616	4 169	4 796	5 509	6 314	7 241	8 289	9 480
Claims paid previous years	1 080	1 175	1 304	1 451	1 591	1 794	2 036	2 318	2 643	3 027	3 463	3 959	4 521	5 159	5 881
Change in claim reserves previous years	1 080	1 175	1 284	1 451	1 591	1 794	2 036	2 318	2 643	3 027	3 463	3 959	4 521	5 159	5 881
Claims Run off	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Commissions	651	708	782	866	956	1 071	1 212	1 360	1 525	1 706	1 906	2 131	2 378	2 653	2 959
Change DAC	28	28	37	42	45	55	67	74	83	91	101	112	124	137	152
Commissions incl. DAC	623	680	745	824	911	1 016	1 145	1 286	1 442	1 615	1 807	2 019	2 254	2 516	2 807
Other acquisition expenses	351	361	421	466	514	580	652	732	821	919	1 027	1 147	1 281	1 438	1 593
Operational expenses	620	635	669	711	772	844	927	1 008	1 103	1 202	1 321	1 447	1 586	1 739	1 908
Total expenses	1 594	1 696	1 834	2 001	2 197	2 440	2 719	3 021	3 366	3 741	4 155	4 611	5 121	5 693	6 308
Underwriting result	146	107	37	28	106	109	90	62	20	35	108	201	314	484	642
Other technical result	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
Investment income	378	423	454	477	546	627	723	834	963	1 110	1 279	1 471	1 691	1 941	2 226
Insurance result	222	317	417	506	653	736	813	896	983	1 075	1 171	1 270	1 372	1 477	1 586
Other income - expenses	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3
Total profit before tax	222	317	417	506	653	736	813	896	980	1 075	1 171	1 270	1 372	1 477	1 586
Taxes	69	98	129	157	203	228	252	278	305	333	363	394	425	458	492
Total profit after tax	153	218	288	348	451	508	561	618	678	742	808	876	947	1 019	1 094
EVA	2	47	96	116	200	212	214	215	214	210	202	198	174	154	178
Period	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Discounted EVA	2	38	72	94	127	123	117	104	94	85	71	64	54	43	33
Sum of discounted EVA (MVA)	2	40	112	206	333	456	573	674	768	851	922	991	1 045	1 088	1 121

Profitability	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
NAV b.o.p	1 406	1 560	1 778	2 066	2 414	2 865	3 373	3 934	4 552	5 231	5 973	6 780	7 657	8 603	9 623
NAV e.o.p	1 560	1 778	2 066	2 414	2 865	3 373	3 934	4 552	5 231	5 973	6 780	7 657	8 603	9 623	10 717
ROE	10,3%	13,1%	15,8%	15,6%	17,1%	16,3%	15,4%	14,6%	13,9%	13,2%	12,7%	12,1%	11,6%	11,2%	10,8%
CoC	10,2%	10,3%	10,8%	9,5%	9,5%	9,5%	9,5%	9,5%	9,5%	9,5%	9,5%	9,5%	9,5%	9,5%	9,5%
Gap ROE vs COC	0,1%	2,8%	5,0%	6,1%	7,6%	6,8%	5,9%	5,1%	4,4%	3,7%	3,2%	2,6%	2,1%	1,7%	1,3%

Value = NAV + MVA	2 527
Net Asset Value (NAV)	1 406
Market Value Added (MVA)	1 121
- explicit cash flows modelling (2002-2016)	1 121
- terminal value	0
- - ROE	9,5%
- CoC	9,5%

Figure 12: Cash flow projection – P&L

A good insight into the value adequacy can provide the value decomposition into its two main components: NAV and MVA (see Figure 13)

