# A theory of risk for two price market equilibria

Dilip Madan
Department of Finance
Robert H. Smith School of Business

Shaun Wang
Department of Risk Management and Insurance
Georgia State University

Philip Heckman
Heckman Actuarial Consultants Ltd.

**Abstract**

This paper is based on a commissioned research study by the Casualty Actuarial Society with a focus on a theoretical framework for a liquidity risk premium and the interaction of illiquidity with credit effects on the valuation of assets and liabilities. The problem addressed is the development of a theory of liability valuation distinct from a theory of asset pricing or valuation. Our proposed solution is to formulate a risk theory for two price economies when markets fail to converge to the law of one price. The traditional one price economy prices the credit, market and some components of liquidity risk using traditional methods that include exponential tilting in the presence of constant absolute risk aversion utilities. The two price economy on the other hand explicitly models illiquidity issues by developing expressions for bid and ask prices that ensure the acceptability of residual risks. The presence of residual risk and its associated market incompleteness is central to the market's inability to converge to the law of one price in two price economies. The resulting two prices are differentiated from classical linear pricing rules as they are nonlinear functions of the cash flows being priced. Specifically bid prices are concave functions of the claims being priced while ask prices are convex functions. Asset and liability valuation then part company as we propose to employ ask prices for evaluating liabilities while assets are to be priced at bid. Two price economies also provide new hedging objectives for corporations with hedging strategies being designed to economize on the commitment of capital needed to cover residual risks. The static two price theory is extended to its dynamic counterpart using recent developments in the theory of non linear expectations. The dynamic model is in discrete time with a tenor reflecting a time horizon at which it is anticipated that genuine counterparties normally arrive. The longer the tenor, the greater the illiquidity and the greater is the spread between bid and ask prices. The dynamic theory is illustrated on pricing a simple compound Poisson gamma insurance loss process. In this context capital minimization as a hedging objective illustrates the construction of optimal reinsurance points. Financial hedging using the securitization of catastrophic loss exceedances and mortality contingent securities for life risk further illustrate the construction of two prices in the context of capital minimizing hedges. Further it is observed that bid prices are sensitive to changes in credit risk but this is not the case for the ask or liability pricing counterpart. A final empirical section addresses issues of measuring the size of cones of acceptability using data on daily high, low and close prices of publicly traded equity.

# 1 Introduction

There is an extensive empirical literature on measuring liquidity, coupled with a substantive theoretical literature on modeling liquidity or more exactly the lack thereof. Broadly speaking, from the benchmark of a perfectly liquid market, illiquidity is viewed in terms of price impact with buyers exposed to price increases while sellers experience price reductions. In the benchmark liquid market there is unlimited trading in both directions at the same price. There are no price impacts associated either with the direction or the size of transactions. Though extensions of this research could focus on size effects this paper is concerned only with the effects of the trade direction, abstracting away size issues. Attention is thereby focused on two prices, the one at which one is guaranteed a purchase or the ask price and the other the one at which one is guaranteed a sale or the bid price. In effect we contemplate an economy in which most transactions of interest are for products not traded on any exchange, for which one may be able to observe the ask price and or the bid price, but importantly there is no possibility of trading in both directions at any observed transaction price. Every transaction is either near or at the ask or near or at the bid. The economy is a two price economy with some information on the two prices and no information on some hypothetical intermediate price that could be considered a candidate price for the one price of an economy where the law of one price prevails. We explicitly model the equilibrium of a two price economy and study the pricing of risk in such a context.

We recognize that every transaction has a price and it also has a buyer and a seller, so how does one decide whether the price observed in the transaction is an ask price or a bid price. Is it not just the price of trading in both directions given that both directions are taken by someone at this transaction price. One has to know more about the transaction. In particular we see one of the two parties as the one that needs to trade and initiates the transaction while the other party generally dictates the terms. We see the party dictating the terms as representing an abstract market while the party that needs to trade is an economic agent seeking a trade with the market. In liquid markets one may have two parties that need to trade in opposite directions and they cross at a prevailing market price. There is a symmetry in the motivations of the two parties. In two price markets only one of the two parties needs to trade and initiates a transaction with the other party being induced to trade by setting the terms of trade. Our interest is in this economic world of two price markets with the consequent implication for marking to two price markets being that assets are to be marked at bid while liabilities are marked at ask.

The terms at which one may liquidate or cash out a position is the real question at issue. In this regard the position could be an asset with a claim to future state contingent cash flows from counterparties in the economy or it could be a liability or promise to make such payments. For swap type contracts state contingent cash flows in both directions may be involved. When the market is liquid and the law of one price prevails one may observe contracts being traded at a sufficiently high frequency and it matters little whether the position is a claim to cash flows or a promise to make such payments, as there is only one price for initiating a transaction in either direction. A lack of such liquidity arises when genuine transactions in opposite directions fail to cross. Cashing out such claims is still possible but claims to receive cash flows are differentiated from promises to make such payments.

The bid ask spread has been modeled in the theoretical literature in terms of the effects of informed traders on market makers (Copeland and Galai (1983), Easley and O'Hara (1987), Glosten and Milgrom (1985)), and with regard to incorporating the order processing and inventory costs of liquidity providers (Ahimud and Mendelson (1980), Demsetz (1968), Ho and Stoll (1981, 1983) and Stoll (1978)). Numerous statistical studies on the bid ask spread (Roll (1984), Choi, Salandro and Shastri (1988), George, Kaul, and Nimalendran (1991), and Stoll (1989), Huang and Stoll (1997)) consider decomposing the spread into order processing, inventory and adverse selection components. The counterparty earning the spread for a variety of reasons in these models is the market maker whose concerns determine these spreads. We view these studies as essentially modeling the process of price formation in liquid markets with most of the empirical studies focused on the relatively liquid equity markets where one observes many transactions at prices at which a reversal of trade direction could possibly take place without a price effect.

Our interest lies more specifically in pure two price economies trading contracts that are not supported by market makers temporarily taking positions that are ultimately reversed for the spread. The positions taken by both parties are generally held to maturity with the interim risks being hedged as best as possible. Hence the motivations for the spread have less to do with order processing, inventory or information asymmetry. They are related more to finding genuine long term counterparties prepared to transact with some commitment. The two prices prevail in equilibrium as a convergence to one price is interrupted by the lack of supply for demanders and or the lack of demand for suppliers. We model the breakdown in a convergence to the law of one price as a fundamental primitive of the two price economy. In the one price economy a slight increase in an offer price unleashes a large supply just as a slight decrease elicits a large demand.

When the law of one price prevails this price is the one at which demand equals supply. The suppliers produce and the demanders consume, the deal is done, and neither party is concerned about the morrow. However, when all consumption and production is in the morrow in that what is traded are claims and promises to state contingent future cash flows there isn't an equilibrium price at which demand equals supply. There are instead two prices with supply running out at the higher price and demand running out at the lower price. For such a fundamentally two price economy there is an observed equilibrium value for both prices with no observations on bidirectional prices. We therefore study the pricing of risk in a novel two price economy equilibrium. For further details on such equilibria we reference Madan and Schoutens (2011), where competition is modeled by the narrowest spread of ask above bid consistent with the aggregate market clearing. This equilibrium spread is however positive and a zero spread is inconsistent with market clearing for it leaves the market holding an unacceptable set of cash flows for the morrow.

The context of our model two price economy is better suited to the study of risk in an insurance setting as such contracts are generally held to maturity by both parties. The two parties do not trade on an exchange and there is no secondary market on which to trade and discover some nonexistent two way price. There is only a directional price. Since we have a genuine two price economy, we essentially have a separation of an asset pricing theory in our economy from its liability valuation counterpart. Our assets are priced in line with the bid price while our liabilities are priced at ask. We comment on comparing the pricing of risk in our economy with the classical one price economy by using for a candidate single price the mid quote of our two prices.

In our two price economy we take a relatively classical view of markets compatible with their role in traditional competitive analysis where markets serve as counterparties to transactions allowing agents to buy or sell at the going price. Note importantly that classically markets were not optimizing agents endowed with a specific objective function to maximize. The only concern for the Walrasian auctioneer was the clearing of markets. Under additional assumptions on the underlying economy such a competitive equilibrium was also Pareto optimal, though in the presence of externalities it may also not be so. Our only point of departure in this perspective is just that the terms of trade depend on the direction of trade, with the market buying at bid and selling at ask. We therefore present a two price theory with separate equations for these two prices.

In this regard consider first the classical market trading in both directions at the going price. The market then accepts to sell at a higher price or buy at a lower price and essentially takes all random cash flows at zero cost if they have a positive expectation under the equilibrium pricing kernel. These are equivalently the positive alpha trades. This is indeed a very large set of risks that are accepted by the classical market and it is unclear that any real market would be that generous. Further we note that the law of one price when coupled with the absence of arbitrage results in prices being linear functionals on the space of random variables.

In reality many risks are parceled to a variety of clienteles to extract the implicit price differentiation that prevails. Under the law of one price there is no point in slicing and dicing risks as the prices for the components always add up to the price of the aggregate. Yet financial markets in the real world are heavily engaged in creating financial products that precisely partition risks recognizing that sum of the parts is not equal to the value of the aggregate. We therefore seek and present a two price theory consistent with many observable nonlinearities for these two prices. In the real world the terms of trade also vary with the size of positions and prices are more complex than what is modeled by our two price theory. We are therefore just initiating a minimal departure from the classical one price world by only entertaining the dependence of price on trade direction.

Our starting point is the classical theory with its equilibrium pricing kernel.

For specific risks this could coincide with the physical probability measure, for example when the risk is not priced in equilibrium. Classically the market accepts all positive alpha trades and this is too large an acceptance set. The economic primitive that differentiates a two price economy from the classical one price economy is the set of zero cost risks accepted by the market. Classically it is, as already noted, a half space of random variables given by the "so-called" positive alpha trades. We model, instead, the set of zero cost risks acceptable to the market by a much smaller set. The modeling of this set of acceptable risks follows Artzner, Delbaen, Eber and Heath (1999), Carr, Geman and Madan (2001) and was further developed in Cherny and Madan (2009, 2010).

The classical perspective is appropriate for pricing state contingent cash flows being traded at high frequency for if a claim is offered at a lower price one buys, immediately sells and collects the difference as an arbitrage profit. When claims are infrequently traded or are being considered from the perspective of being held to a natural maturity one may buy and then one has to wait till the position is naturally resolved. A positive alpha with respect to a single risk neutral measure is now little comfort. Of course if the cash flow is nonnegative then it has a positive alpha with respect to every measure and this is completely acceptable as it is a risk free position. The reality lies in between and we model the trades that will be done as those that have a positive alpha with respect to a whole host of potential measures, including the equilibrium pricing kernel of the one price market.

In this regard the economic primitive for the two price economy recognizes that nonnegative cash flows offered at zero cost are certainly market acceptable and they do meet trade approval. More generally we model the zero cost cash flows acceptable to the market as a set of random variables that is a convex cone containing the nonnegative random variables. Equivalently one has to specify the additional pricing kernels that are to be used to test for a positive alpha, above and beyond the single pricing kernel associated with the one price economy.

As promised, issues related to the size of the trade are abstracted away by taking the set of zero cost cash flows acceptable to the market to be a cone. For a size impact one would replace the cone by a convex set as is done for example in Madan (2010) when studying execution costs. In this paper we restrict attention to two prices only and hence we model acceptability by a convex cone as opposed to a convex set.

For a description of how such a cone is determined in a two price economic equilibrium we cite Madan and Schoutens (2011). The essential equilibrating principle is to let competitive pressures offer the largest possible cone with the lowest ask price and the highest bid price consistent with the aggregate residual risk held by the market being in some small prespecified cone. The smaller cone for the aggregate risk is the market clearing condition of the economy. Since the market is the counterparty for all trades the market is left with an aggregate residual risk given that what is being traded is state contingent cash flows held to a natural maturity. For the two prices to converge down to one price it is imperative that the market as a counterparty accept all cash flows with a positive expectation or alpha under a single risk neutral test measure for acceptability. This makes the cone very large. When the market is more conservative than this and employs more than one test measure the equilibrium will have two prices and all that is observable are these two prices.

There is another view of the classical market that should be addressed. This is that the market only accepts the identically zero cash flow. Hence what it promises to deliver must be received from elsewhere and the only possibility is a complete clearing of cash flows. Now a little reflection would show that receipts dominating payments would be acceptable and this leads us to the cone of nonnegative cash flows being acceptable that finally results in ask and bid prices being related to super and sub replication. What we argue for is the recognition that nonnegativity for acceptability is too strict and in practical economies some exposure to losses is tolerated. One may envisage the financial system as serving the role of the market with losses being absorbed by this system and in the limit by the lender of last resort. These considerations lead us to depart from the stringent view that only zero or a nonnegative cash flow at zero cost is market acceptable, towards the use of a proper convex cone of random variables strictly containing the nonnegative cash flows.

For the construction of explicit examples of such cones we follow Cherny and Madan (2009, 2010) and employ concave distortions of an underlying risk neutral distribution function for the risk. A simple model of risk acceptability by the market as an abstract counterparty is to base the decision of acceptability on the distribution function for the risk. One then has to decide when a risk with a particular distribution function is acceptable. This is done by ensuring that it has a positive expectation under a concave distortion of the distribution function. The distortion reweights upwards the lower quantiles and discounts the upper quantiles. Needless to say, if a cash flow is nonnegative it will have a positive expectation under all distortions and will therefore be highly acceptable. The losses occur at the lower quantiles and the distorted expectation being positive signals that gains deflated by the distortion still compensate losses exaggerated by the distortion, thereby making the proposed distribution function market acceptable. More complex models may well be developed in subsequent research, by incorporating for example some conditioning information, but our choice is a practical and tractable starting point that illuminates the qualitative issues at stake.

In the static context of a single period two date economy, our two price theory shows how low probability events typically insured against produce mid quotes above the probability of the event even when there is no change of probability to a risk neutral one. In the presence of such a change we present additional illiquidity based components for the risk premia that are additional to the risks that are priced in the change of measure to a risk neutral one. The risk neutral measure could account for credit, volatility and some liquidity components of risk, to which we add an explicit illiquidity component associated with the divergences of our two prices. We thereby develop an explicit and parametrically separated accounting for market, credit, liquidity and illiquidity risk, where the latter is reflected in the residual risk charges built into the two prices of two price markets.

Insurance contracts typically extend over multiple periods and it is important to analyze the two price economy over multiple periods. The two prices, bid and ask are known to be nonlinear and we extend these pricing operators to dynamically consistent nonlinear operators by applying the recently developed theory of nonlinear expectations. In this regard we follow Madan and Schoutens (2010) and apply these methods to the pricing of insurance claims modeled by increasing compound Poisson processes. In this context we study the effects of both maturity and the tenor or interperiod length on the two prices.

Madan and Schoutens (2010) generalize the static theory of bid ask prices developed in Cherny and Madan (2010) to building dynamically consistent sequences of bid ask prices in a discrete time model using the recently developed theory of non linear expectations of Cohen and Elliott (2010). A natural question is the choice of tenor in the discrete time model. This should basically match the frequency at which a number of similar risks are traded in the market. This could be a year, a quarter or as frequent as a day or intraday. What is shown is that as the tenor decreases and the trading frequency increases the spreads narrow and approach the liquid market price of the law of one price once we have trading as frequent as intraday. Madan and Schoutens (2010) analyze fixed income and option contracts that have zero spreads at maturity that rise as we come backwards in time. We apply these methods here to insurance losses with a potentially infinite maturity illustrating the effects of various parameters on the resulting valuation sequences.

The hedging objectives in two price economies turn towards the minimization of ask prices or the maximization of bid prices. Equivalently as suggested in Carr, Madan and Vicente Alvarez (2011) one economizes on capital commitments measured by the difference between the ask and the bid price. We contrast our capital minimization hedging criteria with other classical criteria like variance minimization and or the maximization of expected utility. We also apply these new hedging objectives to illustrate the construction of optimal reinsurance points for contracts insuring losses.

Next we address the financial hedging of insurance losses via securitization. This requires the introduction of financial securities that lock into the experience of realized insurance losses at a future date. We study two examples in this connection. The first is a security that locks into the aggregate level of exceedances in catastrophic losses. The second studies the management of life insurance liabilities in a stochastic mortality setting using options on the future mortality rate.

The use of market data to estimate the magnitude of the various constructs introduced in the theoretical development is addressed in a separate empirical section. We recognize that data from liquid markets where the law of one price prevails is not the appropriate data source for studies focused on the law of two prices. The latter however lacks published data sources. We proxy for the two prices by taking the high and low prices over some horizon. Here we report initially on the use of the daily high and low prices.

The outline of the rest of the paper is as follows. Section 2 presents the theory of risk for two price economies in a static two date one period model. Section 3 presents the parametric separation of market, credit, liquidity and illiquidity risk. The dynamically consistent two price valuation principles are presented and illustrated in Section 4. Section 5 contrasts our new capital minimization criterion with the more classical objectives available in the literature. Section 6 illustrates the construction of optimal reinsurance points that economize on capital commitments. Section 7 takes up financial hedging in the context of catastrophic losses. Section 8 analyses financial hedging of life insurance in the presence of stochastic mortality when options on future mortality also trade. The empirical estimation of theoretical constructs is addressed in Section 9. Section 10 concludes.

# 2 Price of risk in two price economies

This section develops the theory of risk for two price economies in the context of a one period two date model. There are three subsections. The first subsection present the results at a general and abstract level. The abstract formulation is followed by a subsection that employs concave distortions to model the primitives for a two price economy. The third subsection considers the effects of existing positions on valuations out of equilibrium.

## 2.1 Two Price Economy Pricing Kernels

Consider a two date one period economy trading state contingent claims paying cash flows at time 1 with prices determined at time 0. The claims traded are random variables on a probability space $(\Omega, \mathcal{F}, P)$ and we suppose that there are some zero cost claims with payouts $H \in \mathcal{H}$ that trade in a liquid market with the same zero cost for trading in both directions. The class of risk neutral measures is then given by

$$\mathcal{R} = \left\{ Q | Q \sim P \text{ and } E^{Q}\left[H\right] = 0, \text{ all } H \in \mathcal{H} \right\}.$$

We suppose that an equilibrium has selected a base risk neutral measure $Q^0$ and the set of classically acceptable risks is then given by the set of positive alpha trades or the set of random variables

$$\mathcal{A}_c = \left\{ X | X \in L^{\infty}\left(\Omega, \mathcal{F}, P\right), E^{Q^0}[X] \geq 0 \right\}.$$

The definition of $\mathcal{A}_c$ recognizes that the classical market will accept to buy any amount at a price below the going market price and agree to sell any amount at a price above the price given by the risk neutral expectation. We may define by $\Lambda_c$ the change of measure density

$$\Lambda_c = \frac{dQ^0}{dP}$$

and equivalently write that the return $R_X$ on $X$ with positive risk neutral price $\pi(X) = (1+r)^{-1}E^{Q^0}[X] > 0$ for a periodic interest rate of $r$, defined by

$$R_X = \frac{X}{\pi(X)} - 1$$

satisfies the condition that

$$E^P[R_X] - r \geq -cov^P(\Lambda_c, R_X),$$

or we have a positive alpha trade.

The point of departure for two price economies from the classical model is the recognition that the half space $\mathcal{A}_c$ is too large an acceptance set for realistic economies. For two price economies the acceptance set for the market is defined by a smaller convex cone containing the nonnegative random variables. It is shown in Artzner, Delbaen, Eber and Heath (1999) that all such cones are defined by requiring a positive expectation under a set of test measures $Q \in \mathcal{M}$. The set of risks accepted by the market is then

$$\mathcal{A} = \left\{X | X \in L^\infty\left(\Omega, \mathcal{F}, Q^0\right), E^Q[X] \geq 0, \text{ all } Q \in \mathcal{M}\right\},$$

where we suppose that our base measure is $Q^0 \in \mathcal{M}$. Madan and Schoutens (2011) determine the set $\mathcal{A}$ in equilibrium as the largest set consistent with the aggregate risk held by the market being in a prespecified small cone containing the nonnegative random variables.

The two prices for a cash flow $X$ of a two price economy are derived from the market's acceptance cone by requiring that the price less the cash flow for a sale by the market or the other way around for a purchase be market acceptable. Cherny and Madan (2010) show that the unhedged bid and ask prices, with a periodic interest rate of $r$, $b(X)$, $a(X)$ respectively are given by

$$\begin{aligned} b(X) &= (1+r)^{-1} \inf_{Q \in \mathcal{M}} E^Q[X] \\ a(X) &= (1+r)^{-1} \sup_{Q \in \mathcal{M}} E^Q[X]. \end{aligned}$$

Note importantly that the two prices of a two price economy are nonlinear functions on the space of random variables with the bid price being concave while the ask price is convex by virtue of the infimum and supremum operations.

