ON THE USE OF CONDITIONAL SPECIFICATION MODELS IN CLAIM COUNT DISTRIBUTIONS: AN APPLICATION TO BONUS-MALUS SYSTEMS

BY

JOSÉ MARÍA SARABIA[1], EMILIO GÓMEZ-DÉNIZ[2]
AND FRANCISCO J. VÁZQUEZ-POLO[2]

ABSTRACT

In this paper a new methodology using the conditional specification technique intoduced by Arnold et al. (1999) is used to obtain bonus-malus premiums. A Poisson distribution for which the parameter is a function of the classical structure parameter is used and a new class of prior distribution arises in a natural way. This model contains, as a particular case, the classical compound Poisson model and is found to be much more robust than earlier ones. An example is given to illustrate our ideas.

KEYWORDS

Conditional Specification Model, Generalized Negative Binomial Distribution, Bonus-Malus systems.

1. INTRODUCTION

Many actuarial papers are based on the distribution of annual claim numbers (Lemaire, 1995, Dionne and Vanasse, 1989; among others). One of the most popular models in this context is the Poisson-Gamma model, which describes the behaviour of premiums to be charged when heterogeneity in the portfolio is considered not to vary with time. The classical Poisson-Gamma model is based on the assumptions that the portfolio is heterogeneous and that all policyholders have constant but unequal underlying risks of having an accident. These assumptions can be modelled assuming:

- given $\Theta = \theta$, the number of claims X are independent and distributed conforming to the Poisson distribution with mean θ,

[1] Department of Economics, University of Cantabria, 39005-Santander, Spain. E-mail: sarabiaj@unican.es
[2] Department of Quantitative Methods in Economics, University of Las Palmas de Gran Canaria, 35017-Las Palmas de G.C., Spain. E-mail: {egomez or fjvpolo}@dmc.ulpgc.es

- Θ has a probability density function (structure function) of the Gamma family
- random variables Θ and X are independent.

Under the above hypothesis, it is straightforward to prove that the unconditional distribution of the number of claims follows a Negative Binomial distribution. In fact, in the context of count regression models, the Negative Binomial distribution can be thought of as a Poisson distribution with unobserved heterogeneity. This, in turn, can be conceptualised as a mixture of two probability distributions, Poisson and Gamma. However, since the Poisson model for count data imposes the restriction that the conditional mean should equal the conditional variance, the Negative Binomial distribution derived can be criticized because the random count process should exhibit overdispersion (i.e. distributions of counts often have variance greater than the mean). Thus, alternative models to the standard Poisson distribution have been proposed; these mainly focus on the well-known generalized Poisson distribution (GPD) introduced in Consul and Jain (1973), and extensively developed in other papers, such as Goovaerts and Kaas (1991), Scollnik (1995), among others. Since empirical evidence (Sichel, 1971 and Willmot, 1987, among others) suggests that overdispersion is a common feature of claim count data, in this paper we take the Poisson-Gamma (or equivalently, the Negative Binomial) model as the benchmark and consider generalization based on it. As is well known, a mixed Poisson distribution can be developed by a conjugate Bayesian model. As in all Bayesian conjugate analysis, the choice of the structure function is motivated by mathematical convenience and perhaps also, in actuarial practice, by the goodness of fit of the Negative Binomial distribution (Panjer and Willmot, 1988). However, Gómez *et al.* (2002) recently investigated the effect of prior elicitation in bonus-malus premiums and showed how the choice of the prior can critically affect the relative premiums.

In this paper we derive a new distribution for prediction of the count data process. As a generalization of this scenario, we seek the most general distribution of a bivariate random variable (X, Θ) such that its conditional distributions are Poisson and Gamma. In this sense, we refer to such a distribution as a generalized negative binomial distribution, which is different from that derived in Gerber (1991). This new class of distribution arises in a natural way, using the conditional specification from an exponential family (CEF) (see Arnold *et al.*, 1999). This new family of distributions is very flexible and contains, as particular cases, many other distributions proposed in the literature. This model is discussed in the next section. Basic properties, like conditionals and marginal distributions, are derived in Section 3, which also includes equations to obtain maximum likelihood estimations. Sections 4 and 5 contain numerical illustrations and conclusions, and a discussion of related work, respectively.

