ON A CLASS OF MEASURES OF DISPERSION WITH APPLICATION TO OPTIMAL REINSURANCE

JAN OHLIN
Stockholm

0. INTRODUCTION AND SUMMARY

In this paper we will investigate the following reinsurance problem: An insurer, whose total claims for a certain period may be regarded as a random variable x with expected value $Ex = m$, wishes to cede part of his business to a reinsurer. A reinsurance treaty will consist of rule for the division of x between the two parties. For any observed value of x it should define uniquely what amount should be borne by the ceding insurer. The amount borne by the reinsurer is then simply the remaining part of x.

We shall assume that the insurer has already decided *how much* of his business he wishes to cede, in the sense that he wants to retain a part of the total risk with expected value $m - c$, where c is a *fixed* constant, $0 < c < m$.

Using the terminology introduced by Kahn in (2) we will describe a reinsurance contract by a transformation (or function) T that for a given x yields the amount Tx borne by the cedent. The random variable x is thus divided into two parts

$$x = Tx + (1 - T)x,$$

and the properties of the reinsurance contract described by T are summarized in the distributions of the two random variables Tx and $(1 - T)x = - Tx$.

The motivation for reinsurance is generally held to be a desire for stability, in other words the cedent wishes to choose a T such that the random fluctuations in Tx are in some sense smaller than those of x. This choice will in our case be performed under the restriction that $ETx = m - c$.

It is clear that we can never talk about an optimal choice of T without defining exactly what criterion we shall use when comparing two transformations, T_1 and T_2. According to the above, the criterion should refer to the properties of the distributions of T_1x and T_2x, so that if one distribution is "more concentrated" around some central value, the corresponding transformation is deemed preferable to the other. However, we still have to define what we mean by "more concentrated".

One way is to consider the variance. This was done by Borch in (1), where he proved that, for given c, the variance of Tx is minimized by a stop loss contract. This result was extended by Kahn in (2). Borch originally approached the problem by considering a reinsurance contract as a transformation of the distribution function of x. By introducing the technique of considering a transformation of x rather than of its distribution function, Kahn not only simplified the proof considerably, but also extended the result to a wider class of transformations than the one originally considered by Borch.

Vajda in (3) investigated the problem from the reinsurer's viewpoint. Apart from the rather obvious condition $0 \leq Tx \leq x$, introduced by Kahn, Vajda made the restriction that $\dfrac{(1 - T)x}{x}$ must be non-decreasing in x, and proved that in this case minimum variance for the reinsurer is realized by a quota contract. The introduction of an extra restriction is necessary in this case, since otherwise the problem would be perfectly symmetrical, and the minimum variance solution for the reinsurer would be of the same type as for the cedent, i.e. a kind of reverse stop loss of little practical interest.

These results are of course open to criticism along the following lines: the choice of the variance as a "criterion of optimality" is somewhat arbitrary, and perhaps another measure of dispersion would have yielded considerably different results. The purpose of this paper is to prove that under the conditions used by the previous authors, the stop loss and quota contracts retain their minimizing properties when the variance is replaced by any member of a rather wide class of measures of dispersion.

Before introducing this class of measures of dispersion, let us consider the variance in a little more detail.

Let x be a random variable with cumulative distribution function $F(x)$. Suppose we associate a "loss" with a deviation of x from a central value μ, and put this loss equal to the square of the deviation. The expected loss will then be

$$E\,(x-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2\, dF(x)$$

We now choose μ as the value that minimizes the expected loss. This is done by putting $\mu = Ex$ and we have thus arrived at the variance, not primarily as a measure of dispersion, but as the expected loss caused by random fluctuations and under the assumption of a quadratic loss function. This interpretation of the variance is very much in line with the previous discussion of criteria for optimal choice of T. However, the assumption of a quadratic loss function is still arbitrary. If we take the absolute value of the deviation instead, the expected loss will be

$$\inf_{\mu} \int_{-\infty}^{\infty} |x-\mu|\, dF(x)$$

It is a well-known fact that in this case minimum is obtained by putting $\mu =$ the median of x. The result is known as the *mean deviation* of x.

Both the loss functions that we have considered, t^2 and $|t|$, are convex. It is natural to demand that a loss function has this property, since it means that the rate of increase of the loss is non-decreasing as we go further away from the central value. Any function φ that is non-negative, convex and equal to zero at $t = 0$ can be used to generate a measure of dispersion (or "expected loss") W_φ by putting

$$W_\varphi\,(x) = \inf_{\mu} E\,\varphi(x-\mu).$$

The properties of such measures of dispersion are discussed in detail in section 2.