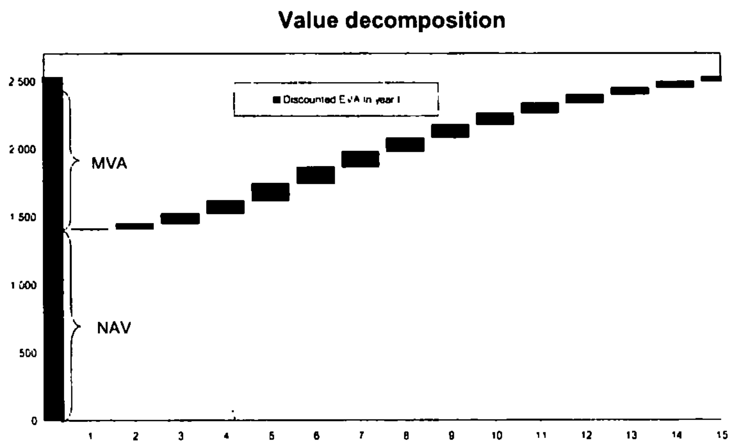


Figure 13: Value decomposition

The share of MVA on the total value accounts for less than 50%. That is based on the assumption that terminal value equals zero (ROE = CoC). What happens if we relax this assumption illustrates Figure 14 (sensitivity of terminal value on the gap between ROE and CoC).

Value = NAV + MVA	**2 527**	**2 672**	**2 816**	**2 961**	**3 105**	**3 250**
% change in Value	0,0%	5,7%	11,4%	17,2%	22,9%	28,6%
Net Asset Value (NAV)	1 406	1 406	1 406	1 406	1 406	1 406
Market Value Added (MVA)	1 121	1 265	1 410	1 554	1 699	1 844
% change in MVA	0,0%	12,9%	25,8%	38,7%	51,6%	64,5%
- explicit cash flows modelling (2002-2016)	1 121	1 121	1 121	1 121	1 121	1 121
- terminal value	0	145	289	434	578	723
- - ROE	9,5%	10,0%	10,5%	11,0%	11,5%	12,0%
- - CoC	9,5%	9,5%	9,5%	9,5%	9,5%	9,5%
- - gap ROE vs. CoC	0,0%	0,5%	1,0%	1,5%	2,0%	2,5%

Figure 14: Terminal value determination

The ROE exceeding CoC by 1% leads to an increase of MVA by more than 25% (2% would lead to an increase of above 50%). Therefore, the procedure how the terminal value is determined critically influences the estimated value.

To complete our analysis of base case scenario, it is also useful to deal with key variables and ratios, as presented on Figure 15.

