

THE DECOMPOSITIONS OF THE DISCOUNTED PENALTY  
FUNCTIONS AND DIVIDENDS-PENALTY IDENTITY  
IN A MARKOV-MODULATED RISK MODEL

BY

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ABSTRACT

In this paper, we study the expected discounted penalty functions and their decompositions in a Markov-modulated risk process in which the rate for the Poisson claim arrivals and the distribution of the claim amounts vary in time depending on the state of an underlying (external) Markov jump process. The main feature of the model is the flexibility modeling the arrival process in the sense that periods with very frequent arrivals and periods with very few arrivals may alternate. Explicit formulas for the expected discounted penalty function at ruin, given the initial surplus, and the initial and terminal environment states, are obtained when the initial surplus is zero or when all the claim amount distributions are from the rational family. We also investigate the distributions of the maximum surplus before ruin and the maximum severity of ruin. The dividends-penalty identity is derived when the model is modified by applying a barrier dividend strategy.

KEYWORDS

Markov-modulated risk model, expected discounted penalty function, maximum surplus before ruin, maximum severity of ruin, dividends-penalty identity.

1. INTRODUCTION

Asmussen (1989) proposed a Markov-modulated risk model in which both the frequency of the claim arrivals and the distribution of the claim amounts are influenced by an external environment process  $\{J(t); t \geq 0\}$ . This model is a generalization of the classical compound Poisson risk model and the primary motivation for this generalization is the enhanced flexibility that it permits for the modeling of the claim arrival process and the claim severity distribution assumed in the classical risk process. The impact of poor weather conditions on the financial performance of automobile insurance portfolios, or of the outbreak of epidemics which impact health insurance portfolios, is well known.

See, for example, Asmussen (1989). Zhu and Yang (2007) refer to states of the process  $\{J(t); t \geq 0\}$  as economic circumstances or political regime switchings. It is therefore appealing to include in the classical risk process assumptions which permit variation in both claim frequencies and claim severities as a result of external environmental factors. The modeling framework that is advocated in this paper achieves this.

Suppose that  $\{J(t); t \geq 0\}$  is a homogeneous, irreducible and recurrent Markov process with finite state space  $E = \{1, 2, \dots, m\}$ . Denote the intensity matrix of  $\{J(t); t \geq 0\}$  by  $\Lambda = (\alpha_{i,j})_{i,j=1}^m$ , with  $\alpha_{i,i} := -\alpha_i$  for  $i \in E$ . Let  $\pi = (\pi_1, \pi_2, \dots, \pi_m)$  be the stationary distribution of  $\{J(t); t \geq 0\}$ .

Let  $N(t)$  be the number of claims occurring in  $(0, t]$ . If  $J(s) = i$  for all  $s$  in a small interval  $(t, t+h]$ , then the number of claims occurring in that interval, i.e.,  $N(t+h) - N(t)$ , has a Poisson distribution with parameter  $\lambda_i (> 0)$ . We assume further that given the process  $\{J(t); t \geq 0\}$ , the process  $\{N(t); t \geq 0\}$  has independent increments. Then

$$\mathbb{P}(N(t+h) = n+1 \mid N(t) = n, J(s) = i \text{ for } t < s \leq t+h) = \lambda_i h + o(h).$$

The process  $\{N(t); t \geq 0\}$  is called a Markov-modulated Poisson process, which is a special case of a Cox process. It also can be seen as a Poisson process with its parameter driven by an external environment process  $\{J(t); t \geq 0\}$ .

We also assume that, given  $J(t) = i$ , the claim amounts have distribution function  $F_i(x)$ , with density function  $f_i(x)$  and finite mean  $\mu_i (i \in E)$ . Moreover, we assume that premiums are received continuously at a positive constant rate  $c$ .

The corresponding surplus process  $\{U(t); t \geq 0\}$  is given by

$$U(t) = u + ct - \sum_{n=1}^{N(t)} X_n, \quad t \geq 0, \quad (1.1)$$

where  $u \geq 0$  is the initial surplus and  $X_n$  is the amount of the  $n$ -th claim. We also assume that the positive loading condition holds, i.e.,

$$\sum_{i=1}^m \pi_i (c - \lambda_i \mu_i) > 0.$$

For notational convenience, let  $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot \mid J(0) = i)$ . Define  $T = \inf\{t \geq 0 : U(t) < 0\}$  to be the time of ruin and let  $w(x, y)$ , for  $x, y \geq 0$ , be non-negative valued of penalty function. Define for  $\delta \geq 0$ ,  $u \geq 0$  and  $i, j \in E$

$$\phi_{i,j}(u) = \mathbb{E}_i[e^{-\delta T} w(U(T-), |U(T)|) I(T < \infty, J(T) = j) \mid U(0) = u]$$

to be the expected discounted penalty (Gerber-Shiu) function at ruin if ruin is caused by a claim in state  $j$ , given the initial surplus  $u$  and the initial environment

$i \in E$ , for the surplus  $U(T-)$  before ruin and the deficit  $|U(T)|$  at ruin, where  $I(\cdot)$  is the indicator function. Then

$$\phi_i(u) = \sum_{j=1}^m \phi_{i,j}(u), \quad u \geq 0, \quad i \in E,$$

is the expected discounted penalty function at ruin, given the initial surplus  $u$  and the initial environment  $i \in E$ . In particular, when  $\delta=0$  and  $w(x,y)=1$ ,  $\phi_{i,j}(u)$  simplifies to  $\Psi_{i,j}(u)$  with the definition

$$\Psi_{i,j}(u) = \mathbb{P}_i(T < \infty, J(T) = j \mid U(0) = u), \quad i, j \in E.$$

Here  $\Psi_{i,j}(u)$  is the ruin probability if ruin is caused by a claim in state  $j$  given that the initial state is  $i$  and hence

$$\Psi_i(u) = \sum_{j=1}^m \Psi_{i,j}(u), \quad u \geq 0, \quad i \in E,$$

is the probability of ruin given that the initial state is  $i$ , and correspondingly  $\Phi_i(u) = 1 - \Psi_i(u)$  is the non-ruin probability given that the initial state is  $i$ .

Models of this type have also been investigated by some authors, e.g., Reinhard (1984), Bäuerle (1996), Schmidli (1997), Wu (1999), Snoussi (2002), Lu and Li (2005), and Lu (2006). Ng and Yang (2006) give closed form solutions for the joint distribution of the surplus before and after ruin when the initial surplus is zero or when the claim amount distributions are phase-type distributed. Li and Lu (2007) study the moments of the present value of the dividend payments and the distribution of the total dividends prior to ruin for the Markov-modulated risk model modified by the introduction of a barrier dividend. Albrecher and Boxma (2005) study the expected discounted penalty function in a semi-Markovian dependent risk model in which at each instant of a claim, the underlying Markov chain jumps to a new state and the distribution of claim depends on this state. This model includes, as special cases, the classical risk model, and the Sparre Andersen model with phase-type inter-arrival times, as well as models with causal dependence between the claim amounts distribution and the claim inter-arrivals distribution.

In this paper, we study the expected discounted penalty functions and their decompositions for the Markov-modulated risk model. Explicit formulas for the expected discounted penalty function at ruin if ruin is caused by a claim in state  $j$ , given initial surplus  $u$  and initial environment  $i$ , are derived when the initial surplus is zero or when all the claim amount distributions are from the rational family. The distributions of the maximum surplus before ruin and the maximum severity of ruin are studied through the ruin probabilities and their decompositions. The dividends-penalty identity is also derived for the model modified by the introduction of a barrier dividend.

