COLLECTIVE THEORY OF RISK AND UTILITY FUNCTIONS

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In a series of studies K. Borch has considered the problem of introducing a utility function into actuarial theory. His work refers mainly to inquiries about the situation during a financial year of an insurance company. By means of the claim distribution for the risks insured during an insurance year the best reinsurance for this insurance year is obtained, the best reinsurance being considered that one which provides the greatest utility.

An obvious generalization of this point of view consists in the examination of the entire future development of the company rather than in the examination of one insurance year. This generalization corresponds to the transition from the classical theory of risk with its fixed insurance structure to the collective theory of risk, which includes the entire future development in its considerations.

In his study "Reformulation of some Problems in the Theory of Risk" in the Proceedings of the Casualty Actuarial Society, Vol. XLIV, K. Borch drew attention to the possibility of such generalizations by considering several insurance years and by investigating the expected dividend payment and the ruin probability for each insurance year. The following note deals with a slightly different method of inquiry which is adapted to the methods of collective theory of risk and is particularly suited to the treatment of this problem. The results are given in broad outline only, detailed derivations being omitted.

We start from a given utility function $n(x)$, which denotes the value of the amount x and assume that a risk situation for this amount x may be defined by a distribution function $G(x)$. The utility of this risk situation may then be described by*)

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*) Compare e.g. K. Borch "Reciprocal reinsurance treaties seen as a two-person co-operative game, Skandinavisk Actuarietidskrift, 1960, Uppsala.

$$N\{G(x)\} = \int_{-\infty}^{+\infty} n(x)\, dG(x) \tag{1}$$

in which negative values of x correspond to the situation in which sums are owed by the company.

Before development it is, however, necessary to include the effect of the time element. Obviously the moment at which an amount is due makes some difference, but our utility function $n(x)$ does not, however, refer to the moment. The question of the valuation of amounts at different moments of time enters into the field of financial mathematics and comparison of two amounts at different times is made possible by appropriate interest assumptions. Generally in the case of $t_2 > t_1$ we have

$$x(t_2) = x(t_1)\, e^{\int_{t_1}^{t_2} \delta(\tau)\, d\tau}$$

$\delta(\tau)$ being the force of interest. To compare two amounts they must first of all be referred to the same moment of time. If time o is chosen as a comparative moment, then

$$x(0) = x(t)\, e^{-\int_0^t \delta(\tau)\, d\tau} \tag{2}$$

Thus the utility of an amount $x(t)$ due at the moment t can be described at the moment o by

$$n\,[x(t) e^{-\int_0^t \delta(\tau)\, d\tau}].$$

Now let $\chi(t)$ be the stochastic process which describes the development of the resources of an insurance company. $\chi(t)$ is a function of time, being increased by premium income, interest etc. and reduced by claim payments, dividend payments etc. The probability $W\{\chi(t) < 0\}$, viz. the probability that the resources of an insurance company reach a negative value, is called the ruin probability and its calculation is one of the applications of collective risk theory. Since the utility function $n(x)$ is also defined for negative values of x, the utility of the ruin may reasonably be measured according to the amount of the deficit.

Considering now the stochastic process $\chi(t)$ the resources of the insurance company are reduced by the claim payments. The moments of time at which claims are paid, are assumed to be distributed in a Poisson form. The probability that there are no claims during the period (o, t) is e^{-t}; the probability of the occurrance of exactly n claims in (o, t) is $\frac{t^n}{n!} e^{-t}$. The claim distribution is defined by a distribution function $F(\xi)$ which in the following, can be considered as dependent on $\chi(t)$ and on t. For each period in which no claim occurs the following relation may be valid:

$$\frac{d\chi(s)}{ds} = m\,[\chi(s), s] > 0. \tag{3}$$

If no claim occurs $\chi(t)$ changes only by premium income, interest payments etc. and it is assumed, that in this case $\chi(t)$ is monotonously increasing. Thus the ruin of a company can only take place at the moment of the occurrence of a claim. The ruin occurs exactly at the moment $\bar{t}$, when the claim exceeds the capital of the company, that is in the case of

$$\xi > \chi(\bar{t})$$

Let $w(x, s, y, t)$ be the probability that a stochastic process, starting with $\chi(s) = x$ at the moment s, will at the moment t show for the first time a value smaller than o and the resulting deficit will be smaller than y. Obviously $w(x, s, \infty, t)$ is the ruin probability for the period (s, t) according to the "classical" collective theory of risk. Now it can be shown, that $w(x, s, y, t)$ for $\chi(s)$ from (3) is the only continuous and bounded solution of the following system of equations:

