ON RISK PROCESSES WITH STOCHASTIC INTENSITY FUNCTION

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1. INTRODUCTION

In this paper we are going to study some properties of a stochastic process, which has been proposed by Cramér (1968) as a model of the claims arising in an insurance company. This process has been studied by Cox in a different context. A few elementary results, concerning moments, are given by Cox and Lewis (1966). The present paper will be a survey of some results derived by the author (1970:1) and (1970:2). For detailed proofs we refer to these papers.

2. DEFINITION OF THE PROCESS

Let $\lambda(t)$ be a real-valued stochastic process, such that $P\{\lambda(t) < 0\} = 0$. We further assume that $E\lambda(t) = 1$ and that $E\lambda^2(t) < \infty$ for every fixed value of t. We denote the covariance

$$\text{Cov}\,\{\lambda(s),\,\lambda(t)\} \text{ by } r(s,\,t).$$

The process $\lambda(t)$ will play the role of an intensity function. That means, that for every fixed realization of the process, the probability of

$$\left.\begin{array}{l} 0 \\ 1 \\ \text{more than } 1 \end{array}\right\} \text{ event in } (t,\, t + \Delta t) = \left\{\begin{array}{l} 1 - \Delta t\lambda(t) + o(\Delta t) \\ \Delta t\lambda(t) + o(\Delta t) \\ o(\Delta t) \end{array}\right.$$

and that the number of events in disjoint intervals are independent.

We now define a point process $N(t)$, where $N(t)$ is the number of events which have occurred in $(0, t])$. With this definition we get

$$P_n(t) = P(N(t) = n) = E\,\frac{\Lambda(t)^n}{n!}\,e^{-(\Lambda t)}.$$

where

$$\Lambda(t) = \int^{t} \lambda(\tau)\,d\tau$$

The integral is assumed to exist almost surely. This process will be called the *N-process*.

We will now define the non-elementary process, which corresponds to the total amount of claims. We then associate a quantity to each event. These quantities are defined by a sequence of independent equally distributed random variables X_1, X_2, X_3, ... with the common distribution function $V(x)$. The quantities are furthermore independent of the process $N(t)$. We now define $v_j = EX^j$ and $v(u) = E(\exp(iuX))$. It is now possible to define a stochastic process by

$$X(t) = \sum_{k=1}^{N(t)} X_k \quad (X(t) = 0 \text{ if } N(t) = 0).$$

This process will be called the *X-process*.

For this process we have

$$F(x, t) = P(X(t) \leq x) = \sum_{n=0}^{\infty} V^{n*}(x)\, E\left\{\frac{\Lambda(t)^n}{n!}\, e^{-\Lambda(t)}\right\}.$$

In the last section we will exemplify with the Poisson-process, the process studied by Ammeter (1948) and the Polya-process studied by Lundberg (1940).

3. Some moment formulae

The following moment formulae are derived by using conditional expectations.

$$EX(t) = v_1 t$$

and

$$\operatorname{Var} X(t) = v_2 t + v_1^2 \operatorname{Var} \Lambda(t).$$

By putting $v_1 = v_2 = 1$ we get the corresponding formulae for the N-process.

4. Limit theorems

4.1. *Some definitions*

Definition 1

The process $\lambda(t)$ will be called *ergodic* if $\lim_{t\to\infty} \operatorname{Var} t^{-1} \Lambda(t) = 0$.

Definition 2

The normal distribution function will be denoted by $\Phi(x)$.

4.2. *Limit distributions of the N-process*

We will now study the limit distribution of the variable

$$\frac{N(t) - t}{\sqrt{\operatorname{Var} N(t)}} \, .$$

The limit distribution will depend on the variance of $\Lambda(t)$. A little vaguely we can express this by saying that the limit distribution depends on the relationship between the randomness of $\Lambda(t)$ and the randomness of $N(t)$ given the value of $\Lambda(t)$.

The following theorems hold.

