A NOTE ON THE RUIN PROBLEM FOR A CLASS OF STOCHASTIC PROCESSES WITH INTERCHANGEABLE INCREMENTS

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SUMMARY

Models for the risk business of an insurance company are often constructed by weighting pure Poisson models. In this paper it is verified that it is possible to calculate the probability of ruin in such weighted models by weighting ruin probabilities of pure Poisson models.

1. INTRODUCTION

In this paper we are going to study a model for the risk business of an insurance company where the claims are located according to a stochastic process $\{N(t);\ 0 \leqslant t < \infty\}$ subordinated to the Poisson process with a directing process $\{\Lambda(t);\ 0 \leqslant t < \infty\}$. This terminology follows Feller [3]. The directing process $\Lambda(t)$ will be a real-valued and non-decreasing process such that $P(\Lambda(0) = 0) = 1$. We will later assume that $\Lambda(t)$ has stationary, and independent increments. Let $\{M(t);\ 0 \leqslant t < \infty\}$ be a Poisson process with intensity 1, i.e.

$$P(M(t) = k) = \frac{t^k}{k!} e^{-t},\ k = 0, 1, 2, \ldots .$$

The process $N(t)$ is then defined by $N(t) = M(\Lambda(t))$.

For each claim the company has to pay a certain amount counted with its proper sign. As usually in the theory of risk these amounts are assumed to be described by a sequence $\{X_n\}_1^\infty$ of independent random variables, each having the distribution function $V(x)$. These variables are further assumed to be independent of the process $N(t)$.

The stochastic process

$$X(t) = \sum_{k=1}^{N(t)} X_k \qquad (X(t) = 0 \text{ if } N(t) = 0)$$

will thus serve as a model for the total amount of claims paid by the company up to time t.

Looking a bit more formally upon this definition we let the stochastic vector process $\{(\Lambda(t), M(s), X_k);\ 0 \leqslant t, s < \infty, k = 1, 2, \ldots\}$ be defined on $(\Omega, \mathfrak{X}, \Pi_{\mathfrak{X}})$ where Ω is a sample space, $\mathfrak{X}$ the σ-algebra of subsets of Ω generated by the process and $\Pi_{\mathfrak{X}}$ a probability measure on $\mathfrak{X}$. Let further $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{V}$ and $\mathfrak{N}$ be the sub-σ-algebras of $\mathfrak{X}$ generated by $\{\Lambda(t);\ 0 \leqslant t < \infty\}$, $\{M(t);\ 0 \leqslant t < \infty\}$ $\{X_k;\ k = 1, 2, \ldots\}$ and $\{(\Lambda(t), M(s));\ 0 \leqslant t, s < \infty\}$ respectively and $\Pi_{\mathfrak{L}}$, $\Pi_{\mathfrak{M}}$, $\Pi_{\mathfrak{V}}$ and $\Pi_{\mathfrak{N}}$ the corresponding marginal measures.

For every $L \varepsilon \mathfrak{L}$, $M \varepsilon \mathfrak{M}$ and $V \varepsilon \mathfrak{V}$ we have $\Pi_{\mathfrak{N}}(L \cap M) = \Pi_{\mathfrak{L}}(L) \cdot \Pi_{\mathfrak{M}}(M)$ and $\Pi_{\mathfrak{X}}(L \cap M \cap V) = \Pi_{\mathfrak{N}}(L \cap M) \cdot \Pi_{\mathfrak{V}}(V)$ because of the independence assumptions.

We will always consider only the separable version of the processes and further all measures will be assumed to be complete.

Let $\mathfrak{N}_0$ and $\mathfrak{X}_0$ be the σ-algebras generated by the above defined processes $\{N(t);\ 0 \leqslant t < \infty\}$ and $\{X(t);\ 0 \leqslant t < \infty\}$ respectively. Let S be a countable but dense set in $[0, \infty)$. Because of the separability assumption $\{N(t) = k\} = \overset{U}{S}\{M(s) = k, \Lambda(t) = s\}$. Thus $\mathfrak{N}_0 \subset \mathfrak{N}$. In the same way it is shown that $\mathfrak{X}_0 \subset \mathfrak{X}$.