$$\begin{gathered} w(x, s, y, s) = 0, \quad x \geqq 0, \\ \lim_{x\to\infty} \; w(x, s, y, t) = 0, \quad t < \infty, \\ \frac{d}{ds} w[x(s), s, y, t] = -\{1 - F[x(s)]\} + w[x(s), s, y, t] - \\ - \int_0^{x(s)} w[x(s) - \xi, s, y, t]\, dF(\xi). \end{gathered} \tag{4}$$

The future development will on the one hand, lead to dividend payments, the utility of which will be treated later, and on the other hand—with a certain probability—to ruin. In so far as the ruin of the company is concerned, the following measure for utility may be introduced:

$$N_R = \int_{t=0}^{\infty} \int_{y=0}^{\infty} n[-y\, e^{-\int_0^t \delta(\tau)d\tau}]\, dw\ (x, 0, y, t). \tag{5}$$

N_R is the weighted average of the utility of ruin at the moment of time t with the deficit y discounted to time 0 and using as weights the probabilities $w(x, 0, y, t)$. This measure corresponds to the measure (1). In both cases the utility of several alternative events occurring with certain probabilities is introduced as the average of the utilities of the individual events weighted with these probabilities. Our next object is the utility of the dividend payment. As $\chi(s)$ was assumed continuous with the exception of the moments when a claim occurs, dividend payments must also be assumed to be continuous. If their frequency function is denoted by $D(s)$, the amount of dividend payment depends only on $\chi(s)$ and on s and thus we have

$$D(s) = D[\chi(s), s].$$

Since $\chi(s)$ represents a stochastic process, $D(s)$ becomes by this relation a stochastic process as well. For a comparison of dividend payments at different moments of time we must use the relation (2) and discount each payment at the moment s to the moment 0. Putting

$$D_0(s) = D(s)\, e^{-\int_0^s \delta(\tau)d\tau}$$

the cash value of the dividends paid during the period (t_1, t_2) will be equal to $\int_{t_1}^{t_2} D_0\ (s)ds$.

Let $v(x, s, y, t)$ be the probability that a stochastic process starting with $\chi(s) = x$ at the moment s will until the moment of time t lead to a dividend payment of the cash value of $\int_s^t D_0(\tau)d\tau < y$ no matter whether or not ruin has occurred until that moment. Dividend payments are assumed to continue until the possible occurrence of ruin.

Putting

$$y(s) = y_0 - \int_0^s D_0(\tau)d\tau$$

it can be shown that $v(x, s, y, t)$ for $\chi(s)$ from (3) is the only bounded solution of the following equation system:

$$v(x, s, y, s) = 1, \quad y > 0, \tag{6}$$

$$\frac{d}{ds} v[x(s), s, y(s), t] = -\{1 - F[x(s)]\} + v[x(s), s, y(s), t] -$$

$$- \int_0^{x(s)} v[x(s) - \xi, s, y(s), t] \, dF(\xi).$$

The utility obtained by dividend payments may now be measured by the expression

$$N_D = \int_0^\infty n(y) \, dv \, (x, 0, y, \infty). \tag{7}$$

The utility of the future development of an insurance is now described by $N = N_R + N_D$ with N_R from (5) and N_D from (7). The probabilities $w(x, 0, y, t)$ and $v(x, 0, y, \infty)$ depend on $F(\xi)$ and on $m[\chi(s), s]$. $m[\chi(s), s]$ is influenced by changes in the dividend payment. The distribution function $F(\xi)$ is influenced by reinsurance arrangements. Thus it can be concluded generally, that the company has to choose its dividend, premium and reinsurance policy in such a way, that utility reaches a maximum value. Since $F(\xi)$ as well as $m[\chi(s), s]$ can be assumed dependent on s, it is possible for the company to adapt its dividend policy, reinsurance policy etc. to the conditions prevailing at any moment.

It must be emphasized that the results have been derived from relatively simple suppositions. The only new assumption as compared with the utility function (1) consisted in the introduction of the relation of equivalence (2), which is well known and well founded in financial mathematics. The stochastic process used may be regarded as a rather general model. Difficulties in the practical application might mainly be involved in the calculation of N and in the determination of its maximum.