## 2. EXPECTED DISCOUNTED PENALTY FUNCTIONS

### 2.1. A system of integro-differential equations

Using the same arguments as in Ng and Yang (2006), we obtain the following integro-differential equations for  $\phi_{i,j}(u)$  by conditioning on the events occurring in a small interval  $[0, h]$ , for  $i \in E$ ,

$$c\phi'_{i,i}(u) = (\lambda_i + \delta)\phi_{i,i}(u) - \lambda_i \left[ \int_0^u \phi_{i,i}(u-x)f_i(x)dx + \omega_i(u) \right] - \sum_{k=1}^m \alpha_{i,k} \phi_{k,i}(u), \quad (2.1)$$

and for  $i \neq j$ ,

$$c\phi'_{i,j}(u) = (\lambda_i + \delta)\phi_{i,j}(u) - \lambda_i \int_0^u \phi_{i,j}(u-x)f_i(x)dx - \sum_{k=1}^m \alpha_{i,k} \phi_{k,j}(u), \quad (2.2)$$

where  $\omega_i(u) = \int_u^\infty w(u, x-u)f_i(x)dx$ .

Let  $\hat{\phi}_{i,j}(s)$ ,  $\hat{f}_i(s)$ , and  $\hat{\omega}_i(s)$  be the Laplace transforms of  $\phi_{i,j}$ ,  $f_i$  and  $\omega_i$ , respectively. Taking Laplace transforms on both sides of Eqs. (2.1) and (2.2), we have, for  $i, j \in E$ ,

$$\left[ s - \frac{\lambda_i + \delta}{c} + \frac{\lambda_i}{c} \hat{f}_i(s) \right] \hat{\phi}_{i,j}(s) + \frac{1}{c} \sum_{k=1}^m \alpha_{i,k} \hat{\phi}_{k,j}(s) = \phi_{i,j}(0) - \frac{\lambda_i}{c} \hat{\omega}_i(s) I(i=j). \quad (2.3)$$

Further for simplicity, define  $S_i(s) = s - \delta/c - (\lambda_i/c)(1 - \hat{f}_i(s))$ , for  $i \in E$ . Then Eq. (2.3) can be rewritten in the following matrix form:

$$\mathbf{A}(s) \hat{\boldsymbol{\phi}}(s) = \boldsymbol{\phi}(0) - \hat{\boldsymbol{\omega}}(s),$$

where  $\mathbf{A}(s) = \text{diag}(S_1(s), S_2(s), \dots, S_m(s)) + \Lambda/c$ ,  $\boldsymbol{\phi}(u) = (\phi_{i,j}(u))_{i,j=1}^m$ ,  $\hat{\boldsymbol{\phi}}(s) = (\hat{\phi}_{i,j}(s))_{i,j=1}^m$ , and  $\hat{\boldsymbol{\omega}}(s) = \text{diag}(\lambda_1 \hat{\omega}_1(s)/c, \lambda_2 \hat{\omega}_2(s)/c, \dots, \lambda_m \hat{\omega}_m(s)/c)$ . It follows that

$$\hat{\boldsymbol{\phi}}(s) = [\mathbf{A}(s)]^{-1} [\boldsymbol{\phi}(0) - \hat{\boldsymbol{\omega}}(s)] = \frac{\mathbf{A}^*(s) \boldsymbol{\phi}(0) - \mathbf{A}^*(s) \hat{\boldsymbol{\omega}}(s)}{\det[\mathbf{A}(s)]}, \quad (2.4)$$

where  $\mathbf{A}^*(s)$  is the adjoint matrix of  $\mathbf{A}(s)$ .

### 2.2. The initial values for $\boldsymbol{\phi}(0)$

Using the same arguments as in Albrecher and Boxma (2005), we can show that the characteristic equation  $\det[\mathbf{A}(s)] = 0$  has exactly  $m$  roots with positive real parts, say,  $\rho_1, \rho_2, \dots, \rho_m$ , which play an important role in determining the initial values for  $\phi_{i,j}(0)$ . We assume that  $\rho_1, \rho_2, \dots, \rho_m$  are distinct in the sequel.

Now we define the divided differences of a matrix  $\mathbf{B}(s)$ , with respect to distinct numbers  $r_1, r_2, \dots$ , recursively as follows:

$$\begin{aligned}\mathbf{B}[r_1, s] &= \frac{\mathbf{B}(s) - \mathbf{B}(r_1)}{s - r_1}, \\ \mathbf{B}[r_1, r_2, s] &= \frac{\mathbf{B}[r_1, s] - \mathbf{B}[r_1, r_2]}{s - r_2}, \\ \mathbf{B}[r_1, r_2, r_3, s] &= \frac{\mathbf{B}[r_1, r_2, s] - \mathbf{B}[r_1, r_2, r_3]}{s - r_3},\end{aligned}$$

and so on. As for the divided differences of a function (Gerber and Shiu (2005)), we have the following formula for the  $(k-1)$ -th divided difference

$$\mathbf{B}[r_1, r_2, \dots, r_k] = \sum_{j=1}^k \frac{\mathbf{B}(r_j)}{\prod_{i=1, i \neq j}^k (r_j - r_i)}.$$

For distinct  $\rho_1, \rho_2, \dots, \rho_m$ , since  $\hat{\phi}_{i,j}(s)$  is finite for  $\Re(s) \geq 0$ , then

$$\mathbf{A}^*(\rho_i) \boldsymbol{\phi}(0) = \mathbf{A}^*(\rho_i) \hat{\omega}(\rho_i), \quad i = 1, 2, \dots, m.$$

Therefore

$$\mathbf{A}^*[\rho_1, \rho_2] \boldsymbol{\phi}(0) = (\mathbf{A}^* \hat{\omega})[\rho_1, \rho_2],$$

where  $(\mathbf{A}^* \hat{\omega})[\rho_1, \rho_2]$  is the divided difference of the product of matrices  $\mathbf{A}^*(s)$  and  $\hat{\omega}(s)$  with respect to  $\rho_1$  and  $\rho_2$ , given by

$$(\mathbf{A}^* \hat{\omega})[\rho_1, \rho_2] = \mathbf{A}^*(\rho_i) \hat{\omega}[\rho_1, \rho_2] + \mathbf{A}^*[\rho_1, \rho_2] \hat{\omega}(\rho_2) \quad (2.5)$$

and recursively,

$$\mathbf{A}^*[\rho_1, \rho_2, \dots, \rho_i] \boldsymbol{\phi}(0) = (\mathbf{A}^* \hat{\omega})[\rho_1, \rho_2, \dots, \rho_i], \quad i = 2, 3, \dots, m,$$

where the matrix  $(\mathbf{A}^* \hat{\omega})[\rho_1, \rho_2, \dots, \rho_m]$  is given by the following formula:

$$(\mathbf{A}^* \hat{\omega})[\rho_1, \rho_2, \dots, \rho_m] = \sum_{i=1}^m \mathbf{A}^*[\rho_1, \dots, \rho_i] \hat{\omega}[\rho_i, \dots, \rho_m]. \quad (2.6)$$

Then we have the following result for  $\boldsymbol{\phi}(0)$ :

$$\boldsymbol{\phi}(0) = \{\mathbf{A}^*[\rho_1, \rho_2, \dots, \rho_m]\}^{-1} (\mathbf{A}^* \hat{\omega})[\rho_1, \rho_2, \dots, \rho_m]. \quad (2.7)$$