2. The case of a directing process with stationary and independent increments

It is shown in Feller [3] that for every non-decreasing process $\Lambda(t)$ with stationary and independent increments (s.i.i. process)

$$E\, e^{iu\Lambda(t)} = e^{t\eta(u)}$$

where

$$\eta(u) = \int_{0+}^{\infty} \frac{e^{iux} - 1}{x}\, \Omega(dx) + ibu$$

where $b \geqslant 0$ and $\int_{0+}^{\infty} \frac{\Omega(dx)}{1 + x} < \infty$

The behaviour of the sample functions of non-decreasing s.i.i. processes is investigated by Walldin [5].

Since $M(t)$ is a s.i.i. process it follows from Feller [3] that also $N(t)$ is a s.i.i. process.

For $N(t)$ we thus have

$$\log E\, e^{iuN(t)} = t \,.\, \sum_{k=1}^{\infty} (e^{iuk} - 1)\, c_k$$

where $c_k \geqslant 0$ for all k and $\sum_{k=1}^{\infty} c_k < \infty$.

$N(t)$ is thus a bunch Poisson (Poisson par grappes) process. The bunches are located according to a Poisson process with intensity $c = \sum_{1}^{\infty} c_k$ and the distribution of the size of the bunches is given by $\frac{c_k}{c}$ for $k = 1, 2, 3, \ldots$.

Theorem

The relation between the measure Ω and the sequence $\{c_k\}_1^\infty$ is given by

$$c_k = \int_{0+}^{\infty} \frac{x^k}{k!} e^{-x} \frac{\Omega(dx)}{x} + \delta_{k,1}\, b$$

where

$$\delta_{k,1} = \begin{cases} 1 \text{ if } k = 1 \\ 0 \text{ if } k \neq 1 \end{cases}$$

Proof

We have $E \exp\{iuN(t)\} = E \exp\{\Lambda(t)\ (e^{iu} - 1\})$.

Since $Re(e^{iu} - 1) \leqslant 0$ we have

$$\int_{0+}^{\infty} \frac{e^{x(e^{iu}} - 1)}{x} \Omega(dx) + b(e^{iu} - 1) = \sum_{k=1}^{\infty} (e^{iuk} - 1)\, c_k$$

Thus the theorem holds since

$$(e^{iu} - 1)b + \sum_{k=0}^{\infty} \int_{0+}^{\infty} \frac{(x\, e^{iu})^k - x^k}{k!} e^{-x} \frac{\Omega(dx)}{x} =$$

$$= (e^{iu} - 1)\, b + \int_{0+}^{\infty} (e^{xe^{iu}} - e^{x})\, e^{-x} \frac{\Omega(dx)}{x} =$$

$$= (e^{iu} - 1)\, b + \int_{0+}^{\infty} \frac{e^{x(e^{iu}} - 1)}{x} \Omega\,(dx)$$

because of dominated convergence since $e^x - 1$ is integrable with respect to $\frac{e^{-x}\Omega(dx)}{x}$.

Example

Put $\Omega(dx) = \alpha e^{-\beta x} dx$ and $b = 0$.

We have

$$\eta(u) = \int_0^\infty \frac{e^{ixu} - 1}{x} \alpha e^{-\beta x} dx = -\alpha \log(1 - \frac{iu}{\beta})$$

and thus

$$E\, e^{iu\Lambda(t)} = \left(1 - \frac{iu}{\beta}\right)^{-\alpha t}$$

$\Lambda(t)$ is then Γ-distributed with the frequency function given by

$$\frac{\beta^{\alpha t}}{\Gamma(\alpha t)} x^{(\alpha t - 1)} e^{-\beta x} \text{ for } x \geqslant 0$$

By direct calculations we get

$$E\, e^{iuN(t)} = \int_0^\infty e^{x(e^{iu}-1)} \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} x^{(\alpha t - 1)} e^{-\beta x} dx = \left(\frac{\beta}{\beta + 1 - e^{iu}}\right)^{\alpha t}$$

from which it follows that

$$P(N(t) = k) = \binom{-\alpha t}{k} \left(\frac{\beta}{\beta + 1}\right)^{\alpha t} \left(\frac{-1}{\beta + 1}\right)^k \text{ for } k = 1, 2, \ldots .$$