1.1. Notation

Let us establish the general notation used below:

Poisson Distribution ($\lambda > 0$):

$$X \sim \mathcal{P}(\lambda) \Longleftrightarrow \Pr(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x = 0,1,2,\dots \tag{1}$$

Gamma Distribution ($\alpha, \beta > 0$):

$$X \sim \mathcal{G}(\alpha,\beta) \Longleftrightarrow f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \ x > 0 \tag{2}$$

Negative Binomial Distribution ($r > 0,\ 0 < p < 1,\ q = 1 - p$):

$$X \sim \mathcal{NB}(r,p) \Longleftrightarrow \Pr(X = x) = \frac{\Gamma(x+r)}{\Gamma(x+1)\Gamma(r)} p^r q^x, \quad x = 0,1,2,\dots \tag{3}$$

2. Conditional specification models

A bivariate distribution can be specified through its conditional distributions. If we assume that conditional distributions belong to certain parametric families of distributions, it is possible to obtain the joint distribution using the methodology described in Arnold, Castillo and Sarabia (1999). See also Arnold, Castillo and Sarabia (2001) for an introduction to this topic. To obtain the joint distribution, it is necessary to determine the resolution of certain functional equations. By means of this methodology, highly flexible multiparametric distributions are obtained. Formally, we seek the more general bivariate distribution (X, Θ) whose conditional distributions satisfy:

$$X | \Theta = \theta \sim \mathcal{P}(\lambda(\theta)) \tag{4}$$

and

$$\Theta | X = x \sim \mathcal{G}(\alpha(x), \beta(x)) \tag{5}$$

where $\lambda(\theta)$: $\mathbb{R}^+ \to \mathbb{R}^+$, and $\alpha(x), \beta(x)$: $\mathbb{N} \to \mathbb{R}^+$. This leads us to the following theorem:

Theorem 1. *The most general bivariate distribution with conditional distributions (4) and (5) is given by:*

$$f(x,\theta) = \frac{1}{x!\theta}\exp\{m_{00} + m_{10}x - m_{01}\theta - m_{11}x\theta + m_{02}\log\theta + m_{12}x\log\theta\} \tag{6}$$

where $x = 0,1,2,\dots$ and $\theta > 0$. The parameter m_{00} is the normalizing constant and is a function of the rest of the parameters. The parameters $\{m_{ij}\}$ must be selected to satisfy $\int_0^\infty f_\Theta(\theta)d\theta < \infty$ or $\sum_{x=0}^{\infty}\Pr(X = x) < \infty$.

Proof: Since both conditional distributions belong to an exponential family (i.e., Poisson and Gamma), this is a particular case of Theorem 4.1 in Arnold, Castillo and Sarabia (1999). □

The distribution (6) was initially considered by Arnold, Castillo and Sarabia (1999), but these authors do not deal with the problems considered here.

3. Basic properties of the model

In this section the basic properties of the Poisson-Gamma conditional model are studied.

Firstly, we note that the normalizing constant is given by:

$$m_{00} = -\log\left\{\int_0^{\infty} \theta^{m_{02}-1}\exp\left\{-m_{01}\theta + \theta^{m_{12}}\exp\{m_{10} - m_{11}\theta\}\right\} d\theta\right\}. \tag{7}$$

Now, any univariate integration rule could be used to obtain (7). For this situation, Gauss-Hermite rules (Davis and Rabinowitz, 1984; among others) could be used to approximate the normalizing constant. The conditions to ensure that (6) is a genuine probability density function are

$$m_{01} > 0,\ m_{02} > 0,\ m_{11} \geq 0,\ m_{12} \geq 0. \tag{8}$$

3.1. Conditional Distributions

The conditional distribution (4) of X given Θ is Poisson with mean:

$$\lambda(\theta) = \exp\{m_{10} - m_{11}\theta + m_{12}\log\theta\}. \tag{9}$$

The conditional distribution (5) of Θ given X is Gamma with parameters:

$$\alpha(x) = m_{02} + m_{12}x, \tag{10}$$

$$\beta(x) = m_{01} + m_{11}x. \tag{11}$$

3.2. Marginal Distributions

The marginal distribution of Θ is obtained from (6) by summation in x. We obtain ($\theta > 0$):

$$\pi(\theta) = \theta^{m_{02}-1}\exp\left\{m_{00} - m_{01}\theta + \theta^{m_{12}}\exp\{m_{10} - m_{11}\theta\}\right\}. \tag{12}$$

