

DYNAMIC MODEL OF INSURANCE COMPANY'S MANAGEMENT

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In this paper, we are interested in a model related to a number of periods of Company's activity.

OPTIMAL POLICY OF DIVIDENDS—OBJECTIVE OF THE INSURANCE COMPANY.

The Company calculates an amount s which then could be given as a supplementary interest to the shareholders. This calculated amount is taken from the risk reserve. Let us assume the risk reserve = S at the beginning of an operating period. If the Company gives an amount s to the shareholders, then the risk reserve is $S - s$. (Borch 1972; Seal 1969).

We must determine the best policy of dividends, that is a rule which determines the payments to be made each year to the shareholders of the Company, maximising a definite criterion.

The problem of dividends must be approached in the "dynamic programming manner". Indeed, the payments of dividends have an effect upon further gains of the Company and its capacity to pay dividends in the future.

The objective of the Company is, for example, maximising the average utility of the dividend payments, which is calculated according to the distribution of claims. (Borch 1964a; Wolff 1966).

DESCRIPTION OF THE MODEL.

The shareholders' order of preferences

We first have to represent the preferences of the shareholders by a collective utility function representing the individual preferences in a rational manner. We will assume that all shareholders have the same preferences regarding the system of dividend payments.

Let $U(s_1, s_2, \dots, s_j, \dots)$ represent the shareholders' order of preferences in various systems of dividend payments $(s_1, s_2, \dots, s_j, \dots)$ where s_j denotes the payments of dividend made by the Company during the year j .

We assume that U is of the following form:

$$U(s_1, s_2, \dots, s_j, \dots) = u(s_1) + v \cdot U(s_2, s_3, \dots, s_j, \dots) \quad (\text{Hakansson 1966})$$

where $u(s_1)$ denotes the shareholders' oneperiod utility function;
 v is a factor expressing the shareholders' preferences for an early dividend ($0 < v < 1$);

$U(s_2, s_3, \dots, s_j, \dots)$ denotes the prospective utility of the dividend program.

We now have, recurring

$$U(s_1, s_2, \dots, s_j, \dots) = \sum_{k=1}^{\infty} v^{k-1} E[u(sk)]$$

where

$F(x) = P(X \leq x)$ is the distribution function of the variable X
 amount of claims to be paid;
 E is the expected value of X .

We assume that $F(x)$ is the same in all operating periods, that is X is distributed identically in all periods.

The objective of the Company

The objective of the Company may be formulated

$$\text{Max } U(s_1, s_2, \dots, s_j, \dots) = \text{Max } \sum_{j=1}^{\infty} v^{j-1} E[u(s_j)]$$

As basis of our calculations, we assumed the first operating period of the Company. We may also consider any period j and then we write:

$$\text{Max } U(s_j, s_{j+1}, s_{j+2}, \dots) = \text{Max } \sum_j v^{j-1} E[u(s_j)]$$

The relation between two successive risk reserves

The relation between the risk reserve of the Company at two succeeding points of decision is given by

$$S_{j+1} = S_j - s_j + k_{1j} \left(P - \int_0^{\infty} x dF(x) \right)$$

We use the following notation:

- S_j = the risk reserve of the Company at the point of decision j ;
- s_j = the amounts of dividends paid at the point of decision j ;
- P = the amount of received premiums during the period j ;
- $k_{1,j}$ = the part of the portfolio, retained by the Company in a quota reinsurance system.

Decision variables and constraints

The Company has two decision variables in each operating period: s_j and $k_{1,j}$. The decisions are taken at the beginning of each period.

We consider the following constraints:

$$0 \leq s_j \leq S_j; \quad 0 \leq k_{1,j} \leq 1.$$

The functional equation

We introduce the function $U_j(S_j)$ denoting the discounted average utility of the dividends s_j, s_{j+1}, \dots evaluated at decision points $j, j+1, \dots$ when initial risk reserve is S_j and an optimal policy is followed with respect to payment of dividends in all subsequent periods.

The principle of optimality in dynamic programming holds that whatever the initial state and the initial decision, future decisions must constitute an optimal policy with respect to the state resulting from the first decision. (Bellman 1961).