Overview of key variables and ratios	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
A) Premium															
Growth rate of premium	9.5%	11.6%	13.2%	13.8%	13.4%	13.1%	12.8%	12.6%	12.4%	12.2%	12.1%	12.0%	11.9%	11.9%	11.8%
Growth rate of no. of new business	12.0%	20.0%	16.0%	12.8%	10.2%	7.5%	7.5%	7.5%	7.5%	7.5%	7.5%	7.5%	7.5%	7.5%	7.5%
Growth rate of avg premium of new business	5.0%	4.6%	4.8%	4.4%	3.8%	3.6%	3.0%	3.8%	3.6%	3.6%	3.8%	3.0%	3.6%	3.6%	3.0%
Cancellation rate (portfolio outflow)	10.0%	10.0%	10.0%	10.0%	10.0%	10.0%	10.0%	10.0%	10.0%	10.0%	10.0%	10.0%	10.0%	10.0%	10.0%
Inflation rate of portfolio as of beginning of period	4.0%	3.7%	3.8%	3.5%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%
Market share	8.8%	9.0%	9.4%	9.9%	10.4%	10.9%	11.3%	11.8%	12.3%	12.8%	13.3%	13.7%	14.2%	14.7%	15.3%
Market growth rate	9.1%	9.2%	9.1%	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%	8.0%
Diff ratio Growth above market level	0.4%	2.3%	4.2%	6.7%	5.4%	5.1%	4.8%	4.5%	4.4%	4.2%	4.1%	4.0%	3.9%	3.8%	3.8%
B) Expenses															
Total expense ratio (incl. DAC) as % of e	33.3%	32.0%	30.6%	29.6%	28.6%	28.1%	27.7%	27.3%	27.0%	26.8%	26.5%	26.2%	26.0%	25.8%	25.6%
Commission rate as % of premium	13.0%	12.6%	12.5%	12.2%	11.9%	11.7%	11.7%	11.6%	11.6%	11.6%	11.5%	11.5%	11.5%	11.4%	11.4%
Other acquisition expenses as % of premium	7.3%	7.2%	7.1%	6.9%	6.7%	6.7%	6.6%	6.6%	6.6%	6.6%	6.5%	6.5%	6.5%	6.5%	6.5%
Variable part of operational expenses	44.4%	48.4%	52.8%	55.7%	58.2%	60.2%	62.1%	64.0%	65.7%	67.4%	69.0%	70.6%	72.1%	73.6%	74.5%
Operational expenses as % of premium	12.9%	12.0%	11.1%	10.5%	10.1%	9.7%	9.4%	9.1%	8.8%	8.6%	8.4%	8.2%	8.1%	7.9%	7.7%
Growth rate of total expenses	8.3%	6.4%	7.5%	9.7%	9.8%	11.1%	11.4%	11.3%	11.2%	11.1%	11.1%	11.0%	11.0%	11.0%	11.0%
Diff ratio growth rate of premium - growth rate of total expenses	1.1%	5.1%	5.7%	4.1%	3.6%	2.0%	1.4%	1.3%	1.2%	1.1%	1.0%	1.0%	0.9%	0.9%	0.8%
C) Claims															
Accident year claims ratio	70.0%	70.0%	70.0%	70.0%	70.0%	70.7%	71.4%	72.1%	72.8%	73.5%	74.2%	74.9%	75.6%	76.3%	77.0%
Calender year claims ratio	70.0%	70.0%	70.0%	70.0%	70.0%	70.7%	71.4%	72.1%	72.8%	73.5%	74.2%	74.9%	75.6%	76.3%	77.0%
Combined ratio	103.3%	102.0%	100.6%	99.6%	98.6%	98.8%	99.1%	99.4%	99.8%	100.3%	100.7%	101.1%	101.6%	102.1%	102.6%
D) Investments															
Risk free rate	4.7%	4.8%	4.5%	4.0%	4.0%	4.0%	4.0%	4.0%	4.0%	4.0%	4.0%	4.0%	4.0%	4.0%	4.0%
Inflation rate	3.7%	3.8%	3.5%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%
Investments yield	6.8%	6.9%	6.6%	6.1%	6.1%	6.1%	6.1%	6.1%	6.1%	6.1%	6.1%	6.1%	6.1%	6.1%	6.1%
Investments result in % of earned premi	7.9%	8.0%	7.6%	7.0%	7.1%	7.2%	7.4%	7.5%	7.7%	7.9%	8.2%	8.4%	8.6%	8.8%	9.0%
Asset leverage	138.0%	138.7%	139.7%	141.4%	144.6%	148.4%	152.6%	156.8%	161.1%	165.4%	169.5%	173.6%	177.4%	181.1%	184.8%
CoC	10.2%	10.3%	10.0%	9.5%	9.6%	9.5%	9.6%	9.5%	9.5%	9.5%	9.6%	9.6%	9.6%	9.6%	9.5%
E) Overall result															
ROE	10.3%	13.1%	15.0%	15.6%	17.1%	16.3%	15.4%	14.6%	13.9%	13.2%	12.7%	12.1%	11.6%	11.2%	10.8%
Diff ratio ROE vs COC	0.1%	2.8%	5.0%	6.1%	7.6%	6.8%	5.9%	5.1%	4.4%	3.7%	3.2%	2.6%	2.1%	1.7%	1.3%

Figure 15: Overview of key variables and ratios

The ratios are divided into the same subcategories as valuation inputs. For instance, we can test the adequacy of premium growth as a difference between market and company growth rates. The peak is reached in 2005 and after that we follow the steady decline leading to a stabilization of market share in the long run. Concerning the ratios related to expenses, we emphasize, besides expense ratio (the strategic target of 25%), the difference ratio between growth rate of premium and growth rate of total expenses. It can be easily interpreted as the contribution of economy of scale. Combined ratio is over the whole projected period around 100%. Its development is determined by steadily decreasing expense ratio and by increasing (from 2007) claims ratio up to 77% in the long run. The changing product mix is partly considered in the increasing asset leverage (because of increasing reserves ratio). It leads to an increasing importance of investment result as measured as % of earned premium. The profitability development has been already discussed above.

2.2.1.4. Sensitivity Analysis

To identify key value drivers, it is worth preceding sensitivity analysis with respect to most important underlying valuation assumptions.

1. Premium growth rate (economy of scale)

The expected growth was deduced from both the expected growth of the whole insurance market (top-down approach) and the company growth above market level. The magnitude of growth has a substantial effect on the achieved economy of scale (through the connection to the fixed part of operational expenses).