In particular, when  $m = 2$ ,

$$\mathbf{A}(s) = \begin{pmatrix} s - \frac{\alpha_1 + \delta}{c} - \frac{\lambda_1}{c} [1 - \hat{f}_1(s)] & \frac{\alpha_1}{c} \\ \frac{\alpha_2}{c} & s - \frac{\alpha_2 + \delta}{c} - \frac{\lambda_2}{c} [1 - \hat{f}_2(s)] \end{pmatrix},$$

and the adjoint matrix of  $\mathbf{A}(s)$  is given as

$$\mathbf{A}^*(s) = \begin{pmatrix} s - \frac{\alpha_2 + \delta}{c} - \frac{\lambda_2}{c} [1 - \hat{f}_2(s)] & -\frac{\alpha_1}{c} \\ -\frac{\alpha_2}{c} & s - \frac{\alpha_1 + \delta}{c} - \frac{\lambda_1}{c} [1 - \hat{f}_1(s)] \end{pmatrix}.$$

Then  $\mathbf{A}^*[\rho_1, \rho_2] = \text{diag}(1 + (\lambda_2/c)\hat{f}_2[\rho_1, \rho_2], 1 + (\lambda_1/c)\hat{f}_1[\rho_1, \rho_2])$ , and by Eq. (2.5), Eq. (2.7) simplifies to

$$\begin{aligned} \boldsymbol{\phi}(0) &= \hat{\omega}(\rho_1) + \{\mathbf{A}^*[\rho_1, \rho_2]\}^{-1} \mathbf{A}^*(\rho_2) \hat{\omega}[\rho_1, \rho_2] \\ &= \begin{pmatrix} \frac{\lambda_1}{c} \hat{\omega}_1(\rho_1) & 0 \\ 0 & \frac{\lambda_2}{c} \hat{\omega}_2(\rho_1) \end{pmatrix} + \begin{pmatrix} \frac{\frac{\lambda_1}{c} [S_2(\rho_2) - \frac{\alpha_2}{c}] \hat{\omega}_1[\rho_1, \rho_2]}{1 + \frac{\lambda_2}{c} \hat{f}_2[\rho_1, \rho_2]} & -\frac{\frac{\alpha_1 \lambda_2}{c^2} \hat{\omega}_2[\rho_1, \rho_2]}{1 + \frac{\lambda_2}{c} \hat{f}_2[\rho_1, \rho_2]} \\ -\frac{\frac{\alpha_2 \lambda_1}{c^2} \hat{\omega}_1[\rho_1, \rho_2]}{1 + \frac{\lambda_1}{c} \hat{f}_1[\rho_1, \rho_2]} & \frac{\frac{\lambda_2}{c} [S_1(\rho_2) - \frac{\alpha_1}{c}] \hat{\omega}_2[\rho_1, \rho_2]}{1 + \frac{\lambda_1}{c} \hat{f}_1[\rho_1, \rho_2]} \end{pmatrix}. \end{aligned}$$

Further, if  $w(x, y) = 1$  and  $\delta = 0$ , we have that  $\rho_1 = \rho$ ,  $\rho_2 = 0$ ,  $\omega_i(u) = \bar{F}_i(u)$ , and  $\phi_{i,j}(u) = \Psi_{i,j}(u)$  for  $i, j = 1, 2$ . In this case, we also get that  $S_i(0) = 0$ ,  $\hat{\omega}_i[\rho, 0] = [\hat{\omega}_i(\rho) - \mu_i] / \rho$  and  $\hat{f}_i[\rho, 0] = -\hat{\omega}_i(\rho)$ . Then

$$\boldsymbol{\Psi}(0) = \begin{pmatrix} \frac{\lambda_1}{c} \hat{\omega}_1(\rho_1) & 0 \\ 0 & \frac{\lambda_2}{c} \hat{\omega}_2(\rho_1) \end{pmatrix} - \frac{1}{c^2 \rho} \begin{pmatrix} \frac{\lambda_1 \alpha_2 [\hat{\omega}_1(\rho) - \mu_1]}{1 - (\lambda_2/c) \hat{\omega}_2(\rho)} & \frac{\lambda_2 \alpha_1 [\hat{\omega}_2(\rho) - \mu_2]}{1 - (\lambda_2/c) \hat{\omega}_2(\rho)} \\ \frac{\lambda_1 \alpha_2 [\hat{\omega}_1(\rho) - \mu_1]}{1 - (\lambda_1/c) \hat{\omega}_1(\rho)} & \frac{\lambda_2 \alpha_1 [\hat{\omega}_2(\rho) - \mu_2]}{1 - (\lambda_1/c) \hat{\omega}_1(\rho)} \end{pmatrix}.$$

where  $\boldsymbol{\Psi}(0) = (\Psi_{i,j}(0))_{i,j=1}^2$  and  $\hat{\omega}_i(\rho) = [1 - \hat{f}_i(\rho)] / \rho$ , for  $i = 1, 2$ .

### 2.3. An explicit expression for $\boldsymbol{\phi}(u)$

By applying the divided differences repeatedly to the numerator of Eq. (2.4), we obtain the following expression for  $\hat{\boldsymbol{\phi}}(s)$ :

$$\hat{\boldsymbol{\phi}}(s) = \frac{\prod_{i=1}^m (s - \rho_i)}{\det[\mathbf{A}(s)]} [\mathbf{A}^*[\rho_1, \rho_2, \dots, \rho_m, s] \boldsymbol{\phi}(0) - (\mathbf{A}^* \hat{\omega})[\rho_1, \rho_2, \dots, \rho_m, s]].$$

By Eq. (2.6), we have

$$\begin{aligned}
(\mathbf{A}^* \hat{\omega})[\rho_1, \rho_2, \dots, \rho_m, s] &= \mathbf{A}^*[\rho_1, \rho_2, \dots, \rho_m, s] \hat{\omega}(s) \\
&\quad + \sum_{i=1}^m \mathbf{A}^*[\rho_1, \dots, \rho_i] \hat{\omega}[\rho_i, \dots, \rho_m, s].
\end{aligned}$$

Then  $\hat{\phi}(s)$  can be rewritten as

$$\begin{aligned}
\hat{\phi}(s) &= \frac{\prod_{i=1}^m (s - \rho_i)}{\det[\mathbf{A}(s)]} \left[ \mathbf{A}^*[\rho_1, \rho_2, \dots, \rho_m, s] (\phi(0) - \hat{\omega}(s)) \right. \\
&\quad \left. - \sum_{i=1}^m \mathbf{A}^*[\rho_1, \dots, \rho_i] \hat{\omega}[\rho_i, \dots, \rho_m, s] \right].
\end{aligned} \tag{2.8}$$

The Laplace transform  $\hat{\phi}(s)$  can be inverted for some special claim amount distributions. Consider the case where the claim amount distributions for  $m$  classes are from the rational family, that is, their Laplace transforms can be expressed as a ratio of polynomials:

$$\hat{f}_i(s) = \frac{p_{k_i-1}^{(i)}(s)}{q_{k_i}^{(i)}(s)}, \quad k_i \in \mathbb{N}^+, \quad i \in E,$$

where  $q_{k_i}^{(i)}$  is a polynomial of degree  $k_i$ , while  $p_{k_i-1}^{(i)}$  is a polynomial of degree  $k_i - 1$  or less; all have leading coefficient 1 and satisfy  $p_{k_i-1}^{(i)}(0) = q_{k_i}^{(i)}(0)$ . Further, equation  $q_{k_i}^{(i)}(s) = 0$  has roots with only negative real parts.