Applying the principle of optimality in dynamic programming, the dividends' decision problem at decision point j can be formulated by the following equation:

$$U_j(S_j) = \underset{\substack{0 \leq s_j \leq S_j \\ 0 \leq k_{1,j} \leq 1}}{\text{Max}} \left[u(s_j) + v \int_0^{\infty} U_{j+1}(S_j - s_j + k_{1,j}(P - x)) dF(x) \right] \quad (1)$$

As a matter of fact, if the Company pays dividend s_j at the beginning of the period j , it then has, at the beginning of the period $j + 1$,

$$S_j + k_{1,j}(P - \int_0^{\infty} x dF(x) - s_j$$

The decisions relating to s_{j+1} , s_{j+2} , s_{j+3} , ... and $k_{1,j+1}$, $k_{1,j+2}$, $k_{1,j+3}$, ... must constitute an optimal policy. Furthermore, the optimal payment s_j and the value of $k_{1,j}$ must certainly maximise the utility of s_j and the discounted average utility of the subsequent payments.

We must have the inequality

$$\int_0^{\infty} [S_j - s_j + k_{1,j}(P - x)] dF(x) \geq 0 \quad (2)$$

We also suppose

$$s_j = s_j(S_j) \quad (3)$$

The problem is therefore to find a dividend function $s_j(S_j)$ and a function $k_{1,j}$ maximising the expression in brackets in (1), verifying the relation (2).

PRACTICAL RESOLUTION OF THE FUNCTIONAL EQUATION.

Hypothesis concerning $u(s)$

We assume that the shareholders' preferences as to a single operating period may be represented by the utility function $u(s) = s^\gamma$ ($0 < \gamma < 1$). This function is a concave function of s ($s > 0$), which implies that the shareholders have a risk aversion.

The functional equation is then

$$U_j(S_j) = \text{Max}_{\substack{0 \leq s_j \leq S_j \\ 0 \leq k_{1,j} \leq 1}} [s_j^\gamma + v \int_0^{\infty} U_{j-1}(S_j - s_j + k_{1,j}(P - x)) dF(x)]$$

We take $\gamma = \frac{1}{2}$.

Special "probability distribution of claims"

We assume that the claim variable takes the values 0 and 2, respectively with the probabilities p and q ($p > q$). We take $P = 1$. (Borch, 1964b).

Method of resolution

The method of resolution is the method of successive approximations. We consider successively a horizon limited to 1 year, 2 years, 3 years, . . . , N years and also an unlimited horizon, if we take N infinite.

The functional equation is then:

$$U_N(S) = \text{Max}_{\substack{0 \leq s_N \leq S \\ 0 \leq k_{1,N} \leq 1}} \{s_N^{1/2} + v [p U_{N-1}(S - s_N + k_{1,N}) + q U_{N-1}(S - s_N - k_{1,N})]\}$$

N denotes the number of years corresponding to the horizon.

We have the relation:

$$(S - s_N + k_{1,N}) p + (S - s_N - k_{1,N}) q \geq 0$$

Indeed, the first member of this inequality is at least equal to $k_{1,N}(p - q) \geq 0$.

Let us replace N by 1, 2, 3, . . . , N finite or infinite.

$$U_1(S) = \text{Max}_{\substack{0 \leq s_1 \leq S \\ 0 \leq k_{1,1} \leq 1}} s_1^{1/2} = S^{1/2}.$$

In effect, U_0 (any amount) = 0; $k_{1,1}$ any figure.

$$U_2(S) = \text{Max}_{\substack{0 \leq s_2 \leq S \\ 0 \leq k_{1,2} \leq 1}} \{s_2^{1/2} + v [p U_1(S - s_2 + k_{1,2}) + q U_1(S - s_2 - k_{1,2})]\}$$

Seeing that $U_1(S) = S^{1/2}$, we have

$$U_2(S) = \text{Max}_{\substack{0 \leq s_2 \leq S \\ 0 \leq k_{1,2} \leq 1}} \{s_2^{1/2} + v [p(S - s_2 + k_{1,2})^{1/2} + q(S - s_2 - k_{1,2})^{1/2}]\}$$

We must determine the values of $k_{1,2}$ and s_2 maximising the expression in braces.