Theorem 1

If $t^{-1} \operatorname{Var} \Lambda(t) \to 0$ as $t \to \infty$, then

$$P\left(\frac{N(t) - t}{\sqrt{\operatorname{Var} N(t)}} \leq x\right) \to \Phi(x).$$

Theorem 2

If $t^{-1} \operatorname{Var} \Lambda(t) \to k, \quad 0 < k < \infty$, and if

$$P\left(\frac{\Lambda(t) - t}{\sqrt{\operatorname{Var} \Lambda(t)}} \leq x\right) \to G(x) \text{ as } t \to \infty, \text{ then}$$

$$P\left(\frac{N(t) - t}{\sqrt{\operatorname{Var} N(t)}} \leq x\right) \to G\left(x \sqrt{1 + \frac{1}{k}}\right) * \Phi(x \sqrt{1 + k}).$$

Theorem 3

If $t^{-1} \operatorname{Var} \Lambda(t) \to \infty$ and if

$$P\left(\frac{\Lambda(t) - t}{\sqrt{\operatorname{Var} \Lambda(t)}} \leq x\right) \to G(x) \text{ as } t \to \infty, \text{ then}$$

$$P\left(\frac{N(t) - t}{\sqrt{\operatorname{Var} N(t)}} \leq x\right) \to G(x).$$

Indication of proof

The theorems are proved by showing that the difference between the characteristic function of $P\left(\frac{N(t)-t}{\sqrt{\operatorname{Var} N(t)}} \leq x\right)$ and the characteristic function of the limit distribution tends to zero as $t \to \infty$.

4.3. *Limit distributions of the X-process*

In treating the questions of limit distributions of $X(t)$ Lundberg (1940) points out in his special case, that there is a fundamental difference if v_1 is equal to zero or not. From the variance formula it seems probable that this will be the case as soon as $\operatorname{Var} \Lambda(t)$ is of the same or a higher order than t. We will anyhow separate the two cases completely.

We now assume that $v_1 \neq 0$ and $v_2 < \infty$. In this case we get the same three different cases as we got for the N-process.

Theorem 4

If $t^{-1} \operatorname{Var} \Lambda(t) \to 0$ as $t \to \infty$, then

$$P\left(\frac{X(t)-v_1 t}{\sqrt{\operatorname{Var} X(t)}} \leq x\right) \to \Phi(x).$$

Theorem 5

if $t^{-1} \operatorname{Var} \Lambda(t) \to k$, $0 < k < \infty$, and if

$$P\left(\frac{\Lambda(t)-t}{\sqrt{\operatorname{Var} \Lambda(t)}} \leq x\right) \to G(x) \text{ as } t \to \infty, \text{ then}$$

$$P\left(\frac{X(t)-v_1 t}{\sqrt{\operatorname{Var} X(t)}} \leq x\right) \text{ tends to}$$

$$\begin{cases} G\left(x\sqrt{1+\dfrac{v_2}{v_1^2 k}}\right) * \Phi\left(x\sqrt{1+\dfrac{v_1^2 k}{v_2}}\right) & \text{if } v_1 > 0 \\[2ex] \left(1-G\left(-x\sqrt{1+\dfrac{v_2}{v_1^2 k}}-0\right)\right) * \Phi\left(x\sqrt{1+\dfrac{v_1^2 k}{v_2}}\right) & \text{if } v_1 < 0. \end{cases}$$

Theorem 6

If $t^{-1} \operatorname{Var} \Lambda(t) \to \infty$ and if

$$P\left(\frac{\Lambda(t)-t}{\sqrt{\operatorname{Var} \Lambda(t)}} \leq x\right) \to G(x) \text{ as } t \to \infty, \text{ then}$$

$$P\left(\frac{X(t)-v_1 t}{\sqrt{\operatorname{Var} X(t)}} \leq x\right) \text{ tends to}$$

$$\begin{cases} G(x) & \text{if } v_1 > 0 \\ 1 - G(-x-0) & \text{if } v_1 < 0. \end{cases}$$

We now assume that $v_1 = 0$ and $v_2 < \infty$. In this case we get the following theorems.

Theorem 7

If $\lambda(t)$ is ergodic then,

$$P\left(\frac{X(t)}{\sqrt{\operatorname{Var} X(t)}} \leq x\right) \to \Phi(x).$$

Theorem 8

If $t^{-1}\sqrt{\operatorname{Var} \Lambda(t)} \to r$, $0 < r < \infty$, and if

$$P\left(\frac{\Lambda(t)-t}{\sqrt{\operatorname{Var} \Lambda(t)}} \leq x\right) \to G(x) \text{ as } t \to \infty, \text{ then}$$

$$P\left(\frac{X(t)}{\sqrt{\operatorname{Var} X(t)}} \leq x\right) \to \int_0^\infty \Phi\left(\frac{x}{\sqrt{y}}\right) dG_1(y)$$

where $G_1(y) = G\left(\frac{y-1}{r}\right)$.