Change in Premium growth rate				
Change	Value	Change in %	MVA	Change in %
0,0%	2 527	0,0%	1 121	0,0%
1,0%	2 756	9,0%	1 349	20,4%
2,0%	2 995	18,5%	1 588	41,7%
3,0%	3 245	28,4%	1 839	64,1%
4,0%	3 507	38,8%	2 101	87,4%
5,0%	3 781	49,6%	2 375	111,9%
-1,0%	2 309	-8,6%	902	-19,5%
-2,0%	2 100	-16,9%	694	-38,1%
-3,0%	1 901	-24,8%	494	-55,9%
-4,0%	1 710	-32,3%	304	-72,9%
-5,0%	1 529	-39,5%	122	-89,1%

The interpretation of results is straightforward:

- If the annual growth were by 2% higher over the whole projected period, the estimated MVA would be higher by 41,7%.
- Or alternatively, the less realistic assumption of higher annual premium growth by 5%, corresponding to market share above 25% in 2016, would imply the value larger by 50%.
- On the other hand, if we supposed the stable market share (= the annual growth lower by 4%), the MVA would drop by approximately ¾ (compare with the worst case scenario 2.2.2).

The projected premium growth and the level of achieved economy of scale substantially impact the estimated value. However, the performed analysis is based on the assumption other things being equal. It means that we neglect some interdependencies (e.g. higher growth would probable imply more aggressive acquisition and underwriting policy).

Now, we explore two variable components of combined ratio: claims ratio, and commission and other acquisition expenses ratio. As previously, we suppose no effects on other parameters (other things being equal).

2. Claims ratio

Change in Claims ratio				
Change	Value	Change in %	MVA	Change in %
0,0%	2 527	0,0%	1 121	0,0%
0,5%	2 388	-5,5%	981	-12,4%
1,0%	2 248	-11,0%	842	-24,9%
1,5%	2 109	-16,6%	702	-37,3%
2,0%	1 969	-22,1%	563	-49,8%
-0,5%	2 666	5,5%	1 260	12,4%
-1,0%	2 806	11,0%	1 400	24,9%
-1,5%	2 945	16,6%	1 539	37,3%
-2,0%	3 085	22,1%	1 679	49,8%

3. Commission and other acquisition expenses ratio

Change in Commission and other acquisition expenses ratio				
Change	Value	Change in %	MVA	Change in %
0.0%	2 527	0,0%	1 121	0,0%
0.5%	2 376	-6,0%	969	-13,5%
1.0%	2 224	-12,0%	818	-27,0%
1,5%	2 073	-18,0%	667	-40,5%
2,0%	1 922	-24,0%	515	-54,0%
-0,5%	2 678	6,0%	1 272	13,5%
-1,0%	2 830	12,0%	1 423	27,0%
-1,5%	2 981	18,0%	1 575	40,5%
-2.0%	3 132	24,0%	1 726	54,0%

It is clear that the valuation outputs are critically sensitive on the variable components of combined ratio. If we compare the sensitivity of both claims and commission and other acquisition expenses ratio, changes in claims ratio have smaller impacts due to compensation on the investments side (higher claims ratio increases reserves and thus investments income).

4. Costs of capital

The determination of an appropriate discount rate is of the central importance in any DCF (EVA) based valuation approach. Mainly in the developing economies we can face with the problem that there is a large amount of uncertainty concerning the specification of an adequate risk premium. That is the reason, why we explore the sensitivity on CoC (more specifically: risk premium), without taking into account other interdependences (e.g. investment yield etc.).

Change in Cost of Capital				
Change	Value	Change in %	MVA	Change in %
0,0%	2 527	0,0%	1 121	0,0%
0,5%	2 344	-7,3%	937	-16,4%
1,0%	2 173	-14,0%	767	-31,6%
1,5%	2 015	-20,3%	609	-45,7%
2.0%	1 868	-26,1%	462	-58,8%
-0,5%	2 724	7,8%	1 318	17,6%
-1.0%	2 937	16,2%	1 531	36,6%
-1,5%	3 167	25,3%	1 760	57,1%
-2.0%	3 414	35,1%	2 007	79,1%

We supposed that the whole amount of country risk premium is already included in the risk free rate. Next, the risk premium was set in a standard way. What happens, if we assume that risk free rate cannot embrace the total country risk premium, shows the table above. For instance, the 1% increase in risk premium cuts the estimated MVA by one third.