From the theorem it follows that

$$c_k = \frac{\alpha}{k(\beta + 1)^k} \text{ for } k = 1, 2, \ldots .$$

Now the probability of ruin may be calculated in the following manner. Assume that $E\, \Lambda(t) = t$ which implies that $E\, N(t) = 1$ and that $v_1 = E\, X_k$ exists. The probability of ruin $\psi(y_0)$ is then the probability that $y_0 + (v_1 + \varkappa)\, t - X(t)$ falls below zero at any time $t \geqslant 0$ where y_0 is the initial value of the risk reserve and $\varkappa$ the safety loading [2].

The bunches of claims will occur according to a Poisson process with intensity c and the amounts to be paid for each bunch will form a sequence of independent random variables each having the distribution function

$$\sum_{k=1}^{\infty} \frac{c_k}{c} V^{k*}(x)$$

where $V^{k*}(x)$ is the k-fold convolution of $V(x)$ with itself. Define $\varkappa_0$ by

$$(v_1 + \varkappa)\, t = (\varkappa_0 + \sum_{k=1}^{\infty} \frac{c_k}{c} k\, v_1)\, ct$$

from which it follows that

$$\varkappa_0 = \frac{v_1 + \varkappa - v_1 \sum_{k=1}^{\infty} c_k\, k}{c}$$

Under the additional assumption that

$$\sum_{k=1}^{\infty} c_k \int_0^{\infty} e^{\sigma x}\, V^{k*}\,(dx) < \infty \text{ for some } \sigma > 0$$

$\psi(y_0)$ may be calculated according to the derivation in Cramér [2] where ct plays the role of the operational time, $\varkappa_0$ of the safety loading and $\sum_{k=1}^{\infty} \frac{c_k}{c} V^{k*}\,(x)$ of the distribution of the amount of each claim.

3. The case of a directing process with interchangeable increments

The directing process $\{\Lambda(t);\ 0 \leqslant t < \infty\}$ is said to have interchangeable increments if for all $n = 2, 3, \ldots$ and all finite $T \in (0, \infty)$ it holds that

$$P\left[\bigcap_{k=1}^{n} \{\Lambda(\frac{kT}{n}) - \Lambda(\frac{(k-1)\,T}{n}) \leqslant x_k\}\right] =$$
$$= P\left[\bigcap_{k=1}^{n} \{\Lambda(\frac{kT}{n}) - \Lambda(\frac{(k-1)\,T}{n}) \leqslant x_{i_k}\}\right]$$

for all $n!$ permutations $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ and for all $(x_1, \ldots, x_n)$.

Bühlmann [1] has shown that for a process with interchangeable increments there exists a nontrivial sub-σ-algebra $\mathfrak{D}$ of $\mathfrak{L}$ such that $\Lambda(t)$ is a s.i.i. process relative to $\mathfrak{D}$. Further there exists a measure $\Pi_{\mathfrak{D}}$ on $\mathfrak{D}$ such that for every $0 \leqslant s < t < \infty$ we have

$$E\, e^{iu(\Lambda(t)-\Lambda(s))} = \int e^{(t-s)\eta(u,\omega)}\, \Pi_{\mathfrak{D}}(d\omega)$$

where

$$\eta(u, \omega) = \int_{0+}^{\infty} \frac{e^{iux} - 1}{x}\, \Omega(dx, \omega) + ib(\omega)u$$

is $\mathfrak{D}$-measurable in ω.

Denote by $\Pi^{\mathfrak{D}}{}_{\mathfrak{L}}$ the measure on $\mathfrak{L}$ relative to $\mathfrak{D}$ and by $\Pi^{\mathfrak{D}}{}_{\mathfrak{L}}$ the measure on $\mathfrak{N}$ relative to $\mathfrak{D}$. Due to the theorem of Kolmogorov $\Pi^{\mathfrak{D}}{}_{\mathfrak{L}}$ and thus $\Pi^{\mathfrak{D}}{}_{\mathfrak{N}}$ are determined.