The market may also quote hedged prices as opposed to unhedged prices and it is shown in Cherny and Madan (2010) that these bid and ask prices $b_h(X)$, $a_h(X)$ respectively are

$$\begin{aligned} b_h(X) &= (1+r)^{-1} \inf_{Q \in \mathcal{M} \cap \mathcal{R}} E^Q[X] \\ a_h(X) &= (1+r)^{-1} \sup_{Q \in \mathcal{M} \cap \mathcal{R}} E^Q[X]. \end{aligned}$$

The hedged bid is then higher than the unhedged bid while the hedged ask is lower than its unhedged counterpart. We shall work here mainly with the unhedged prices, though we also illustrate with some examples the hedged prices after choosing a particular set of hedging instruments. Madan and Schoutens (2011) study both types of equilibria. The hedging price is determined by maximizing the post hedge bid price or minimizing the post hedge ask price. Formally we have (Cherny and Madan (2010)) that

$$\begin{aligned} b_h(X) &= \sup_{H\in\mathcal{H}} b\left(X-H\right) \\ a_h\left(X\right) &= \inf_{H\in\mathcal{H}} a(H-X). \end{aligned}$$

We now investigate the pricing of risk in our two price economy. We may write the bid and ask prices for $X$ as attained at extreme points $Q^{b,X}, Q^{a,X}$ that have densities with respect to the base measure $Q^0$ of

$$\begin{aligned} \Lambda^{b,X} &= \frac{dQ^{b,X}}{dQ^0} \\ \Lambda^{a,X} &= \frac{dQ^{a,X}}{dQ^0} \end{aligned}$$

and we then have that

$$\begin{aligned} b(X) &= (1+r)^{-1}E^P\left[\Lambda^{b,X}\Lambda_c X\right] \\ a(X) &= (1+r)^{-1}E^P\left[\Lambda^{a,X}\Lambda_c X\right] \end{aligned}$$

If we employ a weighted average as a candidate price defining returns $\widetilde{R}_X$ relative to this average by

$$\begin{aligned} \widetilde{R}_X &= \frac{X}{m(X)}-1 \\ m(X) &= \alpha a(X)+(1-\alpha)b(X) \end{aligned}$$

then we infer the risk pricing equation

$$E[\widetilde{R}_X]-r=-cov^P\left(\left(\alpha\Lambda^{a,X}+(1-\alpha)\Lambda^{b,X}\right)\Lambda_c,\widetilde{R}_X\right).$$

Note importantly that by virtue of the nonlinearity of the pricing operators of a two price economy the pricing kernels are no longer independent of the risk being priced. We build on the classical measure change $\Lambda_c$ of a one price economy an additional illiquidity based measure change given by $\left(\alpha\Lambda^{a,X}+(1-\alpha)\Lambda^{b,X}\right)$. The second measure change is precisely an illiquidity based measure change as it comes into existence with a bid ask spread associated with an absence of a convergence to a law of one price. It collapses to the classical liquid market result when the cone of acceptable risks gets to be as large as the traditional half space. We shall see that credit, market and some components of liquidity risk may be classically captured in the measure change $\Lambda_c$ while illiquidity risk is properly viewed as a nonlinear risk captured by a risk dependent measure change.

## 2.2 Acceptance Cones Modeled by Concave Distortions

The market primitive of two price economies is the set of zero cost cash flows accepted by the market. This set is a convex cone of random variables containing the nonnegative random variables. When the acceptance decision for a random variable $X$ is a function solely of its distribution function $F_X(x)$ one may evaluate acceptance as shown in Cherny and Madan (2010) by a positive expectation under a concave distortion of the distribution function. Specifically for a concave distribution function $\Psi(u)$ defined on the unit interval and termed the distortion the random variable $X$ is accepted or belongs to the acceptance cone $\mathcal{A}$,just if

$$\int_{-\infty}^{\infty} x d\Psi\left(F_X(x)\right) \geq 0.$$

It is shown in Cherny and Madan (2010) that the set of approving measures $\mathcal{M}$ are all change of measure densities on the unit interval $Z(u)$ with

$$\int_{-\infty}^{\infty} x Z(F_X(x)) f_X(x) dx \geq 0$$

for all $Z$ for which $L \leq \Psi$, where $L' = Z$.

We mention here two distortions that have been used in earlier work by Cherny and Madan (2010) among other papers and earlier work in the insurance literature Wang (2000). These are the transforms *minmaxvar*, $\Psi^\gamma$ and the Wang transform, $\Phi^\alpha$. They are defined respectively by

$$\begin{aligned}
\Psi^\gamma(u) &= 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma} \\
\Phi^a(u) &= N\left(N^{-1}(u) + a\right)
\end{aligned}$$

Both these transforms have the desirable property of derivatives tending to infinity as $u$ tends to zero and derivatives that tend to zero as $u$ tends to unity. We present in Figure 1 a graph of $\Psi^\gamma$, $\Phi^a$ for $\gamma = 0.5$ and $a = 0.75$. As one can observe the two transforms are fairly close to each other. In light of this observation we shall present results for minmaxvar that is made up of elementary functions and combines two distortions that can be explained in simple terms. In minmaxvar we apply two distortions in sequence that reflect the expectation of the minimum of $\gamma$ independent draws and the other reflects the original distribution being the maximum of $\gamma$ draws from the distorted distribution. In each case the distorted distribution is an inferior one.

In terms of distortions one has exact expressions for bid and ask prices

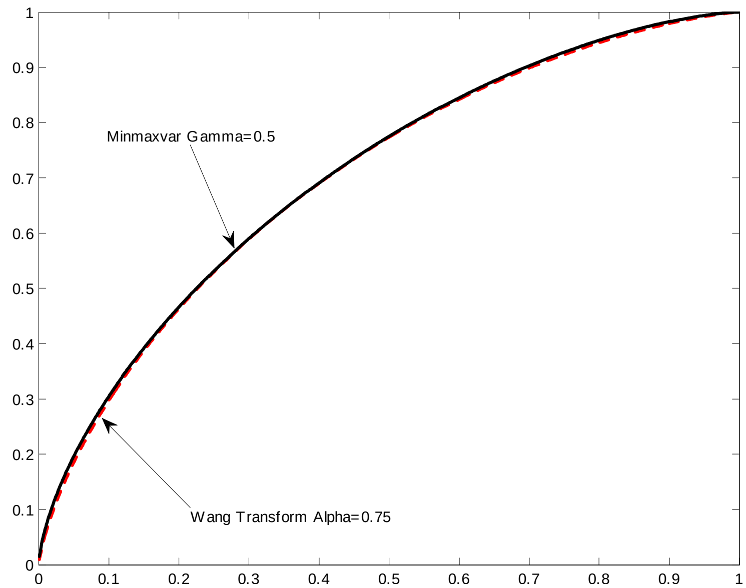


Figure 1: Wang Transform and Minmaxvar transform comparison

(Cherny and Madan (2010)). In this case

$$\begin{aligned} b(X) &= (1+r)^{-1}\int_{-\infty}^{\infty} x\psi(F_X(x))f_X(x)dx \\ a(X) &= -(1+r)^{-1}\int_{-\infty}^{\infty} x\psi\left(F_{-X}(x)\right)f_{-X}(x)dx \\ &= -(1+r)^{-1}\int_{-\infty}^{\infty} x\psi\left(1-F_X(-x)\right)f_X(-x)dx \\ &= (1+r)^{-1}\int_{-\infty}^{\infty} y\psi\left(1-F_X(y)\right)f_X(y)dy \end{aligned}$$

So

$$m_\alpha(X) = (1+r)^{-1}E^{Q^0}\left[(\alpha\psi(F_X(X)) + (1-\alpha)\psi\left(1-F_X(X)\right)X\right]$$

Hence we have that

$$\begin{aligned} E^P[X]-r &= -cov^{Q^0}\left(R_X, (\alpha\psi(F_X(X)) + (1-\alpha)\psi\left(1-F_X(X)\right))\right) \\ &= -cov^P(R_X, (\alpha\psi(F_X(X)) + (1-\alpha)\psi\left(1-F_X(X)\right)\Lambda_c) \end{aligned}$$

The kernel is then $U-shaped$ and we graph in Figure 2 the quantile pricing kernel for $\gamma = .5$ and $\alpha = .5$.

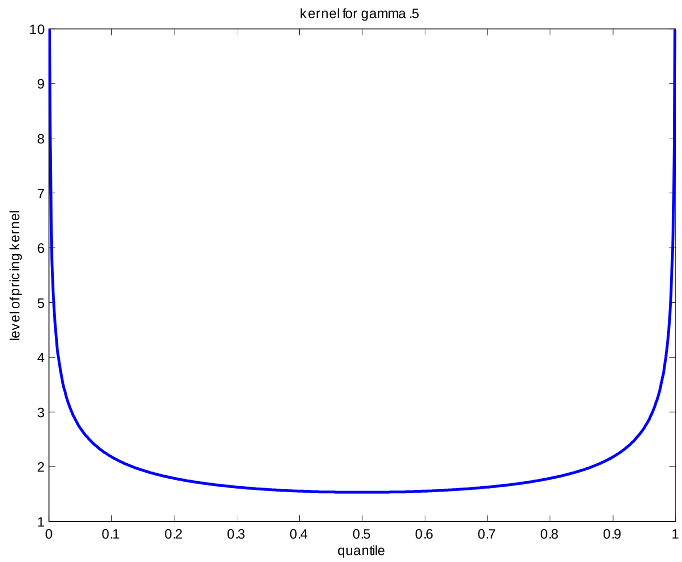


Figure 2: Quantile Risk Pricing Kernel for gamma equal to 0.5

The kernel cannot be uncorrelated with $X$ as it is a deterministic function of $X$. Now if the cash flow is flat at its upper quantiles and has risk exposure or sensitivity at the lower quantile as measured by the derivative of the inverse of the distribution function then the covariation with the kernel is negative and the midquote is above the base expectation. For a cash flow with sensitivity in the upper quantiles the covariation is positive and the midquote is below the base expectation. For most insurance contracts we have sensitivity in the lower quantiles and so we expect the mid quote to rise above the base expectation as may be observed on noting directly that the gap $g(a)$ for a digital at quantile $a$ is

$$g(a) = \Psi(a) + 1 - \Psi(1 - a) - 2a.$$

We graph in Figure 3 this digital gap.

The gap is positive at quantiles below a half and negative for quantiles above a half. Also shown are gaps when the market is biased towards the asking price or the bid price. The value of $\alpha$ when biased to ask is 0.55 and when biased to bid it is 0.45.

We therefore observe importantly from the positive gap that for insurance contracts like catastrophic bonds for which there may be no systematic risk component or correlation with the risk neutral pricing kernel the mid quote will exceed the physical probability of the event on account of a nonlinear illiquidity based risk charge. The nonlinearity manifests itself in the direct dependence of the illiquidity pricing kernel on the distribution function of the risk being priced.

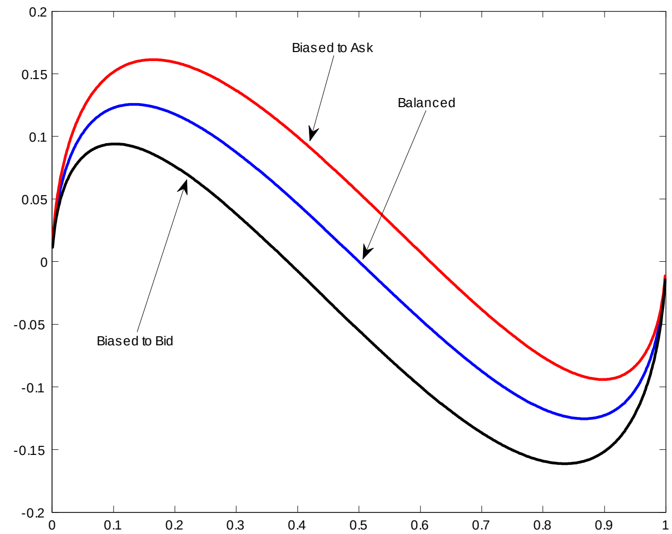


Figure 3: Mid quote base expectation gap at quantiles for digitals

## 2.3 The impact of existing positions on bid and ask prices

Suppose an economic agent has an existing exposure given by the random variable $W$ defined on the probability space and one seeks to determine an agent specific personalized bid and ask price for the random variable $X$. Classically one would consider the determination of a reservation price $p$ such that at the margin the agent is just indifferent to purchasing or selling $t$ units of $X$. The expected utility from this transaction is

$$V(t) = (1+r)^{-1}E\left[U(W + tX - tp(1+r))\right]$$

and the personalized price satisfies $V'(0) = 0$ which yields the result

$$p = (1+r)^{-1}\frac{E\left[U'(W)X\right]}{E\left[U'(W)\right]}.$$

This computation suggests that one works with the risk neutral distribution function for $X$ given by

$$F_X(x) = \frac{E\left[U'(W)\mathbf{1}_{X \le x}\right]}{E[U'(W)]}$$

with the bid and ask prices then obtained by applying the appropriate distorted expectations to this risk neutral distribution. Such a construction is in keeping with the principle of ensuring that the cone of acceptability employed contains not only the nonnegative random variables but is also contained in the relevant half space of positive alpha positions. It is just that out of a full equilibrium personalized pricing kernels vary and we work with the kernel relevant for the agent. In general the utility function employed could allow for preferences to reflect the state and hence we could accommodate state preferences that dependent on $\omega$ as well.

# 3 Synthesizing lognormal market, credit and illiquidity risk

This section develops a synthesizing model combining the three risk components at issue. Let the promised payout on a limited liability claim be given by the strictly positive random variable $X$. Let $\pi$ denote the risk neutral expectation while additionally $b, a$ denote the bid and ask prices and $m$ is a balanced midquote. The probability density of $X$ under the physical probability measure we take to be $p_X(x)$, for $x > 0$. We model the credit event by introducing a strictly positive probability of a default event in which case the payout is zero. The physical probability of default we suppose is $\eta$. In general the risk neutral probability density for $X$ differs from its physical density and we suppose it is $q_X(x)$ while the risk neutral default probability may also be different from $\eta$ and we suppose it is $\lambda$.

The change of measure density restricted to the positive outcomes of $X$ is given by

$$\frac{dQ^0}{dP} = \frac{1-\lambda}{1-\eta}\frac{q_X(x)}{p_X(x)}$$

By way of a simple and relatively classical example suppose the physical distribution is log normal with mean $\mu$ and variance $\sigma^2$. Hence physically the positive random variable $X$ can be described as

$$X = \mu e^{\sigma Z - \frac{\sigma^2}{2}}$$

where $Z$ is a standard normal variate. The expected value however on incorporating the credit event is $(1-\lambda)\mu$.

For a change of measure density with a market price for the risk of the standard normal variate of $\zeta$ we define

$$\Lambda = \exp\left(\zeta Z - \frac{\zeta^2}{2}\right).$$

The risk neutral expectation is then given by

$$(1-\lambda)\mu e^{\sigma\zeta} = (1+r)$$

or

$$\zeta = \frac{1}{\sigma}\ln\left(\frac{1+r}{(1-\lambda)\mu}\right).$$

Risk neutrally we have that

$$X = \frac{1+r}{(1-\lambda)} e^{\sigma Z - \frac{\sigma^2}{2}}.$$

The risk neutral distribution function of the credit sensitive payout is then given by

$$\begin{aligned} F(x) &= \lambda \\ &\quad + (1-\lambda) N\left(\frac{\ln x}{\sigma} + \frac{\sigma}{2} - \ln\left(\frac{1+r}{1-\lambda}\right)\right) \end{aligned}$$

For a one parameter distortion $\Psi^{\gamma}(u)$, $0 \leq u \leq 1$, the bid price is then

$$\begin{aligned} & bp(\sigma, \lambda, \gamma) \\ =\ & (1+r)^{-1} \int_0^{\infty} x d\Psi^{\gamma}(F(x)) \\ =\ & (1+r)^{-1} \int_0^{\infty} x d\Psi\left(\lambda + (1-\lambda) N\left(\frac{\ln x}{\sigma} + \frac{\sigma}{2} - \ln\left(\frac{1+r}{1-\lambda}\right)\right)\right). \end{aligned}$$

We could integrate by parts and write

$$\begin{aligned} & \int_0^{\infty} x d\Psi^{\gamma}\left(\lambda + (1-\lambda) N\left(\frac{\ln x}{\sigma} + \frac{\sigma}{2} - \ln\left(\frac{1+r}{1-\lambda}\right)\right)\right) \\ =\ & x\left(\Psi^{\gamma}\left(\lambda + (1-\lambda) N\left(\frac{\ln x}{\sigma} + \frac{\sigma}{2} - \ln\left(\frac{1+r}{1-\lambda}\right)\right)\right) - 1\right)\Big|_0^{\infty} \\ & - \int_0^{\infty}\left(\Psi^{\gamma}\left(\lambda + (1-\lambda) N\left(\frac{\ln x}{\sigma} + \frac{\sigma}{2} - \ln\left(\frac{1+r}{1-\lambda}\right)\right)\right) - 1\right) dx \\ =\ & \int_0^{\infty}\left(1 - \Psi^{\gamma}\left(\lambda + (1-\lambda) N\left(\frac{\ln x}{\sigma} + \frac{\sigma}{2} - \ln\left(\frac{1+r}{1-\lambda}\right)\right)\right)\right) dx \end{aligned}$$

In which case

$$\begin{aligned} & bp(\sigma, \lambda, \gamma) \\ =\ & (1+r)^{-1} \int_0^{\infty}\left(1 - \Psi^{\gamma}\left(\lambda + (1-\lambda) N\left(\frac{\ln x}{\sigma} + \frac{\sigma}{2} - \ln\left(\frac{1+r}{1-\lambda}\right)\right)\right)\right) dx \\ =\ & \frac{1}{1-\lambda} \int_{-\infty}^{\infty} \left(1 - \Psi^{\gamma}\left(\lambda + (1-\lambda) N(y)\right)\right) \sigma e^{\sigma y - \frac{\sigma^2}{2}} dy \\ =\ & \int_{-\infty}^{\infty} \psi^{\gamma}\left(\lambda + (1-\lambda) N(y)\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sigma)^2}{2}} dy \\ =\ & E\left[\psi^{\gamma}\left(\lambda + (1-\lambda) N(\sigma + Z)\right)\right] \end{aligned}$$

where $\psi^{\gamma}$ is the derivative of $\Psi^{\gamma}$ and $Z$ is a standard normal variate.

The ask price on the other hand is the negative of the distorted expectation of $-X$. We may write this as

$$\begin{aligned}
ap(\sigma,\lambda,\gamma) &= -(1+r)^{-1}\int_0^\infty x d\Psi^\gamma(1-F(x)) \\
&= -(1+r)^{-1}\int_0^\infty x d\Psi^\gamma\left((1-\lambda)\left(1-N\left(\frac{\ln x}{\sigma}+\frac{\sigma}{2}-\ln\left(\frac{1+r}{1-\lambda}\right)\right)\right)\right) \\
&= (1+r)^{-1}\int_0^\infty \left(\Psi^\gamma\left((1-\lambda)\left(\left(1-N\left(\frac{\ln x}{\sigma}+\frac{\sigma}{2}-\ln\left(\frac{1+r}{1-\lambda}\right)\right)\right)\right)\right)\right) \\
&= \frac{1}{1-\lambda}\int_{-\infty}^\infty \Psi^\gamma\left((1-\lambda)\left(1-N(y)\right)\right)\sigma e^{\sigma y-\frac{\sigma^2}{2}}dy \\
&= E\left[\psi^\gamma\left((1-\lambda)(1-N(\sigma+Z)\right)\right]
\end{aligned}$$

The risk neutral expectation is obtained when $\gamma=0$ and in this case the bid price equals the ask price and this is unity the current value of a dollar. More generally we see that the balanced mid quote $m$ satisfies

$$2m = E\left[\psi^\gamma\left(\lambda+(1-\lambda)N(\sigma+Z)\right)+\psi^\gamma\left((1-\lambda)(1-N(\sigma+Z)\right)\right]$$

We may define

$$u=\lambda+(1-\lambda)N(\sigma+Z)$$

and observe that

$$m=\frac{1}{2}E\left[\psi^\gamma(u)+\psi^\gamma(1-u)\right]$$

We are interested in how the three parameters of volatility, default probability and the distortion influence the gap between mid quote and the risk neutral expectation of unity. We present a table of values for the rate of profit defined as $m-1$ as a function of $\sigma,\lambda$ for two different levels of the distortion. We see that the rate of profit is higher for higher volatility and lower default probability and it rises with the level of the distortion or is higher when the cone of acceptable risks is reduced.