To obtain expressions which can be inverted easily, we multiply both numerator and denominator of Eq. 2.8 by  $\prod_{i=1}^m q_{k_i}^{(i)}(s)$ , yielding

$$\begin{aligned}
\hat{\phi}(s) &= \frac{\prod_{i=1}^m (s - \rho_i)}{\det[\mathbf{A}(s)] \prod_{i=1}^m q_{k_i}^{(i)}(s)} \left\{ \left[ \mathbf{A}^*[\rho_1, \rho_2, \dots, \rho_m, s] \prod_{i=1}^m q_{k_i}^{(i)}(s) \right] [\phi(0) - \hat{\omega}(s)] \right. \\
&\quad \left. - \left[ \prod_{i=1}^m q_{k_i}^{(i)}(s) \right] \sum_{i=1}^m \mathbf{A}^*[\rho_1, \dots, \rho_i] \hat{\omega}[\rho_i, \dots, \rho_m, s] \right\}.
\end{aligned} \tag{2.9}$$

First, we look at the denominator in (2.9), denoted by  $D(s)$ :

$$D(s) = \det[\mathbf{A}(s)] \prod_{i=1}^m q_{k_i}^{(i)}(s),$$

which is clearly a polynomial of degree  $m + \sum_{i=1}^m k_i$  with the leading coefficient 1, and therefore equation  $D(s) = 0$  has  $m + \sum_{i=1}^m k_i$  roots in the complex plane. By the fact that equation  $\det[\mathbf{A}(s)] = 0$  has exactly  $m$  roots,  $\rho_1, \rho_2, \dots, \rho_m$ , with positive real parts, we can rewrite  $D(s)$  as

$$D(s) = \prod_{i=1}^m (s - \rho_i) \prod_{i=1}^{K_m} (s + R_i),$$

where  $K_m = \sum_{i=1}^m k_i$ , and all  $R_i$ 's have positive real parts by the definition of the rational distribution. For simplicity, we further assume that these  $R_i$ 's are distinct. Consequently, Eq. (2.9) can be expressed as follows:

$$\begin{aligned} \hat{\phi}(s) = \frac{1}{\prod_{i=1}^{K_m} (s + R_i)} & \left\{ \left[ \mathbf{A}^*[\rho_1, \rho_2, \dots, \rho_m, s] \prod_{i=1}^m q_{k_i}^{(i)}(s) \right] [\boldsymbol{\phi}(0) - \hat{\omega}(s)] \right. \\ & \left. - \left[ \prod_{i=1}^m q_{k_i}^{(i)}(s) \right] \sum_{i=1}^m \mathbf{A}^*[\rho_1, \dots, \rho_i] \hat{\omega}[\rho_i, \dots, \rho_m, s] \right\}. \end{aligned} \quad (2.10)$$

It is easy to see that the elements in matrix  $\mathbf{A}^*[\rho_1, \rho_2, \dots, \rho_m, s] \prod_{i=1}^m q_{k_i}^{(i)}(s)$  are polynomials of degrees which are less than  $K_m$ , and all  $\mathbf{A}^*[\rho_1, \dots, \rho_i]$  for  $i \in E$  are constants. Then we have the following partial fractions:

$$\begin{aligned} \frac{\mathbf{A}^*[\rho_1, \rho_2, \dots, \rho_m, s] \prod_{i=1}^m q_{k_i}^{(i)}(s)}{\prod_{i=1}^{K_m} (s + R_i)} &= \sum_{l=1}^{K_m} \frac{\mathbf{M}^{(l)}}{s + R_l} \\ \frac{\prod_{i=1}^m q_{k_i}^{(i)}(s)}{\prod_{i=1}^{K_m} (s + R_i)} &= 1 + \sum_{l=1}^{K_m} \frac{n_l}{s + R_l}, \end{aligned}$$

where  $\mathbf{M}^{(l)} = (m_{i,j}^{(l)})_{i,j=1}^m$ , for  $l = 1, 2, \dots, K_m$ , are coefficient matrices with

$$\mathbf{M}^{(l)} = \frac{\mathbf{A}^*[\rho_1, \rho_2, \dots, \rho_m, -R_l] \prod_{i=1}^m q_{k_i}^{(i)}(-R_l)}{\prod_{v=1, v \neq l}^{K_m} (R_v - R_l)},$$

while  $n_l$  is the coefficient given by

$$n_l = \frac{\prod_{i=1}^m q_{k_i}^{(i)}(-R_l)}{\prod_{v=1, v \neq l}^{K_m} (R_v - R_l)}, \quad l = 1, 2, \dots, K_m.$$

Thus by partial fraction Eq. (2.10) can be expressed as

$$\begin{aligned} \hat{\phi}(s) = \sum_{l=1}^{K_m} \frac{1}{s + R_l} & \left\{ \mathbf{M}^{(l)} [\boldsymbol{\phi}(0) - \hat{\omega}(s)] - n_l \sum_{i=1}^m \mathbf{A}^*[\rho_1, \dots, \rho_i] \hat{\omega}[\rho_i, \dots, \rho_m, s] \right\} \\ & - \sum_{i=1}^m \mathbf{A}^*[\rho_1, \dots, \rho_i] \hat{\omega}[\rho_i, \dots, \rho_m, s]. \end{aligned} \quad (2.11)$$



To obtain the explicit Laplace inverse of (2.11), we introduce an operator  $T_r$  for a matrix  $\mathbf{B}(y)$  with respect to a complex number  $r$ , to be

$$T_r \mathbf{B}(y) = \int_y^\infty e^{-r(x-y)} \mathbf{B}(x) dx, \quad r \in \mathbb{C}, \quad y \geq 0. \quad (2.12)$$

Here  $\mathbf{B}(y)$  is a matrix with each element being an integrable real-valued function of  $y$ . The composition operators of  $T_r$  can be defined recursively, for example,

$$T_{r_1} T_{r_2} \mathbf{B}(y) = T_{r_2} T_{r_1} \mathbf{B}(y) = \frac{T_{r_1} \mathbf{B}(y) - T_{r_2} \mathbf{B}(y)}{r_2 - r_1}, \quad r_1 \neq r_2 \in \mathbb{C}, \quad y \geq 0.$$

This operator has been used for the integrable real-valued function in some papers, see, for example, Dickson and Hipp (2001) and Li and Garrido (2004). Similar to (10.1) in Gerber and Shiu (2005), the following result holds for the relationship between the operator  $T_r$  and the corresponding divided difference:

$$\left( \prod_{i=1}^m T_{r_i} \right) \mathbf{B}(0) = (-1)^{m-1} \hat{\mathbf{B}}[r_1, r_2, \dots, r_m]. \quad (2.13)$$

Further by the definition of operator  $T_r$  in (2.12), we have that

$$T_s T_r \mathbf{B}(0) = \int_0^\infty e^{-sx} [T_r \mathbf{B}(x)] dx,$$

which shows that the Laplace inverse of matrix  $T_s T_r \mathbf{B}(0)$  is  $T_r \mathbf{B}(x)$ . In general, we have the following formula for the Laplace inverse of matrix  $\left[ T_s \left( \prod_{i=1}^m T_{r_i} \right) \mathbf{B} \right](0)$ :

$$\mathcal{L}^{-1} \left[ T_s \left( \prod_{i=1}^m T_{r_i} \right) \mathbf{B}(0) \right] = \left( \prod_{i=1}^m T_{r_i} \right) \mathbf{B}(x). \quad (2.14)$$

Thus it follows from Eq. (2.11) and by (2.13) and (2.14), the explicit Laplace inversion of  $\hat{\boldsymbol{\phi}}(s)$  is given by

$$\begin{aligned} \boldsymbol{\phi}(u) = & \sum_{i=1}^m (-1)^{m-i} \mathbf{A}^*[\rho_1, \dots, \rho_i] \left( \prod_{k=i}^m T_{\rho_k} \right) \boldsymbol{\omega}(u) + \sum_{l=1}^{K_m} \left\{ e^{-R_l u} \mathbf{M}^{(l)} \boldsymbol{\phi}(0) \right. \\ & \left. - e^{-R_l u} \star \left[ \boldsymbol{\omega}(u) - n_l \sum_{i=1}^m (-1)^{m-i} \mathbf{A}^*[\rho_1, \dots, \rho_i] \left( \prod_{k=i}^m T_{\rho_k} \right) \boldsymbol{\omega}(u) \right] \right\}, \end{aligned}$$

where  $\star$  in above formula is the convolution operator.