Since $\Lambda(t)$ is a s.i.i. process relative to $\mathfrak{D}$ also $N(t)$ is a.s i.i. process relative to $\mathfrak{D}$. Define

$$c_k(\omega) = \int_{1+}^{\infty} \frac{x^k}{k!} e^{-x} \frac{\Omega(dx, \omega)}{x} + \delta_{k,1}\, b(\omega)$$

The sequence $\{c_k(\omega)\}_{k=1}^{\infty}$ is thus a sequence of D-measurable functions. From the theorem of Kolmogorov and from the theorem in section 2 it follows that the restriction $\Pi^{\mathfrak{D}}{}_{\mathfrak{N}_0}$ of $\Pi^{\mathfrak{D}}{}_{\mathfrak{N}}$ to $\mathfrak{N}_0$ may be expressed in terms of the sequence $\{c_k(\omega)\}_{k=1}^{\infty}$.

Assume that $E\ \Lambda(t) = t$. Define the $\mathfrak{D}$-measurable functions

$$c(\omega) = \sum_{k=1}^{\infty} c_k(\omega)$$

$$V(x, \omega) = \sum_{k=1}^{\infty} \frac{c_k(\omega)}{c(\omega)} V^{k*}(x)$$

$$v_1(\omega) = v_1 \sum_{k=1}^{\infty} \frac{c_k(\omega)}{c(\omega)} k$$

and

$$\varkappa_0(\omega) = \frac{v_1(\omega) + \varkappa - v_1 \sum_{k=1}^{\infty} c_k(\omega)\,.\,k}{c(\omega)}$$

If for almost every ω with respect to Π_D

$$\int_0^{\infty} e^{\sigma x}\, V(dx, \omega) < \infty \text{ for some } \sigma > 0$$

the probability of ruin $\psi^{\mathfrak{C}}(y_0)$ relative to $\mathfrak{D}$ may be calculated. Because of the assumption of separability the function

$$I(\omega) = \begin{cases} 1 \text{ if } y_0 + (v_1 + \varkappa)\, t - X(t) < 0 \text{ for some } t \geqslant 0 \\ 0 \text{ elsewhere} \end{cases}$$

is $\mathfrak{X}_0$-measurable and thus furthermore $\mathfrak{X}$-measurable.

Since

$$\psi^{\mathfrak{D}}(y_0) = E(I(\omega)\,|\,\mathfrak{D})$$

$\psi^{\mathfrak{D}}(y_0)$ is $\mathfrak{D}$-measurable and $\psi(y_0)$ is given by

$$\psi(y_0) = \int \psi^{\mathfrak{D}}(y_0)\, d\, \Pi_{\mathfrak{D}}$$

Remark

If $\Lambda(t) = \lambda \cdot t$ where λ is a Γ-distributed random variable $N(t)$ is a Polya-process. In Segerdahl [4] the ruin probability is calculated by a weighting procedure. If $\Omega(dx, \omega) = 0$ almost surely with respect to $\Pi_{\mathfrak{D}}$ and if $\lambda(\omega) = b(\omega)$ our result reduces to the result due to Segerdahl.

4. A NUMERICAL ILLUSTRATION

Consider the case where $N(t)$ is a Polya-process and where

$$V(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geqslant 0 \\ 0 & \text{for } x < 0. \end{cases}$$

In our notations this implies that if λ is a random variable defined on $(\Omega, \mathfrak{D}, \Pi)$ where

$$\Pi_{\mathfrak{D}}\,(\lambda \leqslant x) = \int_0^\infty \frac{y^{h-1}\, h^h}{\Gamma(h)}\, e^{-hy}\, dy$$

then $\Lambda(t) = \lambda \cdot t$ almost surely with respect to $\Pi^{\mathfrak{D}}{}_{\mathfrak{L}}$.

In Cramér [2] it is shown that for this choice of $V(x)$ the ruin probability $\psi(y_0)$ is in the Poisson case (with intensity 1) given by

$$\psi(y_0) = \begin{cases} \dfrac{1}{1+\varkappa}\, e^{-\dfrac{y_0\varkappa}{1+\varkappa}} & \text{for } \varkappa > 0 \\ 1 & \text{for } \varkappa \leqslant 0 \end{cases}$$

Since in the Polya case $v_1(\omega) = 1$, $c(\omega) = \lambda(\omega)$ and $c_k(\omega) = 0$ for $k = 2, 3, \ldots$ almost surely with respect to $\Pi_{\mathfrak{D}}$ it follows that $\varkappa_0(\omega) = \dfrac{1 + \varkappa - \lambda(\omega)}{\lambda(\omega)}$.