The marginal distribution of X is obtained by integrating (6) with respect to θ. We then have:

$$\Pr(X = x) = \frac{e^{m_{00}+m_{10}x}}{x!} \times \frac{\Gamma(m_{02} + m_{12}x)}{(m_{01} + m_{11}x)^{m_{02}+m_{12}x}},\quad x = 0, 1, 2, \ldots \tag{13}$$

Note that (13) is a five-parameter model, and m_{00} is the normalizing constant, and $m_{00} = m_{00}[m_{01}, m_{02}, m_{10}, m_{11}, m_{12}]$. In order to identify the moments of (13), we obtain the moment generating function $G_X(s) = \mathbb{E}(e^{sX})$, which is given by:

$$G_X(s) = \exp\{m_{00} - \tilde{m}_{00}\}, \tag{14}$$

where:

$$\tilde{m}_{00} = m_{00}[m_{01}, m_{02}, m_{10} + s, m_{11}, m_{12}]. \tag{15}$$

The mean and variance of X are:

$$\mathbb{E}(X) = G_X'(0) = -\left\{\frac{\partial \tilde{m}_{00}}{\partial s}\right\}_{s=0}, \tag{16}$$

$$\mathrm{Var}(X) = G_X''(0) - \left\{G_X'(0)\right\}^2 = -\left\{\frac{\partial^2 \tilde{m}_{00}}{\partial s^2}\right\}_{s=0}. \tag{17}$$

Estimation of $m's$ in the generalized Negative Binomial can be accomplished via maximum likelihood. The log-likelihood based on the distribution (13) is such that,

$$\frac{\partial \mathcal{L}}{\partial m_{01}} = t\frac{\partial m_{00}}{\partial m_{01}} - \sum_{i=1}^{t} \frac{m_{02} + m_{12}x_i}{m_{01} + m_{11}x_i} \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial m_{02}} = t\frac{\partial m_{00}}{\partial m_{02}} + \sum_{i=1}^{t} \psi(m_{02} + m_{12}x_i) - \sum_{i=1}^{t} \log(m_{01} + m_{11}x_i) \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial m_{10}} = t\frac{\partial m_{00}}{\partial m_{10}} + \sum_{i=1}^{t} x_i \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial m_{11}} = t\frac{\partial m_{00}}{\partial m_{11}} - \sum_{i=1}^{t} \frac{x_i(m_{02} + m_{12}x_i)}{m_{01} + m_{11}x_i} \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial m_{12}} = t\frac{\partial m_{00}}{\partial m_{12}} + \sum_{i=1}^{t} x_i \psi(m_{02} + m_{12}x_i) - \sum_{i=1}^{t} x_i \log(m_{01} + m_{11}x_i) \tag{22}$$

where:

$$\psi(x) = \frac{d}{dx}\{\log\Gamma(x)\} = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{23}$$

is the digamma function or the Psi function.

The marginal distribution X is obviously a generalization of the Negative Binomial distribution (3), and can be interpreted in terms of mixture distributions:

$$\Pr(X = x) = \int_0^\infty f_{X|\Theta}(x|\theta)\,\pi(\theta)\,d\theta, \tag{24}$$

where $f_{X|\Theta}(x|\theta)$ is given in (4) and $\pi(\theta)$ is given in (12).

Furthermore, empirical Bayes estimates $\hat{\theta}$ of θ are provided by (12) with estimates $\hat{m}_{ij}$ substituted for m_{ij}.

3.3. Sub-Models

The probability distribution given by (13) is a broad-based model and contains as particular cases several common probability distributions; this modelling, thus, includes classical distributions proposed in the literature.

- **Poisson Distribution**. The Poisson distribution corresponds to the choice:

$$m_{11} = m_{12} = 0. \tag{25}$$

and $X \sim \mathcal{P}(e^{m_{10}})$. In this case the random variables X and Θ in (6) are independent.

- **Negative Binomial Distribution**. This model corresponds to the choice:

$$m_{10} = m_{11} = 0, \ m_{12} = 1. \tag{26}$$

In this case $X \sim \mathcal{NB}(m_{02}, 1 - e^{m_{10}} \cdot m_{01}^{-1})$, with $m_{01} > 1$.

