AN OUTLINE OF A GENERALIZATION — STARTED BY E. SPARRE ANDERSEN — OF THE CLASSICAL RUIN THEORY

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1. *Short recapitulation of Sparre Andersen's results*

E. Sparre Andersen [1] [1]) presented to the XV$^{\text{th}}$ International Congress of Actuaries, New York, 1957, a model of a collective risk process with a positive gross risk premium where the epochs of claims formed a renewal process. Let $\Psi(u)$ (where u denotes the original risk reserve) denote the ruin probability in this model. Generalizing the classical result Sparre Andersen deduced the inequality

$$\Psi(u) \leq e^{-Ru}$$

where R is a suitable positive number depending on the distribution function (continuous to the right), $P(y)$, $-\infty < y < \infty$, $P(0) < 1$, for the amounts of claims in case a claim occurs and also depending on the distribution function, $K(t)$, $t \geq 0$, $K(0) = 0$, for the times between the epochs of successive claims, (The times between the epochs of successive claims, the inter-occurrence times, are assumed to be independent and identically distributed random variables. The time between the starting point and the epoch of the first claim is assumed to be independent of and to have the same distribution function as the inter-occurrence times. The amounts of claims are assumed to be independent of each other and of the epochs of claims and to be identically distributed.)

In fact,

$$R = \sup \{\sigma \mid \sigma < Q, f(\sigma) = p(\sigma)\, k(-c\sigma) < 1\}$$

[1]) Numbers in brackets refer to the list of references at the end of the paper.

where

$$p(s) = \int_{-\infty}^{\infty} e^{sy}\, dP(y),$$

$$k(s) = \int_{0}^{\infty} e^{st}\, dK(t),$$

Q is the greatest positive value, for which $p(s)$ is analytic and regular in the strip $0 < Re(s) < Q$ and $c > 0$ is the constant gross risk premium per unit of time.

Thus it is assumed that $Q > 0$, corresponding to the same assumption in the Cramér theory ([3] p. 52). Of course, as in the Cramér theory it is also assumed that

$$p_1 = \int_{-\infty}^{\infty} y\, dP(y) \qquad \text{is finite.}$$

Furthermore, it is assumed that

$$k_1 = \int_{0}^{\infty} t\, dK(t) \qquad \text{is finite,}$$

and that

$$c > \frac{p_1}{k_1}$$

corresponding to the Cramér assumption ([3] p. 46) that $\lambda = c - p_1 > 0$.

Specializing to the Poisson case, $dK(t) = e^{-t}\, dt$, we easily retrieve the Cramér definition ([3] p. 53),

$$R = \sup \{\sigma \mid \sigma < Q \,,\, 1 + c\sigma - p(\sigma) > 0\}$$

since in this case we have

$$k(s) = \frac{1}{1 - s}.$$

At this point we observe that the *net* risk premium in the case of the non-Poisson renewal process is *not* proportional to time. However, as a consequence of the well-known renewal theorem ([4] p. 347), the net risk premium for a time-interval of length *h in the long run* is proportional to h. (If $K(t)$ is arithmetic as e.g. in the deterministic case: $K(t) = \varepsilon(t - k_1)$ some caution is needed)

In fact the formula for the *net* risk premium in the time interval $(T_1, T_2]$ is

$$V(T_1, T_2) = p_1 \sum_{n=1}^{\infty} [K^{n*}(T_2) - K^{n*}(T_1)],$$

where K^{n*} as usual denotes the *nth* convolution of K with itself.

We get

$$V(T_1, T_2) \to \frac{p_1}{k_1} (T_2 - T_1),\ T_2 - T_1 = h,\ T_1 \to \infty.$$

The assumption $c > \frac{p_1}{k_1}$ thus is an assumption that in the long run the safety loading is positive.

Considering the risk reserve

$$X(t) = u + ct - Y(t),$$

where $Y(t)$ is the accumulated amounts of claims, Sparre Andersen wrote this reserve in the form

$$X(t) = u + \sum_{i=1}^{n} (ct_i - y_i) + c(t - t_1 - t_2 - \ldots - t_n)$$

where $t_1 + t_2 + \ldots + t_i$, $i = 1, 2, \ldots$ are the epochs of claims and y_i are the corresponding amounts and

$$t_1 + t_2 + \ldots + t_n \leq t < t_1 + t_2 + \ldots + t_{n+1}.$$

(If $t < t_1$, we have $X(t) = u + ct$). Since ruin only can occur when t is an epoch of claim as a consequence of the assumption $c > 0$, Sparre Andersen could reduce the ruin problem to the consideration of a denumerable number of linear inequalities involving the t_i 's and the y_i's . The existence of the ruin probabilities for a finite or infinite period could thus easily be proved.

Sparre Andersen derived an integral equation for $\Psi(u)$ and proved that there are no other solutions subjected to be bounded by e^{-Ru}.