Indication of proof

From the limit distributions of $N(t)$ the corresponding limit distributions of $X(t)$ follow from the results due to Robbins (1948).

5. Linear estimates of the intensity

Our purpose is to investigate how one observation of the N-process in the interval $(0, T)$ can be used in order to give estimates, $\lambda^*(t)$, of the realization of $\lambda(t)$ which generated the observed N-process. We suppose that $t_1, \ldots, t_{N(T)}$ are the successive times of occurrence for the events in this interval.

An estimate $\lambda^*(t)$ will be said to be the *best estimate* of $\lambda(t)$ if $E\{\lambda^*(t) - \lambda(t)\}^2$ is minimized.

Theorem 9

The best estimate $\lambda^*(t)$ of $\lambda(t)$ is given by

$$\lambda^*(t) = \frac{E_\lambda[\lambda(t)\{\prod_{k=1}^{N(T)} \lambda(t_k)\} e^{-\Lambda(T)}]}{E_\lambda[\{\prod_{k=1}^{N(T)} \lambda(t_k)\} e^{-\Lambda(T)}]}.$$

It is easily understood that for most cases, this estimate will require calculations, which are impossible to perform. We will therefore restrict ourselves to linear estimates. This means that we are going to study estimates of the type

$$\lambda^*(t) = \alpha(t) + \int_0^T \beta_t(s)\, d(N(s) - s).$$

Theorem 10

The best linear estimate $\lambda^*(t)$ of $\lambda(t)$ is given by

$$\lambda^*(t) = 1 + \int_0^T \beta_t(s)\, d(N(s) - s)$$

where $\beta_t(s)$ is the solution of the integral equation

$$\beta_t(s) + \int_0^T \beta_t(\tau)\, r(\tau, s)\, d\tau - r(t, s) = 0.$$

For this estimate we have

$$E\{\lambda^*(t) - \lambda(t)\}^2 = \beta_t(t).$$

Indication of proof

The general linear estimate is given by

$$\lambda^*(t) = \alpha(t) + \int_0^T \beta_t(s)\, d(N(s) - s).$$

We define the eigenfunctions and eigenvalues of $r(s, t)$ by the solutions of the integral equation

$$\phi(t) = \mu \int_0^T \phi(s)\, r(s, t)\, ds.$$

From the theorem of Mercer it follows that

$$r(s, t) = \sum_{k=1}^{\infty} \frac{\phi_k(s)\,\phi_k(t)}{\mu_k}$$

if $r(s, t)$ is continuous in $0 \leq s, t \leq T$.

By expanding $\beta_t(s)$ in terms of the eigenfunctions of $r(s, t)$, it follows that $E\{\lambda^*(t) - \lambda(t)\}^2$ is minimized if

$$\alpha(t) = 1$$

and

$$\beta_t(s) = \sum_{k=1}^{\infty} \frac{\phi_k(s)\,\phi_k(t)}{1 + \mu_k}.$$

This series is the unique solution of the integral equation in the theorem.

6. Linear estimates in a modified process

In certain applications it is impossible to observe the exact time of each event.

Assume that our observations are restricted to $N(\Delta)$, $N(2\Delta)$, ..., $N(n\Delta)$, where Δ is a positive quantity and $n = [T/\Delta]$. In order to avoid trivial complications, we will assume that T is a multiple of Δ.

We now define

$$l_k = 1/\Delta\,\{\Lambda(\Delta k) - \Lambda(\Delta(k-1))\} \qquad (k = 1, \ldots, n)$$

and

$$N_k = N(\Delta k) - N(\Delta(k-1)) \qquad (k = 1, \ldots, n).$$

Defining

$$r_{k,j} = \operatorname{Cov} l_k, l_j$$

we have

$$E\,N_k = \Delta$$

and

$$\operatorname{Cov} N_k, N_j = \Delta\delta_{k,j} + \Delta^2\gamma_{k,j}$$

$$\text{where} \quad \delta_{k,j} = \begin{cases} 1 \text{ if } k = j \\ 0 \text{ if } k \neq j \end{cases}.$$

It is now possible to show that the best linear estimate l_ν^* of l_ν is given by

$$l_\nu^* = 1 + \sum_{k=1}^{n} \beta_{\nu,k} (N_k - \Delta)$$

where the sequence $\{\beta_{\nu,k}\}$ is defined by the solution of

$$\Delta \sum_{j=1}^{n} \beta_{\nu,j}\, r_{j,k} = r_{\nu,k} - \beta_{\nu,k} \qquad (k = 1, \ldots, n).$$

This equation corresponds to the integral equation in Theorem 10.