## 3. THE MAXIMUM SURPLUS BEFORE RUIN

For  $b > u \geq 0$ , define

$$\xi_{i,j}(u; b) = \mathbb{P}_i \left( \sup_{0 \leq t \leq T} U(t) < b, T < \infty, J(T) = j \mid U(0) = u \right), \quad i, j \in E,$$

to be the probability that ruin occurs from initial surplus  $u$  without the surplus process reaching level  $b$  prior to ruin if ruin is caused by a claim in state  $j$  given that the process starts from initial state  $i$ . Alternatively,  $\xi_{i,j}(u; b)$  is the probability that ruin occurs in state  $j$  from initial state  $i$  in the presence of an absorbing barrier at  $b$ . Obviously,  $\xi_{i,j}(u; b) = 0$  for  $b \leq u$ . Then

$$\xi_i(u; b) = \sum_{j=1}^m \xi_{i,j}(u; b), \quad i \in E,$$

is the probability that ruin occurs without the surplus process reaching level  $b$  prior to ruin from initial state  $i$  and initial surplus  $u$ .

For  $0 \leq u \leq b$  and  $i, j \in E$ , define  $\chi_{i,j}(u; b)$  to be the probability that the surplus process attains level  $b$  at state  $j$  from initial state  $i$  and initial surplus  $u$  without first falling below zero. Clearly,  $\chi_{i,j}(b; b) = I(i=j)$  for  $i, j \in E$ . Then

$$\chi_i(u; b) = \sum_{j=1}^m \chi_{i,j}(u; b), \quad i \in E,$$

is the probability that the surplus process attains level  $b$  from initial state  $i$  and initial surplus  $u$  without first falling below zero. Since eventually either ruin occurs without the surplus attaining level  $b$  or the surplus attains level  $b$ , then we have  $\chi_i(u; b) = 1 - \xi_i(u; b)$  for  $i \in E$ . Let  $\chi(u; b) = (\chi_{i,j}(u; b))_{i,j=1}^m$ . It follows from Li and Lu (2007) that

$$\chi(u; b) = \mathbf{v}(u)[\mathbf{v}(b)]^{-1}, \quad 0 \leq u \leq b,$$

where  $\mathbf{v}(u) = (v_{i,j}(u))_{i,j=1}^m$  is an  $m \times m$  matrix with  $v_{i,j}(u)$  being the solution of the following system of integro-differential equations:

$$c v'_{i,j}(u) = \lambda_i v_{i,j}(u) - \lambda_i \int_0^u v_{i,j}(u-x) f_i(x) dx - \sum_{k=1}^m \alpha_{i,k} v_{k,j}(u),$$

with the boundary conditions  $v_{i,j}(0) = I(i=j)$  for  $i, j \in E$ .

By considering whether or not the surplus reaches  $b(>u)$  before ruin, we have

$$\Psi_{i,j}(u) = \xi_{i,j}(u; b) + \sum_{k=1}^m \chi_{i,k}(u; b) \Psi_{k,j}(b), \quad i, j \in E,$$

or in matrix notation,

$$\Psi(u) = \xi(u; b) + \chi(u; b)\Psi(b), \quad (3.1)$$

where  $\xi(u, b) = (\xi_{i,j}(u; b))_{i,j=1}^m$  with  $\xi(b; b) = \mathbf{0}$  and  $\mathbf{0}$  being the  $m \times m$  zero matrix. In particular, when  $m = 1$ , the model simplifies to the classical risk model and since  $\chi(u; b) = 1 - \xi(u; b)$ , then Eq. (3.1) gives

$$\xi(u; b) = \frac{\Psi(u) - \Psi(b)}{1 - \Psi(b)}, \quad 0 \leq u \leq b.$$

This formula can be found in Dickson and Gray (1984). For  $m \in \mathbb{N}^+$ , we will show in the next section that  $\chi(u; b)$  can be expressed in terms of the ruin probability matrix  $\Psi(u)$  and therefore the distribution of the maximum surplus before ruin given in (3.1) can also be expressed in terms of the ruin probability matrix.

#### 4. THE MAXIMUM SEVERITY OF RUIN

In this section, we allow the surplus process to continue if ruin occurs, and consider the insurer's maximum severity of ruin from the time of ruin until the time that the surplus returns to level 0. Since we assume that the positive loading condition holds, it is certain that the surplus process will attain this level after ruin. For the classical risk model, Picard (1994) gives an explicit expression in terms of the ruin probability for the distribution of the maximum severity of ruin. Li and Dickson (2006) study the distribution of the maximum severity of ruin for the Sparre Andersen risk model with Erlang inter-arrival times.

For  $u \geq 0$ , we define  $\bar{T}$  to be the time of the first upcrossing of the surplus process through level 0 after ruin, i.e.,

$$\bar{T} = \inf\{t : t > T, U(t) \geq 0\},$$

and define

$$M_u = \sup\{|U(t)|, T \leq t \leq \bar{T}\}$$

to be the maximum severity of ruin. Let

$$H_{i,j}(z; u) = \mathbb{P}_i(M_u \leq z, T < \infty, J(T) = j), \quad z \geq 0, i, j \in E,$$

denote the distribution function of the maximum severity of ruin if ruin is caused by a claim in state  $j$  given that the process starts from initial state  $i$  and

$$H_i(z; u) = \sum_{j=1}^m H_{i,j}(z; u), \quad i \in E,$$

is the distribution function of the maximum severity of ruin given that the process starts from initial state  $i$ .

If the surplus process starts with an initial surplus  $u$  and initial state  $i$ , then the maximum severity of ruin will be no more than  $z$  if ruin occurs (by a claim in state  $j$ ) with a deficit  $y \leq z$  and if the surplus does not fall below  $-z$  from the level  $-y$ . The probability of the latter event is  $\chi_j(z-y; z)$  since attaining level 0 from level  $-y$  without falling below  $-z$  is equivalent to attaining level  $z$  from level  $z-y$  without falling below 0. Thus

$$H_{i,j}(z; u) = \int_0^z g_{i,j}(u, y) \chi_j(z-y; z) dy,$$

where  $g_{i,j}(u, y) = \partial G_{i,j}(u, y) / \partial y$  with

$$G_{i,j}(u, y) = \mathbb{P}_i(T < \infty, J(T) = j, |U(T)| \leq y), \quad u, y \geq 0, \quad i, j \in E,$$

being the probability that ruin occurs by a claim in state  $j$  and the deficit at ruin is at most  $y$  given that the initial state is  $i$ . Therefore

$$H_i(z; u) = \sum_{j=1}^m H_{i,j}(z; u) = \int_0^z \sum_{j=1}^m g_{i,j}(u, y) \chi_j(z-y; z) dy,$$

or in matrix notation,

$$\mathbf{H}(z; u) = \int_0^z \mathbf{g}(u, y) \boldsymbol{\chi}(z-y; z) \mathbf{1} dy, \quad (4.1)$$

where  $\mathbf{H}(z; u) = (H_1(z; u), H_2(z; u), \dots, H_m(z; u))^T$  is an  $m \times 1$  column vector,  $\mathbf{g}(u, y) = (g_{i,j}(u, y))_{i,j=1}^m$  is an  $m \times m$  matrix, and  $\mathbf{1} = (1, 1, \dots, 1)^T$  is an  $m \times 1$  column vector.