If we introduce $\Phi(u) = 1 - \Psi(u)$ the probability of non-ruin, Sparre Andersen's integral equation can be written

$$\Phi(u) = \int_0^{\infty} dK(v) \int_{-\infty}^{u+cv} \Phi(u + cv - x)\, dP(x).$$

This equation is well-known in the Poisson case (see [4] p. 181). As in this particular case the general equation has the following simple meaning: Let v denote the epoch of the *first* claim. Then non-ruin can only occur if $u + cv$ is not less than the amount x of the claim at v and the probability of non-ruin in this case is $\Phi(u + cv - x)$ since the process starts anew after the claim has occurred but on a new risk reserve level. Taking account of the distribution functions for v and x the equation follows.

2. *Further results*

Let $\Phi(u, T)$ denote the probability of non-ruin in the interval $(0, T]$. Then, in the same simple way as for $\Phi(u)$ we obtain the equation

$$\Phi(u, T) = \int_0^T dK(v) \int_{-\infty}^{u+cv} \Phi\,(u + cv - x, T - v)\, dP(x) + \int_T^\infty dK(v).$$

The type of unicity which Sparre Andersen proved for $T = \infty$ can in the same way be proved here.

The equations for $\Phi(u)$ and $\Phi(u, T)$ can be solved by application of the Wiener-Hopf technique used by Cramér ([3] section 5.8) in the Poisson case. The application in the general case turns out to be simple. There are two reasons for that. First, we have restricted ourselves to the case $c > 0$, second, the above simple equations lend themselves equally well to the application of the Wiener-Hopf technique as the deeper Cramér integral equations ([3] p. 61). (These latter equations do not seem to have direct analogues in the general case. However, assuming $K(t)$ to be exponential one may derive them from the above equations.)

Following Cramér we introduce

$$\overline{\Psi}(u, z) = \int_0^\infty e^{z\tau}\, d_\tau \Psi(u, \tau),\ u \geq 0,\ \xi = Re(z) \leq 0.$$

Letting

$$\overline{\Phi}(u, z) = 1 - \overline{\Psi}(u, z) = 1 + \int_0^\infty e^{z\tau}\, d_\tau\, \Phi(u, \tau),\ u \geq 0,\ \xi \leq 0$$

(by definition $\Phi(u, 0) = 1 - \Psi(u, 0) = 1$) we easily obtain the following equation from the integral equation for $\Phi(u, T)$

$$\Phi(u, z) = \int\limits_0^\infty (1 - e^{zv})\, dK(v) + \int\limits_0^\infty e^{zv}\, dK(v) \int\limits_{\infty}^{u+cv} \bar{\Phi}\, (u + cv - x, z)\, dP(x),\ u \geq 0,\ \xi \leq 0$$

Since $\Phi(u, 0) = \Phi(u, \infty) = \Phi(u)$ we obviously retrieve the integral equation for $\Phi(u)$ if we let $z = 0$ in the last equation.

In analogy with the Cramér treatment we now define

$\bar{\Phi}(u, z) = 0$ for $u < 0$ ($\bar{\Psi}(u, z) = \varepsilon(u) - \bar{\Phi}(u, z)$ for every real u),

$\bar{\Omega}(u, z) = 0$ for $u \geq 0$,

$$\bar{\Omega}(u, z) = \int\limits_0^\infty (1 - e^{zv})\, dK(v) + \int\limits_0^\infty e^{zv}\, dK(v) \int\limits_{-\infty}^{u+cv} \bar{\Phi}\, (u + cv - x, z)\, dP(x)$$

for $u < 0$,

and get

$$\bar{\Phi}(u,z) + \bar{\Omega}(u,z) = \int\limits_0^\infty (1 - e^{zv})\,dK(v) + \int\limits_0^\infty e^{zv}\,dK(v) \int\limits_{\infty}^{u+cv} \bar{\Phi}(u+cv-x,z)\,dP(x)$$

for $-\infty < u < \infty$.

Letting

$$\bar{\varphi}(s, z) = \int\limits_{-\infty}^{\infty} e^{su}\, d_u\, \bar{\Phi}(u, z),\ Re(s) < R,\ \xi \leq 0,$$

$$\bar{\omega}(s, z) = \int\limits_{-\infty}^{\infty} e^{su}\, d_u\, \bar{\Omega}(u, z),\ Re(s) > 0,\ \xi \leq 0,$$

we get

$$\bar{\varphi}(s, z)\, (1 - k(z - cs)\, p(s)) = - \bar{\omega}(s, z).$$

Here we observe that in the Poisson case $dK(v) = e^{-v}\, dv$, we have

$$1 - k(z - cs)\, p(s) = 1 - \frac{p(s)}{1 + cs - z}$$

which function plays an essential role in Cramér's treatment. In fact, Cramér shows the factorization identity ([3] p. 60)

$$1 - \frac{p(s)}{1 + cs - z} = \frac{B(s, z)}{A(s, z)}$$

where

$$\log A(s, z) = \int_0^\infty e^{sx}\, d_x M(x, z), \text{ analytic and regular for } Re(s) < R,$$

$$\log B(s, z) = -\int_\infty^0 e^{sx}\, d_x M(x, z) \text{ analytic and regular for } Re(s) > 0,$$

$$M(x, z) = \sum_{n=1}^{\infty} 1/n! \int_0^\infty v^{n-1} e^{-(1-z)v} (P^{n*}(x + cv) - 1)\, dv.$$

However, in the general case an analogous factorization of $1 - k(z - cs)\, p(s)$ can be effected. The generalized $M(x, z)$ has the form

$$M(x, z) = \sum_{n=1}^{\infty} 1/n \int_0^\infty e^{zv}(P^{n*}(x + cv) - 1)\, dK^{n*}(v)$$

If $K(v)$ is continuous the connections between the generalized $A(s, z)$, $B(s, z)$ and $M(x, z)$ are unchanged. If $K(v)$ is discontinuous some obvious caution is needed.