- **Nested Negative Binomial Distribution**. In this new model we assume:

$$m_{10} = 0, \ m_{12} = 1. \tag{27}$$

This new model depends on three parameters and presents the advantage of including the Binomial Negative distribution as a particular case (nested). The probability mass function is given by:

$$\Pr(X = x) = \frac{e^{m_{00}}}{x!} \times \frac{\Gamma(m_{02} + x)}{(m_{01} + m_{11}x)^{m_{02}+x}}, \quad x = 0, 1, 2, \ldots \tag{28}$$

The Negative Binomial distribution corresponds to the choice $m_{11} = 0$. The likelihood test ratio can be used to test this hypothesis. The parameter vector is $\phi = (m_{01}, m_{02}, m_{11})$. The maximum likelihood estimator $\hat{\phi}$ of ϕ is calculated using the following formulas:

$$\frac{\partial \mathcal{L}}{\partial m_{01}} = t\frac{\partial m_{00}}{\partial m_{01}} - \sum_{i=1}^{t} \frac{m_{02} + x_i}{m_{01} + m_{11}x_i} \tag{29}$$

$$\frac{\partial \mathcal{L}}{\partial m_{02}} = t\frac{\partial m_{00}}{\partial m_{02}} + \sum_{i=1}^{t} \psi(m_{02} + x_i) - \sum_{i=1}^{t} \log(m_{01} + m_{11}x_i) \tag{30}$$

$$\frac{\partial \mathcal{L}}{\partial m_{11}} = t\frac{\partial m_{00}}{\partial m_{11}} - \sum_{i=1}^{t} \frac{x_i(m_{02} + x_i)}{m_{01} + m_{11}x_i} \tag{31}$$

Equations (30)-(32) can be solved simultaneously for $m's$ by standard iterative procedures. We now estimate the vector ϕ with the constraint that $m_{11} = 0$. The vector ϕ with the new constraint is represented by ϕ^*. The critical region for the null hypothesis $H_0 : m_{11} = 0$ is given by:

$$2\{\log \ell(\hat{\phi}; \underset{\sim}{x}) - \log \ell(\hat{\phi}^*; \underset{\sim}{x}) \tag{32}$$

whose distribution is a chi-squared random variable with 1 degree of freedom.

4. Application to a bonus-malus system

In this section, the results obtained above are illustrated with real data from Lemaire (1979). We consider a group in which the claim proneness of a risk is represented by a risk parameter $\lambda(\theta)$. The risks are assumed to be independent, so we take a risk $\lambda(\theta)$ and assume that the distribution of the number of claims for each policyholder is a Poisson distribution with mean $\lambda(\theta) > 0$ as in (9), whose parameter θ varies from one individual to another, reflecting the individual's claim propensity, $\lambda(\theta) = \mathbb{E}(K|\theta)$. This parameter is assumed to be a random variable and presents a structure function given by

$$\pi_0(\theta) = \frac{f(k,\theta)}{f(k\,|\,\theta)} = \theta^{m_{02}-1}\exp\{m_{00} - m_{01}\theta + \theta^{m_{12}}\exp\{m_{10} - m_{11}\theta\}\}.$$

Note that the case $m_{11} = 0$, $m_{12} = 1$ corresponds to the compound Poisson distribution, $K\,|\,\Theta = \theta \sim \mathcal{P}(\theta e^{m_{10}})$, $\Theta\,|\,K = k \sim \mathcal{G}(m_{02}, (m_{01} - e^{m_{10}})\,/\,e^{m_{10}})$ and can be seen as a reparametrization of the classical model.

Consider a policyholder, drawn randomly from the insurance portfolio, who is observed to have the sequence of claims $k_1, k_2, \ldots, k_t$ over t periods. These are assumed to be independent and equidistributed. Let $k = t\bar{k} = \sum_{i=1}^{t} k_i$. The likelihood function is then given by

$$f(k_1, k_2, \ldots, k_t\,|\,\lambda(\theta)) = \frac{1}{\prod_{i=1}^{t} k_i!}\exp\{-t\lambda(\theta)\}(\lambda(\theta))^k. \tag{33}$$

The posterior distribution $\pi_0(\theta\,|\,k, t)$ is conjugated with respect to the likelihood in (33) but now with the updated parameters.

$$\begin{aligned}
m_{00}^* &= m_{00} + m_{10}\,t\bar{k},\\
m_{01}^* &= m_{01} + m_{11}\,t\bar{k},\\
m_{02}^* &= m_{02} + t\bar{k},\\
m_{10}^* &= m_{10} + \log|1 - t|,\\
m_{11}^* &= m_{11},\\
m_{12}^* &= m_{12}.
\end{aligned}$$

Alternative premium calculation principles can be used to compute bonus-malus premiums. In this case we assume the net premium principle (Lemaire 1979, 1985).