TABLE 1
Profit Rates as a function of
Volatility and Default Probability

| $\gamma=.25$ | Default Probability | | | | |
|---|---|---|---|---|---|
| Volatility | .01 | .02 | .03 | .04 | .05 |
| .1 | -.0044 | -.0071 | -.0090 | -.0104 | .0116 |
| .2 | -.0004 | -.0024 | -.0038 | -.0049 | -.0058 |
| .3 | .0054 | .0039 | .0028 | .0020 | .0014 |
| .4 | .0129 | .0118 | .0111 | .0106 | .0103 |
| .5 | .0223 | .0216 | .0212 | .0209 | .0208 |
| $\gamma=.5$ | | | | | |
| .1 | -.0159 | -.0244 | -.0304 | -.0350 | -.0387 |
| .2 | -.0014 | -.0080 | -.0126 | -.0160 | -.0187 |
| .3 | .0189 | .0140 | .0107 | .0083 | .0065 |
| .4 | .0454 | .0420 | .0398 | .0384 | .0375 |
| .5 | .0788 | .0767 | .0755 | .0749 | .0747 |

# 4 Dynamic Two Price Economies

We consider in this section the dynamic valuation of a discrete time stochastic claims or receipts process $X = (X_t, t = 1, \cdots, T)$. The valuation is as at time $t$ and is denoted $V_t^B(X)$, $V_t^A(X)$ depending on whether we are constructing a bid price or an ask price. We suppose that the length of the interperiod time interval is $h$. We suppose the existence of a base risk neutral measure selected by an equilibrium under which one may construct the risk neutral valuation process $V_t^R$ by

$$\begin{aligned} V_t^R &= \sum_{j \leq t} \frac{B(j)}{B(t)} X_j + E^{Q_0}\left[\sum_{j>t} \frac{B(j)}{B(t)} X_j\right] \\ &=_{def} \sum_{j \leq t} \frac{B(j)}{B(t)} X_j + W_t^R \end{aligned}$$

where $B(t)$ is the time zero discount curve supposed fixed in this exercise. Risk neutral valuation is a well understood linear pricing operator and as in the static case it constitutes our starting point. What we shall present are the nonlinear pricing operators for the bid and ask prices. We note in this regard the partitioning of total value into the part of that has been realized and the part that is yet to be realized by defining

$$\begin{aligned} V_t^A(X) &=_{def} \sum_{j \leq t} \frac{B(j)}{B(t)} X_j + W_t^A(X) \\ V_t^B(X) &=_{def} \sum_{j \leq t} \frac{B(j)}{B(t)} X_j + W_t^B(X) \end{aligned}$$

Such nonlinear pricing operators are given by nonlinear expectations that are related to solutions of backward stochastic difference equations. We first briefly present nonlinear expectation operators and their relationship to backward stochastic difference equations and then present our application for the construction of dynamic sequences of bid and ask prices.

For a discrete time finite state Markov chain with states $e_i$ identified with the unit vectors of $\mathbb{R}^M$ for some large integer $M$, Cohen and Elliott (2010) have defined dynamically consistent translation invariant nonlinear expectation operators $\mathcal{E}(.|\mathcal{F}_t)$. The operators are defined on the family of subsets $\left\{\mathbb{Q}_t \subset L^2(\mathcal{F}_T)\right\}$. For completeness we recall here this definition of an $\mathcal{F}_t-$consistent nonlinear expectation for$\{\mathbb{Q}_t\}$. This $\mathcal{F}_t-$consistent nonlinear expectation for$\{\mathbb{Q}_t\}$ is a system of operators

$$\mathcal{E}(.|\mathcal{F}_t) : L^2(\mathcal{F}_T) \to L^2\left(\mathcal{F}_t\right),\ 0 \leq t \leq T$$

satisfying the following properties:

1. For $Q, Q' \in \mathbb{Q}_t$, if $Q \geq Q'$ $\mathbb{P} - a.s.$ componentwise, then

$$\mathcal{E}(Q|\mathcal{F}_t) \geq \mathcal{E}(Q'|\mathcal{F}_t)$$

$\mathbb{P}-a.s.$ componentwise, with for each $i$,

$$e_i\mathcal{E}(Q|\mathcal{F}_t) = e_i\mathcal{E}(Q'|\mathcal{F}_t)$$

only if $e_iQ = e_iQ'$ $\mathbb{P}-a.s.$
2. $\mathcal{E}(Q|\mathcal{F}_t) = Q$ $\mathbb{P}-a.s.$ for any $\mathcal{F}_t-$measurable $Q$.
3. $\mathcal{E}(\mathcal{E}(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}(Q|\mathcal{F}_s)$ $\mathbb{P}-a.s.$ for any $s \leq t$
4. For any $A \in \mathcal{F}_t$, $\mathbf{1}_A\mathcal{E}(Q|\mathcal{F}_t) = \mathcal{E}(\mathbf{1}_AQ|\mathcal{F}_t)$ $\mathbb{P}-a.s.$
Furthermore the system of operators is dynamically translation invariant if for any $Q \in L^2(\mathcal{F}_T)$ and any $q \in L^2(\mathcal{F}_t)$,

$$\mathcal{E}(Q + q|\mathcal{F}_t) = \mathcal{E}(Q|\mathcal{F}_t) + q.$$

Such dynamically consistent translation invariant nonlinear expectations may be constructed from solutions of Backward Stochastic Difference and or Differential Equations (Cohen and Elliott (2010), El Karoui and Huang (1997)). These are equations to be solved simultaneously for processes $Y, Z$ where $Y_t$ is the nonlinear expectation and the pair $(Y, Z)$ satisfy

$$Y_t - \sum_{t \leq u < T} F(\omega, u, Y_u, Z_u) + \sum_{t \leq u < T} Z_u M_{u+1} = Q$$

for a suitably chosen adapted map $F : \Omega \times \{0, \cdots, T\} \times \mathbb{R}^K \times \mathbb{R}^{K \times N} \to \mathbb{R}^K$ called the driver and for $Q$ an $\mathbb{R}^K$ valued $\mathcal{F}_T$ measurable terminal random variable. We shall work in this paper generally with the case $K = 1$. For all $t$, $(Y_t, Z_t)$ are $\mathcal{F}_t$ measurable. Furthermore for a translation invariant nonlinear expectation the driver $F$ must be independent of $Y$ and must satisfy the normalization condition $F(\omega, t, Y_t, 0) = 0$.

The drivers of the backward stochastic difference equations for our bid and ask prices are the risk charges at tenor $h$. We employ the drivers $F^A, F^B$ where

$$F^a(\omega, u, Y_u, Z_u) = h \sup_{Q \in \mathcal{M}} E^Q [Z_u M_{u+1}] \tag{1}$$

$$F^b(\omega, u, Y_u, Z_u) = h \inf_{Q \in \mathcal{M}} E^Q [Z_u M_{u+1}], \tag{2}$$

and the drivers are independent of $Y$. The process $Z_t$ represents the residual risk in terms of a set of spanning martingale differences $M_{u+1}$ and in our applications we solve for the nonlinear expectations $Y_t$ without in general identifying either $Z_t$ or the set of spanning martingale differences. We define risk charges directly for the risk defined for example as the zero mean random variable

$$\frac{B(t+1)}{B(t)} \left(X_{t+1} + W^A(t+1)\right) - E^{Q^0} \left[\frac{B(t+1)}{B(t)} \left(X_{t+1} + W^A(t+1)\right)\right].$$

We therefore apply the recursions

$$
\begin{aligned}
W_t^A(X) &= E^{Q^0}\left[\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^A(t+1)\right)\right] \\
&\quad +h\sup_{Q\in\mathcal{M}}\left(\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^A(t+1)\right)-E^{Q^0}\left[\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^A(t+1)\right)\right]\right) \\
W_t^B(X) &= E^{Q^0}\left[\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^B(t+1)\right)\right] \\
&\quad +h\inf_{Q\in\mathcal{M}}\left(\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^B(t+1)\right)-E^{Q^0}\left[\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^B(t+1)\right)\right]\right)
\end{aligned}
$$

## 4.1 Drivers for nonlinear expectations based on distortions

The driver for a translation invariant nonlinear expectation is basically a positive risk charge for the ask price and a positive risk shave for a bid price applied to a zero mean risk exposure to be held over an interim. We are then given as input the risk exposure ideally spanned by some martingale differences as $Z_uM_{u+1}$ or alternatively a zero risk neutral mean random variable $X$ with a distribution function $F(x)$.

We consider in the rest of the paper drivers based on the distortion *minmaxvar*. In this case

$$
\begin{aligned}
F^B(Z_uM_{u+1}) &= \int_{-\infty}^{\infty} x d\Psi^\gamma(\Theta^B(x)) \\
F^A(Z_uM_{u+1}) &= -\int_{-\infty}^{\infty} x d\Psi^\gamma\left(1-\Theta^A(-x)\right)
\end{aligned}
$$

and in particular

$$
\begin{aligned}
\Theta^B(x) &= Q^0\left(\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^B(t+1)\right)-E^{Q^0}\left[\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^B(t+1)\right)\right]\le x\right) \\
\Theta^A(x) &= Q^0\left(\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^A(t+1)\right)-E^{Q^0}\left[\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^A(t+1)\right)\right]\le x\right).
\end{aligned}
$$

## 4.2 Illustrative Valuation Sequences

We wish to construct an example for the valuation of insurance losses. One may use a compound Poisson process but this is a process on independent and identically distributed increments and dynamic valuation procedures apply to contracts that resolve at their maturity and there is no maturity for a compound Poisson process. We therefore choose a maturity $T$ and consider the valuation of the losses in the compound Poisson process up to this maturity.

So at maturity there are no remaining losses and the bid and ask prices equal the level of realized losses given by $X(T)$ the level of the compound Poisson process at $T$. We therefore have $a_X(T) = b_X(T) = X(T)$. We now work backwards at a tenor of time step $h$ evaluating $a_X(T-nh)$, $b_X(T-nh)$ for $n = 1, \cdots, N$ where $h = \frac{T}{N}$. We suppose that we have a gamma compound Poisson process with arrival rate $\lambda$ and loss sizes gamma distributed with parameters $c, \kappa$ for the gamma distribution. The characteristic function is then given by

$$\phi_X(u) = \exp\left(\lambda\left(\left(\frac{c}{c-iu}\right)^{\kappa} - 1\right)\right)$$

Following the equilibrium pricing model of Madan (2006) for an economy populated with investors with exponential utility we assume that all risks are priced by exponential tilting. If we apply an exponential tilt of $\theta$ we get a risk neutral characteristic function $\widetilde{\phi}_X(u)$ where

$$\begin{aligned} \widetilde{\phi}_X(u) &= \frac{\phi_X(u-i\theta)}{\phi_X(-i\theta)} \\ &= \exp\left(\lambda\left(\frac{c}{c-\theta}\right)^{\kappa}\left(\left(\frac{c-\theta}{c-\theta-iu}\right)^{\kappa} - 1\right)\right) \end{aligned}$$

and this in the same class of processes, being a gamma compound Poisson process with arrival rate

$$\lambda\left(\frac{c}{c-\theta}\right)^{\kappa}$$

that has gone up and jump sizes also gamma distributed with mean

$$\frac{\kappa}{c-\theta}$$

and variance

$$\frac{\kappa}{(c-\theta)^2}$$

so the arrival rate, mean jump size and the variance of the jump sizes have all gone up by a factor reflecting the market price of risk $\theta$.

To model credit effects we incorporate a risk neutral default probability that could be obtained from the CDS rate for the insurer. In this case with probability $\eta h$ losses in the time interval $h$ are not paid while claim payments do occur with probability $(1-\eta h)$. The risk neutral distribution function is over the interval of tenor $h$ is then

$$F_L(x,h) = \eta h + (1-\eta h)F_X(x,h)$$

where the characteristic function for the distribution function $F_X(x,h)$ is

$$\phi_X(u,h) = \exp\left(h\lambda\left(\frac{c}{c-\theta}\right)^{\kappa}\left(\left(\frac{c-\theta}{c-\theta-iu}\right)^{\kappa} - 1\right)\right)$$

This distribution function has incorporated the physical risk, the market price of this risk and credit effects in the construction of the local risk neutral density for the time step $h$.

The nonlinear liquidity effects are now incorporated by the use of distortions for the construction of bid and ask prices. We use minmaxvar with stress level $\gamma$ and use for the bid price drivers

$$F^B(t, Y, Z) = \int_{-\infty}^{\infty} cd\Psi^\gamma(F_L(c, h))$$

while for the ask driver we have

$$\begin{aligned} F^A(t, Y, Z) &= -\int_{-\infty}^{\infty} cd\Psi^\gamma\left(F_{-L}(c, h)\right) \\ &= -\int_{-\infty}^{\infty} cd\Psi^\gamma\left(1 - F_L(-c, h)\right) \\ &= \int_{-\infty}^{\infty} cd\Psi^\gamma\left(1 - F_L(c, h)\right) \end{aligned}$$

We are now ready to illustrate the computation of dynamically consistent bid and ask price sequences and to study the effects on these of credit risk $\eta$, the market price of risk $\theta$, the underlying physical risk $\lambda, c, \kappa$, and the effects of liquidity risk, $\gamma$, as captured by movements in the cone of locally acceptable risks.

## 4.3 A Sample Computation

For a sample computation we take a maturity of $T = 5$ years and consider $N = 20$ with $h = .25$ or a quarterly tenor. Suppose losses arrive at the rate of $\lambda = 100$ per year with gamma distribution for the claims. The mean claim is a quarter of a million dollars with a standard deviation of .1875 or three quarters of the mean. The shape parameter for the gamma distribution is $\kappa = 1.7777$ and the scale parameter $c = 7.1111$.

We suppose that a loss of 10 million is reweighted upwards to 1.5 and this gives a value for $\theta = 0.0405$ or exponential tilting by this $\theta$ at 10 million is $e^{.0405*10} = 1.5$. The risk neutral mean is .2514 the standard deviation is 0.1886 and the arrival rate is 101.02. We suppose the credit risk is 100 basis points and the stress parameter for the cone of locally acceptable risks is 0.4 a typical value we estimated from our analysis of daily stock returns using the High and Low prices as candidates for the bid and ask prices.

The first step in constructing a dynamic valuation sequence for a loss process like our gamma compound Poisson process is to construct the risk neutral grid of potential loss states at the time points $nh$ for $n = 1, \cdots, 20$. For this we take a grid that is uniform in quantiles. We obtain for each year end the distribution function for the risk neutral loss level by inverting the known analytical Laplace transform for the distribution function. We then determine loss levels at the quantiles ranging from 1% to 99% in steps of one percent. We present in Table 2 as a summary the loss levels for 10%, 25%, 50%, 75% and 90% at the five, year ends.

TABLE 2
Loss Levels at Year Ends

| | Year End | | | | |
|---|---|---|---|---|---|
| Quantile | 1 | 2 | 3 | 4 | 5 |
| 10 | 21.41 | 45.13 | 69.25 | 93.56 | 118.01 |
| 25 | 23.22 | 47.74 | 72.46 | 97.29 | 122.19 |
| 50 | 25.31 | 50.71 | 76.11 | 101.51 | 126.91 |
| 75 | 27.48 | 53.76 | 79.84 | 105.81 | 131.71 |
| 90 | 29.50 | 56.58 | 83.27 | 109.76 | 136.11 |

We recognize that bid, ask and expected values equal the realized loss level at the five year maturity as there is no remaining uncertainty. Next we determine bid, ask and expected values on the entire grid by backward recursion. For the backward recursion we suppose we have determined bid, ask and expected values at time $n$ and then for each grid point $j$ at time $n-1$ we with a loss level at this grid point of $x$, we simulate $M = 10000$ loss levels for the next period or quarter. We denote these loss levels by the vector $L$. We then have $M$ readings for a realized loss level at time $n$ of $x+L$. These are transformed to $M$ bid, ask and expected values using the already computed and stored bid, ask and expected value functions for time $n$ on the time $n$ grid. For the expected value at time $n-1$ we just take the mean of the time $n$ expected values and store it at the time $n-1$ grid for loss level $x$ and grid point $j$. Denote this value $E(j, n-1)$.

For the recursion of bid and ask we employ distorted expectations applied to distribution function with credit risk. For the bid price we sort the one step ahead bid values into an increasing sequence of bid values $b$ of length $M$. To incorporate credit risk we recognize that with probability $\eta h$ the result is zero and with probability $(1-\eta h)$ we have the outcome $b_i$ with probability $\eta h + (1-\eta h)i/M$. We evaluate the distortion at these probability points and determine the time $n-1$ bid value as

$$\begin{aligned} B(j,n-1) &= E(j,n-1) + \\ & h\left(\sum_{i=1}^{M} b_i \left(\Psi^{\gamma}\left(\eta h + (1-\eta h)i/M\right) - \Psi^{\gamma}\left(\eta h + (1-\eta h)(i-1)/M\right)\right) - E(j,n-1)\right). \end{aligned}$$

The treatment of the tenor is in line with equation (2).

A similar procedure is implemented for the ask recursion except this time we sort in increasing order the negative of the forward ask values into a sequence $a$ of length $M$. We then form

$$\begin{aligned} A(j,n-1) &= E(j,n-1) - \\ & h\left(\sum_{i=1}^{M} a_i \left(\Psi^{\gamma}\left((1-\eta h)i/M\right) - \Psi^{\gamma}\left((1-\eta h)(i-1)/M\right)\right) - E(j,n-1)\right). \end{aligned}$$

The treatment of the tenor is now in line with equation (1).

This recursion provides us with loss contingent bid and ask value tables for the entire grid from time 1 to time $N$. We exhibit the bid, expected and ask values for a subsample of the grid corresponding to Table 2 in Table 3. In general for a contract converging to a zero spread at maturity the spreads should be rising as we get further away from maturity. However, as the tenor decreases there is a convergence to the risk neutral value also occurring. At the one year tenor the spreads rise as we get further from maturity as reported in Tables 4 and 5. In Table 3 at the quarterly tenor and for the 50% quantile especially the convergence to risk neutrality works against the increase in the spread with the resultant spread being somewhat constant.

The initial bid, ask and expected values are respectively 125.88, 127.25 and 126.99.

In Table 4 we present the same computation now conducted at a tenor of one year.

TABLE 4
Bid Expected and Ask Values Tenor One Year

| | Year End | | | |
|---|---|---|---|---|
| quantile | 1 | 2 | 3 | 4 |
| 10 | 93.73,123.00,129.91 | 98.95,121.40,126.60 | 104.71,120.04, | 111.09,118.96,120.71 |
| 25 | 95.13,124.78,131.70 | 101.14,123.99,129.20 | 107.57,123.22, | 114.67,122.70,124.42 |
| 50 | 96.82,126.92,133.84 | 103.64,126.93,132.11 | 110.88,126.93, | 118.62,126.90,128.66 |
| 75 | 98.55,129.09,136.02 | 106.24,129.97,135.14 | 114.30,130.72, | 122.69,131.18,132.90 |
| 90 | 100.19,131.11,138.05 | 108.64,132.81,138.01 | 117.30,134.07, | 126.44,135.14,136.88 |

The bid and ask and expected values on a one year tenor at the initial date are 90.56, 135.69 and 127.05. The spreads are consistently wider at the lower tenor as a higher level of interperiod risk is being held. The bid and ask prices will converge in the case of a perfectly liquid market with a very small tenor.

In Table 5 we keep the annual tenor of Table 4 but raise the default probability from 100 basis points to 500 basis points. We also report in this case just the bid and ask prices.

TABLE 5
Bid and Ask Prices Annual Tenor High Credit Risk

| | Year End | | | |
|---|---|---|---|---|
| quantile | 1 | 2 | 3 | 4 |
| 10 | 57.88,129.24 | 68.88,125.98 | 82.30,123.14 | 98.50,120.51 |
| 25 | 58.75,130.95 | 70.43,128.60 | 84.59,126.37 | 101.62,124.26 |
| 50 | 59.82,133.05 | 72.18,131.54 | 87.13,129.99 | 105.18,128.49 |
| 75 | 60.86,135.16 | 74.01,134.56 | 89.77,133.67 | 108.74,132.68 |
| 90 | 61.88,137.16 | 75.65,137.32 | 92.19,137.15 | 112.11,136.72 |

The initial bid price is 49.47 while the ask is 134.70. Comparing Tables 4 and 5 we observe that the effect of an increase in the credit risk has a significant effect on bid prices or asset prices. The ask price or the liability valuation is instead relatively unaffected. The two price economy thereby distinguishes a theory of asset prices from the theory of liability valuation especially as far as the effects of credit risk are concerned.

