

FAIR VALUATION OF VARIOUS PARTICIPATION SCHEMES IN LIFE INSURANCE

BY

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ABSTRACT

Fair valuation is becoming a major concern for actuaries, especially in the perspective of IAS norms. One of the key aspects in this context is the simultaneous analysis of assets and liabilities in any sound actuarial valuation. The aim of this paper is to illustrate these concepts, by comparing three common ways of giving bonus in life insurance with profit: reversionary, cash or terminal. For each participation scheme, we compute the fair value of the contract taking into account liability parameters (guaranteed interest rate and participation level) as well as asset parameters (market conditions and investment strategy). We find some equilibrium conditions between all those coefficients and compare, from an analytical and numerical point of view, the systems of bonus. Developments are made first in the classical binomial model and then extended in a Black and Scholes economy.

KEYWORDS

Fair value, participation scheme, asset and liability management

1. INTRODUCTION

If for a long time life insurance could have been considered as a “sleeping beauty”, things have changed dramatically as well from a theoretical point of view as from industrial concerns. Nowadays, the financial risks involved in life insurance products are surely amongst the most important challenges for actuaries. The need to update our actuarial background taking into account the real financial world has been recently emphasized by Hans Bühlman in a recent editorial in *ASTIN BULLETIN* (Bühlman (2002)). The classical way of handling financial revenues in life insurance was characterized by two assumptions: stationarity of the market (no term structure of interest rate) and absence of uncertainty (deterministic approach); all this leading to the famous actuarial paradigm of the technical guaranteed rate: all the future was summarized in one magic number. Clearly things are not so simple and life insurance is a perfect example of stochastic process (even more than non life); the two dimensions of time and

uncertainty are completely involved in the products: long term aspect and financial risk (Norberg (2002)). The purpose of this paper is to focus on the stochastic aspect of the return and to show how to calibrate technical conditions of a life insurance product, using classical financial models. In this context fair valuation is the central topic. The coming introduction of accounting standards for insurance products will undoubtedly increase the importance of fair valuation of life contingencies. Even if mortality risk can deeply influence the profitability of a life insurer (for instance in the annuity market), it seems clear that the financial risk is “the heart of the matter”. Pricing and valuation of life insurance products in a stochastic financial environment started first for equity linked policies with a maturity guarantee (Brennan and Schwartz (1976), Delbaen (1986), Aase and Persson (1994), Nielsen and Sandmann (1995)). The newness of these products explains probably why historically they were the first studied with stochastic financial models. But more classical products, like insurance with profit, are at least as important for the insurance industry and induce also clearly major concerns in terms of financial risk. Different models of valuation of these contracts have been proposed based on the classical neutral approach in finance (Briys and De Varenne (1997), Bacinello (2001, 2003b)). The importance to develop, in this field, good models and to analyse all the embedded options, will surely increase with the IAS world (Grosen and Jorgensen (2000)). Even if for competitive or legal reasons the pricing of these products has very often limited degree of freedom, their valuation requires a complete understanding of various risks involved in each product.

Amongst them, the relation between the guaranteed rate and the participation level is of first importance, in close connection with the asset side. The aim of this paper is to focus on this aspect of participation, for the one hand by introducing some links between the liability conditions and the asset strategy and, for the other hand, by comparing three classical bonus systems. Two problems are presented: first if all the parameters of the contract are fixed (actuarial valuation of an existing product), formulas of fair value are given. Secondly, in order to design new parameters of a product, equilibrium conditions are developed.

The paper is organised as follows. Section 2 presents the main assumptions used in terms of assets and liabilities in a financial binomial environment and introduces three different participation systems: reversionary, cash and terminal bonus. In section 3, a first model on a single period is developed, showing how to compute fair values and equilibrium conditions in function of the chosen investment strategy. In this case, it appears clearly that the three participation schemes are identical. Then section 4 extends the fair valuation computation in a multiple period model; each participation system having then different valuation. Section 5 is devoted to the equilibrium condition in the multiple period; in particular, we show that the conditions are identical for the reversionary and the cash bonus even if in general fair values are different as seen in section 3. This motivates to take a deeper look on the relation between fair values in these two systems. Section 6 shows that cash systems have a bigger valuation when the contract is unbalanced in favour of the insurer and vice versa. Section 7 illustrates numerically the results and section 8 extends

the results in a continuous time framework, using the classical Black and Scholes economy. In this case, we show that explicit formulas for the fair values and the equilibrium value of the participation rate are still available; but implicit relations are only possible for the equilibrium value of the guaranteed rate.

2. NOTATIONS AND DEFINITIONS

We consider a life insurance contract with profit: in return for the initial payment of a premium, the policyholder obtains, at maturity, a guaranteed benefit plus a participation bonus, based on the eventual financial surplus generated by the underlying investments.

Insurer is supposed to be risk neutral with respect to mortality. Furthermore, we assume independence between mortality and financial elements.

The “asset” and the “liability” sides of the product can be characterized as follows:

2.1. Liability side

We consider a pure endowment policy with single premium π issued at time $t = 0$ and maturing at time T . The benefit is paid at maturity at time $t = T$ if and only if the insured is still alive. No surrender option exists before maturity.

We denote by i the guaranteed technical interest rate and by x the initial age.

The survival probability will be denoted by ${}_T p_x$. If we assume without loss of generality that the initial single premium is equal to ${}_T p_x$, the guaranteed benefit paid at maturity (disregarding loadings and taxes) is simply given by:

$$G(T) = \pi \frac{(1+i)^T}{{}_T p_x} = (1+i)^T$$

We introduce then the participation liability; we denote by B the participation rate on the financial surplus ($0 < B \leq 1$).

Three different participation schemes will be compared:

- “*Reversionary bonus*”: a bonus is computed yearly and used as a premium in order to buy additional insurance (pure endowment insurance increasing benefit only at maturity).
- “*Cash bonus*”: the bonus is also computed yearly but is not integrated in the contract. It can be paid back directly to the policyholder or transferred to another contract without guarantee.
- “*Terminal bonus*”: the bonus is only computed at the end of the contract, taking into account the final surplus.