Of course the use of any single measure of dispersion of this type would be open to the same criticism as the use of the variance. Hence we shall not investigate single members but rather the whole class. We shall say that one random variable x is *less dispersed* than another random variable y if

$$W_\varphi\,(x) \leq W_\varphi\,(y)$$

for *any* W_φ, generated by a convex function φ. Thus, if we state that a certain reinsurance contract results in less dispersion than any other in the class of possible contracts, it will mean that it minimizes not only the variance and the mean deviation but *any* measure of dispersion of this type.

The results of section 4 may now be summarized as follows:

Under the conditions considered by Kahn, stop loss reinsurance *minimizes* the dispersion for the ceding insurer.

Under the conditions considered by Vajda, quota reinsurance *minimizes* the dispersion for the reinsurer.

Under the extra conditions that both Tx and $(1 - T)x$ are non-decreasing, stop loss reinsurance *maximizes* the dispersion for the reinsurer and quota reinsurance *maximizes* the dispersion for the ceding insurer.

The last two results emphasize in a drastic way the "peculiar opposition of interests of the two partners of a reinsurance contract" mentioned by Vajda in his paper. Under not very restrictive conditions, what is optimal in our sense to one party is seen to be the opposite of optimal for the other party.

1. Convex Functions

In the following we will make extensive use of some properties of continuous convex functions. We will therefore give a brief review of these properties. For further details the reader is referred to chapter 3 of (4).

A function $f(x)$ is said to be *convex* in the interval (H, K) if the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left(f(x) + f(y)\right)$$

holds for all x, y in (H, K). It can be shown that if $f(x)$ is continuous in (H, K), it also satisfies

$$f(\Sigma\, q_i\, x_i) \leq \Sigma\, q_i f(x_i)\,;\; q_i > 0,\;\; \Sigma q_i = 1\,;\; x_i \in (H, K).$$

By a passage to the limit we obtain

$$f(Ex) \leq E\left(f(x)\right),$$

where E denotes expected value and x is a random variable that only takes values in (H, K).

In this paper we will only consider continuous convex functions. Continuity is not a strong restriction in this case, since convex functions are either very regular or very irregular. This follows from the fact that a function f which is convex in (H, K) is also continuous in (H, K) if it is bounded above in some interval interior to (H, K). If this is true, the function is not only continuous but the derivative $f'(x)$ exists and is continuous everywhere except perhaps for an enumerable set of values of x. Left-hand and right-hand derivatives exist everywhere; the right-hand derivative is not less than the left-hand derivative, and both derivatives are non-decreasing (Theorem 111 in (4)).

Continuous convex functions have two geometrical properties which could have been used for alternative definitions of convexity. Every chord lies entirely above or on the curve. Through every point of the curve at least one line can be drawn which lies wholly under or on the curve. Such a line is called a *line of support*. Whether one or more lines of support can be drawn through a given point depends of course on the behaviour of $f'(x)$. If $f'(x)$ exists there is only one line, the tangent, otherwise any line through the point with a coefficient of inclination K, such that $f'_l \leq K \leq f'_r$, will be a line of support.

In the following we will use the abbreviation *c.c.f.* to denote a function that is continuous and convex on the entire real line $(-\infty, \infty)$.

2. A Class of Measures of Dispersion

Let us now consider $E\, f\,(x - \mu) = \int\limits_{-\infty}^{\infty} f\,(t - \mu)\; d\,F(t)$, where f is a c.c.f. and x a random variable (r.v.) with finite mean and cumulative distribution function (c.d.f.) $F(x)$. Let $l(x) = ax + b$ be a line of support to $f(x)$ at $x = 0$, and introduce the notation φ for the deviation from this line of support, i.e. $\varphi(x) = f(x) - ax - b$. We get

$$\begin{aligned} E\, f(x - \mu) &= E(\varphi(x - \mu) + a(x - \mu) + b) = \\ &= E\, \varphi(x - \mu) + aEx - a\mu + b. \end{aligned}$$

Since Ex is assumed to be finite, $Ef(x - \mu)$ will exist if and only if $E\varphi(x - \mu)$ exists. But, since φ is non-negative, the integral

$\int_{-\infty}^{\infty} \varphi(t - \mu)\, dF$ will either be finite or $= +\infty$. Furthermore, if the integral converges for two values of μ, say $\mu_1 < \mu_2$, the relation

$$0 \leq \varphi(t - (p\mu_1 + q\mu_2)) \leq p\varphi(t - \mu_1) + q\varphi(t - \mu_2),\ p + q = 1,$$

implies that it converges for all $\mu \in [\mu_1, \mu_2]$. On the other hand, if $E\,\varphi(x - \mu)$ is finite for $\mu = \mu_1$ but infinite for $\mu = \mu_2$, where $\mu_2 > \mu_1$ ($\mu_1 > \mu_2$), then $E\varphi\,(x - \mu)$ will be infinite for all $\mu > \mu_2$ ($\mu < \mu_2$). Hence, the domain where $E\varphi\,(x - \mu)$ is finite is always an interval on the μ—axis. This interval may be open, closed or half-closed, and it may be bounded or unbounded.