The posterior mean of $\lambda(\theta)$ under prior π_0 is given by

$$\delta(k,t) = \int_{\Theta} \lambda(\theta)\pi_0(\theta \mid k,t)\, d\theta.$$

Then, the bonus-malus premium to be charged using the net premium principle is given by

$$\delta_{\text{BM}}^{\pi_0}(k,t) = 100 \times \frac{\delta(k,t)}{\delta(0,0)} \tag{34}$$

Additionally, estimation of the bonus-malus premiums under the classical Bühlmann model from the CEF prior distribution can be performed as follows. The classical credibility formula establishes,

$$\mathbb{E}(\lambda(\theta) \mid k_1, k_2, \ldots, k_t) = Z(t)\bar{k} + (1 - Z(t))\, m,$$

where $\lambda(\theta) = \mathbb{E}(K \mid \theta)$, $m = \mathbb{E}_{\pi_0}[\lambda(\theta)]$ and

$$Z(t) = tVar_{\pi_0}[\mathbb{E}(K \mid \theta)] \times \{tVar_{\pi_0}[\mathbb{E}(K \mid \theta)] + \mathbb{E}_{\pi_0}[Var(K \mid \theta)]\}^{-1},$$

with $\mathbb{E}_{\pi_0}$ and Var_{π_0} being the mean and the variance, respectively, with respect to $\pi_0(\theta)$.

In the bonus-malus scenario and using the conditional specification model, the premium adjusted with a credibility formula has the following form,

$$\delta_{\text{BM}}^{\pi_0}(k,t) = \frac{100}{\delta(0,0)} \left\{ Z(t)\bar{k} + [1 - Z(t)]\, m \right\}, \tag{35}$$

where

$$\begin{aligned}
\mathbb{E}(K \mid \theta) &= \lambda(\theta), \\
Var_{\pi_0}[\mathbb{E}(K \mid \theta)] &= \mathbb{E}\left\{[\lambda(\theta)]^2\right\} - \{\mathbb{E}[\lambda(\theta)]\}^2, \\
\mathbb{E}_{\pi_0}[Var(K \mid \theta)] &= \mathbb{E}_{\pi_0}[\lambda(\theta)].
\end{aligned}$$

Table 1 shows fits from the above models to data in Lemaire (1979). We use NB to denote the model derived from a standard Poisson-Gamma distribution or, equivalently, from a Negative Binomial distribution, and use GNB to denote the Generalized Negative Binomial model. The Akaike information criterion (AIC) and the Bayesian information criterion (BIC) are used to compare the estimated models (Leroux, 1992). As is well-known, a model with a minimum AIC value is to be preferred.

Note that to resolve equations (18)-(22) we must identify the partial derivatives of the type $\frac{\partial m_{00}}{\partial m_{ij}}$. To perform this estimation by maximum likelihood, we proceed as follows: firstly, the integral in (7) is approximated by a Gauss-Hermite type squaring formula, which enables us to obtain the approximation as

TABLE 1

OBSERVED AND FITTED DISTRIBUTIONS OF NUMBER OF CLAIMS: NEGATIVE BINOMIAL MODEL (NB), NESTED NEGATIVE BINOMIAL (NNB) MODEL AND GENERALIZED NEGATIVE BINOMIAL (GNB) MODEL

k	Observed	NB	NNB	GNB
0	96978	97086.90	96917	96978
1	9240	9138.01	9316	9239
2	704	696.63	696	712
3	43	48.95	42	43
4	9	3.29	2	2
More than 4	0	0	0	0
Total	106974	106974	106974	106974
$-\log(L)$		187789	73443	44351.7
BIC		375613	146909	88761.3
AIC		375584	146890	88713.4

$AIC = -2\log(L) + 2k$, $BIC = -2\log(L) + k\log(N)$.
L, k and N are the maximized log likelihood, number of parameters and observations, respectively.