# 5 Inhomogeneous Discounted Compound Poisson Insurance Losses

We consider the pricing and valuation of insurance losses given by an inhomogeneous compound Poisson process over the infinite future from some time $T$ backwards at a fixed tenor. We value discounted losses where the discounting is done on a yield curve fixed at the valuation date. For the actual loss distributions contingent on a loss arrival we take three distributions the Gamma, Weibull and Frechet distributions. For each distribution we specify the mean and variance of the loss distribution and solve for the parameters of the distribution consistent with these moments. For the Gamma distribution this transformation is analytical while for the Weibull and Frechet it is numerical. We allow for a credit spread and a stress level for the distortion. We report on pricing the whole loss and report on the dynamic structure of pricing in this case. We then fix some aspects of the loss structure and vary in particular the time stepping tenor, the volatility, the credit spread and the distortion stress level. For each of these four aspects of the loss pricing process we take two settings with 16 cases in all. For each of the three distributions we report on the pricing of a capped and uncapped loss process in the presence of a deductible. The result is two tables with 16 cases for each of two products for each of three underlying loss distributions. The dynamic structure is reported for Gamma aggregate losses with low volatility, a quarterly tenor, a credit spread of 100 basis points and a *minmaxvar* stress level of 0.75 in the first subsection. The next subsection reports on the effects of variations in the product, the distributions, the tenor, volatility, credit spread and stress level.

## 5.1 The Gamma Aggregate Loss

The inhomogeneous arrival rate of losses in all cases reported is of exponential form with a time dependent arrival rate at time $t$ of

$$\lambda(t) = \frac{a}{\tau} e^{-\frac{t}{\tau}}$$

with an aggregate arrival of losses after time $T$ of

$$\lambda_{T,\infty} = a\left(1 - e^{-\frac{t}{\tau}}\right).$$

For the reported results we take $a = 150$ and $\tau = 10$.

We evaluate bid and ask prices at a quarterly tenor with $T = 5$. The loss distribution is gamma with a mean of .25 and a volatility of .75 time the mean or $\sigma = .1875$. The credit spread is 100 basis points and the stress level is 0.75.

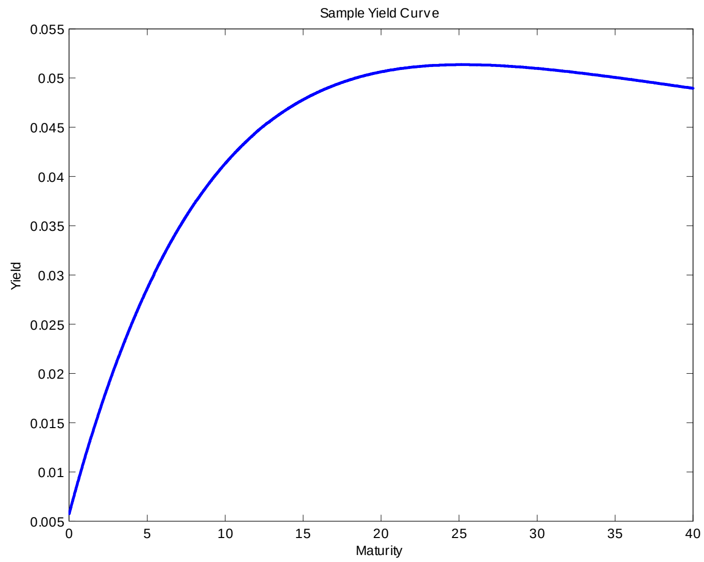


Figure 4: Graph of yield curve used in discounting.

Discounting is done with a Nelson Siegel yield curve with yield at maturity $t$ given by

$$\begin{aligned} y(t) &= a_1 + (a_2 + a_3 t)e^{-a_4 t} \\ a_1 &= .0424 \\ a_2 &= -.0367 \\ a_3 &= .0034 \\ a_4 &= .0686. \end{aligned}$$

We present in Figure 4 a graph of the associated yield curve.

The aggregate losses are subject to a deductible of .15. All losses beyond this level are covered and paid out. For the remaining losses after year five we assume no credit spread with these losses not being subject to default. As we have a random aggregate loss at the final five year point we have a spread between bid and ask prices at this point.

We present tables for the loss quantiles of $10, 25, 50, 75$ and 90 at five year ends, along with bid prices, expected values and ask prices at these grid points along with the spreads of ask to bid, ask to the expected value and the expected value to the bid. These are presented in two tables, one for the grid points, one for all the valuations and spreads.

Table 6
Aggregate Gamma Loss Grid

| | Year | | | | |
|---|---|---|---|---|---|
| quantile | 1 | 2 | 3 | 4 | 5 |
| 10 | 0.8103 | 1.9640 | 3.0987 | 4.1360 | 4.9873 |
| 25 | 1.1795 | 2.5368 | 3.7765 | 4.8892 | 5.5817 |
| 50 | 1.6536 | 3.2214 | 4.6037 | 5.7796 | 6.8332 |
| 75 | 2.2277 | 3.9914 | 5.4968 | 6.7539 | 7.9028 |
| 90 | 2.7762 | 4.7078 | 6.3509 | 7.7045 | 8.9425 |

The bid, expected values and ask prices along with the spreads are presented on this grid in Table 7. The table presents for each of five years six rows for the bid, expected value and ask price followed by the three spreads. Unlike the fixed finite maturity case when all spreads go to zero at maturity we observe quite a substantial spread at the five year point accounting for the risks yet to come. These spreads fall at first and then increase as we move back through time. The spreads are also higher at the lower quantiles. The spread is asymmetric about the expected value with the midquote lying below the expected value. The initial bid, ask and expected values are 12.8241, 13.2889 and 13.1570.

## 5.2 Price Sensitivities

We freeze all but four inputs into loss pricing process. The exception are the tenor, volatility, credit spread and the distortion spread level. For each of these variables we take two settings that makes for 16 cases for which we solve for the grid and the grid conditional bid, ask and expected values. We report in Tables 8, 9 and 10 the sensitivities of the initial bid, ask and expected value to these variations. The three tables employ three different loss distributions, the gamma, Weibull and Frechet distributions. Each table has two subtables for two products the loss with a deductible of .15 capped at .35 and another that pays all losses above .35.

The tenor settings are a quarter and a half year. For volatility we take .1875 and .3125 or .75 and 1.25 times the mean. The credit spread has the levels of 100 or 500 basis points while the distortion stress level is .75 and 1.25.

We first observe that an increase in the stress level raises the ask and lower the bid for the uncapped losses. However, for capped losses both the ask and the bid are reduced. A reduction in the tenor raises ask prices and lowers bid prices for both types of claims. An increase in volatility raises ask prices and lower bid prices for uncapped claims but for capped claims the ask also falls. Finally we observe that bid prices are sensitive to changes in credit risk but ask prices are relatively insensitive to credit risk.

The low volatility Weibull prices are higher than the gamma counterpart while the high volatility Weibull prices are lower. The Frechet prices are substantially lower than the Weibull and gamma counterparts.

## 6 Hedging in Two Price Economies

The analysis and pricing examples studied so far have not explicitly addressed any hedging possibilities or considerations. Typically even with incomplete markets attempts are made to use some hedging instruments to alter risk profiles and better manage the risk exposures. We now take up these issues in the context of two price economies. What is critical in this context is the recognition that replication is not possible, residual risk will have to be held and one has to formulate a criterion guiding the hedge positions. Additionally we note that hedge instruments should have zero means under the base probability measure for otherwise these instruments would become vehicles for investment or speculation instead of being used as hedges. With hedges having zero means one may take target cash flows to be hedged to also have a zero mean and hence the hedging criterion should be receptive of negative as well as positive cash flows.

A classical criterion often used in studies related to hedging in incomplete markets is variance minimization or the equivalent of quadratic hedging, given zero means (Föllmer and Schweizer (1990), Schweizer (1992, 1995, 2001)). This criterion has no parameter with which to reflect some degree of aggressiveness or otherwise in hedge design. An often studied alternative criterion, especially in the context of indifference pricing (Carmona (2009), Musiela and Zariphopolou (2004a, 2004b)), is the maximization of expected utility.

In the context of two price markets it is noted in Carr, Madan and Vicente Alvarez (2011) and Madan and Schoutens (2011) that competitive pressures lead us to minimize ask prices and maximize bid prices. The maximization of bid prices is equivalent to the minimization of their negative and this suggests the minimization of the difference between ask and bid prices. Carr, Madan and Vicente Alvarez (2011) note that for a liability to be acceptable one must deliver the ask price. If the liability is credited as at least being worth the bid price then the capital required to support the exposure to residual risk is this difference between the ask and bid prices. Minimizing this difference is then tantamount to economizing on capital requirements. Given that bid and ask prices reflect stress parameters embedded in distortions capital minimization becomes a hedging criterion with a parameter allowing for the expression of different levels of aggressiveness in hedge design.

This section compares this new hedging criterion of capital minimization with other classical criteria. For this purpose we construct a set of sample static hedging exercises. We formulate a target cash flow given by the random variable $Y$ that has to hedged with a zero mean hedging cash flow $X$. The residual cash flow is then

$$R = Y - aX$$

for a position of $a$ units in the hedge instrument.

One may choose $a$ to minimize the variance of $R$ and this results in the least squares hedge given by the beta of $Y$ with respect to $X$.

Another criterion often recommended as already noted is the maximization of expected utility. Most popularly one uses exponential utility at some level of risk aversion or the related certainty equivalent. We introduce here also the proposals to minimize ask prices, maximize bid prices or minimize capital requirements defined as the difference between the ask and the bid.

In the exercise to be conducted here we compare the various criteria and their hedges. For the law of the underlying risk we take it to be an exponential of a variance gamma variate with a negative skew and some kurtosis as is typical for risk neutral stock price distributions (Madan and Seneta (1990), Madan, Carr and Chang (1998)). The variance gamma law results on time changing Brownian with drift by a gamma process with unit mean rate and volatility $\nu$. If the drift of the Brownian motion is $\theta$ and its volatility is $\sigma$ we get a three parameter law for the variance gamma variate with parameters $\sigma, \nu, \theta$. The parameter $\theta$ controls the skewness of the distribution while $\sigma, \nu$ control the volatility and kurtosis.

We work with four parameter sets for $\sigma, \nu, \theta$ at

$$.25, .75, -.3$$
$$.25, .75, -.6$$
$$.25, 1.5, -.3$$
$$.25, 1.5, -.6$$

that reflect varying levels of skewness and kurtosis. The target cash flow is

$$Y = \delta \Delta S + \frac{\gamma}{2} (\Delta S)^2$$

with $\delta$ ranging from $-1$ to 1 in steps of .2 while $\gamma$ ranges from $-20$ to 20 in steps of 5. There are 11 values for $\delta_i$ $i = 1, \cdots, 11$ and 8 values for $\gamma_j$, $j = 1, \cdots, 8$ excluding the zero gamma position. This gives us 88 possible target cash flows.

We choose the hedge position to minimize the variance of the residual cash flow as one criterion. The other criteria are minimize capital using minmaxvar at stress level .75 and 1.5. We also consider minimizing the ask price and maximizing the bid price as potential criteria. We tried unsuccessfully to allow for maximizing expected utility but this was not possible for a fixed level of absolute risk aversion as the relevant level of risk aversion needs to be changed with the scale of the cash flows. One can fix relative risk aversion as this is a pure number but then one cannot manage negative cash flows that must occur given zero means. So reluctantly we gave up on expected utility as a practical hedging criterion.

In the presence of a negative skew the presence of gamma in the target cash flow induces a reduction in the hedge position below the delta. We regress for each of the four parametric contexts and for each hedge criterion the hedge position on the delta and gamma of the target cash flow including a constant term. We first observe this result analytically for variance minimization.

The variance of the residual cash flow is

$$E(R^2) - (E[R])^2 = (\delta - a)^2 E[(\Delta S)^2] + \frac{\gamma^2}{4} E\left[(\Delta S)^4\right] - \gamma(\delta - a) E\left[(\Delta S)^3\right] - \frac{\gamma^2}{4}\left(E\left[(\Delta S)^2\right]\right)^2.$$

The variance minimizing hedge satisfies the equation

$$-2(\delta - a)E[(\Delta S)^2] + \gamma E\left[(\Delta S)^3\right] = 0$$

or

$$a = \delta + \frac{E\left[(\Delta S)^3\right]}{2E[(\Delta S)^2]}\gamma \tag{3}$$

and we see that the constant term should be zero, the coefficient of the delta should be unity while that of $\gamma$ should be negative in the presence of a negative skewness.

The variance minimization criterion has no parameter allowing one to express any preferences related to hedging. Expected utility has a risk aversion parameter that is difficult to set for exponential utility as already noted and when it is a pure number like relative risk aversion the utility function cannot be evaluated for negative cash flows and this is problematic for hedging exercises. The stress level of the distortions serves as a preference parameter for capital and ask price minimization and bid price maximization. This parameter is also a pure number. We therefore compare variance minimization with capital and ask price minimization and bid price maximization.

We conducted hedges for 88 target cash flows in four contexts for volatility, skewness and kurtosis of the stock price. We then regressed the hedge position on the target delta, gamma and a constant term. The coefficient of the delta was unity in all cases and is not reported. We report the coefficients for the constant term and the gamma in the case of variance minimization for the four contexts. For capital, ask and bid we report the same two coefficients for two separate stress levels of .75 and 1.5. We do this in four separate tables for the four parametric contexts.

Case 1 ($\sigma = .25$, $\nu = .75$, $\theta = -.3$)

| | Constant | Gamma |
|---|---|---|
| Variance | 0 | -.1368 |
| Capital Low Stress | 0 | -.0963 |
| Ask Low Stress | -.8390 | -.0807 |
| Bid Low Stress | .8383 | -.0807 |
| Capital High Stress | 0 | -.1111 |
| Ask High Stress | -.9975 | -.0838 |
| Bid Low Stress | .9974 | -.0838 |

Case 2 ($\sigma = .25$, $\nu = .75$, $\theta = -.6$)

| | Constant | Gamma |
|---|---|---|
| Variance | 0 | -.2218 |
| Capital Low Stress | 0 | -.1885 |
| Ask Low Stress | -1.8712 | -.1753 |
| Bid Low Stress | 1.8714 | -.1753 |
| Capital High Stress | 0 | -.2359 |
| Ask High Stress | -1.9148 | -.1605 |
| Bid Low Stress | 1.9148 | -.1605 |

Case 3 ($\sigma = .25$, $\nu = 1.5$, $\theta = -.3$)

| | Constant | Gamma |
|---|---|---|
| Variance | 0 | -.2026 |
| Capital Low Stress | 0 | -.1494 |
| Ask Low Stress | -1.3283 | -.1258 |
| Bid Low Stress | 1.3280 | -.1258 |
| Capital High Stress | 0 | -.1787 |
| Ask High Stress | -1.5369 | -.1284 |
| Bid Low Stress | 1.5374 | -.1284 |

Case 4 ($\sigma = .25$, $\nu = 1.5$, $\theta = -.6$)

| | Constant | Gamma |
|---|---|---|
| Variance | 0 | -.2835 |
| Capital Low Stress | 0 | -.2654 |
| Ask Low Stress | -2.2146 | -.2102 |
| Bid Low Stress | 2.2146 | -.2102 |
| Capital High Stress | 0 | -.3153 |
| Ask High Stress | -2.4031 | -.2018 |
| Bid Low Stress | 2.4030 | -.2018 |

We make the following observations on these results. The coefficient on Gamma for variance minimization is always close to the theoretical value given by equation (3). It appears that variance minimization is a form of high stress hedging with delta reductions that match capital minimization for some stress level above unity. So it would reduce the hedge position too far in response to skewness. Ask minimization reduces hedge positions by a constant while bid maximization raises them by the same constant. Capital minimization is a symmetric objective that has a zero constant term like variance. The asymmetry embodied in minimizing the ask and maximizing the bid is also present in expected utility maximization and may not be a desirable hedging feature. The relative symmetry of capital minimization as a hedging criterion is welcome and it provides us with a symmetric hedging criterion with a preference parameter in the criterion that is a pure number. The rest of the paper employs capital minimization as the preferred criterion for the design of hedges.

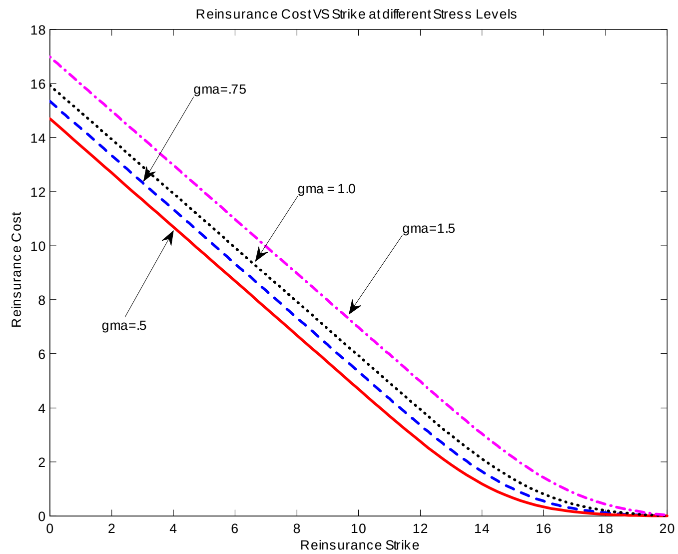


Figure 5: Reinsurance cost at the minmaxvar ask price for a variety of stress levels.

# 7 Capital economizing optimal reinsurance hedges

Given the popularity of reinsurance as a hedge in the insurance industry we now address the issue of optimally choosing the reinsurance attachment point, that is one that minimizes the cost of provision. This cost includes the price of reinsurance plus the capital set aside for the part of the loss that is held and not reinsured. We illustrate here the construction of such optimal reinsurance attachment points in the context of our inhomogeneous compound Poisson loss process. We work in this section with an annual tenor.

The reinsurance market we suppose covers all present values in excess of the reinsurance strike of $A$ by paying at $T$ the sum

$$(L(\infty) - A)^{+}.$$

The reinsurer takes all losses that exceed the present value of $A$ on the current yield curve. We use the same yield curve as in section 5. The reinsurer prices this contract at an ask price based on the *minmaxvar* distortion at the stress level of .75. The pricing is that of a straight call on $L(\infty)$ with no dynamic recursion on a specified tenor. We expect the price of this reinsurance coverage to decrease as we raise the attachment point, a typical property of call option prices. For the base case of section 5 we present in Figure (5) the cost of this reinsurance as a function of the attachment point. We assume the contracts covered all losses with a deductible of .15 and no caps.

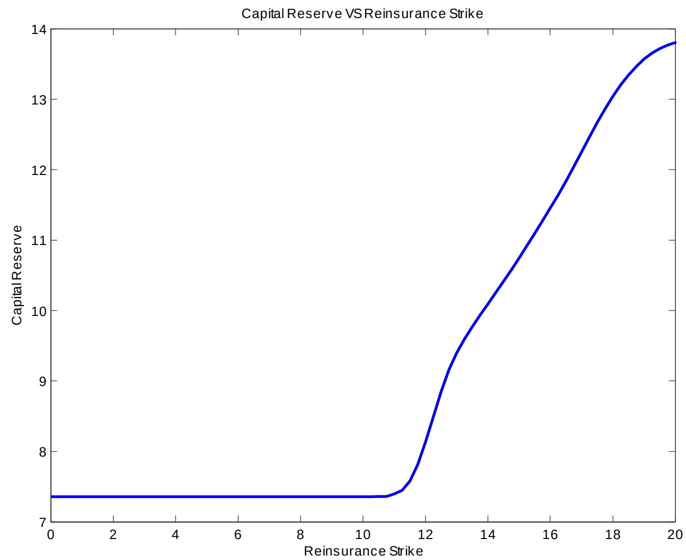


Figure 6: Risk Sensitive Capital Cost as a function of the reinsurance strike.

As we raise the reinsurance attachment point however we have to cover the losses ourselves and this raises the risk exposure we hold and our associated capital costs as measured by the difference between the bid and the ask price. We compute these prices on a five year annual tenor recursion. We graph in Figure (6) this capital cost a function of the reinsurance attachment point. We see that these costs are fairly small at the lower strikes reflecting primarily a spread created by the sensitivity of bid prices to the default probability. The required reserves begin to rise significantly once the attachment point rises past 10.