For the classical risk model ( $m = 1$ ), Picard (1994) shows that the integral in (4.1) can be expressed in terms of the ruin probability. Now we aim at calculating the integral in (4.1) for  $m \in \mathbb{N}^+$ . To achieve this, first we need to express  $\boldsymbol{\chi}(u; b)$  in terms of  $\boldsymbol{\Psi}(u)$ .

Let  $\bar{\boldsymbol{\Psi}}(u) = (\bar{\Psi}_{i,j}(u))_{i,j=1}^m = \mathbf{I} - \boldsymbol{\Psi}(u)$  with  $\mathbf{I}$  being the  $m \times m$  identity matrix. Setting  $\delta = 0$  and  $w(x, y) = 1$  in Eqs. (2.1) and (2.2), we have, for  $i, j \in E$ , that

$$c \bar{\boldsymbol{\Psi}}'_i(u) = \lambda_i \bar{\Psi}_{i,j}(u) - \lambda_i \int_0^u \bar{\Psi}_{i,j}(u-x) f_i(x) dx - \sum_{k=1}^m \alpha_{i,k} \bar{\Psi}_{k,j}(u) + \alpha_{i,j}. \quad (4.2)$$

Let  $\hat{\bar{\Psi}}_{i,j}(s) = \int_0^\infty e^{-su} \bar{\Psi}_{i,j}(u) du$  be the Laplace transform of  $\bar{\Psi}_{i,j}(u)$ . Taking Laplace transforms on both sides of Eq. (4.2) yields, for  $i, j \in E$ , that

$$\left[ s - \frac{\lambda_i}{c} (1 - \hat{f}_i(s)) \right] \hat{\bar{\Psi}}_{i,j}(s) + \frac{1}{c} \sum_{k=1}^m \alpha_{i,k} \hat{\bar{\Psi}}_{k,j}(s) = \bar{\Psi}_{i,j}(0) + \frac{\alpha_{i,j}}{cs}. \quad (4.3)$$

Eq. (4.3) can be expressed in the following matrix form:

$$\mathbf{A}(s)\hat{\Psi}(s) = \bar{\Psi}(0) + \frac{\Lambda}{cs},$$

where  $\hat{\Psi}(s) = (\hat{\Psi}_{i,j}(s))_{i,j=1}^m$ , and  $\mathbf{A}(s)$  is defined in Section 2 with  $\delta = 0$ . Since  $\chi(u; b) = \mathbf{v}(u)[\mathbf{v}(b)]^{-1}$  with  $\hat{\mathbf{v}}(s) = (\hat{v}_{i,j}(s))_{i,j=1}^m = [\mathbf{A}(s)]^{-1}$ , then we have

$$\hat{\Psi}(s) = \hat{\mathbf{v}}(s)\bar{\Psi}(0) + \frac{\hat{\mathbf{v}}(s)}{s} \frac{\Lambda}{c}. \quad (4.4)$$

Inverting (4.4) yields

$$\bar{\Psi}(u) = \mathbf{v}(u)\bar{\Psi}(0) + \left( \int_0^u \mathbf{v}(x) dx \right) \frac{\Lambda}{c}. \quad (4.5)$$

Taking derivatives with respect to  $u$  on both sides of (4.5) we obtain the following first order differential equation for the matrix  $\mathbf{v}(u)$ :

$$\mathbf{v}'(u) + \mathbf{v}(u) \frac{\Lambda [\bar{\Psi}(0)]^{-1}}{c} = \bar{\Psi}'(u) [\bar{\Psi}(0)]^{-1}, \quad u \geq 0,$$

with boundary condition  $\mathbf{v}(0) = \mathbf{I}$ . Solving it gives

$$\mathbf{v}(u) = \bar{\Psi}(u) [\bar{\Psi}(0)]^{-1} - \int_0^u \bar{\Psi}(x) [\bar{\Psi}(0)]^{-1} \frac{\Lambda [\bar{\Psi}(0)]^{-1}}{c} e^{-\frac{\Lambda [\bar{\Psi}(0)]^{-1}}{c}(u-x)} dx. \quad (4.6)$$

For simplicity, let  $\tilde{\Psi}(u) = \bar{\Psi}(u) [\bar{\Psi}(0)]^{-1}$  and  $\Delta = (\Lambda/c) [\bar{\Psi}(0)]^{-1}$ . Then Eq. (4.6) can be rewritten as

$$\mathbf{v}(u) = \tilde{\Psi}(u) - \int_0^u \tilde{\Psi}(x) \Delta e^{-\Delta(u-x)} dx. \quad (4.7)$$

Since  $\chi(u; b) = \mathbf{v}(u)[\mathbf{v}(b)]^{-1}$  with  $\mathbf{v}(u)$  being given by (4.7), then Eq. (4.1) can be rewritten as

$$\begin{aligned} \mathbf{H}(z; u) &= \int_0^z \mathbf{g}(u, y) \mathbf{v}(z-y) [\mathbf{v}(z)]^{-1} \mathbf{1} dy \\ &= \int_0^z \mathbf{g}(u, y) \tilde{\Psi}(z-y) [\mathbf{v}(z)]^{-1} \mathbf{1} dy \\ &\quad - \int_0^z \mathbf{g}(u, y) \left( \int_0^{z-y} \tilde{\Psi}(z-y-x) \Delta e^{-\Delta x} dx \right) dy [\mathbf{v}(z)]^{-1} \mathbf{1} \\ &= \int_0^z \mathbf{g}(u, y) \tilde{\Psi}(z-y) dy [\mathbf{v}(z)]^{-1} \mathbf{1} \\ &\quad - \int_0^z \left( \int_0^{z-x} \mathbf{g}(u, y) \tilde{\Psi}(z-y-x) dy \right) \Delta e^{-\Delta x} dx [\mathbf{v}(z)]^{-1} \mathbf{1}. \end{aligned} \quad (4.8)$$

For further evaluation, note that

$$\Psi_{i,j}(u+z) = \sum_{k=1}^m \int_0^z g_{i,k}(u,y) \Psi_{k,j}(z-y) dy + \int_z^\infty g_{i,j}(u,y) dy,$$

or in matrix form

$$\Psi(u+z) = \int_0^z \mathbf{g}(u,y) \Psi(z-y) dy + \int_z^\infty \mathbf{g}(u,y) dy,$$

which is equivalent to

$$\mathbf{I} - \bar{\Psi}(u+z) = \int_0^z \mathbf{g}(u,y) [\mathbf{I} - \bar{\Psi}(z-y)] dy + \int_z^\infty \mathbf{g}(u,y) dy. \quad (4.9)$$