a function of the other parameters m_{ij}. The partials are then obtained and the non-linear system resolved by the numerical routines of the Mathematica software package. In all the cases resolved, we test the convergence conditions of the series and the integrals necessary for the marginal distributions. The estimation of the parameters $m's$ by the maximum likelihood method produces the following values:

- Poisson-Gamma: $m_{10} = m_{11} = 0$, $m_{12} = 1$, $\hat{m}_{02} = 1.6131$, $\hat{m}_{01} = 17.1384$.
- Generalized Negative Binomial: $\hat{m}_{12} = 1.47011$, $\hat{m}_{02} = 1$, $\hat{m}_{01} = 3.1$, $\hat{m}_{10} = 10^{-3}$, $\hat{m}_{11} = 1.45883001$.
- Nested Negative Binomial: $m_{10} = 0$, $m_{12} = \hat{m}_{02} = 1$, $\hat{m}_{01} = 7.58$, $\hat{m}_{11} = 1.3$.

The observed distribution in Table 1 provides a fairly good fit for both models. Nevertheless, the χ^2-test of goodness of fit for the three models considered is very poor ($\chi^{NB}_{obs} = 11.97$, $\chi^{NNB}_{obs} = 24.59$, $\chi^{GNB}_{obs} = 25.27$, respectively). Probably, the asymptotic character of the standard χ^2-test and the limited number of classes in the example considered here would largely explain this lack of fit (observe that the class "more than 4" contains almost all the value of the χ^2 statistic). We then have to take into account the differences in the log-likelihood results and the BIC and AIC values. Table 1 reveals that the GNB model performs very well in fitting the distribution, compared to the NB model. The highest gain in log-likelihood is obtained from NB, although the data support adding additional terms and the preferred model is a GNB model. Moreover, when using BIC and/or AIC to discriminate between models, GNB is preferred to NB.

Obviously, the model presented is somewhat more complex than the negative binomial and therefore it might appear that we run the risk of overfitting (which is not strictly the objective of this paper), particularly taking into account that in the example presented we only consider five or six classes for a model with five parameters. Of course, the methodology presented can easily be applied to cases with more than 6 or 7 classes, in which, obviously, the goodness of fit would be much greater. As shown below, the GNB and NNB models present additional advantages.

TABLE 2

BONUS-MALUS PREMIUMS UNDER THE NET PREMIUM PRINCIPLE IN THE CLASSICAL NB MODEL

	k							
t	0	1	2	3	4	5	6	$Z_{NB}(t)$
0	100	–	–	–	–	–	–	0
1	94.165	152.540	210.916	269.291	–	–	–	0.058348
2	88.973	144.131	199.288	254.445	309.601	364.758	–	0.110263
3	84.324	136.600	188.875	241.150	293.424	345.699	397.974	0.156753
4	80.137	129.817	179.496	229.175	278.854	328.523	378.212	0.198626

TABLE 3

BONUS-MALUS PREMIUMS UNDER THE NET PREMIUM PRINCIPLE IN THE NNB MODEL

	k						
t	0	1	2	3	4	5	6
0	100	–	–	–	–	–	–
1	94.818	147.456	179.741	201.013	–	–	–
2	89.884	142.641	175.830	197.954	213.415	224.646	–
3	85.205	137.864	171.858	194.808	210.944	222.692	231.506
4	80.787	133.146	167.837	191.582	208.391	220.666	229.884

TABLE 4

BONUS-MALUS PREMIUMS UNDER THE NET PREMIUM PRINCIPLE IN THE GNB MODEL

	k						
t	0	1	2	3	4	5	6
0	100	–	–	–	–	–	–
1	94.364	152.673	178.283	192.009	–	–	–
2	88.901	148.665	175.753	190.348	199.213	205.070	–
3	83.646	144.548	173.603	188.601	198.005	204.198	208.531
4	78.623	140.337	170.336	186.766	196.737	203.283	207.847

Note the following comments, always in comparison to the NB model:

- In the NNB model. there is a slight increase in premiums for good-risk policyholders ($k = 0$). On the other hand, to guarantee the financial stability of the insurance company, there is a significant fall in the premiums payable by bad-risk policyholders ($k > 0$), except in the case of $k = 1$ for $t = 3$ and 4.
- A similar effect is observed with the GNB model. All premiums fall significantly for the policyholders located in $k > 1$, and also for the good-risk group $k = 0$ for $t > 1$. Again, to balance the former effect, there is a slight rise in the premiums payable by other policyholders.