We denote by B , B_C and B_T the corresponding participation rates.

2.2. Asset side

We assume perfectly competitive and frictionless markets, in a discrete time framework with one risky asset and one risk less asset.

The annual compounded risk free rate is supposed to be constant and will be denoted by r .

The risky asset is supposed to follow a binomial evolution (Cox-Ross-Rubinstein (1979)); two returns are possible at each period: a good one denoted by u and a bad one denoted by d .

In order to avoid any arbitrage opportunity, we suppose as usual:

$$d < 1 + r < u$$

These values can be also expressed alternatively in terms of risk premium and volatility:

$$u = 1 + r + \lambda + \mu \tag{1}$$

$$d = 1 + r + \lambda - \mu$$

where

λ : risk premium ($\lambda > 0$)

μ : volatility ($\mu > \lambda$)

On this market, the insurer is supposed to invest a part of the premium in the risky asset and the other part in the risk less asset. A strategy is then defined by a coefficient γ (with constraint $0 < \gamma \leq 1$) giving the risky part of the investment.

The return generated by a strategy γ is a random variable, taking one of two possible values denoted by u_γ and d_γ and defined as follows:

$$u_\gamma = \gamma u + (1 - \gamma)(1 + r) = 1 + r + \gamma(\lambda + \mu) \tag{2}$$

$$d_\gamma = \gamma d + (1 - \gamma)(1 + r) = 1 + r + \gamma(\lambda - \mu) \tag{3}$$

The case $\gamma = 0$ will not be taken into account.

3. A FIRST ONE PERIOD MODEL

We start with the very special case $T = 1$. Then by definition the three ways of giving bonus, as defined in section 2.1, are identical.

A contract is characterized by its vector of technical and financial parameters:

$$v = (i, B, \gamma) \tag{4}$$

The other parameters (r, u, d) can be seen as constraints of the market.

3.1. Fair valuation

Using the standard risk neutral approach in finance, the fair value of the contract can be expressed as the discounted expectation of its future cash flows, under the risk neutral measure, and taking into account the survival probability.

The risk neutral probabilities, associated respectively with the good return and the bad return, are given by:

$$\begin{aligned}
 p_1 &= \frac{1+r-d}{u-d} = \frac{\mu-\lambda}{2\mu} \\
 p_2 &= 1-p_1 = \frac{u-(1+r)}{u-d} = \frac{\mu+\lambda}{2\mu}
 \end{aligned}
 \tag{5}$$

The corresponding liabilities are respectively:

$$\begin{aligned}
 L_1 &= 1+i+B[u_\gamma-(1+i)]^+ = 1+i+B[r-i+\gamma(\lambda+\mu)]^+ \\
 L_2 &= 1+i+B[d_\gamma-(1+i)]^+ = 1+i+B[r-i+\gamma(\lambda-\mu)]^+
 \end{aligned}
 \tag{6}$$

The fair value of a contract v is then defined by:

$$\overline{FV}_0(v) = {}_1p_x FV_0(v)$$

with $FV_0(v)$ defined as the “financial” fair value of the benefit and given by:

$$\begin{aligned}
 FV_0(v) &= \frac{1}{1+r} \sum_{j=1}^2 p_j L_j = \\
 &= \frac{1}{1+r} \left[\frac{\mu-\lambda}{2\mu} [1+i+B[r-i+\gamma(\lambda+\mu)]^+] + \frac{\mu+\lambda}{2\mu} [1+i+B[r-i+\gamma(\lambda-\mu)]^+] \right] \\
 &= \frac{1}{1+r} \left[1+i+B \left[\frac{\mu-\lambda}{2\mu} [r-i+\gamma(\lambda+\mu)]^+ + \frac{\mu+\lambda}{2\mu} [r-i+\gamma(\lambda-\mu)]^+ \right] \right]
 \end{aligned}
 \tag{7}$$

The “financial” fair value consists of two parts:

$$FV_0(v) = GFV_0(v) + PFV_0(v)$$

with

$$GFV_0(v): \text{ fair value of the guarantee} = \frac{1+i}{1+r}
 \tag{8}$$

$PFV_0(v)$: fair value of the participation corresponding to a call option and given by:

$$PFV_0(v) = \frac{B}{1+r} \left[\frac{\mu-\lambda}{2\mu} P_1 + \frac{\mu+\lambda}{2\mu} P_2 \right]
 \tag{9}$$

with

$$\begin{aligned} P_1 &= [r - i + \gamma(\lambda + \mu)]^+ \\ P_2 &= [r - i + \gamma(\lambda - \mu)]^+ \end{aligned} \quad (10)$$

To go further, we have to make some assumptions on the values of P_1 and P_2 . Of course, we have $P_1 \geq P_2$.

The following situations can happen for a given contract:

$$\text{Case 1: } P_1 = P_2 = 0$$

$$\text{Case 2: } P_2 = 0 < P_1$$

$$\text{Case 3: } 0 < P_2 < P_1$$

Case 1: $P_1 = P_2 = 0$

In this case, the technical guaranteed rate is so big that no participation can be given.

In particular, $P_1 = 0$ implies:

$$r - i + \gamma(\lambda + \mu) \leq 0 \Rightarrow i \geq r + \gamma(\lambda + \mu) > r \quad (11)$$

The contract becomes purely deterministic, and the fair value is then:

$$FV_0(v) = \frac{1+i}{1+r} \quad (12)$$

Case 2: $P_2 = 0 < P_1$

This case can be considered as the realistic assumption:

- If the risky asset is “up”, there is a surplus and participation.
- But if the risky asset is “down”, the guarantee is playing and there is no participation.

