# ON THE CALCULATION OF THE RUIN PROBABILITY FOR A FINITE TIME PERIOD

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In many risk theoretical questions a central problem is the numerical evaluation of a convolution integral and much effort has been devoted over the years to mathematical and computational aspects. The paper presented to this colloquium by O. Thorin shows the subject to be topical but the present note stems from a recent paper by H. L. Seal, "Simulation of the ruin potential of non-life insurance companies", published in the Transactions of the Society of Actuaries, Volume XXI page 563.

In this paper, amongst other topics, Seal has presented some simulations of ruin probabilities over a finite time interval. Some years ago (Journal Institute of Actuaries Students' Society, Volume 15, 1959). I pointed out that the form of ruin probability could be expressed as a successive product of values of a distribution function. To my knowledge no one has attempted to see if this product form was capable of development and the numerical values in Seal's paper prompted me to spend a little time on the problem. In the time I had available it has not been possible to do more than experiment, but the conclusions reached may be of interest to other workers in this field They showed that calculation is feasible but laborious. However, the knowledge that it can be done may suggest methods of improving the techniques.

Seal's first simulation example is the calculation of ruin probabilities when the distribution of the interval of time between the claims is negative exponential and the individual claim distribution is also negative exponential. Instead of following Seal's method we can determine the "gain per interval" and find that if $\lambda$ is the security loading the frequency function for the gain $z$ is

$$
\begin{aligned}
f(z)\, dz &= \frac{e^{z}}{2+\lambda}\, dz \qquad -\infty < z < 0 \\
&= \frac{e^{-z/(1+\lambda)}}{2+\lambda}\, dz \qquad 0 < z < \infty
\end{aligned}
$$

i.e. a Laplace distribution.

The cusp at $z = 0$ and the exponential tails show immediately that numerical work with this function will be troublesome and provide a severe test of any approximation techniques. The following notes are concerned solely with a model of this form.

Some preliminary experiments led to the suggestion that the "non-ruin" probability at the end of operational time $t$ when $u = 0$ could be expressed as

$$P_t^\lambda = \left(1 - \frac{1}{2+\lambda}\right)\left(1 - \frac{1}{2+\lambda} \cdot \frac{1}{2+\lambda}\right)$$

$$\left(1 - \frac{1}{2+\lambda} \cdot \frac{2.1}{2+\lambda.3+\lambda}\right) \dots \left(1 - \frac{1}{2+\lambda} \cdot \frac{(t-1)!}{2+\lambda \dots t+\lambda}\right)$$

or

$$P_t^0 = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2.2}\right)\left(1 - \frac{1}{2.3}\right) \dots \left(1 - \frac{1}{2} \cdot \frac{1}{2t}\right)$$

and direct comparison with Seal's values confirmed the suggestion. For $u \neq 0$ no simple product form was apparent.

The probability of "non-ruin" is, of course, the truncated convolution integral

$$^{u}P_t = \int_{-u}^{\infty} f(z_1) \int_{u-z_1}^{\infty} f(z_2) \dots \int_{u-z_1-\dots-z_{t-1}}^{\infty} f(z_t)\, dz_1 \dots dz_t \qquad (\alpha)$$

which can be evaluated in the Laplace case. The values for $t = 1, 2$ being:

$$^{u}P_1^\lambda = 1 - \frac{e^{-u}}{2+\lambda}, \qquad ^{u}P_2^\lambda = 1 - e^{-u}\left\{\frac{5+5\lambda+\lambda^2}{(2+\lambda)^3} + \frac{u}{(2+\lambda)^2}\right\}$$

For $\lambda = 0$ the expressions are simpler, i.e.

$$^{u}P_1^0 = 1 - \frac{e^{-u}}{2}, \qquad ^{u}P_2^0 = 1 - \frac{e^{-u}}{8}(2u+5)$$

$$^{u}P_3^0 = 1 - \frac{e^{-u}}{16}(u^2+6u+11), \qquad ^{u}P_4^0 = 1 - \frac{e^{-u}}{384} \cdot (4u^3+42u^2+174u+279)$$

The recurrence formula between $P_t^0$ and $P_{t+1}^0$ has not been investigated to see if the polynomials in $u$ could be approximated in any way.

These early values were required for tests of subsequent experiments.

From the expression ($\alpha$) for ${}^{u}P_t$ we can write

$$ {}^{u}P_t = \int_{-u}^{\infty} f(z_1)\, {}^{u+z_1}P_{t-1}\, dz_1 \qquad (\beta) $$

and from the integral mean value theorem

$$ \begin{aligned} {}^{u}P_t &= {}^{u+\bar{z}}P_{t-1} \,.\, {}^{u}P_1 \qquad -u < z_r < \infty \\ &= {}^{u+\bar{z}_1}P_1 \,.\, {}^{u+\bar{z}_2}P_2 \ldots {}^{u}P_t \end{aligned} $$

As has been pointed out (Ref. 1 page 133) formula ($\beta$) provides a recurrence formula but the quadratures soon become unreasonable by normal methods.