By using the formula that  $\int_0^\infty \mathbf{g}(u,y) dy = \Psi(u)$ , Eq. (4.9) can be simplified to

$$\int_0^z \mathbf{g}(u,y) \bar{\Psi}(z-y) dy = \bar{\Psi}(u+z) - \bar{\Psi}(u),$$

and consequently,

$$\int_0^z \mathbf{g}(u,y) \tilde{\Psi}(z-y) dy = \tilde{\Psi}(u+z) - \tilde{\Psi}(u). \quad (4.10)$$

Finally, by substituting (4.7) and (4.10) into (4.8) we have

$$\begin{aligned} \mathbf{H}(z;u) &= \left\{ \tilde{\Psi}(u+z) - \tilde{\Psi}(u) - \int_0^z [\tilde{\Psi}(u+z-x) - \tilde{\Psi}(u)] \Delta e^{-\Delta x} dx \right\} \\ &\quad \times \left[ \tilde{\Psi}(z) - \int_0^z \tilde{\Psi}(z-x) \Delta e^{-\Delta x} dx \right]^{-1} \mathbf{1}, \end{aligned}$$

where  $\tilde{\Psi}(u) = \bar{\Psi}(u)[\bar{\Psi}(0)]^{-1} = [\mathbf{I} - \Psi(u)][\mathbf{I} - \Psi(0)]^{-1}$ .

**Remarks:**

- As in the classical risk model,  $\mathbf{H}(z;u)$  depends only on the ruin probability matrix  $\Psi(u)$ .
- When  $m = 1$ , the model reduces to the classical risk model;  $\Lambda = 0$ , and then  $\Delta = 0$ . In this case,  $\mathbf{H}(z;u)$  simplifies to

$$\mathbf{H}(z;u) = [\bar{\Psi}(u+z) - \bar{\Psi}(u)][\bar{\Psi}(z)]^{-1},$$

which can be found in Picard (1994). Here  $\bar{\Psi}(u)$  is the survival probability for the classical risk model.

### 5. THE DIVIDENDS-PENALTY IDENTITY

In this section, as in Lin et al. (2003) and Gerber et al. (2006), we derive the dividends-penalty identity for the Markov-modulated risk model. Now we consider the surplus process (1.1) modified by the payment of dividends according to a barrier strategy: when the surplus exceeds a constant barrier  $b(\geq u)$ , dividends are paid continuously so the surplus stays at level  $b$  until a new claim occurs. Let  $U_b(t)$  be the surplus process with initial surplus  $U_b(0) = u$  under the above barrier strategy and define  $T_{u,b} = \inf\{t \geq 0 : U_b(t) < 0\}$  to be the time of ruin. Let  $\delta > 0$  be the force of interest for valuation and define

$$D_{u,b} = \int_0^{T_{u,b}} e^{-\delta t} dD(t), \quad 0 \leq u \leq b,$$

to be the present value of all dividends paid until the time of ruin  $T_{u,b}$  given that the initial surplus is  $u$ , where  $D(t)$  is the aggregate dividends paid by time  $t$ . Define

$$V_{i,j}(u; b) = \mathbb{E}_i[D_{u,b} I(J(T_{u,b}) = j)], \quad 0 \leq u \leq b, i, j \in E,$$

to be the expected present value of the dividend payments before ruin if ruin is caused by a claim in state  $j$  given the initial state is  $i$ . Then

$$V_i(u; b) = \sum_{j=1}^m V_{i,j}(u; b), \quad 0 \leq u \leq b, i \in E,$$

is the expected present value of the dividend payments before ruin given that the initial state is  $i$ .

Let  $\mathbf{V}(u; b) = (V_{i,j}(u; b))_{i,j=1}^m$  be an  $m \times m$  matrix. It follows from Li and Lu (2007) that

$$\mathbf{V}(u; b) = \mathbf{v}_\delta(u) [\mathbf{v}'_\delta(b)]^{-1}, \quad 0 \leq u \leq b, \tag{5.1}$$

where  $\mathbf{v}_\delta(u) = (v_{i,j}(u; \delta))_{i,j=1}^m$  is an  $m \times m$  matrix with  $v_{i,j}(u; \delta)$  satisfying the system of homogenous integro-differential equations

$$cv'_{i,j}(u; \delta) = (\lambda_i + \delta)v_{i,j}(u; \delta) - \lambda_i \int_0^u v_{i,j}(u-x; \delta) f_i(x) dx - \sum_{k=1}^m \alpha_{i,k} v_{k,j}(u; \delta), \tag{5.2}$$

with boundary conditions  $v_{i,j}(0; \delta) = I(i=j)$  for  $i, j \in E$ .

For the modified surplus process  $\{U_b(t); t \geq 0\}$ , we define  $\phi_{i,j}(u; b)$  for  $0 \leq u \leq b$  and  $i, j \in E$  to be the expected discounted penalty function at ruin if ruin is caused by a claim in state  $j$ , given initial surplus  $u$  and initial environment  $i \in E$ . In the definition of  $\phi_{i,j}(u; b)$ , we use the same  $\delta$  and penalty function  $w(x, y)$  as in  $\phi_{i,j}(u)$ . Then

$$\phi_i(u; b) = \sum_{j=1}^m \phi_{i,j}(u; b), \quad u \geq 0, \quad i \in E,$$

is the expected discounted penalty (Gerber-Shiu) function at ruin, given initial surplus  $u$  and initial environment  $i$ . In particular, when  $\delta = 0$  and  $w(x, y) = 1$ ,  $\phi_{i,j}(u; b)$  simplifies to  $\Psi_{i,j}(u; b)$  with the definition

$$\Psi_{i,j}(u; b) = \mathbb{P}_i(T_{u,b} < \infty, J(T_{u,b}) = j | U_b(0) = u), \quad i, j \in E,$$

where  $\Psi_{i,j}(u; b)$  is the ruin probability if ruin is caused by a claim in state  $j$  given that the initial state is  $i$  and hence

$$\Psi_i(u; b) = \sum_{j=1}^m \Psi_{i,j}(u; b), \quad i \in E,$$

is the probability of ruin given that the initial state is  $i$ .  $\Phi_i(u; b) = 1 - \Psi_i(u; b)$  is the non-ruin probability.

To construct the dividends-penalty identity, we define  $\tau_b$  to be the first time that the surplus  $U_b(t)$  reaches  $b$ , and for  $\delta > 0$ , define

$$L_{i,j}(u; b) = \mathbb{E}_i[e^{-\delta\tau_b} I(\tau_b < T_{u,b}, J(\tau_b) = j) | U_b(0) = u], \quad 0 \leq u \leq b, \quad i, j \in E.$$

$L_i(u; b)$  can be interpreted as the expected present value of one dollar payable at the time of reaching the barrier  $b$  in state  $j$  without ruin occurring, given that the initial environment state is  $i$  and the initial surplus is  $u$ . Alternatively, it can be viewed as the Laplace transform of the time to reach the dividend barrier  $b$  without ruin occurring, with respect to the parameter  $\delta$ .

Using the same arguments as in Li and Lu (2007), we can show that  $L_{i,j}(u)$  for  $i, j \in E$  and  $0 \leq u < b$  satisfy the following system of homogenous integro-differential equations

$$cL'_{i,j}(u; b) = (\lambda_i + \delta)L_{i,j}(u; b) - \lambda_i \int_0^u L_{i,j}(u-x; b) f_i(x) dx - \sum_{k=1}^m \alpha_{i,k} L_{k,j}(u; b),$$

with boundary conditions  $L_{i,j}(b; b) = I(i=j)$ .