The condition of first order with respect to s_2 is:

$$\frac{\delta \{ \dots \}}{\delta s_2} = \frac{1}{2} s_2^{-1/2} - \frac{1}{2} v [(S - s_2 + k_{1,2})^{-1/2} \cdot p + (S - s_2 - k_{1,2})^{-1/2} \cdot q]$$

Let it be equal to zero. We have:

$$(S - s_2 + k_{1,2})^{-1/2} \cdot p + (S - s_2 - k_{1,2})^{-1/2} \cdot q = \frac{s_2^{-1/2}}{v} \quad (1)$$

Doing the same with respect to $k_{1,2}$, we have:

$$\frac{\delta \{ \dots \}}{\delta k_{1,2}} = \frac{1}{2} v \cdot [(S - s_2 + k_{1,2})^{-1/2} \cdot p - (S - s_2 - k_{1,2})^{-1/2} \cdot q] = 0 \quad (2)$$

or

$$k_{1,2} = (S - s_2) \cdot \frac{p^2 - q^2}{p^2 + q^2} \quad (3)$$

The relation (1) may be written, because of (2)

$$2 (S - s_2 - k_{1,2})^{-1/2} \cdot q = \frac{s_2^{-1/2}}{v}$$

Replacing $k_{1,2}$ by the value (3), we have:

$$s_2 = \frac{S}{1 + 2v^2 \cdot (p^2 + q^2)} \quad (4)$$

The values of $k_{1,2}$ and s_2 give $U_2(S)$. Indeed, the expression $Ah^2 + 2Bhk + Ck^2$, where

$$A = \frac{\delta^2 \{ \dots \}}{\delta s_2^2}; \quad B = \frac{\delta^2 \{ \dots \}}{\delta s_2 \delta k_{1,2}}; \quad C = \frac{\delta^2 \{ \dots \}}{\delta k_{1,2}^2}$$

has compoundroots and $A < 0$.

The value of $U_2(S)$ is given by

$$U_2(S) = S^{1/2} \cdot [1 + 2v^2 (p^2 + q^2)]^{1/2} \quad (5)$$

The condition $0 \leq s_2 \leq S$ is always verified by (4).

The condition $0 \leq k_{1,2} \leq 1$ is, because (3) and (4)

$$0 \leq \frac{2 S v^2 (p^2 - q^2)}{[1 + 2v^2 (p^2 + q^2)]} \leq 1$$

The first inequality is always verified ($p > q$).

The second inequality is

$$S \leq \frac{1 + 2v^2 (p^2 + q^2)}{2v^2 (p^2 - q^2)}$$

$$U_3(S) = \text{Max}_{\substack{0 \leq s_3 \leq S \\ 0 \leq k_{1,3} \leq 1}} \{s_3^{1/2} + v[pU_2(S - s_3 + k_{1,3}) + qU_2(S - s_3 - k_{1,3})]\}$$

As $U_2(S) = S^{1/2} \cdot [I + 2v^2 (p^2 + q^2)]^{1/2}$, we have

$$U_3(S) = \text{Max}_{\substack{0 \leq s_3 \leq S \\ 0 \leq k_{1,3} \leq 1}} \{s_3^{1/2} + v[p(S - s_3 + k_{1,3})^{1/2} + q(S - s_3 - k_{1,3})^{1/2}] \cdot [I + 2v^2 (p^2 + q^2)]^{1/2}\}$$

Repeating the operations for $U_3(S)$, we have

$$k_{1,3} = (S - s_3) \cdot \frac{p^2 - q^2}{p^2 + q^2}$$

$$s_3 = \frac{S}{I + 2v^2 (p^2 + q^2) + [2v^2 (p^2 + q^2)]^2}$$

$$U_3(S) = S^{1/2} \{I + 2v^2 (p^2 + q^2) + [2v^2 (p^2 + q^2)]^2\}^{1/2}$$

The condition (a) $0 \leq s_3 \leq S$ is always verified.