The condition can be written as follows:

$$r + \gamma(\lambda - \mu) \leq i < r + \gamma(\lambda + \mu) \quad (13)$$

In this situation, the fair value becomes:

$$FV_0(v) = \frac{1}{1+r} \left[1 + i + B \frac{\mu - \lambda}{2\mu} (r - i + \gamma(\lambda + \mu)) \right] \quad (14)$$

Case 3: $0 < P_2 < P_1$

In this case, the technical guaranteed rate is so low that even in the down situation of the market, there is a surplus:

$$i < r - \gamma(\mu - \lambda) \quad (15)$$

The fair value becomes then:

$$\begin{aligned}
 FV_0(v) &= \frac{1}{1+r} \left[1+i+B \left[\frac{\mu-\lambda}{2\mu} (r-i+\gamma(\lambda+\mu)) + \frac{\mu+\lambda}{2\mu} (r-i+\gamma(\lambda-\mu)) \right] \right] \\
 &= \frac{1}{1+r} [1+i+B(r-i)]
 \end{aligned}
 \tag{16}$$

The fair value of the participation is just based on the difference between the risk free rate and the guaranteed rate.

3.2. Equilibrium

A vector v of parameters is said to be equilibrated if the corresponding initial fair value is equal to the single premium paid at time $t=0$:

$$\overline{FV}_0(v) = \pi = {}_1p_x \tag{17}$$

that is equivalent to $FV_0(v) = 1$. Since relation (7) implies:

$$FV_0(v) \geq \frac{1+i}{1+r}$$

we have a first general equilibrium condition given by:

$$i \leq r \tag{18}$$

So, in order to obtain equilibrium, the technical guaranteed rate must always be lower or equal to the risk free rate.

We try now to look at equilibrium situations in the three cases presented in section 3.1.

Case 1: $P_1 = P_2 = 0$

In this case, relation (11) shows that $i > r$, and no equilibrium is possible.

Case 2: $P_2 = 0 < P_1$

Taking into account simultaneously relation (18) and (13), we must have:

$$r + \gamma(\lambda - \mu) \leq i < r \tag{19}$$

The relation can be written in terms of equilibrium values of each of the parameters of the contract:

– *Guaranteed rate* (function of the participation rate and the strategy coefficient):

$$i = r - B\gamma \frac{\mu^2 - \lambda^2}{2\mu - B(\mu - \lambda)} \tag{20}$$

with conditions: $0 < B \leq 1$ and $0 < \gamma \leq 1$.

The condition (19) is well respected, taking into account these two conditions.

– *Participation rate* (function of the guaranteed rate and the strategy coefficient):

$$B = \frac{2\mu}{\mu - \lambda} \left[\frac{r - i}{r - i + \gamma(\lambda + \mu)} \right] \quad (21)$$

with conditions: $0 < \gamma \leq 1$ and $r - \gamma(\mu - \lambda) \leq i < r$.

– *Strategy coefficient* (function of the guaranteed rate and the participation rate):

The purpose here is to see if, for a couple of coherent values of the technical parameters i and B , there exists an asset strategy generating an equilibrium situation:

$$\gamma = \frac{r - i}{B} \frac{2\mu - B(\mu - \lambda)}{\mu^2 - \lambda^2} \quad (22)$$

with conditions: $0 < B \leq 1$

The condition on i is obtained by the condition $0 < \gamma \leq 1$:

a) $\gamma > 0$ for $i < r$ and $B < \frac{2\mu}{\mu - \lambda}$ (i.e. for all B since $\frac{2\mu}{\mu - \lambda} > 1$)

b) $\gamma \leq 1$ if $i \geq r - \frac{B(\mu^2 - \lambda^2)}{2\mu - B(\mu - \lambda)}$

Case 3: $0 < P_2 < P_1$

The equilibrium condition on the fair value becomes then, by (16):

$$1 = \frac{1}{1 + r} [1 + i + B(r - i)] \quad \text{or} \quad B(r - i) = r - i$$

Because in this case $i < r$, this implies $B = 1$

All this development shows that the real interesting situation with a non-trivial solution is the case 2.

4. FAIR VALUATION IN MULTIPLE PERIOD MODELS

We extend here the computation of fair values for a general maturity T .

The three participation schemes defined in section 2.1. must now be studied separately.

4.1. Reversionary bonus

Taking into account the binomial structure of the returns, the total benefit to be paid at maturity is a random variable given by $B(T) = L_1^j L_2^{T-j}$, where j is

the number of “up” cases of the risky asset amongst the T years and L_1 and L_2 are the corresponding total returns. The fair value is then given by:

$$\begin{aligned} \overline{FV}_0^T(v) &= {}_T P_x \left(\frac{1}{1+r} \right)^T \sum_{j=0}^T C_T^j p_1^j p_2^{T-j} L_1^j L_2^{T-j} = {}_T P_x \left(\frac{1}{1+r} \right)^T [p_1 L_1 + p_2 L_2]^T \\ &= {}_T P_x [FV_0(v)]^T \end{aligned} \tag{23}$$

where C_T^j denotes the binomial coefficient and $FV_0(v)$ is the financial fair value on one year (cf. (7)).