If we introduce the further restriction that $\varphi\,(t) > 0$ for some $t > 0$ and some $t < 0$, it is easily seen that $\lim_{t \to \pm\infty} \varphi(t) = +\infty$. This implies that $\lim_{\mu \to \pm\infty} \int_{-\infty}^{\infty} \varphi(t - \mu)\, dF(t) = +\infty$. From this we conclude that as soon as $E\,\varphi(x - \mu)$ is finite for some μ, then $\inf_{\mu} E\,\varphi(x - \mu)$ is finite and is obtained for some finite value of μ, say $\mu = \mu_0$.

We shall now introduce a class of measures of dispersion.

Definition

Let x be a r.v. with c.d.f. F, and φ a function with the following properties:

(a) φ is continuous and convex on $(-\infty, \infty)$,

(b) $\varphi\,(t) \geq 0$, $\varphi\,(0) = 0$,

(c) $\varphi\,(t) > 0$ for some $t > 0$ and some $t < 0$.

The *measure of dispersion* W_φ *generated by* φ is then defined by

$$W_\varphi(x) = \inf_{\mu} \int_{-\infty}^{\infty} \varphi\,(t - \mu)\, dF(t) = E\varphi(x - \mu_0)$$

This definition calls for a few comments. If φ and F are such that $E\,\varphi\,(x - \mu)$ is infinite for all μ, the value of $W_\varphi(x)$ is of course interpreted as $+\infty$. If condition (c) is not fulfilled, if $\varphi\,(t) \equiv 0$ for, say all $t \leq 0$, then

$$\lim_{\mu \to +\infty} \int_{-\infty}^{\infty} \varphi(t - \mu)\, dF(t) = 0 \text{ and hence } W_\varphi(x) = 0$$

for all r.v:s x. This would not cause any trouble in what follows, since we will mainly be considering inequalities of the type $W_\varphi(x) \leq W_\varphi(y)$ for all φ. Such inequalities would still hold true (as trivial equalities) but, since W_φ :s of this type are of little interest as measures of dispersion, they are excluded by imposing condition (c).

As already stated, both the *variance* and the *mean deviation* belong to the class of measures of dispersion just defined, since they can be regarded as generated by $\varphi(t) = t^2$ and $\varphi(t) = |t|$, and both these functions satisfy conditions (a)—(c) above.

The following terminology will be used. The abbreviation "m.o.d. W_φ" will be used for the rather lengthy expression "measure of dispersion W_φ generated by the continuous convex function φ satisfying conditions (a)—(c)". If, for two random variables x and y, the inequality

$$W_\varphi(x) \leq W_\varphi(y)$$

holds for any m.o.d. W_φ, we will say that x is *less dispersed* than y or that y is *more dispersed* than x. Strictly speaking we should also require that $W_\varphi(x) < W_\varphi(y)$ for some m.o.d. W_φ, but this will obviously always be the case unless there exists a constant a, such that x and $y + a$ have the same distribution.

3. Three Lemmas

In this section we shall prove three lemmas giving sufficient conditions for one r.v. to be less dispersed than another r.v.

Lemma 1: Let x and y be r.v:s. If

$$E\,f(x) \leq E\,f(y) \text{ for any c.c.f. } f,$$

then x is less dispersed than y.

(Note: It can be proved that if $Ex = Ey$, the reverse is also true, so that the condition is not only sufficient but also necessary. The proof is rather lengthy, and since the result is not needed here, it is given in an Appendix at the end of the paper).

Proof: Consider any m.o.d. W_φ and let μ_x and μ_y be values of μ for which inf $E\,\varphi(x - \mu)$ and inf $E\varphi(y - \mu)$ are obtained. Then

$$W_\varphi(y) - W_\varphi(x) = E\varphi(y - \mu_y) - E\varphi(x - \mu_x) \geq$$
$$E\varphi(y - \mu_y) - E\varphi(x - \mu_y) \geq 0$$

That the last member is non-negative follows from the assumption with $f(t) = \varphi(t - \mu_y)$. Hence the lemma is proved.

The second lemma shows that a simple relation between two c.d.f:s is sufficient for Lemma 1 to be applicable.