We present in Figure (7) the total cost as a function of the reinsurance strike with an optimal strike at 12.

# 8 Financial Hedging of Catastrophic Losses

We now consider the hedging of insurance losses via securitization as opposed to reinsurance. For this purpose we introduce a financial security that pays at $T$ the sum of loss exceedances over the level $B$ or

$$H(T) = \sum_{i=1}^{N(T)} (X_i - B)^+ .$$

Our formulation of financial securities reflecting insurance losses follows the work of Norberg (2010), Norberg and Savina (2011), though we use Lévy and diffusion

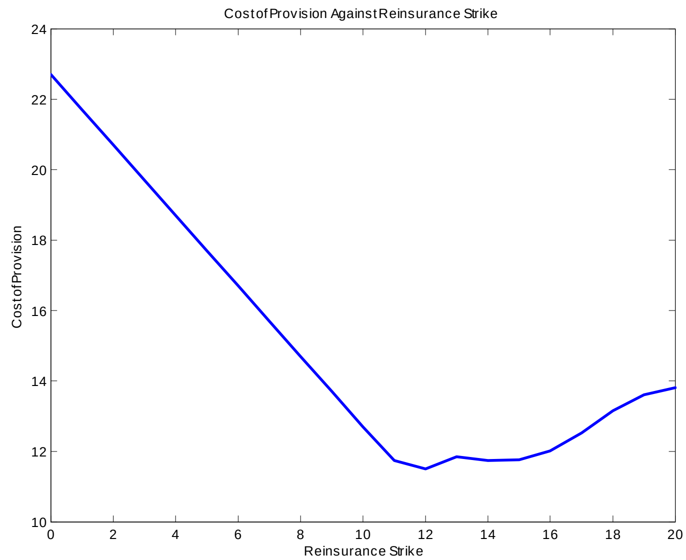


Figure 7: Optimal Reinsurance Strike minimizing the cost of provision defined as capital cost plus the cost of reinsurance.

based risks in place of Markov chains. Such processes may be approximated by Markov chains as is done for example in Mijtović and Pistorius (2009). We suppose an inhomogeneous arrival rate in the interval $(t, T)$ of the form

$$a\left(e^{-\frac{t}{\tau}} - e^{-\frac{T}{\tau}}\right).$$

The jump distribution is a gamma density $g(x)$ where

$$g(x) = \frac{c^{\gamma} x^{\gamma-1} e^{-cx}}{\Gamma(\gamma)}.$$

The arrival rate of jumps over the size $B$ however are

$$\begin{aligned} & a\left(e^{-\frac{t}{\tau}} - e^{-\frac{T}{\tau}}\right) \int_B^{\infty} \frac{c^{\gamma} x^{\gamma-1} e^{-cx}}{\Gamma(\gamma)} dx \\ = \quad & a\left(e^{-\frac{t}{\tau}} - e^{-\frac{T}{\tau}}\right) gammainc(cB, \gamma, 'upper') \end{aligned}$$

and the density is

$$\frac{c^{\gamma} x^{\gamma-1} e^{-cx}}{gammainc(cB, \gamma, 'upper')} \mathbf{1}_{x>B}$$

All a risk neutral law does to a pure jump process is to change its compensator (Jacod and Shiryaev (1980)) from

$$\nu(dx, ds)$$

to

$$Y(x,s)\nu(dx,ds)$$

for some positive adapted process $Y(s)$. Let us analyze under the hypothesis that

$$Y(x,s) = Ae^{\theta(x-a)}$$

where $\theta > 0, A > 0$ are market prices of risk.

If we now change measure to

$$Ae^{\theta(x-B)}a\left(e^{-\frac{t}{\tau}} - e^{-\frac{T}{\tau}}\right)c^{\gamma}x^{\gamma-1}e^{-cx}\mathbf{1}_{x>B}$$

We have an arrival rate of

$$\begin{aligned} & a\left(e^{-\frac{t}{\tau}} - e^{-\frac{T}{\tau}}\right)Ae^{-B}\frac{c^{\gamma}}{(c-\theta)^{\gamma}}(c-\theta)^{\gamma}x^{\gamma-1}e^{-(c-\theta)x}\mathbf{1}_{x>B} \\ = \; & a\left(e^{-\frac{t}{\tau}} - e^{-\frac{T}{\tau}}\right)Ae^{-B}\frac{c^{\gamma}}{(c-\theta)^{\gamma}}gammainc((c-\theta)B,\gamma,'upper')\frac{(c-\theta)^{\gamma}x^{\gamma-1}e^{-(c-\theta)x}\mathbf{1}_{x>B}}{gammainc((c-\theta)B,\gamma,'upper')} \end{aligned}$$

When we compute $S(t)$ it is

$$\begin{aligned} S(t) &= e^{-r(T-t)}\sum_{i=1}^{N(t)}(X_i - B)^{+} + \\ & a\left(e^{-\frac{t}{\tau}} - e^{-\frac{T}{\tau}}\right)Ae^{-B}\frac{c^{\gamma}}{(c-\theta)^{\gamma}}e^{-r(T-t)}\int_B^{\infty}(x-B)(c-\theta)^{\gamma}x^{\gamma-1}e^{-(c-\theta)x}dx \end{aligned}$$

This is

$$\begin{aligned} S(t) &= e^{-r(T-t)}\sum_{i=1}^{N(t)}(X_i - a)^{+} + \\ & a\left(e^{-\frac{t}{\tau}} - e^{-\frac{T}{\tau}}\right)Ae^{-B}\frac{c^{\gamma}}{(c-\theta)^{\gamma}}e^{-r(T-t)} \times \\ & \left(\frac{\gamma}{c-\theta}gammainc\left((c-\theta)B,\gamma+1,'upper\right) - B\; gammainc\left((c-\theta)B,\gamma,'upper'\right)\right) \end{aligned}$$

Now if we model the market price of risk as stochastic following a geometric Brownian motion process (Black and Scholes (1973), Merton (1973)) we get a stock price of the form

$$\begin{aligned} S(t) &= e^{-r(T-t)}\sum_{i=1}^{N(t)}(X_i - B)^{+} + \\ & a\left(e^{-\frac{t}{\tau}} - e^{-\frac{T}{\tau}}\right)e^{-B}\frac{c^{\gamma}}{(c-\theta)^{\gamma}}e^{-r(T-t)} \times \\ & \left(\frac{\gamma}{c-\theta}gammainc\left((c-\theta)B,\gamma+1,'upper\right) - B\; gammainc\left((c-\theta)B,\gamma,'upper'\right)\right) \times \\ & A(0)e^{\sigma W(t) - \frac{\sigma^2 t}{2}}. \end{aligned}$$

So the stock price is adapted to a Brownian motion and the exceedances. This gives us a specific jump diffusion for the financial asset that is correlated with the insurance loss process as some of the insurance losses affect directly the price of this financial asset. Hedging with the financial asset exposes us directly to financial risk that we did not have and is captured by $\sigma$ the volatility of the market price of risk.

We suppose that this particular financial asset trades in a market where the law of one price holds but the market is open and trades occur at a regular intervals with a time step $h$. The hedged loss process is then

$$L(mh) = \sum_{i=1}^{N(mh)} X_i + \sum_{j=0}^{(m-1)h} a_j \left(S((j+1)h) - S(jh)e^{rh}\right)$$

where the hedge position is $a_j$ at time $jh$. We may seek to find hedge functions of the form $a_j(S)$ with the hedge adapted to the level of the financial asset. The hedge is financed and the cash flow to the hedge asset has a zero risk neutral expectation. We distort the same risk neutral measure for the bid and ask prices to determine the local capital minimizing hedge position. One would expect to do more hedging when the market price of risk is low and less when it is high.

## 8.1 Details of hedge design

Suppose the financial security has a maturity of 10 years and we perform a quarterly backward recursion beginning at $T = 5$. This computation is repeated at each quarter end $n$ and state $i$ and following the recursion all the way back to time 0 yields the optimal dynamic quarterly financial hedging scheme for the insurance product under study. The exercise may then be repeated for different settings to study the effects of managing insurance losses via a securitization regime. We shall consider minimizing here just the local risk for which the only state variable of interest will be the level of the Brownian motion in the market price of risk. First we consider the static case and then go on to the dynamic case.

### 8.1.1 The Static Case

Consider first a one period situation where the loss is

$$\sum_{i=N(s)}^{N(t)} X_i$$

The change in the stock price is

$$\Delta S = \sum_{i=N(s)}^{N(t)} (X_i - a)^+ - E\left[\sum_{i=N(s)}^{N(t)} (X_i - a)^+\right] + E\left[\sum_{i=N(s)}^{N(t)} (X_i - a)^+\right]\left(e^{\sigma\sqrt{t-s}z - \frac{\sigma^2(t-s)}{2}} - 1\right)$$

The first part is the jump martingale for the effects of the securitized losses on the price of the financial security while the second represents the effects of changes in the market price of risk. The static hedged position with hedge position $\alpha$ is

$$C = \sum_{i=N(s)}^{N(t)} X_i + \alpha\Delta S$$

We take a risk neutral process with asymptotic arrivals $a = 150$. The average arrival time is $\tau = 12.5$. The mean of the claims is .2540 with a volatility of .3175. The exceedance level is set at .5. The volatility of the market price of risk is 10%.

We now consider as potential hedging criteria two classical criteria like the variance and certainty equivalents under exponential utility. In addition we consider capital minimization defined as ask less bid using minmaxvar. We consider two levels of risk aversion for the utility function, .5 and 2. We also take two stress levels for the distortion, .75 and 1.5.

We present in Figure 8 a graph of the five criterion functions in our context of inhomogeneous Poisson with compound gamma losses and a hedge position in a financial security locking into cumulated exceedances over .5. We observe that certainty equivalents are quite asymmetric in their effects on the hedging criterion. Variance as already noted is symmetric but lacks a parameter. We shall continue with capital minimization in the dynamic exercise.

The optimal hedge positions are as follows.

| Criterion | Hedge |
|---|---|
| Variance | -1.7 |
| CE Low Risk Aversion | -1.65 |
| CE High Risk Aversion | -1.54 |
| Capital Low Stress | -1.61 |
| Capital High Stress | -1.53 |

As the risk aversion or the stress level rises the hedge position is reduced in absolute value.

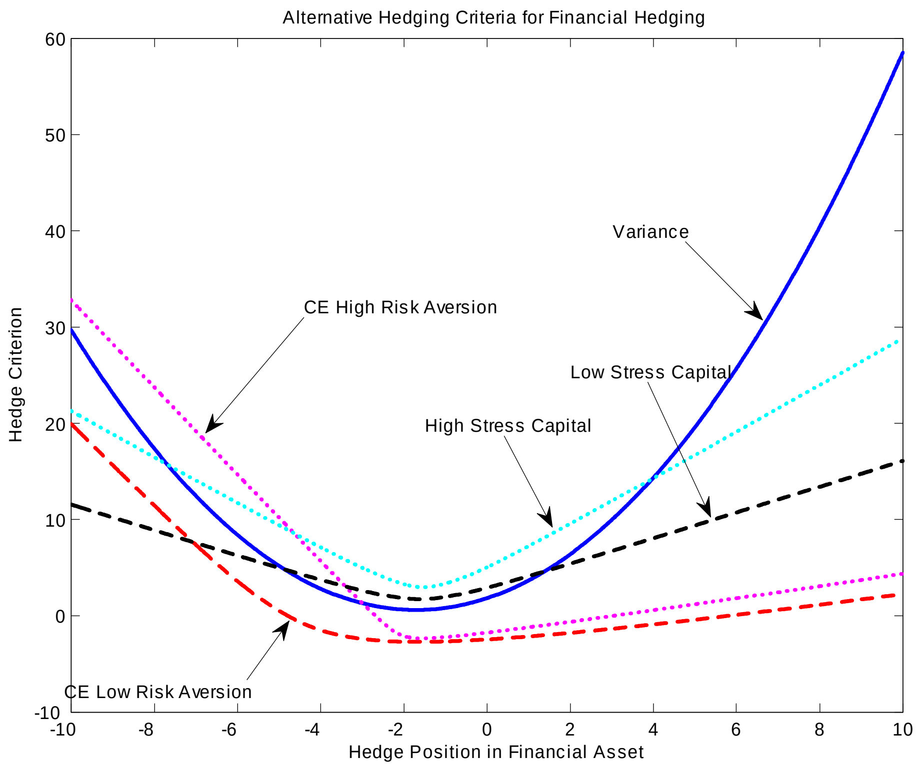


Figure 8: Alternative Hedging Criteria for Inhomogeneous Poisson Compound Gamma Losses hedged by security tracking cumulated exceedances

## 8.2 Dynamic Financial Hedging using a Securitized Loss Process for the Hedging Asset

We determine here the hedge $\alpha$ to minimize the capital for the local risk in present value terms

$$e^{r(t+h)}\sum_{i=N(t)}^{N(t+h)} e^{-rs_i}X_i + \alpha\left(S(t+h) - S(t)e^{rh}\right)$$

$$= \sum_{i=N(t)}^{N(t+h)} e^{-rs_i}X_i$$

$$+\alpha\left(\begin{array}{c} e^{-rT}\left(\sum_{i=N(t)}^{N(t+h)}(X_i-B)^+ - E_t\left(\sum_{i=N(t)}^{N(t+h)}(X_i-B)^+\right)\right) \\ +a\left(e^{-\frac{t}{\tau}} - e^{-\frac{T}{\tau}}\right)e^{-B}\frac{c^\gamma}{(c-\theta)^\gamma}e^{-rT}\times \\ \left(\frac{\gamma}{c-\theta}gammainc\left((c-\theta)B,\gamma+1,'upper\right) - B\ gammainc\left((c-\theta)B,\gamma,'upper'\right)\right)\times \\ A(0)e^{\sigma W(t)-\frac{\sigma^2 t}{2}}\left(e^{\sigma(W(t+h)-W(t))-\sigma^2 h/2} - 1\right) \end{array}\right)$$

So the only state variable relevant to the hedge is the level of $W(t)$. The rest is the risk going forward much like our static model. We span the levels of $A(nh)$ with $A(0) = 1.5$ by $A_{ni}$ and for each of these levels we determine $\alpha_i$. The hedge is just a function of the market price of risk process. We first determine this function at each quarter.

For the parameter settings specified I graph in Figure 9 the hedge position as a function of the market price of risk.

### 8.2.1 Dynamic Recursion for Bid and Ask using Financial Hedge Strategy

We begin at time 5 and determine the terminal bid and ask values as a function of loss levels to date without a hedge as usual.

We now determine matrices $BMV(21,21,20)$, $AMV(21,21,20)$ and $EMV(21,21,20)$ for 21 states of loss levels, 21 levels for the market price of risk and 19 quarters as follows. At the twentieth quarter all columns are the same as we have no hedge. Assume we have the values one time step later.

At time step $n$ with loss level $L_i$ and market price of risk $A_j$ we generate the next loss present value loss as

$$L' = L_i + yL + \alpha_{jn}\left(\begin{array}{c} e^{-rT}\left(\sum_{i=N(t)}^{N(t+h)}(X_i-B)^+ - E_t\left(\sum_{i=N(t)}^{N(t+h)}(X_i-B)^+\right)\right) \\ +a\left(e^{-\frac{t}{\tau}} - e^{-\frac{T}{\tau}}\right)e^{-B}\frac{c^\gamma}{(c-\theta)^\gamma}e^{-rT}\times \\ \left(\frac{\gamma}{c-\theta}gammainc\left((c-\theta)B,\gamma+1,'upper\right) - B\ gammainc\left((c-\theta)B,\gamma,'upper'\right)\right)\times \\ A(0)e^{\sigma W(t)-\frac{\sigma^2 t}{2}}\left(e^{\sigma(W(t+h)-W(t))-\sigma^2 h/2} - 1\right) \end{array}\right)$$

We also generate the next market price of risk as

$$A' = A_j e^{\sigma(W(t+h)-W(t))-\sigma^2 h/2}.$$

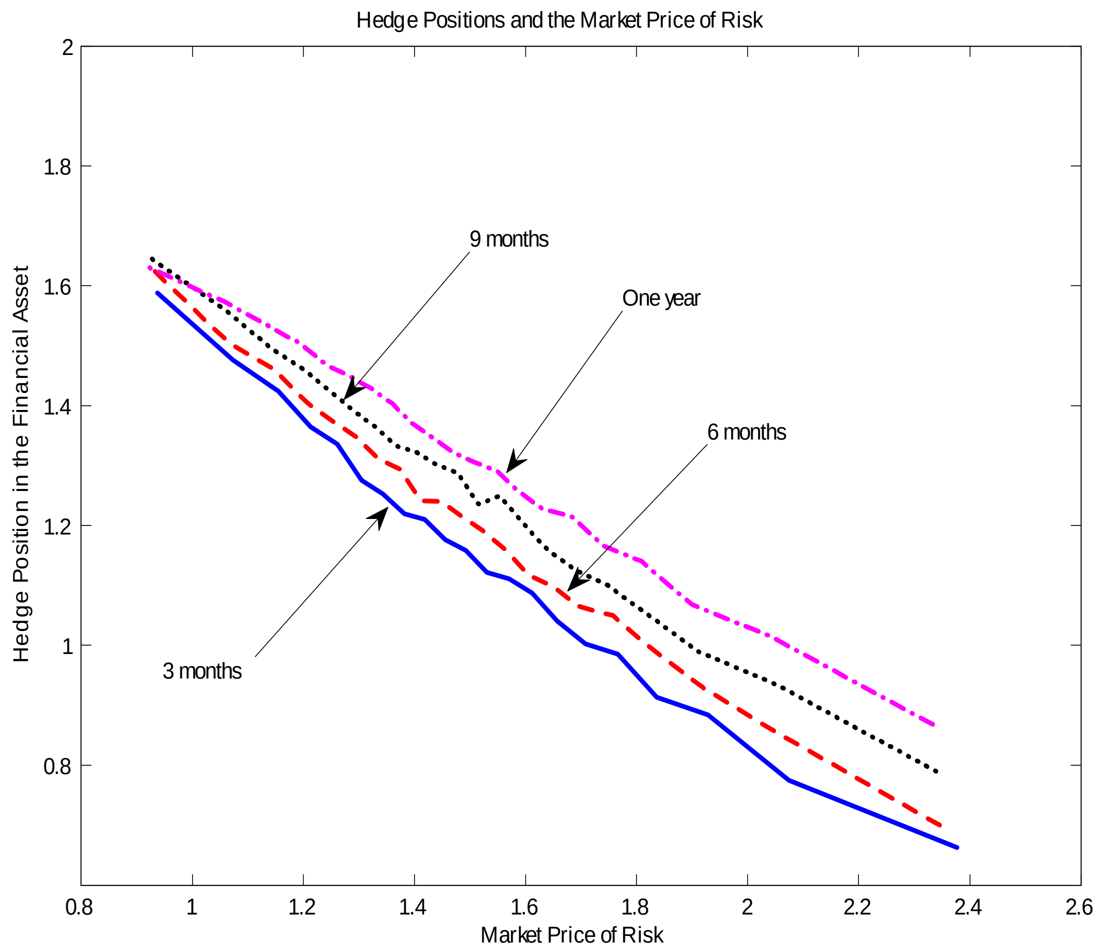


Figure 9: Capital Minimizing Hedge Positions in the Financial Security as a function of time and the market price of risk. The mean price was set at 1.5

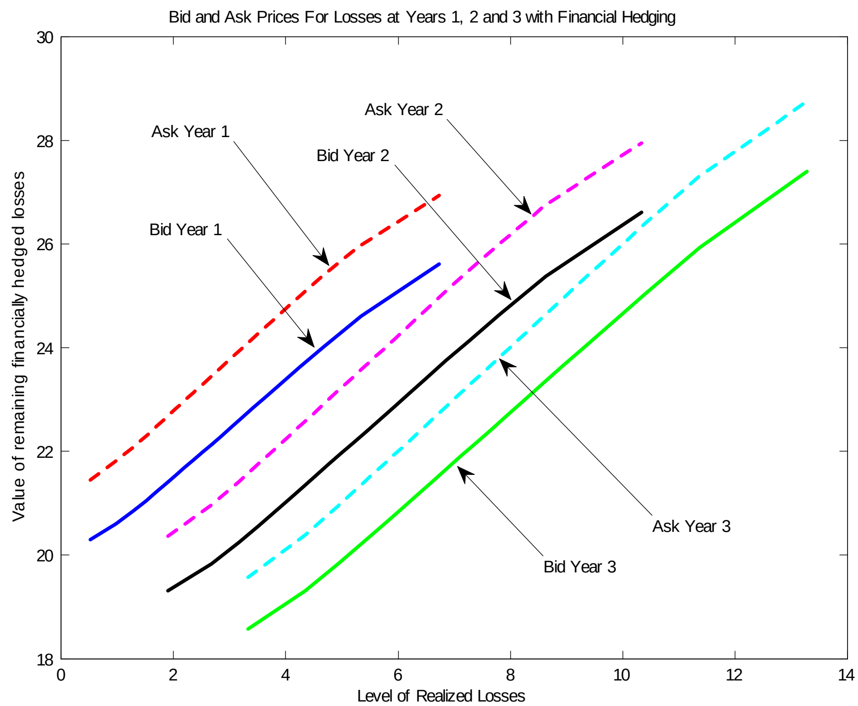


Figure 10: Bid and Ask Prices as functions of realized losses at years 1, 2 and 3 as a function of realized losses.