4.2. Cash bonus

In this case, each year, the rate of bonus is applied only to the reserve accumulated at rate i and taking into account the survival probabilities. For an initial single premium equal to $\pi = {}_T P_x$, the reserve $V(t)$ to use at time t is given by:

$$V(t) = \pi(1+i)^{t-1} / {}_t P_x = {}_T P_x (1+i)^{t-1} / {}_t P_x$$

The part of the liabilities to be paid at time t ($\forall t = 1, 2, \dots, T$) in case of survival as cash bonus is a random variable given by:

- in “up” case: $C_1(t) = B_c P_1 ({}_T P_x (1+i)^{t-1} / {}_t P_x)$
- in “down” case: $C_2(t) = B_c P_2 ({}_T P_x (1+i)^{t-1} / {}_t P_x)$

The fair value can be expressed as discounted expected value of all futures cash flows under the risk neutral measure and taking into account the survival probabilities:

$$\begin{aligned} \overline{FV}_0^T(v) &= {}_T P_x \left(\frac{1+i}{1+r} \right)^T + \sum_{t=1}^T \left(\frac{1}{1+r} \right)^t {}_t P_x (p_1 C_1(t) + p_2 C_2(t)) = \\ &= {}_T P_x \left(\frac{1+i}{1+r} \right)^T + B_c {}_T P_x \sum_{t=1}^T (p_1 P_1 + p_2 P_2) \frac{(1+i)^{t-1}}{(1+r)^t} \\ &= {}_T P_x FV_0^T(v) \end{aligned}$$

where $FV_0^T(v)$ is the “financial” fair value and is given by:

$$FV_0^T(v) = \left(\frac{1+i}{1+r} \right)^T + B_c (p_1 P_1 + p_2 P_2) \frac{1}{(1+r)^T} \frac{(1+r)^T - (1+i)^T}{r-i} \tag{24}$$

p_1, p_2 and P_1, P_2 being defined in (5) and (10)

In the particular case $P_2 = 0; P_1 > 0$, this gives:

$$FV_0^T(v) = \frac{1}{(1+r)^T} \left[(1+i)^T + B_c \frac{(\mu - \lambda)(r - i + \gamma(\lambda + \mu))}{2\mu} \frac{(1+r)^T - (1+i)^T}{r-i} \right] \tag{25}$$

The corresponding value for the reversionary bonus is (cf. (23)):

$$FV_0^T(v) = \frac{1}{(1+r)^T} \left[1 + i + B \frac{(\mu - \lambda)(r - i + \gamma(\lambda + \mu))}{2\mu} \right]^T \tag{26}$$

As expected, for $T = 1$ these two values are identical. A deeper comparison between (25) and (26) will be developed in section 6, after calculation of equilibrium values in section 5.

4.3. Terminal bonus

The bonus is only computed at the end of the contract, comparing the final technical liability $(1 + i)^T$ with the asset value at maturity.

If we denote by $A_\gamma(T)$ the terminal assets at time $t = T$, using strategy γ , the fair value can be written as follows:

$$\begin{aligned} \overline{FV}_0^T(v) &= \frac{1}{(1+r)^T} {}_T p_x \left[(1+i)^T + B_T E_Q [A_\gamma(T) - (1+i)^T]^+ \right] \\ &= {}_T p_x FV_0^T(v) \end{aligned} \tag{27}$$

where Q is the risk neutral measure and E_Q denotes expectation under Q .

Alternatively, we can express this fair value in terms of option price:

$$FV_0^T(v) = \frac{(1+i)^T}{(1+r)^T} + B_T C(A_\gamma; T; (1+i)^T)$$

where $C(A_\gamma; T; (1+i)^T)$ is the price of a call option on the asset A_γ , with maturity T and strike price $(1+i)^T$

Taking into account the binomial structure of the model, the general form of this price is given by:

$$C(A_\gamma; T; (1+i)^T) = \frac{1}{(1+r)^T} \sum_{j=0}^T \left[u_\gamma^j d_\gamma^{T-j} - (1+i)^T \right]^+ C_T^j p_1^j p_2^{T-j}$$

The fair value is then given explicitly by:

$$FV_0^T(v) = \frac{(1+i)^T}{(1+r)^T} + \frac{B_T}{(1+r)^T} \sum_{j=0}^T \left[u_\gamma^j d_\gamma^{T-j} - (1+i)^T \right]^+ C_T^j p_1^j p_2^{T-j} \tag{28}$$

As example, let us look at the model on two periods of time ($T = 2$).

If we denote by a the minimal number of jumps in order to give participation:

$a = \inf \{ j \in N : (1+r + \gamma(\lambda + \mu))^j (1+r + \gamma(\lambda - \mu))^{2-j} > (1+i)^2 \}$, the following situations can happen:

➤ *First situation: $a > 2$*

The technical guaranteed rate is so big that no participation can be given $[(1 + i)^2 \geq u_y^2]$. The fair value is just:

$$FV_0^2(v) = \left[\frac{1+i}{1+r} \right]^T$$

➤ *Second situation: $a = 2$*

The only case where bonus can be given is the situation of two “up” jumps of the risky asset:

$$u_y d_y \leq (1+i)^2 < u_y^2 \tag{29}$$

The fair value becomes then:

$$FV_0^2(v) = \frac{1}{(1+r)^2} \left[(1+i)^2 + B_T \left(\frac{\mu - \lambda}{2\mu} \right)^2 [u_y^2 - (1+i)^2] \right]$$

➤ *Third situation: $a = 1$*

As soon as there is one “up” jump on the two periods, bonus is given:

$$d_y^2 \leq (1+i)^2 < u_y d_y \tag{30}$$

The fair value becomes then:

$$FV_0^2(v) = \frac{1}{(1+r)^2} \left[(1+i)^2 + B_T \left[\left(\frac{\mu - \lambda}{2\mu} \right)^2 (u_y^2 - (1+i)^2) + 2 \frac{\mu^2 - \lambda^2}{(2\mu)^2} (u_y d_y - (1+i)^2) \right] \right] \tag{31}$$

➤ *Fourth situation: $a = 0$*

A bonus is given each year whatever are the returns of the asset:

$$d_y^2 > (1+i)^2 \tag{32}$$

The fair value is simply:

$$FV_0^2(v) = \frac{(1+i)^2}{(1+r)^2} + B_T \left[1 - \left(\frac{1+i}{1+r} \right)^2 \right] \tag{33}$$

The first and the last situations can be considered as degenerate. If we want to avoid these limit situations, we must have:

$$d_y^2 \leq (1+i)^2 < u_y^2 \tag{34}$$

We are then in situation 2 or situation 3.