The condition (b) is now:

$$S \leq \frac{I + [2v^2 + (2v^2)^2 \cdot (p^2 + q^2)] \cdot (p^2 + q^2)}{[2v^2 + (2v^2)^2 \cdot (p^2 + q^2)] \cdot (p^2 - q^2)} \delta \leq \quad (I)$$

Since we have used the formula

$$U_2(S - s_3 + k_{1,3}) = (S - s_3 + k_{1,3})^{1/2} \cdot [I + 2v^2 (p^2 + q^2)]^{1/2}$$

we must consider yet another condition

$$S - s_3 + k_{1,3} \leq \frac{I + 2v^2 (p^2 + q^2)}{2v^2 (p^2 - q^2)}$$

or

$$S \leq (I) \times \frac{I + 2v^2 (p^2 + q^2)}{2p^2 \cdot 2v^2} \delta \leq \quad (2)$$

$$U_N(S) = S^{1/2} \{I + 2v^2 (p^2 + q^2) + [2v^2 (p^2 + q^2)]^2 + \dots + [2v^2 (p^2 + q^2)]^{N-1}\}^{1/2}$$

$$k_{1,N} = (S - s_N) \cdot \frac{p^2 - q^2}{p^2 + q^2}$$

$$s_N = \frac{S}{I + 2v^2 (p^2 + q^2) + [2v^2 (p^2 + q^2)]^2 + \dots + [2v^2 (p^2 + q^2)]^{N-1}}$$

We have $N - 1$ conditions concerning S :

$$S \leq \frac{1 + [2v^2 + (2v^2)^2 \cdot (p^2 + q^2) + \dots + (2v^2)^{N-1} \cdot (p^2 + q^2)^{N-2}] \cdot (p^2 + q^2)}{[2v^2 + (2v^2)^2 \cdot (p^2 + q^2) + \dots + (2v^2)^{N-1} \cdot (p^2 + q^2)^{N-2}] \cdot (p^2 - q^2)} \delta \leq (1)$$

$$S \leq (1) \cdot \frac{1}{(2p^2) \cdot (2v^2)} \cdot \frac{(2v^2)^{N-1} \cdot (p^2 + q^2)^{N-1} - 1}{(2v^2)^{N-1} \cdot (p^2 + q^2)^{N-2} - 1} \delta \leq (2)$$

...

$$S \leq (1) \cdot \frac{1}{(2p^2)^{N-2} \cdot (2v^2)^{N-2}} \cdot \frac{(2v^2)^{N-1} \cdot (p^2 + q^2)^{N-1} - 1}{2v^2 \cdot (p^2 + q^2) - 1} \delta \leq (N-1)$$

To consider an infinite horizon, let us take N infinite.

We have

$$s_\infty = \lim_{N \rightarrow \infty} S_N = \frac{S \cdot [2v^2(p^2 + q^2) - 1]}{\lim_{N \rightarrow \infty} [2v^2(p^2 + q^2)]^N - 1}$$

The value of s_∞ depends on $\lim_{N \rightarrow \infty} [2v^2(p^2 + q^2)]^N$

We have:

$$a) \quad 2v^2(p^2 + q^2) < 1$$

$$\frac{1}{2} < p < (u/2) \quad S \leq \frac{1}{2v^2(p^2 - q^2)}$$

$$s_\infty = S [1 - 2v^2(p^2 + q^2)]; k_{1,\infty} = 2Sv^2(p^2 - q^2); U_\infty(s) = S^{1/2} \cdot$$

$$\cdot \frac{1}{[1 - 2v^2(p^2 + q^2)]^{1/2}}$$

$$(u/2) < p < \frac{v + (1 - v^2)^{1/2}}{2v} \quad S = 0 \quad s_\infty = k_{1,\infty} = U_\infty(S) = 0$$

$$b) \quad 2v^2(p^2 + q^2) \geq 1 \text{ or } p \geq \frac{v + (1 - v^2)^{1/2}}{2v}$$

$$S = 0 \quad s_\infty = k_{1,\infty} = U_\infty(S) = 0$$

$s_\infty = k_{1,\infty} = U_\infty(S) = 0$ means the whole portfolio must be reinvested; there are no dividends and the dividends' utility is zero.

Remark

If the conditions concerning S are not verified, we must take values of k_1 other than those giving $U(S)$.

For example, if $N = 2$ and if $S > \frac{1 + 2v^2(p^2 + q^2)}{2v^2(p^2 - q^2)}$, the calculated value of $k_{1,2}$ would be bigger than 1.

In this case, we may take the value $k_{1,2}$ of the reinsurance treaty and determine the value of s_2 giving the dividends' average utility.

It will now be of prime interest:

to determine the appropriate utility functions,

to suppress some hypotheses,

to consider more variables in order to obtain better fitted models.

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