Tables 5 and 6 show the Bonus-Malus premiums adjusted with a credibility formula. Both tables show, in comparison with the NB model, that the premiums payable by good-risk policyholders ($k = 0$) rise slightly, while those for bad-risk policyholders ($k > 0$) fall significantly. The variation is always greater with the GNB model, because

$$Z_{NNB}(t) < Z_{GNB}(t) < Z_{NB}(t),$$

TABLE 5

BONUS-MALUS PREMIUMS AND CREDIBILITY FACTOR, $Z_{NNB}(t)$, UNDER THE NET PREMIUM PRINCIPLE IN THE NNB MODEL ADJUSTED WITH A CREDIBILITY FORMULA

	k							
t	**0**	**1**	**2**	**3**	**4**	**5**	**6**	$Z_{NNB}(t)$
0	100	–	–	–	–	–	–	0
1	94.966	144.620	194.275	243.929	–	–	–	0.050339
2	90.414	137.689	184.964	232.239	279.513	326.788	–	0.095853
3	86.279	131.392	176.505	221.617	266.730	311.842	356.955	0.137205
4	82.506	125.646	168.785	211.925	255.925	298.204	341.344	0.174939

TABLE 6

BONUS-MALUS PREMIUMS AND CREDIBILITY FACTOR, $Z_{GNB}(t)$, UNDER THE NET PREMIUM PRINCIPLE IN THE GNB MODEL ADJUSTED WITH A CREDIBILITY FORMULA

	k							
t	**0**	**1**	**2**	**3**	**4**	**5**	**6**	$Z_{GNB}(t)$
0	100	–	–	–	–	–	–	0
1	94.598	148.099	201.600	255.101	–	–	–	0.054014
2	89.750	140.510	191.269	242.028	292.787	343.546	–	0.102494
3	85.375	133.660	181.945	230.230	278.514	326.799	375.084	0.146246
4	81.407	127.447	173.448	219.528	219.528	311.608	317.608	0.185930

and thus the NNB model gives greater weight to the group premium than to individual claim experiences.

Lemaire (1979) remarked on the problem of overcharges arising in insurance markets in which Bonus-Malus systems are applied. It was seen that, because a posteriori distributions for different k and t overlap, there occur overcharges among groups of policyholders who pay more, although their claim frequency is lower than the mean of other policyholders who pay less. In Lemaire's study, this problem is corrected by resolving a problem of restricted optimisation. The NNB and GNB models reduce overcharges in a direct way.

Thus, if we limit ourselves to the example adopted by Lemaire (1979), calculating the area under the a posteriori distribution for $k = 3$, $t = 3$ between $\theta = 0$ and $\theta =$ mean of the a posteriori distribution for $k = 0$, $t = 3$ in the three models presented in this article, we obtain

	Posterior mean	**Area**
NB	0.100256	0.0694900
NNB	0.115209	0.0580713
GNB	0.282739	0.0513790

It can be seen that the GNB model produces the smallest area, and therefore the least overcharge for the case in question. This reduction, obviously, produces a lower penalization when the number of claims made by high-risk policyholders increases (Table 7).

Finally, we calculated the average efficiency ψ (see Lemaire and Zi, 1994) given by

$$\psi = \int_{\Theta} \psi(\theta)\pi(\theta)d\theta,$$

where

$$\psi(\theta) = \frac{\partial\delta(\theta)}{\partial\theta}\frac{\theta}{\delta(\theta)} = \frac{\partial\log\delta(\theta)}{\partial\log(\theta)},$$

is the efficiency or the elasticity of the stationary premium with $\delta(\theta)$ representing the risk premium. In our case, i.e. $\delta(\theta) = \exp\{m_{10} - m_{11}\theta + m_{12}\log\theta\}$, under a quadratic loss function we have $\psi(\theta) = -m_{11}\theta + m_{12}$. The ψ values were computed and found to be $\psi_{\text{NB}} = 1$, $\psi_{\text{NNB}} = 0.816437$ and $\psi_{\text{GNB}} = 0.968422$.