Remark: It is easy to see that we will be automatically in situation 3 ($a = 1$) if:

$$\mu^2 < \lambda^2 + 2\lambda(1 + r) \quad (35)$$

and

$$i < r$$

- i. The first condition (35) is independent of the characteristics of the contract; it just means that the volatility has not to be too important.
- ii. The second condition [$i < r$], related to the contract, seems to be quite reasonable (cf. (18) on one period).

5. EQUILIBRIUM RELATION IN MULTIPLE PERIOD MODELS

The aim of this section is to generalize on T periods the equilibrium relations obtained in section 3.2. for one period, using the explicit formulas of fair value obtained in section 4.

5.1. Reversionary bonus

Formula (23) shows clearly that:

$$FV_0^T(v) = 1 \Leftrightarrow FV_0(v) = 1$$

Equilibrium results obtained in section 3.2. are unchanged in a multiple period model with reversionary bonus.

5.2. Cash bonus

Formulas (25) and (26), for instance, show that the fair values are normally different using the reversionary bonus or the cash bonus. Nevertheless, we will see that the equilibrium values of the parameters of the contract are the same. Using (24), the contract will be equilibrated in the cash bonus scheme if:

$$(1 + r)^T = (1 + i)^T + B_C(p_1P_1 + p_2P_2) \frac{[(1 + r)^T - (1 + i)^T]}{(r - i)} \quad (36)$$

Like in section 3.2., we can consider different cases, depending on the values of P_1 and P_2 .

Case 1: $P_1 = P_2 = 0$

No equilibrium is possible like in the reversionary scheme.

Case 2: $P_2 = 0 < P_1$ (central assumption)

Using the values (5) and (10) of P_1 and P_2 , formula (36) becomes:

$$(1+r)^T = (1+i)^T + B_C \frac{\mu - \lambda}{2\mu} (r - i + \gamma(\lambda + \mu)) \frac{[(1+r)^T - (1+i)^T]}{(r-i)}$$

or
$$r - i = B_C \frac{\mu - \lambda}{2\mu} [r - i + \gamma(\lambda + \mu)]$$

or
$$i = r - \gamma B_C \frac{\mu^2 - \lambda^2}{2\mu - B_C(\mu - \lambda)} \tag{37}$$

identical to relation (20) obtained for the reversionary bonus.

Case 3: $0 < P_2 < P_1$

Formula (36) becomes then:

$$(1+r)^T = (1+i)^T + B_C [(1+r)^T - (1+i)^T]$$

which implies like in the reversionary case: $B_C = 1$

Conclusion: The equilibrium conditions on the parameters of the contract are the same for the reversionary bonus and for the cash bonus.

5.3. Terminal bonus

Formula (28) gives as equilibrium condition:

$$1 = \frac{(1+i)^T}{(1+r)^T} + \frac{B_T}{(1+r)^T} \sum_{j=0}^T [u_\gamma^j d_\gamma^{T-j} - (1+i)^T]^+ C_T^j p_1^j p_2^{T-j} \tag{38}$$

From this relation, it is already possible to obtain an equilibrium value for the participation rate, as a function of the guaranteed rate and the strategy coefficient (to be compared with formula (21) for the reversionary bonus):

$$B_T = \frac{(1+r)^T - (1+i)^T}{\sum_{j=0}^T [u_\gamma^j d_\gamma^{T-j} - (1+i)^T]^+ C_T^j p_1^j p_2^{T-j}} \tag{39}$$

On the other hand, if the participation rate is known, it is possible to express the equilibrium value of the guaranteed rate as follows:

$$\begin{aligned}
 (1+i)^T &= \frac{(1+r)^T - B_T \sum_{j=a}^T u_\gamma^j d_\gamma^{T-j} C_T^j p_1^j p_2^{T-j}}{1 - B_T \sum_{j=a}^T C_T^j p_1^j p_2^{T-j}} = \\
 &= \frac{(1+r)^T - B_T \sum_{j=a}^T C_T^j u_\gamma^j d_\gamma^{T-j} \left(\frac{\mu-\lambda}{2\mu}\right)^j \left(\frac{\mu+\lambda}{2\mu}\right)^{T-j}}{1 - B_T \sum_{j=a}^T C_T^j \left(\frac{\mu-\lambda}{2\mu}\right)^j \left(\frac{\mu+\lambda}{2\mu}\right)^{T-j}}
 \end{aligned}
 \tag{40}$$

where a is defined as usual by:

$$a = \inf\{k \in N : u_\gamma^k d_\gamma^{T-k} > (1+i)^T\}$$

Unfortunately, this relation is not explicit because the coefficient a depends on the level of i . The relation can be computed, assuming a certain value for a and then check afterwards if the condition on a is fulfilled. As example, let us see again what happens on two periods.

We first concentrate on the standard situation $a = 1$; this means that constraint (30) has to be fulfilled. Starting from (40), we will get an equilibrium candidate for i and check afterwards this constraint.

For $T = 2$ and $a = 1$, we get:

$$(1+i)^2 = \frac{(1+r)^2 - B_T \left[\left(\frac{\mu-\lambda}{2\mu}\right)^2 u_\gamma^2 + 2 \frac{\mu^2 - \lambda^2}{(2\mu)^2} u_\gamma d_\gamma \right]}{1 - B_T \left[\left(\frac{\mu-\lambda}{2\mu}\right)^2 + 2 \frac{\mu^2 - \lambda^2}{(2\mu)^2} \right]}
 \tag{41}$$

with the following condition to check:

$$d_\gamma^2 \leq (1+i)^2 < u_\gamma d_\gamma$$

Similarly, we can try to find a candidate for the situation $a = 2$; now constraint (29) has to be checked.