Lemma 2: Let x_1 and x_2 be two r.v:s with the same finite mean $Ex_1 = Ex_2$, and F_1 and F_2 their c.d.f:s.

If there exists a t_0 such that,

$$F_1(t) \leq F_2(t) \quad \text{for } t < t_0$$
$$F_1(t) \geq F_2(t) \quad \text{for } t > t_0,$$

then $E\,f(x_1) \leq E\,f(x_2)$ for any c.c.f. f.

Proof: Let f be any c.c.f.

$$E\,f(x_2) - E\,f(x_1) = \int_{-\infty}^{\infty} f(t)\,dF_2(t) - \int_{-\infty}^{\infty} f(t)\,dF_1(t) = \int_{-\infty}^{\infty} f(t)\,\{dF_2 - dF_1\}.$$

Since $$\int_{-\infty}^{\infty} t\,\{dF_2 - dF_1\} = \int_{-\infty}^{\infty} \{dF_2 - dF_1\} = 0,$$

the integral is not changed if we replace $f(t)$ by its deviation from a line of support at t_0, $g(t) = f(t) - at - b$.

Hence $$E\,f(x_2) - E\,f(x_1) = \int_{-\infty}^{\infty} g(t)\,\{dF_2 - dF_1\},$$

where $g(t)$ has the following properties:

$$g(t) \text{ is a c.c.f.}$$
$$g(t) \geq 0,\ g(t_0) = 0$$
$$g'(t) \leq 0, \text{ for } t < t_0$$
$$g'(t) \geq 0, \text{ for } t > t_0$$

Assume first that f (and hence g) is integrable both with respect to dF_1 and dF_2. We may then integrate by parts and get

$$\int_{-\infty}^{\infty} g(t)\,\{dF_2 - dF_1\} = g(t)\,(F_2 - F_1)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g'(t)\,\{F_2 - F_1\}dt$$

That g is integrable implies that

$$\left.\begin{aligned}&\lim_{t\to-\infty} g(t)\,F_i(t) = 0\\&\lim_{t\to\infty} g(t)\,(1 - F_i(t)) = 0\end{aligned}\right\}\quad i = 1, 2$$

Hence the first term vanishes, and, since g' and $F_2 - F_1$ are of unequal sign, we get

$$E\,f(x_2) - E\,f(x_1) = -\int_{-\infty}^{\infty} g'(t)\,(F_2 - F_1)dt \geq 0.$$

This completes the proof of the lemma for the case that both $E\,f(x_2)$ and $E\,f(x_1) < \infty$.

Assume now that $E\,f(x_1)$ (and hence $E\,g(x_1)$) $= +\infty$. We define a new function $g_A(t)$ as follows. Let $l_{-A}(t)$ and $l_A(t)$ be lines of support to $g(t)$ at $t = -A$ and $t = A$.

Put

$$\begin{aligned}g_A(t) &= g(t) && \text{for } |t| \leq A\\ g_A(t) &= l_{-A}(t) && \text{for } t < -A\\ g_A(t) &= l_A(t) && \text{for } t > A.\end{aligned}$$

$g_A(t)$ will be integrable both with respect to dF_1 and dF_2, and it follows from the above that

$$E\,g_A(x_1) \leq E\,g_A(x_2)$$

Since $g_A(t) \to g(t)$ monotonically as $A \to \infty$, we get $E\,g_A(x) \to E\,g(x)$ and may conclude that $E\,g(x_2)$ (and hence $E\,f(x_2)$) $= +\infty$.

Accordingly, we may regard the inequality

$$E\,f(x_1) \leq E\,f(x_2)$$

as proved in the following sense: either both expectations are finite and the inequality holds, or $E\,f(x_2) = +\infty$, in which case the inequality is trivially true, whether $E\,f(x_1)$ is finite or not.

The third lemma gives a sufficient condition based on a relation between the variables themselves, rather than between their c.d.f:s.

Lemma 3: Let y be a r.v. and $x_1(y)$ and $x_2(y)$ two non-decreasing functions of y, such that $E\,x_1(y)$ and $Ex_2(y)$ are finite and equal. If there exists a y_0, such that

$$x_1(y) \geq x_2(y) \quad \text{for } y < y_0$$
$$x_1(y) \leq x_2(y) \quad \text{for } y > y_0,$$

then $x_1(y)$ is less dispersed than $x_2(y)$.