Let  $\mathbf{L}(u; b) = (L_{i,j}(u; b))_{i,j=1}^m$  be an  $m \times m$  matrix. Following from the fact  $\mathbf{L}(b; b) = \mathbf{I}$ , we have

$$\mathbf{L}(u; b) = \mathbf{v}_\delta(u) \mathbf{L}(0; b) = \mathbf{v}_\delta(u) [\mathbf{v}_\delta(b)]^{-1}. \quad (5.3)$$



Using the same arguments as in Gerber et al. (2006), we have for  $0 \leq u < b$  that

$$\phi_{i,j}(u; b) - \phi_{i,j}(u) = \sum_{k=1}^m L_{i,k}(u; b) [\phi_{k,j}(b; b) - \phi_{k,j}(b)], \quad (5.4)$$

with boundary conditions

$$\phi'_{i,j}(b-; b) = 0, \quad i, j \in E.$$

Formula (5.4) can be obtained by reasoning as in Gerber et al. (2006): consider a particular sample path of the surplus process starting at  $u < b$  and in state  $i$ . The penalties at ruin if the ruin is caused by a claim in state  $j$ , with and without the dividend barrier, can be different only if the surplus reaches the level  $b$  in state  $k$  before ruin for  $k \in E$ . The boundary conditions  $\phi'_{i,j}(b-; b) = 0$  can be explained by heuristic reasoning as in Gerber et al. (2006).

In matrix notation, Eq. (5.4) and its boundary conditions can be expressed as

$$\begin{aligned} \boldsymbol{\phi}(u; b) - \boldsymbol{\phi}(u) &= \mathbf{L}(u; b) [\boldsymbol{\phi}(b; b) - \boldsymbol{\phi}(b)], \\ \boldsymbol{\phi}'(b-; b) &= \mathbf{0}, \end{aligned}$$

where  $\boldsymbol{\phi}(u; b) = (\phi_{i,j}(u; b))_{i,j=1}^m$ ,  $\boldsymbol{\phi}'(b-; b) = (\phi'_{i,j}(b-; b))_{i,j=1}^m$ , and  $\mathbf{0}$  is the  $m \times m$  zero matrix. Then

$$\begin{aligned} \boldsymbol{\phi}(b; b) - \boldsymbol{\phi}(b) &= -[\mathbf{L}'(b; b)]^{-1} \boldsymbol{\phi}'(b) = -[\mathbf{v}'_{\delta}(b) [\mathbf{v}_{\delta}(b)]^{-1}]^{-1} \boldsymbol{\phi}'(b) \\ &= -\mathbf{v}'_{\delta}(b) [\mathbf{v}'_{\delta}(b)]^{-1} \boldsymbol{\phi}'(b). \end{aligned} \quad (5.5)$$

Finally by Eqs. (5.1), (5.3), and (5.5) we have the following dividends-penalty identity in matrix form:

$$\begin{aligned} \boldsymbol{\phi}(u; b) &= \boldsymbol{\phi}(u) - \mathbf{L}(u; b) \mathbf{v}_{\delta}(b) [\mathbf{v}'_{\delta}(b)]^{-1} \boldsymbol{\phi}'(b) \\ &= \boldsymbol{\phi}(u) - \mathbf{v}_{\delta}(u) [\mathbf{v}_{\delta}(b)]^{-1} \mathbf{v}_{\delta}(b) [\mathbf{v}'_{\delta}(b)]^{-1} \boldsymbol{\phi}'(b) \\ &= \boldsymbol{\phi}(u) - \mathbf{V}(u; b) \boldsymbol{\phi}'(b), \quad 0 \leq u \leq b. \end{aligned} \quad (5.6)$$

We remark that  $\boldsymbol{\phi}(u)$  can be obtained explicitly for rational claim amounts as in Section 2. In particular, when  $\delta = 0$  and  $w(x, y) = 1$ , Eq. (5.6) simplifies to

$$\boldsymbol{\Psi}(u; b) = \boldsymbol{\Psi}(u) - \mathbf{V}(u; b) \boldsymbol{\Psi}'(b), \quad 0 \leq u \leq b. \quad (5.7)$$

Eq. (5.7) can be used to show that ruin is certain under the barrier strategy, i.e.,  $\Phi_i(u; b) = 0$  for  $i \in E$  and  $0 \leq u \leq b$ . Let  $\boldsymbol{\Phi}(u) = (\Phi_1(u), \Phi_2(u), \dots, \Phi_m(u))^T$  and  $\boldsymbol{\Phi}(u; b) = (\Phi_1(u; b), \Phi_2(u; b), \dots, \Phi_m(u; b))^T$  be  $m \times 1$  column vectors. Then

$$\begin{aligned} \boldsymbol{\Phi}(u; b) &= \mathbf{1} - \boldsymbol{\Psi}(u; b) \mathbf{1} = \mathbf{1} - \boldsymbol{\Psi}(u) \mathbf{1} + \mathbf{V}(u; b) \boldsymbol{\Psi}'(b) \mathbf{1} \\ &= \boldsymbol{\Phi}(u) - \mathbf{V}(u; b) \boldsymbol{\Phi}'(b), \quad 0 \leq u \leq b. \end{aligned} \quad (5.8)$$

Lu and Li (2005) show that  $\Phi_i(u)$  for  $i \in E$  satisfies the following system of integro-differential equations

$$c\Phi'_i(u) = \lambda_i\Phi_i(u) - \lambda_i \int_0^u \Phi_i(u-x)f_i(x)dx - \sum_{k=1}^m \alpha_{i,k} \Phi_k(u). \quad (5.9)$$

The solutions of the above system of equations are uniquely determined by the initial conditions  $\Phi_i(u)$  for  $i \in E$ . It follows from (5.2) that  $\sum_{j=1}^m v_{i,j}(u)\Phi_j(0)$  for  $i \in E$  has the same initial values as  $\Phi_i(u)$  and satisfies the system of integro-differential equations (5.9) when  $\delta = 0$ . Then we conclude that  $\Phi_i(u) = \sum_{j=1}^m v_{i,j}(u)\Phi_j(0)$ , or  $\Phi(u) = \mathbf{v}_0(u)\Phi(0)$ , and Eq. (5.8) simplifies to

$$\begin{aligned} \Phi(u; b) &= \mathbf{v}_0(u)\Phi(0) - \mathbf{V}(u; b)\mathbf{v}'_0(b)\Phi(0) \\ &= \mathbf{v}_0(u)\Phi(0) - \mathbf{v}_0(u)[\mathbf{v}'_0(b)]^{-1}\mathbf{v}'_0(b)\Phi(0) = \mathbf{0}, \quad 0 \leq u \leq b, \end{aligned}$$

where  $\mathbf{0}$  is the zero  $m \times 1$  column vector.

#### CONCLUDING REMARKS

By using matrix notation, we have shown how the evaluation of Gerber-Shiu's expected discounted penalty function for the classical risk model can be extended to a Markov-modulated risk model with claim amount distributions belonging to the rational family. We have generalized the results on the maximum surplus before ruin and maximum severity of ruin for the classical risk model to those for the Markov-modulated risk model. The dividends-penalty identity was first given in Lin et al. (2003) for the classical compound Poisson risk model. Gerber et al. (2006) extend the identity to the risk model with independent and stationary increments. The matrix form dividends-penalty identity is derived in this paper for the Markov-modulated risk model.

All the results obtained in Section 3-5 can be evaluated by the expected discounted penalty functions and the (decomposed) ruin probabilities studied in Section 2. Finally, all the results obtained in this paper can be extended to the case where premium rate  $c$  varies according to the state of the external environment process  $\{J(t); t \geq 0\}$ .

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