5. Risk free rate

The change in risk free rate affects in our contemplation both CoC and investments yield. The positive effect on investment income is compensated by the opposite effect resulting from costs of holding capital and discounting.

Change in Risk free rate				
Change	Value	Change in %	MVA	Change in %

0,0%	2 527	0,0%	1 121	0,0%
0,5%	2 509	-0,7%	1 102	-1,6%
1,0%	2 483	-1,7%	1 077	-3,9%
1,5%	2 452	-3,0%	1 046	-6,7%
2,0%	2 415	-4,4%	1 009	-10,0%
-0,5%	2 537	0,4%	1 130	0,9%
-1,0%	2 536	0,4%	1 130	0,8%
-1,5%	2 525	-0,1%	1 118	-0,2%
-2,0%	2 500	-1,1%	1 093	-2,4%

You can see that the effect of changed risk free rate is due to mentioned compensation negligible. However, we completely overlook the relationship with inflation rate and claims reserves payoff.

6. Investment yield

Last but not least, we explore the sensitivity on investment yield. We suppose changes in investment yield without any interrelationships and dependencies with other variables (e.g risk free rate, respectively CoC).

Change in Investment yield				
Change	Value	Change in %	MVA	Change in %
0,0%	2 527	0,0%	1 121	0,0%
0,5%	2 703	7,0%	1 297	15,7%
1,0%	2 879	13,9%	1 473	31,4%
1,5%	3 055	20,9%	1 649	47,1%
2,0%	3 231	27,9%	1 825	62,9%
-0,5%	2 351	-7,0%	945	-15,7%
-1,0%	2 175	-13,9%	768	-31,4%
-1,5%	1 999	-20,9%	592	-47,1%
-2,0%	1 822	-27,9%	416	-62,9%

Conclusion:

We find the mainly **growth rates of premium** together with the achieved level of **economy of scale** as the key factors substantially influencing the estimated value. Furthermore, the discount rate (**CoC**) is of a key importance here. The components of profit margin, as represented here by the variable components of combined ratio and the investment yield (no dependencies with CoC assumed), are also very relevant factors. But we dare to believe that under the standard market environment they tend to be more or less determined by the industry environment in the middle term.

2.2.1.5. Application of Relative Valuation (Decomposition of V/P Ratio)

We apply the relative valuation utilizing the analysis of V/P ratio, as discussed in Section 4.4.2, in order to check the reasonability and correctness of the results given by the EVA-based valuation approach. We proceed in the following way. First, we substitute the inputs utilized by the EVA-based valuation approach into the decomposed relative measure (V/P ratio). Next, we compare the resulting decomposed V/P ratio with the EVA-based valuation. We go out from the thesis that if the valuation assumptions are roughly the same, both methods should bring similar results. If it is not the case, there must be some discrepancies in the valuation assumptions.

A) Relative valuation
Following equation (4.7) and incorporating tax rate, we obtain:

V/P = (1 / CoC) * [(1 – combined ratio) + AL*IY)] * (1 - tax rate).

The weighted averages over the 15-year horizon of explicit cash flow modeling are utilized as the respective inputs.

Avg combined ratio	100,9%
Avg. IY	6,2%
Avg. AL	165.9%
Avg CoC	9,5%
Avg. tax rate	31,0%
V/P ratio according to relative valuation	**67,4%**

B) EVA-based valuation
In this case, we relate the estimated value to the expected premium in t = 1.

Premium (expected in t = 1)	5 011
Value	2 527
V/P ratio as a result of EVA based valuation	**50,4%**

While the estimated value according to the EVA-based valuation approach corresponds to V/P ratio of 50,4%, utilizing decomposition we get V/P ratio of 67,4%. Where does the difference come from?

Obviously, there is a difference in the considered time horizon While the decomposition implicitly assumes the infinite horizon (for details see Section 4.4.2), the EVA-based valuation takes into account the only first 15 years of explicit cash flow modeling (terminal value is given at zero). To reach the comparability of both approaches, we must apply the same assumption concerning the infinite horizon. In our case, it means to calculate the terminal value in the EVA-based valuation approach by projecting ROE and CoC into infinite (the average values from cash flow modeling are to be applied).