Using the same approach, the equilibrium value is given by:

$$(1+i)^2 = \frac{(1+r)^2 - B_T \left(\frac{\mu-\lambda}{2\mu}\right)^2 u_\gamma^2}{1 - B_T \left(\frac{\mu-\lambda}{2\mu}\right)^2}
 \tag{42}$$

with the following condition to check: $u_\gamma d_\gamma \leq (1+i)^2 < u_\gamma^2$

6. ANALYTICAL COMPARISON BETWEEN REVERSIONARY AND CASH SYSTEMS

Reversionary and cash systems seem not so different; the only difference is the integration of the bonus inside the contract. Moreover, we saw in section 5 that the equilibrium conditions are the same, even if the fair value formulas look quite different. From now on, and without loss of generality, we will work without mortality effect.

The aim of this section is to prove the following relation:

$FV(\text{reversionary}) > FV(\text{cash})$ if and only if the participation rate is greater than its equilibrium value.

In order to get this relation, we have to compare the valuations in the two participation schemes using the same parameters:

– For the reversionary bonus (cf. (23)):

$$FV_0^T(v) = \left(\frac{1}{1+r}\right)^T [p_1L_1 + p_2L_2]^T = \left(\frac{1}{1+r}\right)^T [(1+i) + BK]^T$$

with $K = [p_1P_1 + p_2P_2]$

– For the cash bonus (cf. (24)):

$$FV_0^T(v) = \left(\frac{1+i}{1+r}\right)^T + B_cK \frac{1}{(1+r)^T} \frac{(1+r)^T - (1+i)^T}{r-i}$$

Developing the reversionary formula gives:

$$FV_0^T(v) = \frac{1}{(1+r)^T} \left[(1+i)^T + \sum_{j=1}^T C_T^j (BK)^j (1+i)^{T-j} \right]$$

The condition to have a bigger fair value for the reversionary system becomes then:

$$\frac{(1+i)^T}{(1+r)^T} + \frac{1}{(1+r)^T} \sum_{j=1}^T C_T^j (BK)^j (1+i)^{T-j} > \left(\frac{1+i}{1+r}\right)^T + BK \frac{1}{(1+r)^T} \frac{(1+r)^T - (1+i)^T}{r-i}$$

Or:

$$\sum_{j=1}^T C_T^j (BK)^{j-1} (1+i)^{T-j} > \frac{(1+r)^T - (1+i)^T}{r-i}$$

Or assuming $K \neq 0$ (i.e. $P_1 > 0$):

$$\frac{\sum_{j=0}^T C_T^j (BK)^j (1+i)^{T-j} - (1+i)^T}{BK} = \frac{(BK + 1+i)^T - (1+i)^T}{BK} > \frac{(1+r)^T - (1+i)^T}{r-i} \tag{43}$$

Let us consider the function:

$$G(x) = \frac{(x+1+i)^T - (1+i)^T}{x} \quad \text{with } T \geq 2$$

For $x = r - i$, we have: $G(r - i) = \frac{(1+r)^T - (1+i)^T}{r - i}$

On the other hand, it is easy to show that for $x > 0$, the function G is strictly increasing. So when condition $BK > r - i$ is fulfilled, the fair value for the rever- sionary bonus is bigger than the fair value for the cash bonus and vice versa. This last condition can be written as follows:

$$B[p_1P_1 + p_2P_2] > r - i \quad (44)$$

Taking into account the different cases studied in section 3.1, condition (44) becomes:

Case 1: $P_1 = P_2 = 0$: not relevant here ($K \neq 0$).

Case 2: $P_2 = 0 < P_1$:

$$Bp_1P_1 = B \frac{\mu - \lambda}{2\mu} [r - i + \gamma(\lambda + \mu)] > r - i$$

or $B > \frac{2\mu}{\mu - \lambda} \left[\frac{r - i}{r - i + \gamma(\lambda + \mu)} \right]$

which means that the participation rate is bigger than its equilibrium value (cf. (21)).

Case 3: $0 < P_2 < P_1$:

$$B(p_1P_1 + p_2P_2) = B(r - i) > r - i$$

Or $B > 1$ that is the equilibrium value in that case.

7. NUMERICAL ILLUSTRATION

In this section, we will compare the three participation schemes in terms of fair values and equilibrium values of the parameters in a two periods model and without mortality effect.

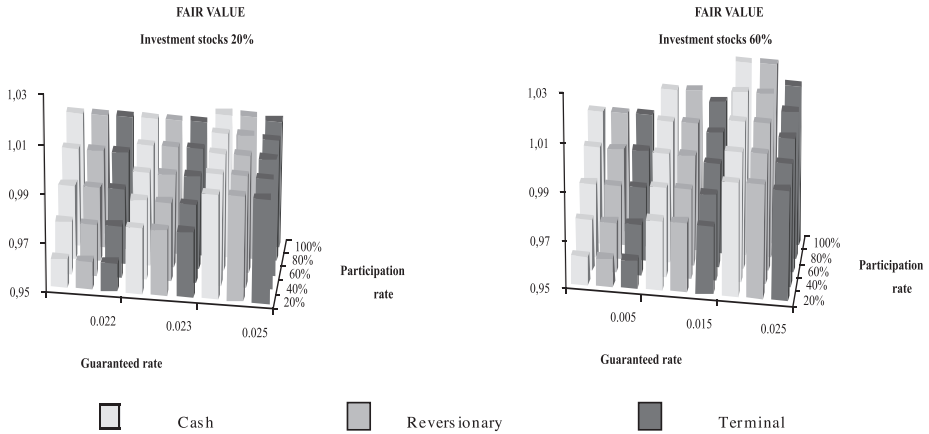
In terms of *financial market*, we will use a central scenario based on the fol- lowing values:

$$r = \text{riskfree rate} = 0.03$$

$$\lambda = \text{risk premium} = 0.02$$

$$\mu = \text{volatility} = 0.06$$

FIGURE 1



In terms of *investment strategy* underlying the product, we will mainly compare two choices:

$\gamma = 0.20$: “conservative” strategy

$\gamma = 0.60$: “aggressive” strategy

7.1. Fair values for different guarantees and participation levels

Figure 1 shows for each chosen strategy (conservative or aggressive) the relation between the guaranteed rate, the participation level and the fair value of the contract. The fair value is clearly an increasing function of the participation level, whatever is the participation scheme.