Consider now

$$ \begin{aligned} {}^{u}P_2 &= \int_{u}^{\infty} f(z_1) \int_{u-z_1}^{\infty} f(z_2)\, dz_1\, dz_2 \\ &= \int_{-u}^{\infty} f(z_1)\, {}^{u+z_1}P_1\, dz_1 = {}^{u+\bar{z}}P_1 \,.\, {}^{u}P_1 \end{aligned} $$

We know that

$$ \bar{z} = m + \frac{\mu_2}{2}\frac{f''}{f'} + \frac{\mu_3}{6}\frac{f'''}{f'} + \ldots $$

where $m$, $\mu_2$ are the successive moments of $f(z)$ between $a$ and $\infty$ and the $f$'s are successive differential coefficients of $f(z)$ when $z = m$. Since the moments of $f(z)$ are common to each of the values of $P_t$ attempts were made to see if it was possible to calculate the successive values of $\bar{z}$, without, however, leading to any apparent progress.

Quadrature formula based on selected intervals in the range $-u$ to $\infty$ soon become troublesome because of the increase in range at each stage.

Accordingly investigations were made into the use of some form of weighted Gaussian type formula. These were investigated by me in some detail in 1947 (see J.I.A. Volume 73, page 356) in the form

$$ \int_{a}^{b} f(x)\,\phi(x)\,dx = \sum_{r=1}^{n} a_r f(x_r) \int_{a}^{b} \phi(x)\,dx + R_n \int_{a}^{b} \phi(x)\,dx $$

where the $a_r$ and $x_r$ are determined as functions of the successive moments of $\phi(x)$.

This formula, apart from the remainder term, avoids the differentials of $f(x)$ and is in a suitable form for the present problem since the $a_r$ and $x_r$, can be determined once and used in each successive integration. The technique also has the advantage that the order of approximation can be controlled by using various values of $n$ to test the convergence of the approximations, coupled with knowledge of the numerical coefficients in the remainder terms.

The computation routine would thus be

(a) Having regard to the known shape of $f(z)$, and the inferred shape of $P_t$ decide on the type of formula and the value of $n$
(b) For a range of values of $u$, determine the $a_r$ and $x_r$
(c) Calculate values of ${}^uP_2$ from ${}^uP_1$ for the selected values of $u$
(d) Repeat (c) by interpolation on ${}^uP_2$ etc.

The interpolation in stage (d) could be troublesome if the mesh of $u$-values is coarse but various options are available. One is to express ${}^uP_t$ as ${}^zP_1$, and interpolate on the values of $z$. Another is to express ${}^uP_t$ as a product of $P_1$ values and interpolate on $z$ from ${}^uP_t =$ $= {}^zP_1 \, . \, P_{t-1}$.

It will be noted, of course, that the technique provides an approximation to the distribution of ${}^uP_t$ for a range of values of $u$.

Turning now to the calculations, the following are limited to the case $\lambda = 0$. We know that $f(z)$ is of Laplace form so that it is almost essential to split the range of integration at $z = 0$ and evaluate ${}^uP_t$ in the form ${}^uP_t = \int_{-u}^{0} f(z_1)\, {}^{u+z_1}P_t \, dz_1 + \int_{0}^{\infty} f(z_1)\, {}^{u+z_1}P_t \, dz_1$.

For the L.H. integral a finite range formula is needed whilst an infinite range is needed for the R.H. integral. For low values of $t$ it was evident that a 3 term integration formula was required on each part, because of the exponential tail. Moments of $\int_{-u}^{0} f(z_1)\, dz_1$ and $\int_{0}^{\infty} f(z_1)\, dz_1$ were calculated for values of $u = 2(2)\,10, \infty$.

For example

$$\int_{u}^{0} f(z)\; P(z)\; dz = \sum_{1}^{3} a_r\; P(z_r) \int_{-u}^{0} f(z)\; dz + R_3 \int_{-u}^{0} f(z)\; dz$$

$$R_3 = \frac{21.1\, f^{(6)}(\bar{z})}{6!} \qquad -a < z < 0$$

where for $u = 10$

| $r$ | $a_r$ | $z_i$ |
|---|---|---|
| 1 | 69536 | 4009 |
| 2 | .29111 | 2 2015 |
| 3 | 01353 | 5 9027 |

Values of ${}^uP_2$ and ${}^uP_3$ calculated by this technique, together with their true values are as follows:

| $u$ | *Calculated* ${}^uP_2$ | *True* | *Calculated* ${}^uP_3$ | *True* |
|---|---|---|---|---|
| 16 | | 999,999 | | |
| 14 | | 999,997 | | |
| 12 | | 999,978 | | |
| 10 | .999,873 | .999,858 | .999,562 | .999,519 |
| 8 | 999,159 | 999,119 | 997,479 | .997,421 |
| 6 | .994,792 | .994,733 | .987,216 | .987,141 |
| 4 | .970,267 | .970,237 | .941,823 | .941,619 |
| 2 | .847,837 | .847,748 | .772,089 | .771,622 |
| 0 | | 375,000 | | |

Some difficulty was experienced for $u = 2$ in interpolating on $z$ from ${}^uP_2 = {}^zP_1$. It will be noted that the errors are all of the same sign, the calculated values being in excess of the true values. It would be possible to improve the approximations by calculating values from 1, 2 and 3 term formula and use of the remainder term, and by using more significant figures in certain stages of the calculations.

It will be appreciated that the amount of arithmetic required is considerable but not unexpected in an experimental investigation of this type. In principle much of the calculation could be programmed for a computer and, of course, higher order formulae and a closer mesh used for the ${}^uP$ values.

## Reference

BEARD, R. E., PENTIKAINEN, T.; PESONEN, E., *Risk Theory*, Methuen, London, 1969