Proof: Let $F_1(x)$ and $F_2(x)$ be the c.d.f:s of x_1 and x_2, and put

$$x_0 = x_2(y_0)$$

For $x < x_0$ we get

$$F_2(x) = P(x_2(y) \leq x) = P\,(x_1(y) \leq x) + P\,(x_2(y) \leq x < x_1(y)) \geq \geq F_1(x).$$

Likewise for $x > x_0$

$$F_1(x) = P\,(x_1(y) \leq x) = P\,(x_2(y) \leq x) + P(x_1(y) \leq x < x_2(y)) \geq \geq F_2(x)$$

Hence F_1 and F_2 satisfy the conditions of Lemma 2, and consequently $x_1(y)$ is less dispersed than $x_2(y)$ and the lemma is proved.

4. Applications to Reinsurance

We shall now apply the results of sections 2 and 3 to the reinsurance problem presented in the introduction. We will use the technique introduced by Kahn in (2). Thus, a reinsurance scheme will be described by a measurable transformation T of the random variable x (representing the total claims during the period), such that Tx represents the amount borne by the ceding insurer and, consequently, $(1 - T)x = x - Tx$ the amount borne by the reinsurer.

Obviously, to be meaningful, the analysis has to be carried out with certain restrictions on T. First of all, we only consider transformations such that $E\,(x - Tx)$, i.e. the (net) reinsurance premium, equals a fixed constant c. Second, we demand that the amount borne by the insurer shall never exceed the total claims x. These two conditions define the class of what Kahn calls "admissible transformations". Since we want to study the problem not only from the viewpoint of the ceding insurer, but also from that of the reinsurer, we have to introduce extra restrictions. Otherwise we would, due to the symmetry of the problem, arrive at exactly the same type of solution in both cases—only with the rôles reversed.

This leads us to consider the classes of transformations which satisfy the following four sets of conditions. Here *admissible under A* corresponds to Kahn's *admissible*, and *admissible under C* to the extra restriction introduced by Vajda in (3).

A. A measurable transformation T is said to be *admissible under A* if

$$0 \leq Tx \leq x$$

$c = \int_0^\infty (x - Tx)\, dF(x)$, where c is a fixed constant such that

$$0 < c < m = Ex.$$

B. A measurable transformation T is said to be *admissible under B* if it is admissible under A and furthermore Tx and $x - Tx$ are both non-decreasing in x.

C. A measurable transformation T is said to be *admissible under C* if it is admissible under A and furthermore $\frac{(1 - T)x}{x} = \frac{x - Tx}{x}$ is non-decreasing in x.

D. A measurable transformation T is said to be *admissible under D* if it is admissible under C and furthermore Tx is non-decreasing in x.

Obviously $D \rightarrow B \rightarrow A$ and $D \rightarrow C \rightarrow A$.

Two transformations are of particular interest:

The Stop Loss Transformation T^*

T^* is defined by
$T^*x = x$ for $x < n_0$
$T^*x = n_0$ for $x \geq n_0$, where n_0 is chosen such that

$$c = \int_{n_0}^\infty (x - n_0)\, dF(x)$$

The Quota Transformation $\hat{T}$.

$\hat{T}$ is defined by

$$\hat{T} = \frac{m - c}{m} x \quad \text{for all } x.$$

It is easily seen that both T^* and $\hat{T}$ are admissible under all four sets of conditions.

We shall now state and prove a theorem on "optimality" properties of these two transformations.

Theorem: Let T_A, T_B, T_C and T_D be any transformations admissible under A, B, C and D respectively. Let T^* and $\hat{T}$ be the stop loss and quota transformations defined above. Then the following statements are true:

(1) T^*x is less dispersed than $T_A x$

(2) $(1 - T^*)x$ is more dispersed than $(1 - T_B)x$

(3) $(1 - \hat{T})x$ is less dispersed than $(1 - T_C)x$

(4) $\hat{T}x$ is more dispersed than $T_D x$

(Note: (1) is an extension to any m.o.d. W_φ of the Borch-Kahn theorem on minimum variance and (3) a similar extension of the Vajda theorem.)

Proof: Let F, F^* and F_A be the c.d.f:s of x, T^*x and $T_A x$.

Then $F_A(t) \geq F(t)$, because

$F_A(t) = P(T_A x \leq t) = P(x \leq t) + P(T_A x \leq t < x) \geq F(t)$.

Now consider statement (1). Obviously

$$F^*(t) = F(t) \quad \text{for } t < n_0$$
$$F^*(t) = 1 \quad \text{for } t \geq n_0$$

Hence

$$F^*(t) \leq F_A(t) \quad \text{for } t < n_0$$
$$F^*(t) \geq F_A(t) \quad \text{for } t > n_0.$$

Since $ET^*x = ET_A x = m - c$, we may apply Lemma 2 and conclude that (1) is true.