7.2. Equilibrium values

Like seen in section 5.2, the equilibrium conditions on the parameters of the contract are the same for the reversionary bonus and for the cash bonus. Figure 2 compares, for the aggressive investment strategy, the equilibrium values of the participation rate and of the guaranteed rate between reversionary and terminal bonus.

7.3. Fair value and equilibrium value

Table 1 compares for various values of the technical parameters, chosen in relation with their equilibrium values, the fair values in the three participation schemes for the aggressive investment strategy (60% in risky asset). Figures in bold and italic correspond to situations where the initial fair value is bigger than the paid premium.

FIGURE 2

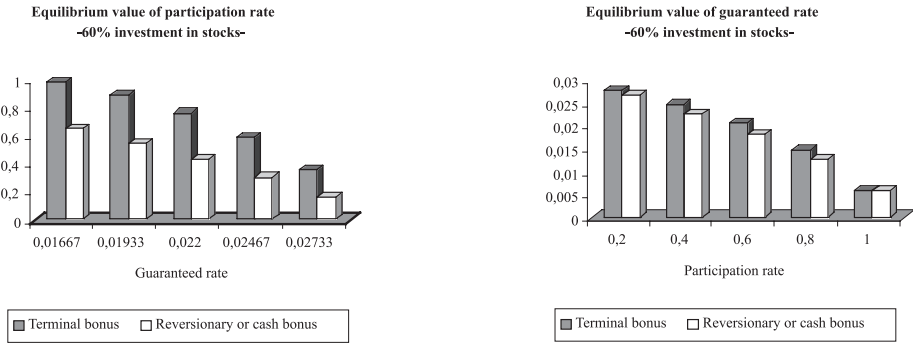


TABLE 1

$\gamma = 60\%$		$i = 0.025$	$i = 0.015$	$i = 0.006$
Equilibrium values of interest rate	<i>B</i>	TERMINAL BONUS		
0,0276	20%	0,9954851	0,9783926	0,9631527
0,0245	40%	1,0006553	0,9856995	0,9723645
0,0204	60%	1,0058256	0,9930063	0,9815763
0,0147	80%	1,0109958	1,0003131	0,9907881
0,006	100%	1,0161661	1,0076199	1
		REVERSIONARY BONUS		
0,02657	20%	0,9971541	0,9791391	0,9615420
0,02261	40%	1,0040169	0,9872255	0,9710858
0,018	60%	1,0109033	0,9953452	0,9806767
0,01254	80%	1,0178132	1,0034982	0,9903148
0,006	100%	1,0247467	1,0116844	1
		CASH BONUS		
0,02657	20%	0,9971591	0,9791818	0,9616363
0,02261	40%	1,0040032	0,9872777	0,9712272
0,018	60%	1,0108473	0,9953737	0,9808181
0,01254	80%	1,0176915	1,0034696	0,9904091
0,006	100%	1,0245357	1,0115656	1

Clearly, the difference between the different bonus schemes would be much more pronounced in models on more than two periods.

8. GENERALIZATION IN CONTINUOUS TIME MARKET

The principles of comparison between the different participation schemes can be easily adapted in a continuous time financial market. We develop here the

model, using the Black and Scholes environment. The classical assumptions on the market are supposed to be fulfilled. Two kinds of assets are supposed to exist:

- the risk less asset X_1 , linked to the risk free rate:

$$dX_1(t) = \bar{r}X_1(t)dt$$

where $\bar{r} = \ln(1 + r)$ is the instantaneous risk free rate.

- the risky fund X_2 , modelled by a geometric Brownian motion:

$$dX_2(t) = \eta X_2(t)dt + \sigma X_2(t)dw(t)$$

where w is a Wiener process.

Once again, the risk neutral probability measure will be denoted by Q .

The reference portfolio of the insurer consists of a constant proportion γ invested in the risky fund and a proportion $(1 - \gamma)$ invested in the risk free asset. The evolution equation of this portfolio denoted by X_γ is then given by:

$$dX_\gamma(t) = (\gamma\eta + (1 - \gamma)\bar{r})X_\gamma(t)dt + \gamma\sigma X_\gamma(t)dw(t) \tag{45}$$

8.1. The one period model

We extend here the results of section 3 obtained in a binomial environment. On one period, the three kinds of participation schemes are identical. We compute the fair values for a given contract $v = (i, B, \gamma)$ and obtain equilibrium conditions on the coefficients in order to have a fair contract.

The fair value is given now by:

$$\overline{FV}_0(v) = {}_1p_x \left(\frac{1+i}{1+r} + B c(i, \gamma, 1) \right) \tag{46}$$

where $c(i, \gamma, 1)$ represents the price of a call option on the reference portfolio X_γ for one period and for a strike price equal to the guarantee $(1 + i)$.

In the Black and Scholes environment this price is given by:

$$\begin{aligned} c(i, \gamma, 1) &= \frac{1}{1+r} E_Q \left((X_\gamma(1) - (1+i))^+ \right) \\ &= \Phi(d_1(i, \gamma, 1)) - \frac{1+i}{1+r} \Phi(d_2(i, \gamma, 1)) \end{aligned}$$

where

$$\begin{aligned} d_1(i, \gamma, 1) &= \left(\ln \left(\frac{1+r}{1+i} \right) + \frac{1}{2} \gamma^2 \sigma^2 \right) / \gamma\sigma \\ d_2(i, \gamma, 1) &= d_1(i, \gamma, 1) - \gamma\sigma \end{aligned}$$

Φ is the cumulative distribution function of a standard normal variable.