Avg. Diff. ratio: ROE vs COC	3.3%
Terminal value	959
Value adjusted by terminal value	3 486
Adjusted V/P ratio as a result of EVA based valuation	**69,6%**

Conclusion:
The V/P ratio according to relative valuation yields 67,4%. The corresponding value from the EVA-based valuation approach extended over the infinite horizon (to insure the consistency) gives the very similar result of 69,6%. The comparability of both results confirms that the valuation assumptions were applied in the correct way and there are no logical inconsistencies.

2.2.2. Worst Case Scenario

Although the assumptions in the base case scenario, in the mainly underwriting part, were conservative and realistic enough, the sensitivity analysis has disclosed high dependence on the premium growth rate and the achieved economy of scale (quite strong assumptions about market and company growth). That is the reason, why management puts the question, what happens if the expected growth perspectives will not be accomplished.

To analyze this issue, we construct the so-called **worst-case scenario**. It represents the combination of several adverse but still reasonable assumptions, but in no case it represents a catastrophic scenario (the emphasis on reasonability is here of a key importance).

2.2.2.1. Determination of NAV

Recalling Section 2.2 1.1, there are two main uncertain areas in the determination of NAV: adjustments to receivables and claims reserves adequacy. In the worst case, they could reduce NAV by 300 mio. Thus, NAV is estimated at 1 106 mio, according to the worst case scenario.

2.2.2.2. Cash Flow Modeling

We believe that the worst-case scenario should be primarily investigated with regard to the strategic assumptions. We will consider the following adverse development:

- **Lower growth of the insurance market**
  Within the projected period the insurance market will grow at lower rates.
- **The company's growth in line with the market**
  Fast no increase of the market share is expected. We assume very moderate growth of No. of New business.
- **Portfolio structure**
  The share of liability products will grow more moderately. It implies the shorter payoff pattern and consequently the lower asset leverage as compared with the base case scenario.
- **Costs savings**
  The space for costs savings through the improved operational efficiency will not be achieved to such a large extent as initially expected.
- **Economy of scale**
  The above aspects imply that the large expectations concerning economy of scale will not be fully realized.
- **Other underwriting and investments assumptions**
  All other underwriting and investments (including CoC) assumption will be for the sake of simplicity kept just the same.

To sum up, the presented worst-case scenario focuses primarily on the adverse development on the production and operational side (economy of scale). The other underwriting and investments parameters follow the assumptions from the base scenario. Explanation:

- Just the production and operational expenses (economy of scale) parameters represent key strategic inputs, which determine the main objective behind the given acquisition strategy and therefore should be critically tested under the adverse development
- On the other hand, underwriting and investments parameters are to a large extent determined externally (by insurance markets - hard vs. soft market, competition, and

by financial markets) and are therefore only partly influenceable by management. Thus, they are of a less importance here. To investigate the magnitude of underwriting and investments parameters, we refer to sensitivity analysis (2.2.1.4).