Now consider statement (2). Since $T_B x$ is non-decreasing, and $T_B x \leq T^*x$ for all x would imply $T_B x = T^*x$ a.s. (almost surely = with probability one), we may conclude that $T_B x > T^*x$ for some x. Obviously this $x > n_0$, since otherwise $T_B x > x$ contrary to assumptions. Hence there must exist a finite $x_0 > n_0$, such that

$$T_B x \leq T^*x \quad \text{for } x < x_0$$
$$T_B x \geq T^*x \quad \text{for } x > x_0$$

Writing this as

$$(1 - T_B)x \geq (1 - T^*)x \text{ for } x < x_0$$
$$(1 - T_B)x \leq (1 - T^*)x \text{ for } x > x_0,$$

and applying Lemma 3, we conclude that $(1 - T_B)x$ is less dispersed than $(1 - T^*)x$. Hence (2) is true.

To prove (3) we consider the behaviour of $(1 - T_C)x$ and $(1 - \hat{T})x$. Assume first that $(1 - T_C)x_1 \leq (1 - \hat{T})x_1 = \frac{c}{m} x_1$, in other words, that $\frac{(1 - T_C)x_1}{x_1} \leq \frac{c}{m}$. Since $\frac{(1 - T_C)x}{x}$ is non-decreasing, this implies that $(1 - T_C)x \leq (1 - \hat{T})x$ for all $x \leq x_1$. Similarly, $(1 - T_C)x_2 \geq (1 - \hat{T})x_2$ implies that $(1 - T_C)x \geq (1 - \hat{T})x$ for all $x \geq x_2$. Since $(1 - T_C)x \geq$ or $\leq (1 - \hat{T})x$ for all x are trivial cases (they both imply $T_C x = \hat{T}x$ a.s.), we may conclude that there exists a finite $x_0 > 0$, such that

$$(1 - \hat{T})x \geq (1 - T_C)x \quad \text{for } x < x_0$$
$$(1 - \hat{T})x \leq (1 - T_C)x \quad \text{for } x > x_0.$$

Since both $(1 - \hat{T})x$ and $(1 - T_C)x$ are non-decreasing we may apply Lemma 3 and conclude that $(1 - \hat{T})x$ is less dispersed than $(1 - T_C)x$. Hence (3) is true.

To prove (4) we apply the same reasoning with T_D instead of T_C to state that either $T_D x = \hat{T}x$ a.s. or there exists a finite $x_0 > 0$, such that

$$T_D x \geq \hat{T}x \qquad \text{for } x < x_0$$
$$T_D x \leq \hat{T}X \qquad \text{for } x > x_0$$

Since T_D is non-decreasing, we may once again apply Lemma 3 and conclude that $T_D x$ is less dispersed than $\hat{T}x$. This completes the proof of the theorem.

5. Individual versus Collective Reinsurance

In section 4 we only considered fully collective reinsurance forms, i e. the amount borne by the ceding insurer was assumed to depend only on the total sum of claims. We shall now show that this is not really a restriction by proving that neither the ceding insurer nor the reinsurer can gain anything, in terms of achieving small dispersion, by allowing Tx to depend on the claims on the individual policies.

Consider a company with a portfolio consisting of k policies, and let the claims on the individual policies be a k-dimensional r.v. $(x_1, x_2, \ldots, x_k)$ with the simultaneous distribution function $F(x_1, \ldots, x_k)$. An individual reinsurance treaty is now described by a real-valued measurable transformation T. For each value of $(x_1, \ldots, x_k)$, $T(x_1, \ldots, x_k)$ represents the amount borne by the ceding insurer. The restrictions in condition A of section 4 are replaced by their obvious counterparts:

$$0 \leq T(x_1, \ldots, x_k) \leq \sum_1^k x_i$$

$$c = \int_{E^{(k)}} \{\Sigma x_i - T(x_1, \ldots, x_k)\} \, dF(x_1, \ldots, x_k)$$

where $E^{(k)}$ denotes the k-dimensional Euclidian space and c a fixed constant such that $0 < c < m = E\Sigma x_i$. The extra restrictions in conditions B — D will now be that $T(x_1, \ldots, x_k)$, $\Sigma x_i - T(x_1, \ldots, x_k)$ and/or $\dfrac{\Sigma x_i - T(x_1, \ldots, x_k)}{\Sigma x_i}$ are non-decreasing in Σx_i.