We use these to interpolate for $V^a_{n+1}, V^b_{n+1}$ from the stored grid. We then take expectations, residuals, distorted expectation penalties to determine $V^a_{ij}, V^b_{ij}$ and then go back down to time zero. This develops our locally financially hedged dynamic recursion.

We implemented this dynamic recursion and present in Figure 10 the Bid and Ask prices at years 1, 2 and 3 when the market price of risk is at the median.

# 9 Life Insurance with Stochastic Mortality

We now consider the hedging of life insurance risk via securitization in a stochastic mortality context. Similar models for the evolution of mortality risk have been employed in Dahl (2004) and Dahl and Møller (2006). The financial security we introduce will be a call option on the realized mortality rate at a specific future date. We model first the mortality rate and then a financial security based on this rate. We allow for jumps in the mortality rate and take them to be exponentially distributed with mean $\zeta$ and arrival rate $\lambda$.

The process for the mortality rate at time $t$, $y(t)$ is given by a solution to the Ornstein-Uhlenbeck ($OU$) equation

$$dy = -\kappa y dt + dU(t)$$

where the driving process $U(t)$ is the exponential compound Poisson process. For the computation of survival probabilities we are interested in the law of

$$Y(t) = \int_0^t y(u) du.$$

We therefore wish to access the joint law of $Y(t), y(t)$ via the joint characteristic function

$$\Phi_t(a, b) = E\left[\exp\left(iaY(t) + iby(t)\right)\right].$$

We may write

$$\begin{aligned} U(t) &= (x * \mu_U)_t \\ &= \int_0^t \int_0^\infty x \mu_U(dx, ds) \end{aligned}$$

where $\mu_U$ is the integer valued random measure associated with the jumps of $U$.

We also have that

$$y(t) = y(0)e^{-\kappa t} + \int_0^t \int_0^\infty e^{-\kappa(t-u)} x \mu_U(dx, du)$$

It follows that

$$\begin{aligned} Y(t) &= y(0)\frac{1-e^{-\kappa t}}{\kappa} + \int_0^t \int_0^s \int_0^\infty e^{-\kappa(s-u)} x \mu_U(dx, du) ds \\ &= y(0)\frac{1-e^{-\kappa t}}{\kappa} + \int_0^t \int_0^\infty \frac{1-e^{-\kappa(t-u)}}{\kappa} x \mu_U(dx, du) \end{aligned}$$

Hence we have that

$$\begin{aligned} iaY(t) + iby(t) &= \left( ia\frac{1-e^{-\kappa t}}{\kappa} + ibe^{-\kappa t} \right) y(0) \\ &\quad + \int_0^t \int_0^\infty \left( ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ibe^{-\kappa(t-u)} \right) x \mu_U(dx, du) \end{aligned}$$

We then have that

$$\begin{aligned} \Phi_t(a, b) &= \exp\left( \left( ia\frac{1-e^{-\kappa t}}{\kappa} + ibe^{-\kappa t} \right) y(0) \right) \times \\ &\quad E\left[ \exp\left( \int_0^t \int_0^\infty \left( ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ibe^{-\kappa(t-u)} \right) x \mu_U(dx, du) \right) \right] \end{aligned}$$

Now define the compensated jump martingale

$$n(t) = \left(\exp\left(\left(ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ibe^{-\kappa(t-u)}\right)x\right) - 1\right) * (\mu_U(dx,du) - k_U(x)dxdu)$$

where $k_U$ is the Lévy measure for $U(t)$. The stochastic exponential

$$N = \mathcal{E}(n)$$

is a martingale and

$$\begin{aligned}
N(t) &= \exp\left(\int_0^t\int_0^\infty \left(ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ibe^{-\kappa(t-u)}\right)x\mu_U(dx,du)\right)\times \\
&\quad \exp\left(-\int_0^t\int_0^\infty \left(\exp\left(\left(ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ibe^{-\kappa(t-u)}\right)x\right) - 1\right)k_U(x)dx\right) \\
&= \exp\left(\int_0^t\int_0^\infty \left(ia\frac{1-e^{-\kappa(t-u)}}{\kappa} + ibe^{-\kappa(t-u)}\right)x\mu_U(dx,du)\right)\times \\
&\quad \exp\left(-\int_0^t \psi_U\left(a\frac{1-e^{-\kappa(t-u)}}{\kappa} + be^{-\kappa(t-u)}\right)du\right)
\end{aligned}$$

It follows that

$$\Phi_t(a,b) = \exp\left(\int_0^t \psi_U\left(a\frac{1-e^{-\kappa(t-u)}}{\kappa} + be^{-\kappa(t-u)}\right)du\right)$$

Now make the change of variable

$$\begin{aligned}
v &= e^{-\kappa(t-u)} \\
dv &= \kappa v du
\end{aligned}$$

and write

$$\Phi_t(a,b) = \exp\left(\int_{e^{-\kappa t}}^1 \frac{\psi_U(\frac{a}{\kappa}(1-v)+bv)}{\kappa v}dv\right)$$

In the special case of the exponential compound Poisson we have

$$\psi_U(u) = \lambda\left(\frac{1}{1-iu\zeta} - 1\right)$$

and we wish to integrate

$$\begin{aligned}
&\frac{\lambda}{\kappa v}\left(\frac{1}{1-i\zeta\left(\frac{a}{\kappa}(1-v)+bv\right)} - 1\right) \\
= &\frac{\lambda i\zeta\left(\frac{a}{\kappa}(1-v)+bv\right)}{\kappa v\left(1-i\zeta(\frac{a}{\kappa}(1-v)+bv\right)}
\end{aligned}$$

This is of the form $(a+bx)/(dx+ex^2)$ and

$$\int \frac{a+bx}{dx+ex^2} = \frac{1}{ed}\left((bd-ea)\log(d+ex) + ea\log(x)\right)\right).$$

Hence we have access to the joint characteristic function of $Y(t)$ and $y(t)$. For our example

$$\begin{aligned} a &= \frac{\lambda ai\zeta}{\kappa}; b = \lambda bi\zeta - \frac{\lambda ai\zeta}{\kappa} \\ d &= \kappa - ai\zeta; e = -bi\zeta\kappa + ai\zeta \end{aligned}$$

We evaluate at $x = 1$ and $x = e^{-\kappa t}$ and take the difference for the joint characteristic function.

## 9.1 Pricing term life under this stochastic mortality model

As observed the price of term life of maturity $t$ is given by

$$\begin{aligned} w(t) &= -1000000 \times \int_0^t e^{-ru} S'(u) du. \\ &= 1000000 \times \left(1 - e^{-rt} S(t)\right) - r \int_0^t S(u) e^{-ru} du \end{aligned}$$

and we need to compute

$$S(t) = E\left[\exp\left(-\int_0^t y(s)ds\right)\right]$$

But this is just the characteristic function of integrated term life taken at $1i$. The continuous coupon rate is

$$c(t) = \frac{rw(t)}{1 - \exp(-rt)}.$$

For an interest rate of 3% with a mean reversion rate of 1.5 and one arrival of jump every 10 years with a mean of 10 basis points and an initial mortality rate of 20 basis points the term life premiums for the five years are as follows.

| | |
|---|---|
| 1 | 1064 |
| 2 | 1305 |
| 3 | 1362 |
| 4 | 1377 |
| 5 | 1382 |

If the mean reversion rate is reduced to .5 the result is as follows.

| | |
|---|---|
| 1 | 1608 |
| 2 | 2575 |
| 3 | 3157 |
| 4 | 3507 |
| 5 | 3717 |

## 9.2 Pricing the mortality security

The future value of the security should be a martingale and hence a conditional expectation. We consider here an option written on the mortality rate at time $T$, $y(T)$. This requires that we price the claim

$$(y(T)-a)^+$$

This is given by

$$w(a)=\int_a^\infty (y-a)g(y)dy$$

where $g$ is the density of $y$. We have in closed form

$$\phi(u)=\int_0^\infty e^{iuy}g(y)dy.$$

We follow Carr and Madan (1999) and consider the Fourier transform

$$\begin{aligned}
&\int_0^\infty e^{iua}w(a)da \\
=\;& \int_0^\infty e^{iua}\int_a^\infty (y-a)g(y)dyda \\
=\;& \int_0^\infty dyg(y)\int_0^y e^{iua}\left(y-a\right)da \\
=\;& \int_0^\infty dyg(y)\left[\frac{y}{iu}\left(e^{iuy}-1\right)-\int_0^y ae^{iua}da\right] \\
=\;& \int_0^\infty dyg(y)\left[\frac{y}{iu}\left(e^{iuy}-1\right)-\left(\frac{ye^{iuy}}{iu}-\frac{1}{iu}\int_0^y e^{iua}da\right)\right] \\
=\;& \int_0^\infty dyg(y)\left[\frac{y}{iu}\left(e^{iuy}-1\right)-\left(\frac{ye^{iuy}}{iu}+\frac{1}{u^2}\left(e^{iuy}-1\right)\right)\right] \\
=\;& \int_0^\infty dyg(y)\left[\frac{1}{u^2}-\frac{e^{iuy}}{u^2}-\frac{y}{iu}\right] \\
=\;& \frac{1-\phi(u)}{u^2}-\frac{E[y]}{iu} \\
=\;& \gamma(u)
\end{aligned}$$

Fourier inversion will give us the option price that we can use for hedging. We will work with the forward price as this is equivalent to trading the futures contract and abstract from issues related to discounting.

We know that the whole density is concentrated in the domain of basis points. Hence we may work with

$$h=10000y$$

with

$$\phi_h(u)=E[e^{iuh}]=\phi_y(10000u).$$

We now wish to evaluate

$$\begin{aligned} w(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\gamma(u)du \\ &= \frac{1}{\pi}\,\mathrm{real}\left(\int_{0}^{\infty} e^{-iuk}\gamma(u)du\right) \\ &\approx \frac{1}{\pi}\,\mathrm{real}\left(\sum_{j=1}^{N} e^{-iu_j\eta k}\gamma(u_j)\right) \end{aligned}$$

where we have $u_j = (j-1)\eta$.

Now the $FFT$ (Fast Fourier Transform) computes

$$w(k) = \sum_{j=1}^{N} e^{-1i\frac{2\pi}{N}(j-1)(k-1)}x(j) \quad \text{for } k = 1, \cdots, N$$

We could take

$$k_i = \lambda(i-1)$$

so our strikes go from zero to $(N-1)\lambda$. We than have

$$\begin{aligned} w(k_i) &\approx \frac{1}{\pi}\,\mathrm{real}\left(\sum_{j=1}^{N} e^{-1iu_j\eta k_i}\gamma(u_j)\right) \\ &= \frac{1}{\pi}\,\mathrm{real}\left(\sum_{j=1}^{N} e^{-1i\eta\lambda(j-1)(i-1)}\gamma(u_j)\right) \end{aligned}$$

With $\eta\lambda = \frac{2\pi}{N}$ we get

$$w(k_i) \approx \frac{1}{\pi}\,\mathrm{real}\left(\sum_{j=1}^{N} e^{-1i\frac{2\pi}{N}(j-1)(i-1)}\gamma(u_j)\right)$$

So for our choice of $u_j$ and $k_i$ we apply the $FFT$ to the sequence $\gamma(u_j)$ and we obtain $\pi w(k_i)$ in the real part.

## 9.3 Other Stochastic Mortality Processes

We develop here for possible future applications the joint law of mortality and integrated mortality for some mean reverting processes driven by other processes.

### 9.3.1 Gamma driven mortality rate

Suppose that instead of an exponential compound Poisson driver we have for $U(t)$ a gamma process that has infinite activity with many small jumps and some bigger ones. In this case

$$E[\exp(iuU(t))] = \left(\frac{c}{c-iu}\right)^{\gamma}$$

and

$$\begin{aligned}\psi_U(u) &= \gamma\left(\ln(c) - \ln(c - iu)\right) \\ &= \int_0^\infty \left(e^{iux} - 1\right) k_U(x) dx\end{aligned}$$

We need to evaluate

$$\exp\left(\int_0^t \int_0^\infty \left(\exp\left(\left(ia\frac{1 - e^{-\kappa(t-u)}}{\kappa} + ibe^{-\kappa(t-u)}\right) x\right) - 1\right) k_U(x) dx du\right)$$

and this is

$$\exp\left(\int_0^t \gamma\left(\ln(c) - \ln(c - i\left(a\frac{1 - e^{-\kappa(t-u)}}{\kappa} + be^{-\kappa(t-u)}\right)\right) du\right)$$

Now make the change of variable

$$\begin{aligned}v &= e^{-\kappa(t-u)} \\ dv &= \kappa v du\end{aligned}$$

which gives us

$$\frac{\gamma}{\kappa}\int_{e^{-\kappa t}}^1 \frac{\left(\ln(c) - \ln(c - i(\frac{a}{\kappa}(1 - v) + bv)\right)}{v} dv$$

We now need to integrate for the joint characteristic function this expression which is of the form

$$\frac{\ln(c) - \ln(c + \alpha + \beta x)}{x}.$$

This has a solution with

$$\begin{aligned}\int \frac{\ln(c) - \ln(c + \alpha + \beta x)}{x} &= Li_2\left(-\frac{\beta x}{\alpha + c}\right) - \log(x) \times \\ &\quad \left((\alpha + \beta x + c) - \log\left(\frac{\beta x}{\beta + c} + 1\right)\right) + \\ &\quad \log(c)\log(x)\end{aligned}$$

Hence defining

$$\alpha = \frac{-ia}{\kappa}; \beta = i\left(\frac{a}{\kappa} - b\right)$$

we have that

$$\Phi_t(a, b) = \exp\left(\left(ia\frac{1 - e^{-\kappa t}}{\kappa} + ibe^{-\kappa t}\right) y(0)\right) \times \exp\left(G(1) - G(e^{-\kappa t})\right)$$

where

$$G(x) = \frac{\gamma}{\kappa}\left(\begin{array}{c} Li_2\left(-\frac{\beta x}{\alpha+c}\right) - \log(x)\times \\ \left((\alpha+\beta x+c) - \log\left(\frac{\beta x}{\beta+c}+1\right)\right) + \\ \log(c)\log(x) \end{array}\right)$$

Hence we can accommodate gamma driven mean reverting mortality rates.

The term life premiums for at a 3% interest rate, with $\kappa = 1.5$ , mean jump of .005 a volatility of 5% and an initial mortality rate 20 basis points is as follows.

| | |
|---|---|
| 1 | 2493 |
| 2 | 3266 |
| 3 | 3567 |
| 4 | 3712 |
| 5 | 3793 |

### 9.3.2 Inverse Gaussian

An increasing process is given by the time taken by Brownian motion with drift $c$ to reach level $t$. This time is denoted $T_t^c$ and

$$E[\exp(iuT^c)] = \exp\left(-\sqrt{c^2-2iu}+c\right)$$

in which case for $U(t) = T_t^c$ we have that

$$\psi_U(u) = c - \sqrt{c^2 - 2iu}$$

and the function we wish to integrate is

$$\frac{c - \sqrt{c^2 - 2i\left(\left(\frac{\alpha}{\kappa}(1-v)+bv\right)\right)}}{\kappa v}$$

which is of the form

$$\frac{c-\sqrt{c^2+\alpha+\beta x}}{x}$$

and

$$\begin{aligned}\int \frac{c-\sqrt{c^2+\alpha+\beta x}}{x} &= -2\sqrt{\alpha+\beta x+c^2} + \\ & 2\sqrt{-\alpha-c^2}\tan^{-1}\left(\frac{\sqrt{\alpha+\beta x+c^2}}{\sqrt{-\alpha-c^2}}\right) + \\ & c\log(\beta x)\end{aligned}$$

Hence we have a variety of possible mortality models based on exponential compound Poisson (ECP), Gamma (G), Inverse Gaussian (IG) along with $CIR$.

We could report on the differences in term life premia in these three models and how to choose between them.

In the IG case we have

$$
\begin{aligned}
\Phi_t(a,b) &= \exp\left(\left(ia\frac{1-e^{-\kappa t}}{\kappa}+ibe^{-\kappa t}\right)y(0)\right)\times \\
&\quad \exp\left(K(1)-K(e^{-\kappa t})\right) \\
K(x) &= \frac{1}{\kappa}\left(\begin{array}{c} -2\sqrt{\alpha+\beta x+c^2}+ \\ 2\sqrt{-\alpha-c^2}\tan^{-1}\left(\frac{\sqrt{\alpha+\beta x+c^2}}{\sqrt{-\alpha-c^2}}\right)+ \\ c\log(\beta x)\end{array}\right) \\
\alpha &= -2ia/\kappa \\
\beta &= 2ia/\kappa-2ib
\end{aligned}
$$

We therefore have CIR, ECP, GD, IGD possibilities for the spot rate and the mortality rate that may be calibrated to the yield curve and the term structure of term life premia.

For a 3% interest and with $\kappa = 1.5$, $c = 500$ and $y(0) = .002$ the termlife premia are as follows.

| | |
|---|---|
| 1 | 1671 |
| 2 | 2162 |
| 3 | 2338 |
| 4 | 2416 |
| 5 | 2460 |

### 9.3.3 CIR case

Finally we consider the case when $y(t)$ is just the CIR process in which case $\Phi_t(a,b)$ is well known (See for example (Lamberton and Lapeyre (1996)).

With an interest rate of 3%, $\kappa = 1.5$, $\theta = .003$, $\lambda = .1$ and $y(0) = .002$ the termlife premia are as follows.

| | |
|---|---|
| 1 | 2475 |
| 2 | 2669 |
| 3 | 2759 |
| 4 | 2807 |
| 5 | 2835 |

### 9.3.4 Results on Exponential Compound Poisson Mortality Process

I use a CIR rates process with mean reversion 0.3712, long term rate 0.0477, volatility 0.0599 and initial spot rate 0.0004. For the mortality process the parameters are mean reversion 1.5, with a jump arrival rate of one jump every two years of $\lambda = .5$, with a mean jump of 30 basis points and an initial mortality of 20 basis points.

The term life premiums on this mortality process with a 3% interest rate and five annual maturities are as follows.

| Term | Premium |
|---|---|
| 1 | 1511 |
| 2 | 1936 |
| 3 | 2080 |
| 4 | 2142 |
| 5 | 2175 |

I employ an annual tenor for five years. We first construct a grid of possible spot rates and mortality rates that may be attained in the coming five years. I take 9 levels for each at the deciles of the marginal distribution for the rate and mortality processes. I present the third, fifth and seventh deciles at the year ends one, three and five for the level of the spot rate and the mortality rate.

Spot Rate Levels

| | Years | | |
|---|---|---|---|
| Deciles | 1 | 3 | 5 |
| 3 | 1.22 | 2.61 | 3.27 |
| 5 | 1.46 | 3.11 | 3.90 |
| 7 | 1.72 | 3.67 | 4.59 |

Mortality Rates

| | Years | | |
|---|---|---|---|
| Deciles | 1 | 3 | 5 |
| 3 | 9 | 10 | 10 |
| 5 | 14 | 15 | 15 |
| 7 | 24 | 25 | 25 |

I assume that each year we have 4000 subscribers with all deaths being replaced by new subscribers on a million dollar 5 year term life insurance with a premium of 2171 per policy. One is thereby collecting 8.69 million dollars in premiums each year.

There are two hedging assets available each year and they are a constant maturity 3 year bond and the forward price of a call option on the level of mortality in year 10 struck at 25 basis points. At each year end working back from year 4 down to year 1, we have 81 states represented by 9 deciles for the spot rate and 9 deciles for the mortality rate. This gives us $324 = 81 * 4$ points of computation. At each of these points we compute the expected value, ask price and bid price of the hedged business along with the variance minimizing hedge positions in the constant maturity 3 year bond and the 10 year mortality call option. We take year five as a terminal year with expected values equal to bid values and ask values and all these values are zero. They represent the present value of the business continuing past year five and since we terminate in five years this is zero.