Finally the fair value can be written as follows:

$$\overline{FV}_0(v) = {}_1p_x \left(\frac{1+i}{1+r} + B \left(\Phi(d_1(i, \gamma, 1)) - \frac{1+i}{1+r} \Phi(d_2(i, \gamma, 1)) \right) \right) \quad (47)$$

The equilibrium condition given by (17) can be expressed, in this model, as an explicit equilibrium value B of the participation rate, for a given guaranteed rate i and a given strategy coefficient γ (equivalent of formula (21) in the binomial model):

$$B = \frac{r - i}{(1+r)\Phi(d_1(i, \gamma, 1)) - (1+i)\Phi(d_2(i, \gamma, 1))} \quad (48)$$

Implicit relations can only be obtained in this model if we want to solve it for the two other parameters (equilibrium value respectively for the guaranteed rate and for the strategy coefficient).

8.2. Fair value in multiple period models

8.2.1. Reversionary bonus

Exactly as in the discrete case and taking into account the structure of the return process, the fair value for a contract of T periods using a reversionary bonus is given by:

$$\overline{FV}_0^T(v) = {}_T p_x \left(\frac{1+i}{1+r} + B \left(\Phi(d_1(i, \gamma, 1)) - \frac{1+i}{1+r} \Phi(d_2(i, \gamma, 1)) \right) \right)^T \quad (49)$$

8.2.2. Cash Bonus

The fair value is expressed as the discounted expected value of all future cash flows under the risk neutral measure Q and the survival probabilities:

$$\overline{FV}_0^T(v) = {}_T p_x \left(\frac{1+i}{1+r} \right)^T + \sum_{t=1}^T {}_t p_x \frac{E_Q(CB(t))}{(1+r)^t}$$

where $CB(t)$ is the cash bonus paid at time t :

$$E_Q(CB(t)) = B_c \left((1+i)^{t-1} \frac{{}_T p_x}{{}_t p_x} \right) (1+r)^T c(i, \gamma, 1)$$

Finally, the fair value is given by:

$$\overline{FV}_0^T(v) = {}_T p_x \left(\left(\frac{1+i}{1+r} \right)^T + B_c (1+r)^T c(i, \gamma, 1) \frac{1}{(1+r)^T} \frac{(1+r)^T - (1+i)^T}{r-i} \right) \quad (50)$$

8.2.3. Terminal Bonus

The fair value will have the same structure as in the one period model:

$$\overline{FV}_0^T(v) = {}_T P_x \left(\left(\frac{1+i}{1+r} \right)^T + B_T c(i, \gamma, T) \right) \tag{51}$$

with:

$$\begin{aligned} c(i, \gamma, T) &= \left(\frac{1}{1+r} \right)^T E_Q \left((X_T(T) - (1+i)^T)^+ \right) \\ &= \Phi(d_1(i, \gamma, T)) - \left(\frac{1+i}{1+r} \right)^T \Phi(d_2(i, \gamma, T)) \end{aligned}$$

where

$$\begin{aligned} d_1(i, \gamma, T) &= \sqrt{T} \left(\ln \left(\frac{1+r}{1+i} \right) + \frac{1}{2} \gamma^2 \sigma^2 \right) / \gamma \sigma \\ d_2(i, \gamma, T) &= d_1(i, \gamma, T) - \gamma \sigma \sqrt{T} \end{aligned}$$

8.3. Equilibrium relation in multiple period models

8.3.1. *Reversionary Bonus*

According to formula (49) and like in the binomial model, the equilibrium condition is the same as in the one period model.

8.3.2. *Cash Bonus*

Using formula (50), the equilibrium condition becomes for the cash bonus:

$$1 = \left(\frac{1+i}{1+r} \right)^T + B_C (1+r) c(i, \gamma, 1) \frac{(1+r)^T - (1+i)^T}{(1+r)^T (r-i)}$$

or:
$$B_C = \frac{r-i}{(1+r) c(i, \gamma, 1)} = \frac{r-i}{(1+r) \Phi(d_1(i, \gamma, 1)) - (1+i) \Phi(d_2(i, \gamma, 1))}$$

which is again equal to the equilibrium value on one period (cf. (48)).

8.3.3. *Terminal Bonus*

Using formula (51), the equilibrium condition for the terminal bonus becomes:

$$B_T = \frac{(1+r)^T - (1+i)^T}{(1+r)^T \Phi(d_1(i, \gamma, T)) - (1+i)^T \Phi(d_2(i, \gamma, T))} \tag{52}$$

8.4. Comparison between reversionary and cash systems

A same methodology as in section 6 can be used in order to obtain a ranking between fair values for reversionary and cash bonus when the parameters are

not in equilibrium. Indeed using respectively formulas (49) and (50), the fair values can be written as follows:

– in the reversionary case:

$$\overline{FV}_0^T(v) = {}_T P_x \frac{1}{(1+r)^T} (1+i + BK^*)^T$$

with: $K^* = (1+r)c(i, \gamma, 1)$

– in the cash case :

$$\overline{FV}_0^T(v) = {}_T P_x \left(\left(\frac{1+i}{1+r} \right)^T + B_c K^* \frac{1}{(1+r)^T} \frac{(1+r)^T - (1+i)^T}{(r-i)} \right)$$

which have exactly the same form as in the binomial case. So the same conclusion can be drawn.

9. CONCLUSION

In this paper, we have developed various formulations in order to compare the fair value for life insurance products based on three participation schemes: reversionary, cash and terminal bonus, taking into account simultaneously the asset side and the liability side in a multiple period model.

We have shown that the fair value depends on the investment strategy (and on the associated risk), on the participation level and on the guaranteed rate but also on the bonus system chosen. We have found some explicit equilibrium conditions between all these parameters.

A deep comparison has been made between the three participation schemes, as well in terms of computation of the fair value as in the equilibrium conditions. Using first a binomial model, we have obtained closed forms and given clear interpretations on the link between the market conditions, the volatility of the assets and the parameters of the product. A same approach, leading to similar conclusions, has been proposed in a time continuous model. The model could be also extended in order to take into account other aspects like surrender options, periodical premiums or the longevity risk.

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