Thus

$$\psi^{\mathfrak{D}}(y_0) = \begin{cases} \dfrac{\lambda(\omega)}{1+\varkappa}\, e^{-\dfrac{1+\varkappa-\lambda(\omega)}{1+\varkappa}\, y_0} & \text{for } \lambda(\omega) < 1 + \varkappa \\ 1 & \text{for } \lambda(\omega) \geqslant 1 + \varkappa \end{cases}$$

Define the function

$$\Gamma(x, \alpha, \beta) = \int_0^x \frac{y^{\alpha-1}\, \beta^\alpha}{\Gamma(\alpha)}\, e^{-\beta y}\, dy.$$

It then follows that

$$\psi(y_0) = 1 - \Gamma(1 + \varkappa, h, h) + \left(\frac{h}{h - \frac{y_0}{1+\varkappa}}\right)^{h+1} \frac{e^{-y_0}}{1+\varkappa} \Gamma\left(1 + \varkappa, h + 1, h - \frac{y_0}{1+\varkappa}\right).$$

In tables 1, . . . , 4 this function is calculated for $\varkappa = 0.0$, 0.1, 0.2 and 0.3 and for $h = 1$, 2, 5, 10, 25, 50 and 100. These values are further compared with the ruin probabilities in the Poisson case which is indicated by $h = \infty$.

Table 1

$\varkappa = 0$	$\psi(y_0)$							
y_0	$h = 1$	$h = 2$	$h = 5$	$h = 10$	$h = 25$	$h = 50$	$h = 100$	$h = \infty$
0	0.6321	0.7293	0.8245	0.8749	0.9205	0.9437	0.9601	1.0000
10	0.4042	0.4589	0.5236	0.5717	0.6380	0.6905	0.7429	1.0000
20	0.3862	0.4329	0.4838	0.5189	0.5671	0.6079	0.6533	1.0000
30	0.3801	0.4240	0.4696	0.4992	0.5379	0.5703	0.6077	1.0000
40	0.3771	0.4195	0.4624	0.4890	0.5223	0.5495	0.5809	1.0000
50	0.3752	0.4168	0.4580	0.4829	0.5128	0.5364	0.5635	1.0000
60	0.3740	0.4150	0.4551	0.4787	0.5063	0.5275	0.5514	1.0000
70	0.3731	0.4137	0.4530	0.4758	0.5017	0.5210	0.5425	1.0000
80	0.3725	0.4128	0.4515	0.4735	0.4982	0.5161	0.5358	1.0000
90	0.3720	0.4120	0.4502	0.4718	0.4954	0.5123	0.5305	1.0000
100	0.3716	0.4114	0.4493	0.4704	0.4932	0.5092	0.5262	1.0000
$\to \infty$	0.3679	0.4060	0.4405	0.4579	0.4734	0.4812	0.4867	1.0000

Table 2

$\varkappa =$ 0.1	$\psi(y_0)$							
y_0	$h = 1$	$h = 2$	$h = 5$	$h = 10$	$h = 25$	$h = 50$	$h = 100$	$h = \infty$
0	0.6065	0.6976	0.7858	0.8315	0.8713	0.8897	0.9008	0.9091
10	0.3694	0.4080	0.4425	0.4588	0.4686	0.4663	0.4545	0.3663
20	0.3512	0.3815	0.4009	0.4016	0.3847	0.3578	0.3186	0.1476
30	0.3451	0.3725	0.3864	0.3812	0.3533	0.3151	0.2626	0.0595
40	0.3420	0.3680	0.3791	0.3709	0.3373	0.2934	0.2340	0.0240
50	0.3402	0.3653	0.3748	0.3648	0.3278	0.2806	0.2172	0.0097
60	0.3390	0.3635	0.3719	0.3607	0.3215	0.2721	0.2064	0.0039
70	0.3381	0.3623	0.3698	0.3578	0.3171	0.2661	0.1988	0.0016
80	0.3375	0.3613	0.3683	0.3556	0.3137	0.2617	0.1933	0.0006
90	0.3369	0.3605	0.3671	0.3539	0.3112	0.2583	0.1890	0.0003
100	0.3365	0.3599	0.3661	0.3526	0.3091	0.2556	0.1857	0.0001
$\to \infty$	0.3329	0.3546	0.3575	0.3405	0.2910	0.2322	0.1584	0.0000