For every transformation T we shall now define a transformation $\bar{T}$ that is fully collective in the sense that the result depends only on $x = \Sigma x_i$. We do this by

$$\bar{T}x = E\{T(x_1, \ldots, x_k) \mid \Sigma x_i = x\},$$

i.e. $\bar{T}x$ equals the conditional expectation of $T(x_1, \ldots, x_k)$, given that $\Sigma x_i = x$. If T is admissible under A, B, C or D, the same will obviously hold for $\bar{T}$, since

$$0 \leq \bar{T}x \leq x, \text{ and}$$

$$c = \int_0^\infty (x - \bar{T}x) \, dG(x),$$

where $G(x) = \int_{\Sigma x_i \leq x} dF(x_1, \ldots, x_k)$ is the c.d.f. of $x = \Sigma x_i$, and any extra condition under B, C or D will also be satisfied by $\bar{T}x$, regarded as a function of x.

Theorem: Let T be any admissible transformation and $\bar{T}$ the corresponding collective transformation. Then $\bar{T}x$ is less dispersed than $T(x_1, \ldots, x_k)$ and $x - \bar{T}x$ less dispersed than $\Sigma x_1 - T(x_1, \ldots, x_k)$.

Proof: Consider any m.o.d. W_φ. The convexity of φ implies that

$$E\,\{\varphi(T(x_1, \ldots, x_k) - \mu) \mid \Sigma\, x_i = x\} \geq \varphi(\bar{T}x - \mu)$$

for every μ. By taking expected value of both members we get that

$$E\,\varphi(T(x_1, \ldots, x_k) - \mu) \geq E\varphi\,(\bar{T}x - \mu)$$

for every μ, in particular for the μ that minimizes the left member. Hence

$$W_\varphi\,(T\,(x_1, \ldots, x_k)\,) \geq W_\varphi\,(\bar{T}x).$$

Exactly the same line of reasoning leads to

$$W_\varphi(\Sigma x_i - T(x_1, \ldots, x_k)\,) \geq W_\varphi\,(x - \bar{T}x)$$

and this completes the proof of the theorem.

We have thus proved that any transformation, admissible under A, B, C or D can be replaced by a fully collective transformation, admissible under the same conditions, which yields a result that is less dispersed for both parties. It should be noted that it was not necessary, in the proof of the above theorem, to assume that the claims on the individual policies are independent r.v:s

(Note: In this section we have used the word individual to denote any reinsurance form that is not fully collective, i.e. that is dependent, however negligibly, on the claims on the individual policies. Individual in this sense thus includes all forms of individual or half-collective reinsurance.)

Acknowledgements

This paper is a revised and extended version of a paper presented to the 6th ASTIN Colloquium 1966. The work was carried out as part of a research project financed by the Skandia Insurance Company.

The author also wishes to express his gratitude to Prof. U. Grenander for his suggestion to approach the problem via convex functions, and to his colleagues B. Ajne and J. Grandell for many helpful and stimulating discussions on the subject.

References

[1] Borch, Karl: "An Attempt to Determine the Optimum Amount of Stop Loss Reinsurance", Transactions of the XVIth International Congress of Actuaries, Vol. 2, pp. 597-610.
[2] Kahn, Paul Markham: "Some Remarks on a Recent Paper by Borch", The ASTIN Bulletin, Vol. I, Part V, pp. 265-272.
[3] Vajda, Stefan: Minimum Variance Reinsurance", The ASTIN Bulletin, Vol. II, Part II, pp. 257-260
[4] Hardy-Littlewood-Polya: Inequalities, Cambridge 1934.

Appendix

In Lemma 1 of section 3 we gave a sufficient condition for one r.v. to be less dispersed than another. We shall now prove that this condition is also necessary.

It will be convenient to use the following terminology:

We say that a c.c.f. $f(t)$ is *linear outside* (A, B) if it is linear for all $t < A$ and all $t > B$.

We say that a c.c.f. $f(t)$ is *asymptotically linear* if $\lim f'(t)$ exists and is finite for both $t \to -\infty$ and $t \to +\infty$. If a function is linear outside some bounded interval, it is obviously also asymptotically linear.

We will need the following lemma.

Lemma: Let x be a r.v. with finite mean and $f(t)$ an asymptotically linear c.c.f., such that $f'(t)$ exists and is continuous everywhere. Furthermore, let $g(t, z)$ be f's deviation at $t + z$ from the line of support at z, i.e.

$$g(t, z) = f(t + z) - f'(z)\, t - f(z).$$

Then we can always choose $z = z_0$, such that

$$\inf_{\mu} Eg(x - \mu, z_0) = Eg(x - z_0, z_0).$$

Proof: Let $F(x)$ be the c.d.f. of x and consider

$$Eg(x - \mu, z) = \int_{-\infty}^{\infty} f(x - \mu + z)\, dF(x) - f'(z)\,(Ex - \mu) - f(z)$$