For the best linear estimate we have

$$E\{l_\nu^* - l_\nu\}^2 = \beta_{\nu,\nu}.$$

If we are interested in estimating the whole sequence $l_1, \ldots, l_n$, it is reasonable to use

$$1/n \sum_{\nu=1}^{n} \beta_{\nu,\nu}$$

as a measure of the efficiency of the estimates.

Assume that l_k is defined for all integers k and that $r_{k,j} = r_{k-j}$. From the general theory of stationary stochastic processes it follows that r_{k-j} has the representation

$$r_{k-j} = \int_{-\pi}^{\pi} e^{i(k-j)x}\, dF(x)$$

where F is a non-decreasing bounded function.

We further assume that

1. $F'(x)$ is bounded almost everywhere for $-\pi \leq x \leq \pi$.
2. $F(x)$ has at most finitely many discontinuities.

Under these assumptions the following theorem holds.

Theorem 11

If the process l_k is stationary and if the given regularity assumptions are fulfilled, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} \beta_{\nu,\nu} = \int_{-\pi}^{\pi} \frac{F'(x)\, dx}{1 + 2\pi\Delta F'(x)}.$$

Indication of proof

In order to prove the theorem, spectral representation of matrices as well as some facts about Toeplitz matrices, given by Grenander and Szegö (1958), are used.

7. Some examples

We are going to illustrate the results on three well-known models.

1. The Poisson-model.
2. The model due to Ammeter (1948).
3. The Polya-model.

With our formulation, these models can be described in the following way

1. $P(\lambda(t) = 1) = 1$	for all values of t.
2. $P(\lambda(t) = \lambda_{[t/\tau]}) = 1$	for all values of t. τ is a positive constant and $\lambda_0, \lambda_1, \ldots$ a sequence of independent random variables, with common distribution function $U(\lambda)$.
3. $P(\lambda(t) = \lambda) = 1$	for all values of t. λ is a random variable with distribution function $U(\lambda)$.

In both ex. 2 and ex. 3 the distribution function $U(\lambda)$ will be assumed to be a Γ-distribution with the frequency function given by

$$u(\lambda) = \begin{cases} \dfrac{h^h \lambda^{h-1} e^{-\lambda h}}{\Gamma(h)} & \text{if } \lambda \geq 0 \\ 0 & \text{if } \lambda < 0 \end{cases}.$$

7.1. *Limit distributions*

In these examples, the following variance formulae hold

1. $\operatorname{Var} \Lambda(t) = 0$
2. $\operatorname{Var} \Lambda(t) = \left[\dfrac{t}{\tau}\right] \dfrac{\tau^2}{h} + \left(t - \tau\left[\dfrac{t}{\tau}\right]\right)^2 \dfrac{1}{h}$
3. $\operatorname{Var} \Lambda(t) = \dfrac{t^2}{h}$.

The quantity k, defined by $k = \lim_{t\to\infty} t^{-1}$ Var $\Lambda(t)$, takes the following values

1. $k = 0$
2. $k = \tau/h$
3. $k = \infty$.

To be able to apply the limit theorems we must calculate

$$\lim_{t\to\infty} P\left(\frac{\Lambda(t) - t}{\sqrt{\operatorname{Var} \Lambda(t)}} \leq x\right) \text{ in exs. 2 and 3.}$$

In ex. 2 it follows from the central limit theorem that

$$\lim_{t\to\infty} P\left(\frac{\Lambda(t) - t}{\sqrt{\operatorname{Var} \Lambda(t)}} \leq x\right) = \Phi(x).$$

In ex. 3 we have

$$P\left(\frac{\Lambda(t) - t}{\sqrt{\operatorname{Var} \Lambda(t)}} \leq x\right) = P\left(\frac{t\cdot\lambda - t}{t/\sqrt{h}} \leq x\right) =$$
$$= P(\sqrt{h}(\lambda - 1) \leq x) = G(x)$$

where
$$G(x) = U\left(\frac{x}{\sqrt{h}} + 1\right).$$

Assume that the claim distribution is such that $v_2 < \infty$.