Table 3

$\varkappa = 0.2$	$\psi(y_0)$							
y_0	$h = 1$	$h = 2$	$h = 5$	$h = 10$	$h = 25$	$h = 50$	$h = 100$	$h = \infty$
0	0.5823	0.6670	0.7470	0.7864	0.8169	0.8280	0.8323	0.8333
10	0.3376	0.3615	0.3686	0.3564	0.3180	0.2755	0.2305	0.1574
20	0.3194	0.3350	0.3267	0.2988	0.2344	0.1709	0.1088	0.0297
30	0.3133	0.3261	0.3126	0.2793	0.2064	0.1368	0.0717	0.0056
40	0.3103	0.3216	0.3056	0.2698	0.1930	0.1213	0.0564	0.0011
50	0.3084	0.3190	0.3014	0.2641	0.1853	0.1127	0.0486	0.0002
60	0.3072	0.3172	0.2987	0.2604	0.1803	0.1073	0.0440	0.0000
70	0.3064	0.3159	0.2967	0.2578	0.1768	0.1036	0.0411	0.0000
80	0.3057	0.3150	0.2952	0.2558	0.1742	0.1009	0.0390	0.0000
90	0.3052	0.3143	0.2941	0.2543	0.1722	0.0989	0.0375	0.0000
100	0.3048	0.3137	0.2932	0.2531	0.1707	0.0973	0.0363	0.0000
$\to \infty$	0.3012	0.3084	0.2851	0.2424	0.1573	0.0844	0.0280	0.0000

Table 4

$\varkappa = 0.3$	$\psi(y_0)$							
y_0	$h = 1$	$h = 2$	$h = 5$	$h = 10$	$h = 25$	$h = 50$	$h = 100$	$h = \infty$
0	0.5596	0.6378	0.7090	0.7414	0.7628	0.7681	0.7692	0.7692
10	0.3086	0.3194	0.3032	0.2691	0.2026	0.1502	0.1122	0.0765
20	0.2905	0.2931	0.2624	0.2146	0.1297	0.0686	0.0304	0.0076
30	0.2844	0.2844	0.2490	0.1972	0.1083	0.0477	0.0138	0.0008
40	0.2815	0.2801	0.2425	0.1889	0.0987	0.0395	0.0088	0.0001
50	0.2797	0.2775	0.2386	0.1840	0.0934	0.0353	0.0067	0.0000
60	0.2785	0.2758	0.2361	0.1808	0.0900	0.0329	0.0056	0.0000
70	0.2776	0.2766	0.2343	0.1786	0.0877	0.0312	0.0050	0.0000
80	0.2770	0.2737	0.2329	0.1770	0.0860	0.0301	0.0046	0.0000
90	0.2765	0.2730	0.2319	0.1757	0.0848	0.0292	0.0043	0.0000
100	0.2761	0.2724	0.2311	0.1747	0.0837	0.0286	0.0041	0.0000
$\to \infty$	0.2725	0.2674	0.2237	0.1658	0.0754	0.0236	0.0028	0.0000

REFERENCES

[1] BÜHLMANN H., Austauschbare stochastische Variabeln und ihre Grenzwertsätze, *Univ. of Cal. Publ. in Stat. 3*, 1960.

[2] CRAMÉR H., Collective risk theory, *Jubilee volume of Försäkringsaktiebolaget Skandia*, Stockholm, 1955.

[3] FELLER W., *An introduction to probability theory and its applications Vol. II*, John Wiley & Sons, New York, 1966.

[4] SEGERDAHL C.-O., Lecture notes, University of Stockholm, 1967.

[5] WALLDIN K.-E., *Aspects of operations research in theory and practice*, Almqvist & Wiksell, Uppsala, 1969.