Since Ex is finite and f asymptotically linear, the integral exists for all μ and may be differentiated under the sign of integration·

$$\frac{\partial Eg(x - \mu, z)}{\partial \mu} = -\int_{-\infty}^{\infty} f'(x - \mu + z)\, dF(x) + f'(z).$$

Since $f'(z)$ is continuous and non-decreasing, there exists at least one root to the equation

$$f'(z) = \int_{-\infty}^{\infty} f'(x)\, dF(x).$$

Let z_0 be such a root. Then $\mu = z_0$ satisfies the equation

$$\frac{\partial Eg(x - \mu, z_0)}{\partial \mu} = 0,$$

and since $Eg(x - \mu, z_0)$ is continuous and convex in μ (see section 2), this must correspond to a minimum. Hence

$\inf_{\mu} Eg(x - \mu, z_0) = Eg(x - z_0, z_0)$, which was to be proved.

We shall now prove the following theorem which is the reverse of Lemma 1 of section 3.

Theorem: Let x and y be r.v:s with finite and equal means.

If x is less dispersed than y, then

$Ef(x) \leq Ef(y)$ for any c.c.f. $f(t)$ such that $Ef(x)$ is finite.

Proof: Assume that there exists a c.c.f. $h(t)$ such that $Eh(x) > Eh(y)$, with $Eh(x)$ finite. The theorem will be proved if we can show that this implies that there exists a c.c.f. φ_0 such that

$$W_{\varphi_0}(x) > W_{\varphi_0}(y).$$

Assume first that $h(t)$ is asymptotically linear and that $h'(t)$ exists and is continuous everywhere.

We put

$$\varphi(t, z) = h(t + z) - h'(z)\, t - h(z).$$

According to the lemma just proved, we can choose $z = z_0$, such that

$$\inf_{\mu} E\varphi(x - \mu, z_0) = E\varphi(x - z_0, z_0).$$

We now define $\varphi_0(t)$ as $\varphi(t, z_0)$ and get

$$W_{\varphi_0}(x) = \inf_{\mu} E\varphi(x - \mu, z_0) = E\varphi(x - z_0, z_0) =$$
$$= Eh(x) - h'(z_0)\,(Ex - z_0) - h(z_0) > Eh(y) -$$
$$h'(z_0)\,(Ey - z_0) - h(z_0) = E\varphi(y - z_0, z_0) \geq \inf_{\mu} E\varphi(y - \mu, z_0) = W_{\varphi_0}(y)$$

Hence there exists a c.c.f. φ_0, such that

$$W_{\varphi 0}(x) > W_{\varphi 0}(y),$$

and the theorem is proved for this case.

The theorem will be proved for the general case if we can show that the existence of a c.c.f. $h(t)$, such that $Eh(x) > Eh(y)$, with $Eh(x)$ finite, implies that there exists an asymptotically linear c.c.f. $h_1(t)$ with $h_1'(t)$ continuous everywhere, that also satisfies the inequality $Eh_1(x) > Eh_1(y)$, with $Eh_1(x)$ finite. We do this by modifying the original $h(t)$ in two steps. First, we make $h(t)$ linear outside the interval $(-A, A)$ by replacing it with lines of support at $t = \pm A$ outside the interval (cf. the proof of Lemma 2, section 3.). By choosing A sufficiently large, we can make the resulting decrease in $Eh(x)$ arbitrarily small and, since $Eh(y)$ will certainly not increase, we can make sure that the strict inequality still holds. Second, we approximate $h(t)$ inside the interval $(-A, A)$ with a function that has a continuous derivative everywhere. The easiest way to do this is perhaps to divide the interval $(-A, A)$ in small intervals and replace $h(t)$ by the convex polygon formed by the chords over those intervals. After that we "round off" the corners of the polygon by replacing the chords in the vicinity of each corner by small circular arcs that make second order contact with the chords. In a bounded interval, the slopes of all lines of support to a c.c.f. and the slopes of all chords are bounded, both below and above. This means that by choosing the intervals and the radii of the circular arcs sufficiently small, we can make the maximum deviation of the approximating function from the original one arbitrarily small. Hence the resulting increase (the approximating curve will never fall below $h(t)$) in $Eh(y)$ can be kept so small that the strict inequality still holds. Hence we have managed to construct a c.c.f. $h_1(t)$, consisting of straight lines outside $(-A, A)$ and line segments joined by circular arcs inside $(-A, A)$, such that $Eh_1(x) > Eh_1(y)$. That $Eh_1(x)$ is finite and $h_1'(t)$ exists everywhere follows directly from the method of construction. This completes the proof of the theorem.