In the case $v_1 \neq 0$ we have the following limit distributions.

$$\lim_{t\to\infty} P\left(\frac{(X(t) - v_1 t}{\operatorname{Var} X(t)} \leq x\right) = \begin{cases} \Phi(x) \text{ in exs. 1 and 2} \\ G(x) \text{ in ex. 3 if } v_1 > 0 \\ 1 - G(-x-0) \text{ in ex. 3 if } v_1 < 0. \end{cases}$$

Now assume that $v_1 = 0$. In ex. 3 we have

$$\lim_{t\to\infty} t^{-1}\sqrt{\operatorname{Var} \Lambda(t)} = \frac{1}{\sqrt{h}}.$$

Since the processes in exs. 1 and 2 are ergodic, the limit distributions are given by Theorem 7. In ex. 3 we have

$$\lim_{t\to\infty} P\left(\frac{X(t)}{\sqrt{\operatorname{Var} X(t)}} \leq x\right) = \int_0^\infty \Phi\left(\frac{x}{\sqrt{y}}\right) dG_1(y)$$

where

$$G_1(y) = G\left(\frac{y-1}{r}\right) = G(\sqrt{h}(y-1)) = U(y).$$

This result is given by Lundberg (1940).

7.2. Estimation of the intensities

Only in exs. 2 and 3 there is any estimation problem. From the calculations by Lundberg (1940) it follows, that the best estimate $\lambda^*(t)$ of $\lambda(t)$ is given by

$$\lambda^*(t) = \begin{cases} \dfrac{h + N([t/\tau]\,\tau + \tau) - N([t/\tau]\,\tau)}{h + \tau} & (\text{for } t < [T/\tau]\,\tau) \ \text{ in ex. } 2 \\[2ex] \dfrac{h + N(T)}{n + T} & \text{in ex. } 3. \end{cases}$$

Since these best estimates are linear, they are at the same time the best linear estimates. Ex. 2 is however not included in the general treatment of linear estimates, since the theorem of Mercer requires a continuous covariance function.

It follows, however, from Lundberg (1940), that the best estimate is linear, only when the distribution $U(x)$ is a Γ-distribution. The best linear estimate, however, is dependent only on the two first moments of $U(x)$.

We now turn over to the modified process. Since we will only illustrate linear estimates, the examples may be given in terms of the covariances. It is natural to study the following examples.

$$2'.\ r_{k,j} = \begin{cases} 1/h & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

$$3'.\ r_{k,j} = 1/h \text{ for all } k \text{ and } j.$$

In *ex.* 2′ we have the equations

$$\Delta(1/h)\,\beta_{\nu,k} = (1/h)\,\delta_{\nu,k} - \beta_{\nu,k} \qquad (\nu, k = 1, \ldots, n)$$

or

$$\beta_{\nu,k} = \begin{cases} \dfrac{1}{h + \Delta} & \text{if } k = \nu \\ 0 & \text{if } k \neq \nu \end{cases}$$

and thus the estimates

$$l_\nu^* = 1 + \frac{N_\nu - \Delta}{h + \Delta} = \frac{h + N_\nu}{h + \Delta} \qquad (\nu = 1, \ldots, n)$$

are the best linear estimates.

For these estimates we have

$$E(l_\nu^* - l_\nu)^2 = \frac{1}{h + \Delta}.$$

In *ex.* 3′ we have

$$\Delta(1/h) \sum_{j=1}^{n} \beta_{\nu,j} = (1/h) - \beta_{\nu,k} \qquad (k, \nu = 1, \ldots, n)$$

or

$$\beta_{\nu,k} = \frac{1}{h + n\Delta}$$

and thus

$$l_\nu^* = \frac{h + \sum_{k=1}^{n} N_k}{h + n\Delta}.$$

For these estimates we have

$$E(l_\nu^* - l_\nu)^2 = \frac{1}{h + n\Delta}.$$

In *ex.* 2′ the spectral distribution is absolutely continuous, with spectral density given by $f(x) = 1/2\pi h$.

As an illustration of Theorem 11 we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\nu=1}^{n} \beta_{\nu,\nu} = \int_{-\pi}^{\pi} \frac{\frac{1}{2\pi h}}{1 + \frac{2\pi\Delta}{2\pi h}}\, dx = \frac{1}{h + \Delta}.$$

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