Beginning in year 4, for each of the 81 points on the grid we first compute the mortality rate specific survival probability for a year. For a mortality rate of $g$ this is given by

$$E\left[\exp\left(-\int_0^1 y(u)du\right)|y(0)=g\right].$$

We then simulate 10000 readings on the possible deaths among our population of 4000 individuals. The cash flow for the annual period is the total premium collected less the number of million dollar payouts $N$ associated with each death.

For the purposes of our hedge we define an aggregate cash flow to be hedged as the expected value of this payout plus the expected value of the business at year end. To compute this value we simulate 10000 final states for the spot rate $r' = r(t+1)$ and the mortality rate $y' = y(t+1)$ from the conditional distribution for $r(t+1)$ given $r(t) = r$ and $y(t+1)$ given $y(t) = y$. We then interpolate from our stored grid of expected values one year later as a function of final states the expected value $V' = V(t+1)$ given the pair $(r', y')$.

We next simulate 10000 values for

$$a = \int_0^1 r(u)du, \text{ given } r(t) = r, \text{ and } r(t+1) = r'$$

The cash flow to be hedged is

$$C = \exp(-a) * (4000 * prem - 1000000 * N + V')$$

and we have 10000 readings on this cash flow.

We now construct the cash flows on our hedging assets. For this we have to evaluate the price $P(r,3)$ of a 3 year pure discount bond when the spot rate is $r$ and the price $P'(r',2)$ one year later when the spot rate is $r'$ and the maturity is now 2 years. The cash flow to the bond hedge is

$$P(r',2) - P(r,3).$$

Similarly we determine $C(25, 10-t, y)$ and $C(25, 10-(t+1), y')$ the prices for the 10 year 25 basis point strike call to get the cash flow on the option as

$$C(25, 10-(t+1), y') - C(25, 10-t, y).$$

This gives us 10000 readings on our two hedging assets that are stored in a matrix $H$ that is 2 by 10000.

We next determine the residual cash flow for positions $x$ by

$$R = C + x'H$$

and we select $x$ to minimize the variance of $R$ and the capital required for holding $R$ using minmaxvar at stress level 0.75. These are hedge positions in the three year pure discount bond and the mortality security.

Using the capital minimizing hedge $x^*$ we define the optimal residual cash flow

$$R^* = C + x^{*\prime}H$$

and the expected value of the business at the grid point $(t, r, y)$ is the mean of $R^*$. The bid and ask prices are given by

$$b(t,r,y) = DistortedExpectation(R^*, .75)$$
$$a(t,r,y) = -DistortedExpectation(-R^*, .75).$$

We present the bid, ask, expected values and the bond, option hedge positions at the third fifth and seventh deciles for years 1 and 3. These are presented in five pairs of tables with each pair referring to years one and three.

Bid Values (millions)

| Year One | | | |
|---|---|---|---|
| Mortality | Rate Decile | | |
| Decile | 3 | 5 | 7 |
| 3 | 8.76 | 8.72 | 8.73 |
| 5 | 8.11 | 8.04 | 8.11 |
| 7 | 6.42 | 6.47 | 6.51 |
| Year Three | | | |
| 3 | 4.33 | 4.33 | 4.29 |
| 5 | 3.57 | 3.49 | 3.56 |
| 7 | 2.30 | 2.31 | 2.34 |

Ask Values (millions)

| Year One | | | |
|---|---|---|---|
| Mortality | Rate Decile | | |
| Decile | 3 | 5 | 7 |
| 3 | 14.35 | 14.34 | 14.27 |
| 5 | 13.96 | 13.83 | 13.89 |
| 7 | 12.63 | 12.60 | 12.79 |
| Year Three | | | |
| 3 | 9.72 | 9.76 | 9.95 |
| 5 | 9.13 | 9.26 | 9.18 |
| 7 | 8.37 | 8.23 | 8.33 |

Expected Values (millions)

| Year One | | | |
|---|---|---|---|
| Mortality | Rate Decile | | |
| Decile | 3 | 5 | 7 |
| 3 | 11.77 | 11.70 | 11.69 |
| 5 | 11.21 | 11.14 | 11.71 |
| 7 | 9.75 | 9.74 | 9.81 |
| Year Three | | | |
| 3 | 7.09 | 7.07 | 7.11 |
| 5 | 6.38 | 6.43 | 6.43 |
| 7 | 5.40 | 5.37 | 5.42 |

Bond Position (thousands)

| Year One | | | |
|---|---|---|---|
| Mortality | Rate Decile | | |
| Decile | 3 | 5 | 7 |
| 3 | −19.43 | −18.94 | −19.71 |
| 5 | −19.07 | −18.35 | −18.56 |
| 7 | −17.24 | −17.22 | −17.10 |
| Year 3 | | | |
| 3 | −7.48 | −7.45 | −7.47 |
| 5 | −6.79 | −6.77 | −6.80 |
| 7 | −5.77 | −5.77 | −5.80 |

Mortality Option Position(thousands)

| Year One | | | |
|---|---|---|---|
| Mortality | Rate Decile | | |
| Decile | 3 | 5 | 7 |
| 3 | 7.65 | 7.24 | 8.02 |
| 5 | 7.86 | 7.21 | 7.39 |
| 7 | 7.48 | 7.48 | 7.29 |
| Year 3 | | | |
| 3 | 0.39 | 0.38 | 0.36 |
| 5 | 0.42 | 0.35 | 0.37 |
| 7 | 0.38 | 0.41 | 0.39 |

The exercise may be repeated for gamma, inverse Gaussian and CIR driven mortality processes.

# 10 Empirical Results

We wish to ascertain empirically relevant values for estimates of credit risk and especially liquidity risk parameters like $\gamma$ introduced in our theoretical development. Estimates for credit may to some extent be available from the market for credit default swaps. To gain some insights into these matters we proxy the daily high as a supremum of valuations and treat it like an ask price while the daily low we treat like a bid price. The daily return we suppose here is log normal with a volatility estimated from returns. We then apply our analytical procedures for bid and ask prices to infer the credit and liquidity risk parameters.

We recognize that the prices of publicly traded assets like stocks are not ideal candidates for data on contracts relevant to two price equilibria. More appropriately on would seek information on divesting insurance liabilities, selling real estate or cancelling long term contracts. However, such information is hard to come by. One may nonetheless learn about our liquidity parameters from data on publicly traded assets by taking as conservative readings on bid and ask the daily, weekly of monthly low or high price. For an initial investigation we consider as a first step the daily low and high prices.

Let $p_t$ denote a price series like the closing price on a stock. We have in addition data for $b_t, a_t$ for bid and ask prices that we proxy as stated by the daily high and low price. We scale positions for to a dollar investment with the risk of

$$r_{t+1} = \frac{p_{t+1} - p_t}{p_t},$$

if we unwind next day at market close.

This risk has a volatility exposure at any time. In addition there is a credit exposure associated with the stock going to zero and the return being $-1$. To allow for both we model risk neutrally under a lognormal return model that

$$r_{t+1} = \left\{ \begin{bmatrix} -1 & probability\ \lambda \\ e^{\theta + \sigma z - \frac{\sigma^2}{2}} - 1 & probability\ (1-\lambda) \end{bmatrix} \right. ,$$

where $z$ is a standard normal variate.

Exact expressions for bid and ask prices may be deduced following the methods of section 3. We now write

$$\begin{aligned} \frac{b_{it}}{p_t} &= E\left[\psi^{\gamma_{it}}\left(\lambda_{it} + (1-\lambda_{it})N(\sigma_{it} + z)\right)\right] + u_{it} \\ \frac{a_{it}}{p_{it}} &= E\left[\psi^{\gamma_{it}}\left((1-\lambda_{it}) - (1-\lambda_{it})N(\sigma_{it} + z)\right)\right)] + v_{it} \end{aligned}$$

We seek to estimate $\lambda_{it}, \gamma_{it}$ by minimizing

$$\eta(\lambda_{it}, \gamma_{it}) = \sqrt{\frac{u_{it}^2 + v_{it}^2}{2}},$$

with $\sigma_{it}$ estimated from 21 past daily returns.

For data on 38 stocks starting in January 2007 to the end of 2010 we estimated $\sigma_{it}, \lambda_{it}, \gamma_{it}$ for each name and each day. We then smoothed these parameters using exponential smoothing with a 10% weight on the most recent observation and 90% on the previous average. We also correlated a handful of $\lambda$ estimates with data on the CDS for these names and observed that they correlate well with the CDS. Hence the $\lambda$ estimates are estimates of credit. Clearly $\sigma$ is volatility. The movements in the cone of acceptable risks is a measure of our liquidity parameter.

We present in Figure 11 a graph of the parameter values averaged over the 38 names and exponentially smoothed across time.

# 11 Conclusion

A theory of risk for two price economies is overlaid on an underlying one price economy. The latter economy is seen as pricing credit, market and some components of liquidity risk using a traditional linear pricing rule that is known to

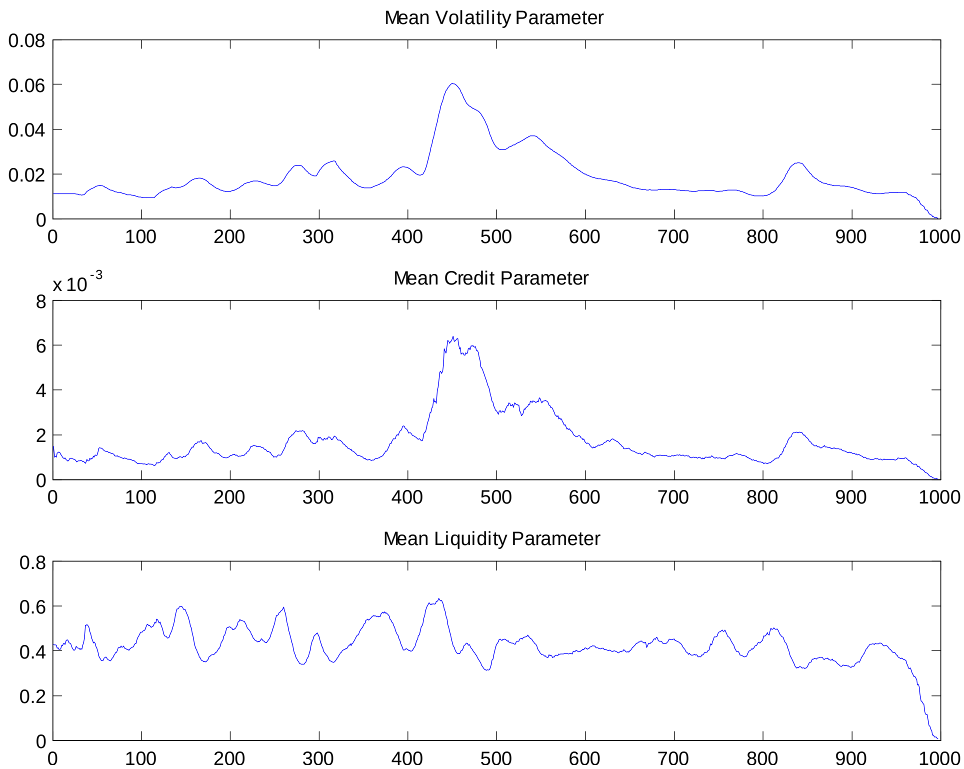


Figure 11: Graph of average parameter values for volatility, credit and liquidity. The parameter values have been exponentially smoothed across time and the averaged across stocks.

price risks using for example exponential tilting in the presence of exponential utilities. The two price economy is concerned with the failure of markets to converge to the law of one price and goes on to develop explicit equations for bid and ask prices with a view to ensuring the acceptability of residual unhedgeable risks in incomplete markets. The acceptability approach results in nonlinear pricing operators that are concave for bid prices and convex for ask prices. The former is an infimum of test valuations while the latter is a supremum of a similar set of valuations. Explicit closed forms for the two prices result when the cone of acceptable risks is modeled using parametric concave distortions of distribution functions for the residual risk. The final pricing theory separates out the physical risks, the market prices for systematic risks, the effects of credit and liquidity risk and the nonlinear effects of illiquidity captured by movements in the cone of acceptable risks. With assets marked at bid and liabilities valued at ask prices the theory allows a separation of liability valuation from an associated asset pricing theory.

The static two price theory is then extended to its dynamic counterpart by leveraging recent advances made in the theory of non linear expectations and its association with solutions of backward stochastic difference and differential equations. We introduce explicit drivers for these equations that price the local liquidity risk nonlinearly using concave distortions after the underlying one price economy has dealt with the market, credit and possibly some liquidity risk. The dynamic theory is illustrated on pricing a simple insurance loss modeled as a gamma compound Poisson process. We observe that spreads fall as the pricing tenor is reduced. Additionally it is noted that bid prices are sensitive to changes in credit risk but this is not the case for the ask price counterpart.

For the hedging of risks we introduce the new criterion of capital minimization defined as the difference between the ask and bid prices. This criterion is contrasted variance minimization and expected utility maximization. Like variance minimization it is relatively symmetric but additionally it incorporates a preferential parameter that is a pure number. Expected utility maximization on the other hand is problematic in that it is difficult to deal with losses when using relative risk aversion that is a pure number, while exponential utility that deals with losses, has a parameter that must be changed with the scale of cash flows and is not a pure number.

We provide three hedging examples. The first addresses capital minimization and the determination of optimal reinsurance attachment points in this context. The other two illustrate securitization of insurance losses. We consider for this purpose the dynamic financial hedging of catastrophic losses and the use of mortality indexed securities for the hedging of life insurance risks.

A final empirical section addresses issues of measuring the size of cones of acceptability using data on daily high, low and close prices. In this exercise bid prices are proxied by the daily low while the daily high proxies for the ask. An estimation of credit and liquidity parameters is then conducted separately for each stock on each day with the underlying market risk being modeled as log normal with volatility estimated from past returns.

## References


[1] Amihud, Y., and H. Mendelson (1980), "Dealership Market: Market Making with Inventory," *Journal of Financial Economics*, 8, 31-53.

[2] Artzner, P., F. Delbaen, J.M. Eber, and D. Heath (1999), "Coherent Measures of Risk," *Mathematical Finance*, 9, 203-228.

[3] Black, F. and M. Scholes (1973), "The pricing of Options and Corporate Liabilities," *The Journal of Political Economy*, 81, 637-659.

[4] Carmona, R. (2009), *Indifference Pricing:Theory and Applications*, Princeton University Press, Princeton, NJ.

[5] Carr, P. and D. Madan (1999), "Option valuation using the fast Fourier transform," *Journal of Computational Finance*, 2, 61-73.

[6] Carr, P., H. Geman, and D. Madan (2001), "Pricing and Hedging in Incomplete Markets," *Journal of Financial Economics*, 62, 131-167.

[7] Carr, P., D. Madan and J. J. Vicente Alvarez (2011), "Markets, Profits, Capital, Leverage and Return," forthcoming *Journal of Risk*, SSRN Paper No. 1679503.

[8] Cherny, A. and D. Madan (2009), "New Measure for Performance Evaluation," *Review of Financial Studies*, 22, 2571-2606.

[9] Cherny, A. and D. Madan (2010), "Markets as a counterparty: An Introduction to Conic Finance," (with Alexander Cherny), International Journal of Theoretical and Applied Finance, 13, 1149-1177.

[10] Choi, J. Y., D. Salandro, and K. Shastri (1988), "On the Estimation of Bid-Ask Spreads: Theory and Evidence," *Journal of Financial and Quantitative Analysis*, 23, 219-230.

[11] Cohen, S. and R. J. Elliott (2010), "A General Theory of Finite State Backward Stochastic Difference Equations," arXiv: 0810.4957v2.

[12] Copeland, T.C. and D. Galai (1983), "Information effects on the bid ask spread," *Journal of Finance*, 1457-1469.

[13] Dahl, M. (2004), "Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts," *Insurance: Mathematics & Economics,* 35, 113-136.

[14] Dahl, M., and Møller, T. (2006), "Valuation and hedging of life insurance liabilities with systematic mortality risk," *Insurance: Mathematics & Economics,* 39, 193-217.

[15] Demsetz, H. (1968), "The Cost of Transacting," *Quarterly Journal of Economics*, 82, 33-53.

[16] Easley, D. and M. O'Hara (1987), "Price, trade size and information in securities markets," *Journal of Financial Economics*, 19, 69-90.

[17] El Karoui, N. and S. J. Huang (1997), *Backward Stochastic Differential Equations*, Pitman Research Notes in Mathematics, Longman.

[18] Föllmer, H. and Schweizer, M. (1990), "Hedging of contingent claims under incomplete information," *Applied Stochastic Analysis*, 5, 389-414.

[19] George, T. J., G. Kaul, and M. Nimalendran (1991), "Estimation of the Bid-Ask Spreads and its Components: A New Approach," *Review of Financial Studies*, 4, 623-656.

[20] Glosten, L. R., and P. R. Milgrom (1985), "Bid, Ask and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders," *Journal of Financial Economics*, 14, 71-100.

[21] Ho, T., and H. R. Stoll (1981), "Optimal Dealer Pricing under Transactions and Return Uncertainty," *Journal of Financial Economics*, 9, 47-73.

[22] Ho, T., and H. R. Stoll (1983), "The Dynamics of Dealer Markets under Competition," *Journal of Finance*, 38, 1053-1074.

[23] Huang, R. D. and H. R. Stoll (1997), "The components of the bid ask spread: A general approach," *Review of Financial Studies*, 10, 995-1034.

[24] Jacod, J. and A. Shiryaev (1980), *Limit Theorems for Stochastic Processes,* Springer-Verlag, Berlin.

[25] Lamberton, D. and B. Lapeyre (1996), *Introduction to Stochastic Calculus Applied to Finance*, Chapman and Hall, London.

[26] Madan, D. B. (2006) "Equilibrium Asset Pricing with Non-Gaussian Returns and Exponential Utility." Quantitative Finance, 6, 455-463.

[27] Madan, D. B. (2010), "Execution Costs and Efficient Execution Frontiers," forthcoming Annals of Financial Economics, SSRN Paper No. 1679524.

[28] Madan, D., P. Carr and E. Chang (1998), "The Variance Gamma Process and Option Pricing," *European Finance Review*, 2, 79-105.

[29] Madan, D. B. and W. Schoutens (2011), "Structured Products Equilibria in Conic Two Price Markets," SSRN Paper No. 1839364.

[30] Madan, D. B. and W. Schoutens (2010), "Tenor Specific Pricing," SSRN Paper No. 1749407.

[31] Madan, D. and E. Seneta (1990), "The variance gamma (V.G.) model for share market returns," *Journal of Business*, 63, 511-524.

[32] Merton, R. (1973) "Theory of Rational Option Pricing," *The Bell Journal of Economics and Management Science*, 4, 141-183.

[33] Mijatović, A. and M. Pistorius (2009), "Continuously monitored barrier options under Markov processes," forthcoming *Mathematical Finance.*

[34] Musiela, M. and T. Zariphopoulou (2004a), "An example of indifference prices under exponential preferences," *Finance and Stochastics*, 8, 229-239.

[35] Musiela, M. and T. Zariphopoulou (2004b), "A valuation algorithm for indifference prices in incomplete markets," *Finance and Stochastics*, 8, 399-414.

[36] Norberg, R. (2010), "Optimal hedging of demographic risk in life insurance and pensions," Working Paper, London School of Economics.

[37] Norberg, R and O. Savina (2011), "A Quadratic Hedging Approach to Comparison of Catastrophe Indices," Working Paper, London School of Economics.

[38] Roll, R. (1984), "A Simple Implicit Measure of the Effective Bid-Ask Spread in an Efficient Market," *Journal of Finance*, 39, 1127-1139.

[39] M. Schweizer (1992), "Mean-variance hedging for general claims", *Annals of Applied Probability,* 2, 171-179.

[40] M. Schweizer (1995), "Variance-optimal hedging in discrete time", *Mathematics of Operations Research* 20, 1-32.

[41] Schweizer, M. (2001), "A Guided Tour through Quadratic Hedging Approaches," in E. Jouini, J. Cvitanić and M. Musiela (eds), *Option Pricing, Interest Rates and Risk Management*, Cambridge University Press, 538–574.

[42] Stoll, H. R. (1978), "The Supply of Dealer Services in Securities Markets," *Journal of Finance*, 33, 1133-1151.

[43] Stoll, H. R. (1989), "Inferring the Components of the Bid-Ask Spread: Theory and Empirical Tests," *Journal of Finance*, 44, 115-134.

[44] Wang, S. (2000), "A Class of Distortion Operators for Pricing Financial and Insurance Risks," *The Journal of Risk and Insurance*, 67,15-36.