2.2.2.3. Valuation Outputs

P&L Statement	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Premiums written	4 981	5 431	5 941	6 481	7 035	7 612	8 277	8 889	9 597	10 353	11 163	12 000	12 959	13 954	15 071
Change in unearned premium reserves	-202	-225	-256	-270	-273	-291	-310	-331	-354	-376	-405	-433	-464	-499	-530
Premiums earned	4 779	5 206	5 685	6 211	6 754	7 317	7 917	8 558	9 243	9 975	10 758	11 567	12 495	13 455	14 487
Claims paid accident year	1 489	2 148	2 324	2 516	2 710	2 936	3 181	3 439	3 716	4 008	4 321	4 662	5 013	5 354	5 801
Claims reserves accident year	1 350	1 497	1 666	1 800	2 017	2 236	2 412	2 731	3 015	3 324	3 601	4 000	4 430	4 674	5 355
Claims paid previous years	1 080	1 174	1 274	1 386	-1 512	1 648	1 804	-1 996	2 190	2 412	2 653	2 918	-3 207	3 521	-3 865
Change in claim reserves previous years	1 080	1 174	1 274	1 386	1 512	1 648	1 804	1 996	2 190	2 412	2 653	2 918	3 207	3 521	3 865
Claims Run off	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Commissions	648	688	734	779	822	888	957	1 031	1 110	1 195	1 284	1 380	1 482	1 597	1 709
Change in DAC	26	20	23	23	21	30	32	37	39	42	45	46	51	56	58
Commissions incl. DAC	621	668	711	757	801	858	925	994	1 071	1 152	1 239	1 335	1 431	1 537	1 650
Other acquisition expenses	349	371	395	420	443	478	515	558	536	643	632	743	798	852	920
Operational expenses	631	670	714	760	804	852	902	955	1 011	1 071	1 138	1 209	1 284	1 364	1 450
Total expenses	1 601	1 709	1 820	1 936	2 048	2 184	2 339	2 504	2 680	2 867	3 009	3 284	3 514	3 753	4 020
Underwriting result	167	141	116	73	22	-41	75	117	162	221	293	373	464	460	689
Other technical result	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
Investment income	377	416	439	447	473	544	600	661	730	805	886	976	1 077	1 185	1 304
Insurance result	211	271	325	374	471	503	525	545	564	582	594	606	612	616	617
Other income / expenses	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Total profit before tax	211	271	325	375	471	503	525	545	564	582	594	606	612	616	617
Taxes	66	84	101	116	146	156	163	169	175	180	184	188	190	191	191
Total profit after tax	145	187	224	259	325	347	362	376	389	402	410	418	422	425	426
EVA	25	49	63	68	127	117	98	77	54	29	1	11	60	106	115
Period	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Discounted EVA	23	40	52	61	81	68	52	37	24	12	0	11	21	-30	37
Sum of discounted EVA (MVA)	23	63	114	176	257	325	377	414	438	450	450	439	418	388	351

Profitability	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
NAV b.o.p	1 106	1 252	1 439	1 661	1 921	2 247	2 594	2 956	3 332	3 721	4 123	4 533	4 950	5 373	5 798
NAV e.o.p	1 252	1 419	1 661	1 921	2 247	2 594	2 956	3 332	3 721	4 123	4 533	4 950	5 373	5 798	6 223
ROE	12.3%	13.5%	14.4%	14.4%	15.6%	14.3%	13.0%	12.0%	11.0%	10.2%	9.5%	8.8%	8.2%	7.6%	7.1%
CoC	10.2%	10.3%	10.0%	9.5%	9.5%	9.5%	9.5%	9.5%	9.5%	9.5%	9.5%	9.5%	9.5%	9.5%	9.5%
Gap ROE vs. COC	2.1%	3.6%	4.4%	4.9%	6.1%	4.8%	3.5%	2.5%	1.5%	0.7%	0.0%	-0.7%	-1.3%	-1.9%	-2.4%

Value = NAV + MVA	1 457
Net Asset Value (NAV)	1 106
Market Value Added (MVA)	351
- explicit cash flows modelling (2002-2016)	351
terminal value	0
- - ROE	9.5%
- - CoC	9.5%

Figure 16: P&L – the worst case scenario

According to the worst case scenario, MVA drops by more than 2/3 to 351 mio, as compared with the base case scenario. That is explained by lower premium growth than would be otherwise necessary to spread fix costs and to achieve positive contributions of economy of scale. The estimated value of 1457 mio can be understood as the lower boundary for negotiations

2.2.3. Summary

Based on the performed valuation analysis, we can come back to the issues arisen by management and try to give the appropriate recommendation and conclusions.

The base case scenario represents the outgoing base for acquisition negotiations. We do not recommend going too much above this estimated amount. Nevertheless, if we considered some more optimistic assumptions, for instance concerning the determination of terminal value as illustrated on Figure 14, we could arrive at the upper boundary. The lower boundary is determined by the worst case scenario.

The key value drivers:

- Economy of scale
  - Premium growth
    - Growth of the whole insurance market
    - Company growth
  - Improvements in the operational efficiency
- Discount rate as embodied in the assumption about the stable economic and legal environment

This list of value drivers corresponds to the main uncertainties inherent in the valuation:

- The stable economic environment, growth of GDP
- The growth of the insurance market
- The standardization and competitiveness of insurance market
- The uncertainties inherent in the determined NAV
- Company growth perspectives
- Cost cuttings