With the generalized $A(s, z)$ and $B(s, z)$ we get

$$\frac{\bar{\varphi}(s, z)}{A(s, z)} = -\frac{\bar{\omega}(s, z)}{B(s, z)}, \quad 0 < Re(s) < R.$$

Observing that the left member is analytic, regular and bounded for $Re(s) \leq R - \varepsilon$ and that the right member has the same property in $Re(s) \geq \varepsilon$, where ε is an arbitrary positive number we conclude that both members represent a constant for fixed z.

Thus we get

$$\frac{\bar{\varphi}(s, z)}{A(s, z)} = \frac{\bar{\varphi}(0, z)}{A(0, z)} = \frac{1}{A(0, z)}, \; Re(s) < R, \xi \leq 0$$

or

$$\bar{\varphi}(s, z) = \frac{A(s, z)}{A(0, z)}.$$

From this identity it is now possible to deduce analogues of Cramér's explicit expressions for $\Psi(u)$, $\bar{\Psi}(u, z)$, and $\Psi(u, T)$ ([3] pp. 67-68). In order to secure absolute convergence in the expressions for $\bar{\Psi}(u, z)$ we assume some condition of the following type

$$k(z - cs) = O(\tau^{-\alpha}),\; \tau = Im(s) \to \pm\infty,\; \alpha > 0,$$

which i.a. is satisfied by each Γ-distribution for a suitable choice of α.

After this precaution we can write down the following analogues of Cramér's formulas ([3] p. 67 formulas (99) and (102))

$$\bar{\Psi}(u, z) = \frac{1}{2\pi i A(0, z)} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{-su} \frac{k(z - cs)\, P(s)\, B(s, z)}{s(1 - k(z - cs)\, p(s))}\, ds,$$

$$u \geq 0, \xi \leq 0, 0 < \sigma < R,$$

($z = 0$ gives $\Psi(u)$)

$$\Psi(u, T) = \frac{1}{2\pi i} \lim_{Y \to \infty} \int_{-Y}^{Y} \frac{1 - e^{-i\eta T}}{\eta} \bar{\Psi}(u, i\eta)\, d\eta.$$

It is also possible to deduce (after some precaution) an asymptotic formula for $\Psi(u)$ analogous to Cramér's corresponding formula ([3] p. 68)

$$\Psi(u) = C\, e^{-Ru} + O(e^{-(R+\theta)u}),\ u \to \infty$$

where $\theta > 0$ and

$$C = \frac{f(R)\, B(R, 0)}{A(0, 0)\, R f'(R)},\ f(s) = k(-cs)\, p(s).$$

3. *Final remarks*

A complete account of the considered generalization including detailed proofs will be given in a forthcoming paper in the Skandinavisk Aktuarietidskrift [6]. I will also draw the attention to three recent papers by Brans [2] where he has treated the general problem of a risk process, where the epochs of claims form a renewal process. Brans—like Prabhu [5] in the Poisson case—uses queue-theoretic methods.

References

[1] ANDERSEN, E. SPARRE, 1957: On the collective theory of risk in the case of contagion between the claims. *Transactions XVth International Congress of Actuaries, New York*, vol II, 219-229

[2] BRANS, J P., 1966-1967: Le problème de la ruine en théorie collective du risque. Cas non markovien. Première partie. Le montant total des sinistres. Existence et mesurabilité de la ruine. Deuxième partie Cas des opérations à capitaux-risques négatifs. Le processus $\{Z_N(t), t \geq 0 \mid Z_N(0) = u\}$. Troisième partie. Cas des opérations à capitaux-risques positifs Le processus $\{Z_P(t), t \geq 0 \mid Z_P(0) = u\}$. *Cahiers du Centre d'Études de Recherche Opérationelle, Bruxelles, 8*, 159-178, *9*, 5-31, 117-122.

[3] CRAMÉR, H., 1955. Collective risk theory. *Jubilee volume of Försäkringsaktiebolaget Skandia.*
[4] FELLER, W., 1966: *An Introduction to Probability Theory and its Applications*, vol. II Wiley
[5] PRABHU, N. U , 1961: On the ruin problem of collective risk theory. *The Annals of Mathematical Statistics*, *32*, 757-764
[6] THORIN, O , 1970: Some remarks on the ruin problem in case the epochs of claims form a renewal process. *Skandinavisk Aktuarietidskrift*, LIII, 29-50