TABLE 7

Bid Ask Expected Values and Spreads for the Gamma Base Case by Quantile

| Year | | Quantile 10 | 25 | 50 | 75 | 90 |
|---|---|---|---|---|---|---|
| | Bid | 11.90498 | 12.27231 | 12.73143 | 13.29975 | 13.83543 |
| | Expected Value | 12.21273 | 12.58712 | 13.05349 | 13.63365 | 14.17865 |
| 1 | Ask | 12.3372 | 12.70822 | 13.17027 | 13.75103 | 14.29139 |
| | Spread AB | 0.036305 | 0.03552 | 0.034469 | 0.033931 | 0.032956 |
| | Spread AE | 0.010192 | 0.009621 | 0.008946 | 0.00861 | 0.007951 |
| | Spread EB | 0.02585 | 0.025652 | 0.025297 | 0.025105 | 0.024807 |
| | | | | | | |
| | Bid | 11.52025 | 12.08137 | 12.75072 | 13.50975 | 14.21195 |
| | Expected Value | 11.81326 | 12.3846 | 13.06576 | 13.83888 | 14.55467 |
| 2 | Ask | 11.92762 | 12.497 | 13.17364 | 13.94248 | 14.65804 |
| | Spread AB | 0.035362 | 0.034402 | 0.033169 | 0.032031 | 0.031389 |
| | Spread AE | 0.009681 | 0.009076 | 0.008256 | 0.007486 | 0.007102 |
| | Spread EB | 0.025435 | 0.025099 | 0.024708 | 0.024363 | 0.024115 |
| | | | | | | |
| | Bid | 11.29471 | 11.96491 | 12.77222 | 13.65102 | 14.48782 |
| | Expected Value | 11.57506 | 12.25795 | 13.07979 | 13.97607 | 14.82823 |
| 3 | Ask | 11.67807 | 12.35874 | 13.17489 | 14.07403 | 14.92075 |
| | Spread AB | 0.033941 | 0.032915 | 0.031527 | 0.030987 | 0.029883 |
| | Spread AE | 0.008899 | 0.008222 | 0.007271 | 0.007009 | 0.00624 |
| | Spread EB | 0.024821 | 0.024492 | 0.024081 | 0.023812 | 0.023496 |
| | | | | | | |
| | Bid | 11.14011 | 11.88006 | 12.75517 | 13.70882 | 14.64695 |
| | Expected Value | 11.41312 | 12.16693 | 13.05824 | 14.02994 | 14.98594 |
| | Ask | 11.5085 | 12.26222 | 13.14827 | 14.11794 | 15.07066 |
| 4 | Spread AB | 0.033068 | 0.032168 | 0.030819 | 0.029844 | 0.028929 |
| | Spread AE | 0.008357 | 0.007832 | 0.006895 | 0.006273 | 0.005654 |
| | Spread EB | 0.024507 | 0.024147 | 0.023761 | 0.023424 | 0.023144 |
| | | | | | | |
| | Bid | 9.972634 | 10.86337 | 11.87038 | 12.89932 | 13.94159 |
| | Expected Value | 11.22448 | 12.09447 | 13.09085 | 14.13663 | 15.16792 |
| 5 | Ask | 12.44939 | 13.31479 | 14.33001 | 15.34393 | 16.3859 |
| | Spread AB | 0.248355 | 0.225659 | 0.207207 | 0.189515 | 0.175325 |
| | Spread AE | 0.109129 | 0.100899 | 0.094658 | 0.085403 | 0.0803 |
| | Spread EB | 0.125528 | 0.113326 | 0.102816 | 0.09592 | 0.087962 |

Table 8

Dynamic Two Price Constructions for Gamma Capped Losses

| Horizon | Vol percent | Credit | Stress | Bid | E-Value | Ask | S-AB | S-AE | S-EB |
|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 18.75 | 100.00 | 0.75 | 11.1136 | 11.3988 | 11.4915 | 0.0340 | 0.0081 | 0.0257 |
| 0.25 | 18.75 | 100.00 | 1.25 | 10.7757 | 11.3999 | 11.4451 | 0.0621 | 0.0040 | 0.0579 |
| 0.25 | 18.75 | 500.00 | 0.75 | 10.8329 | 11.3986 | 11.4901 | 0.0607 | 0.0080 | 0.0522 |
| 0.25 | 18.75 | 500.00 | 1.25 | 10.3315 | 11.4051 | 11.4545 | 0.1087 | 0.0043 | 0.1039 |
| 0.25 | 31.25 | 100.00 | 0.75 | 10.2122 | 10.4787 | 10.5781 | 0.0358 | 0.0095 | 0.0261 |
| 0.25 | 31.25 | 100.00 | 1.25 | 9.9042 | 10.4836 | 10.5573 | 0.0659 | 0.0070 | 0.0585 |
| 0.25 | 31.25 | 500.00 | 0.75 | 9.9547 | 10.4793 | 10.5784 | 0.0627 | 0.0095 | 0.0527 |
| 0.25 | 31.25 | 500.00 | 1.25 | 9.4853 | 10.4749 | 10.5505 | 0.1123 | 0.0072 | 0.1043 |
| 0.50 | 18.75 | 100.00 | 0.75 | 10.2353 | 11.3982 | 11.7929 | 0.1522 | 0.0346 | 0.1136 |
| 0.50 | 18.75 | 100.00 | 1.25 | 9.2187 | 11.3973 | 11.6924 | 0.2683 | 0.0259 | 0.2363 |
| 0.50 | 18.75 | 500.00 | 0.75 | 9.2860 | 11.3950 | 11.7800 | 0.2686 | 0.0338 | 0.2271 |
| 0.50 | 18.75 | 500.00 | 1.25 | 8.0859 | 11.3971 | 11.6746 | 0.4438 | 0.0243 | 0.4095 |
| 0.50 | 31.25 | 100.00 | 0.75 | 9.3910 | 10.4831 | 10.8957 | 0.1602 | 0.0394 | 0.1163 |
| 0.50 | 31.25 | 100.00 | 1.25 | 8.4475 | 10.4697 | 10.8392 | 0.2831 | 0.0353 | 0.2394 |
| 0.50 | 31.25 | 500.00 | 0.75 | 8.5291 | 10.4891 | 10.9057 | 0.2786 | 0.0397 | 0.2298 |
| 0.50 | 31.25 | 500.00 | 1.25 | 7.4227 | 10.4806 | 10.8346 | 0.4596 | 0.0338 | 0.4120 |

Dynamic Two Price Constructions for Gamma Uncapped Losses

| Horizon | Vol percent | Credit | Stress | Bid | E-Val | Ask | S-AB | S-AE | S-EB |
|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 18.75 | 100.00 | 0.75 | 4.1914 | 4.3071 | 4.3918 | 0.0478 | 0.0197 | 0.0276 |
| 0.25 | 18.75 | 100.00 | 1.25 | 4.0580 | 4.2988 | 4.4024 | 0.0849 | 0.0241 | 0.0593 |
| 0.25 | 18.75 | 500.00 | 0.75 | 4.0836 | 4.3039 | 4.3893 | 0.0749 | 0.0199 | 0.0539 |
| 0.25 | 18.75 | 500.00 | 1.25 | 3.8919 | 4.3010 | 4.4065 | 0.1322 | 0.0245 | 0.1051 |
| 0.25 | 31.25 | 100.00 | 0.75 | 8.4960 | 8.7311 | 8.9124 | 0.0490 | 0.0208 | 0.0277 |
| 0.25 | 31.25 | 100.00 | 1.25 | 8.2326 | 8.7198 | 8.9508 | 0.0872 | 0.0265 | 0.0592 |
| 0.25 | 31.25 | 500.00 | 0.75 | 8.2752 | 8.7249 | 8.9049 | 0.0761 | 0.0206 | 0.0544 |
| 0.25 | 31.25 | 500.00 | 1.25 | 7.8942 | 8.7254 | 8.9554 | 0.1344 | 0.0264 | 0.1053 |
| 0.50 | 18.75 | 100.00 | 0.75 | 3.8189 | 4.3067 | 4.6608 | 0.2204 | 0.0822 | 0.1277 |
| 0.50 | 18.75 | 100.00 | 1.25 | 3.4475 | 4.3035 | 4.7369 | 0.3740 | 0.1007 | 0.2483 |
| 0.50 | 18.75 | 500.00 | 0.75 | 3.4732 | 4.3063 | 4.6447 | 0.3373 | 0.0786 | 0.2399 |
| 0.50 | 18.75 | 500.00 | 1.25 | 3.0296 | 4.3009 | 4.7362 | 0.5633 | 0.1012 | 0.4196 |
| 0.50 | 31.25 | 100.00 | 0.75 | 7.7396 | 8.7295 | 9.4567 | 0.2219 | 0.0833 | 0.1279 |
| 0.50 | 31.25 | 100.00 | 1.25 | 6.9920 | 8.7375 | 9.6757 | 0.3838 | 0.1074 | 0.2497 |
| 0.50 | 31.25 | 500.00 | 0.75 | 7.0313 | 8.7204 | 9.4510 | 0.3441 | 0.0838 | 0.2402 |
| 0.50 | 31.25 | 500.00 | 1.25 | 6.1428 | 8.7223 | 9.6396 | 0.5692 | 0.1052 | 0.4199 |

Table 9

Dynamic Two Price Constructions for Weibull Capped Losses

| Horizon | Vol percent | Credit | Stress | Bid | E-Val | Ask | S-AB | S-AE | S-EB |
|---|---|---|---|---|---|---|---|---|---|
| 20.00 | 18.75 | 100.00 | 0.75 | 11.3927 | 11.6849 | 11.7797 | 0.0340 | 0.0081 | 0.0257 |
| 20.00 | 18.75 | 100.00 | 1.25 | 11.0449 | 11.6848 | 11.7338 | 0.0624 | 0.0042 | 0.0579 |
| 20.00 | 18.75 | 500.00 | 0.75 | 11.1056 | 11.6851 | 11.7774 | 0.0605 | 0.0079 | 0.0522 |
| 20.00 | 18.75 | 500.00 | 1.25 | 10.5860 | 11.6857 | 11.7338 | 0.1084 | 0.0041 | 0.1039 |
| 20.00 | 31.25 | 100.00 | 0.75 | 10.0318 | 10.2944 | 10.3931 | 0.0360 | 0.0096 | 0.0262 |
| 20.00 | 31.25 | 100.00 | 1.25 | 9.7278 | 10.2958 | 10.3689 | 0.0659 | 0.0071 | 0.0584 |
| 20.00 | 31.25 | 500.00 | 0.75 | 9.7822 | 10.2977 | 10.3947 | 0.0626 | 0.0094 | 0.0527 |
| 20.00 | 31.25 | 500.00 | 1.25 | 9.3233 | 10.2956 | 10.3630 | 0.1115 | 0.0066 | 0.1043 |
| 10.00 | 18.75 | 100.00 | 0.75 | 10.4949 | 11.6874 | 12.0820 | 0.1512 | 0.0338 | 0.1136 |
| 10.00 | 18.75 | 100.00 | 1.25 | 9.4488 | 11.6845 | 11.9894 | 0.2689 | 0.0261 | 0.2366 |
| 10.00 | 18.75 | 500.00 | 0.75 | 9.5202 | 11.6813 | 12.0735 | 0.2682 | 0.0336 | 0.2270 |
| 10.00 | 18.75 | 500.00 | 1.25 | 8.2870 | 11.6804 | 11.9867 | 0.4465 | 0.0262 | 0.4095 |
| 10.00 | 31.25 | 100.00 | 0.75 | 9.2153 | 10.2904 | 10.7057 | 0.1617 | 0.0404 | 0.1167 |
| 10.00 | 31.25 | 100.00 | 1.25 | 8.2993 | 10.2901 | 10.6463 | 0.2828 | 0.0346 | 0.2399 |
| 10.00 | 31.25 | 500.00 | 0.75 | 8.3715 | 10.2960 | 10.7035 | 0.2786 | 0.0396 | 0.2299 |
| 10.00 | 31.25 | 500.00 | 1.25 | 7.2850 | 10.2880 | 10.6463 | 0.4614 | 0.0348 | 0.4122 |

Dynamic Two Price Constructions for Weibull Uncapped Losses

| Horizon | Vol percent | Credit | Stress | Bid | E-Value | Ask | S-AB | S-AE | S-EB |
|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 18.75 | 100.00 | 0.75 | 4.2256 | 4.3419 | 4.4253 | 0.0473 | 0.0192 | 0.0275 |
| 0.25 | 18.75 | 100.00 | 1.25 | 4.0998 | 4.3426 | 4.4453 | 0.0843 | 0.0236 | 0.0592 |
| 0.25 | 18.75 | 500.00 | 0.75 | 4.1202 | 4.3428 | 4.4255 | 0.0741 | 0.0190 | 0.0540 |
| 0.25 | 18.75 | 500.00 | 1.25 | 3.9299 | 4.3428 | 4.4435 | 0.1307 | 0.0232 | 0.1051 |
| 0.25 | 31.25 | 100.00 | 0.75 | 8.2807 | 8.5107 | 8.7063 | 0.0514 | 0.0230 | 0.0278 |
| 0.25 | 31.25 | 100.00 | 1.25 | 8.0248 | 8.5004 | 8.7607 | 0.0917 | 0.0306 | 0.0593 |
| 0.25 | 31.25 | 500.00 | 0.75 | 8.0709 | 8.5091 | 8.7016 | 0.0781 | 0.0226 | 0.0543 |
| 0.25 | 31.25 | 500.00 | 1.25 | 7.6962 | 8.5055 | 8.7689 | 0.1394 | 0.0310 | 0.1052 |
| 0.50 | 18.75 | 100.00 | 0.75 | 3.8531 | 4.3439 | 4.6761 | 0.2136 | 0.0765 | 0.1274 |
| 0.50 | 18.75 | 100.00 | 1.25 | 3.4806 | 4.3448 | 4.7529 | 0.3656 | 0.0939 | 0.2483 |
| 0.50 | 18.75 | 500.00 | 0.75 | 3.5046 | 4.3432 | 4.6755 | 0.3341 | 0.0765 | 0.2393 |
| 0.50 | 18.75 | 500.00 | 1.25 | 3.0583 | 4.3411 | 4.7218 | 0.5439 | 0.0877 | 0.4195 |
| 0.50 | 31.25 | 100.00 | 0.75 | 7.5320 | 8.5030 | 9.2755 | 0.2315 | 0.0908 | 0.1289 |
| 0.50 | 31.25 | 100.00 | 1.25 | 6.8042 | 8.5005 | 9.4792 | 0.3931 | 0.1151 | 0.2493 |
| 0.50 | 31.25 | 500.00 | 0.75 | 6.8571 | 8.5073 | 9.2771 | 0.3529 | 0.0905 | 0.2407 |
| 0.50 | 31.25 | 500.00 | 1.25 | 5.9911 | 8.5119 | 9.5508 | 0.5942 | 0.1221 | 0.4208 |

Table 10

Dynamic Two Price Constructions for Frechet Capped Losses

| Horizon | Vol percent | Credit | Stress | Bid | E-Value | Ask | S-AB | S-AE | S-EB |
|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 18.75 | 100.00 | 0.75 | 9.2855 | 9.5220 | 9.5992 | 0.0338 | 0.0081 | 0.0255 |
| 0.25 | 18.75 | 100.00 | 1.25 | 9.0007 | 9.5205 | 9.5618 | 0.0623 | 0.0043 | 0.0578 |
| 0.25 | 18.75 | 500.00 | 0.75 | 9.0493 | 9.5202 | 9.5954 | 0.0604 | 0.0079 | 0.0520 |
| 0.25 | 18.75 | 500.00 | 1.25 | 8.6267 | 9.5212 | 9.5625 | 0.1085 | 0.0043 | 0.1037 |
| 0.25 | 31.25 | 100.00 | 0.75 | 8.6918 | 8.9164 | 8.9958 | 0.0350 | 0.0089 | 0.0258 |
| 0.25 | 31.25 | 100.00 | 1.25 | 8.4304 | 8.9195 | 8.9730 | 0.0644 | 0.0060 | 0.0580 |
| 0.25 | 31.25 | 500.00 | 0.75 | 8.4700 | 8.9134 | 8.9931 | 0.0618 | 0.0089 | 0.0524 |
| 0.25 | 31.25 | 500.00 | 1.25 | 8.0755 | 8.9145 | 8.9627 | 0.1099 | 0.0054 | 0.1039 |
| 0.50 | 18.75 | 100.00 | 0.75 | 8.5597 | 9.5230 | 9.8428 | 0.1499 | 0.0336 | 0.1125 |
| 0.50 | 18.75 | 100.00 | 1.25 | 7.7116 | 9.5273 | 9.7679 | 0.2667 | 0.0253 | 0.2355 |
| 0.50 | 18.75 | 500.00 | 0.75 | 7.7671 | 9.5247 | 9.8487 | 0.2680 | 0.0340 | 0.2263 |
| 0.50 | 18.75 | 500.00 | 1.25 | 6.7595 | 9.5219 | 9.7718 | 0.4456 | 0.0262 | 0.4087 |
| 0.50 | 31.25 | 100.00 | 0.75 | 7.9998 | 8.9179 | 9.2574 | 0.1572 | 0.0381 | 0.1148 |
| 0.50 | 31.25 | 100.00 | 1.25 | 7.2077 | 8.9190 | 9.1988 | 0.2762 | 0.0314 | 0.2374 |
| 0.50 | 31.25 | 500.00 | 0.75 | 7.2631 | 8.9187 | 9.2469 | 0.2731 | 0.0368 | 0.2279 |
| 0.50 | 31.25 | 500.00 | 1.25 | 6.3234 | 8.9193 | 9.2066 | 0.4559 | 0.0322 | 0.4105 |

Dynamic Two Price Constructions for Frechet Uncapped Losses

| Horizon | Vol percent | Credit | Stress | Bid | E-Value | Ask | S-AB | S-AE | S-EB |
|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 18.75 | 100.00 | 0.75 | 2.8798 | 2.9615 | 3.0875 | 0.0721 | 0.0425 | 0.0284 |
| 0.25 | 18.75 | 100.00 | 1.25 | 2.8008 | 2.9676 | 3.2059 | 0.1446 | 0.0803 | 0.0596 |
| 0.25 | 18.75 | 500.00 | 0.75 | 2.8061 | 2.9608 | 3.0857 | 0.0996 | 0.0422 | 0.0551 |
| 0.25 | 18.75 | 500.00 | 1.25 | 2.6775 | 2.9605 | 3.1987 | 0.1947 | 0.0805 | 0.1057 |
| 0.25 | 31.25 | 100.00 | 0.75 | 4.2115 | 4.3317 | 4.5442 | 0.0790 | 0.0491 | 0.0286 |
| 0.25 | 31.25 | 100.00 | 1.25 | 4.0944 | 4.3385 | 4.7514 | 0.1605 | 0.0952 | 0.0596 |
| 0.25 | 31.25 | 500.00 | 0.75 | 4.1039 | 4.3289 | 4.5356 | 0.1052 | 0.0478 | 0.0548 |
| 0.25 | 31.25 | 500.00 | 1.25 | 3.9264 | 4.3402 | 4.7399 | 0.2072 | 0.0921 | 0.1054 |
| 0.50 | 18.75 | 100.00 | 0.75 | 2.6126 | 2.9637 | 3.4938 | 0.3373 | 0.1789 | 0.1344 |
| 0.50 | 18.75 | 100.00 | 1.25 | 2.3582 | 2.9570 | 3.8876 | 0.6486 | 0.3147 | 0.2539 |
| 0.50 | 18.75 | 500.00 | 0.75 | 2.3766 | 2.9613 | 3.4695 | 0.4599 | 0.1716 | 0.2460 |
| 0.50 | 18.75 | 500.00 | 1.25 | 2.0847 | 2.9667 | 3.8768 | 0.8597 | 0.3068 | 0.4231 |
| 0.50 | 31.25 | 100.00 | 0.75 | 3.8158 | 4.3285 | 5.1817 | 0.3580 | 0.1971 | 0.1344 |
| 0.50 | 31.25 | 100.00 | 1.25 | 3.4463 | 4.3217 | 5.9877 | 0.7374 | 0.3855 | 0.2540 |
| 0.50 | 31.25 | 500.00 | 0.75 | 3.4622 | 4.3214 | 5.2410 | 0.5138 | 0.2128 | 0.2482 |
| 0.50 | 31.25 | 500.00 | 1.25 | 3.0415 | 4.3266 | 5.6815 | 0.8680 | 0.3132 | 0.4225 |