TABLE 7

PERCENTAGE OF PREMIUM INCREASE WHEN THE POLICYHOLDER PASSES FROM $k = 1$ TO $k = 2$

t	***NB***	***NNB***	***GNB***
2	30.64%	19.24%	15.11%
3	31.04%	20.48%	16.77%
4	31.40%	21.70%	17.84%

The two models, NNB and GNB, present very high efficiency, with values close to one.

5. Conclusions

As is well known, the classical model in which the Poisson distribution is mixing distribution with a Gamma yields a Negative Binomial claim count distribution. In this paper we use the conditional specification model as an alternative, and equally general, mixing distribution. We obtain a discrete distribution that we call the Generalized Negative Binomial distribution; this has some useful properties. The results and applicability of the proposed model show that the new class of model is simple to estimate. The distributional fit is quite good and reasonable predictions are obtained. Moreover, a bonus-malus table is obtained in a direct way.

Naturally, the results shown in this paper can be applied to other premium calculation principles and an analysis of Bayesian robustness could be performed to analyse the sensitivity of the premiums obtained. Gómez et. al (2002) showed how sensitive the Bonus-Malus premium can be when we move away from a single prior. Both the NNB and the GNB model are very robust, and this is a subject for future studies.

Acknowledgements

The authors are grateful to the Editor and to the referee for their careful, valuable and constructive comments. JMS is funded by Ministerio de Ciencia y Tecnología, Spain (project BEC2000-1186). EGD and FJVP are funded by Ministerio de Ciencia y Tecnología, Spain, under project BEC2001-3774.

References

Arnold, B.C, Castillo, E. and Sarabia, J.M. (1999). *Conditional Specification of Statistical Models*. Springer Series in Statistics, Springer Verlag, New York.

Arnold, B.C., Castillo, E. and Sarabia, J.M. (2001) Conditionally Specified Distributions: An Introduction (with discussion). *Statistical Science*, **16(3)**, 249-274.

Consul, P.C. and Jain, G.C. (1973) A generalization of the Poisson distribution. *Technometrics*, **15(4)**, 791-799.

Davis, P. and Rabinowitz, P. (1984) *Methods of numerical integration*. San Diego: Academic Press.

Dionne, G. and Vanasse, C. (1989) A generalization of actuarial automobile insurance rating models: the Negative Binomial distribution with a regression component. *ASTIN Bulletin*, **19**, 199-212.

Gerber, H. (1991) From the generalized gamma to the generalized negative binomial distribution. *Insurance: Mathematics & Economics*, **10**, 303-309.

Gómez, E., Pérez, J., Hernández, A. and Vázquez-Polo, F.J. (2002) Measuring sensitivity in a Bonus-Malus System. *Insurance: Mathematics & Economics*, **31**, 105-113.

Goovaerts, M.J. and Kaas, R. (1991) Evaluating compound generalized Poisson distributions recursively. *ASTIN Bulletin*, **21**, 193-197.

LEMAIRE, J. (1979) How to define a Bonus-Malus system with an exponential utility function. *Astin Bulletin*, **10**, 274-282.

LEMAIRE, J. (1985) Automobile Insurance (Actuarial Models). Kluwer-Nijhoff Publishing. Boston/Dordrecht/Lancaster.

LEMAIRE, J. (1995) Bonus-Malus Systems in Automobile Insurance. Kluwer Academic Publishers. Boston/Dordrecht/Londres.

LEMAIRE, J. and ZI, H. (1994) High deductibles instead of Bonus-Malus: can it work? *Astin Bulletin*, **24(1)**, 75-88.

LEROUX, B.G. (1992) Consistent estimation of a mixing distribution. *Annals of Statistics*, **20(3)**, 1350-1360.

PANJER, H.H. and WILLMOT, G.E. (1988) Motivating claims frequency models, in *Proceedings of the twenty-third international congress of actuaries, Helsinki*, **3**, 269-284.

SCOLLNIK, D.P.M. (1995) The Bayesian analysis of generalized Poisson models for claim frequency data utilising Markov chain Monte Carlo, in *Proceedings of the 29th Actuarial Research Conference*, Actuarial Research Clearing House.

SICHEL, H. (1971) On a family of discrete distribution particularly suited to represent long-tailed frequency data, in *Proceedings of the Third Symposium on Mathematical Statistics*, ed. N.F. Laubscher, Pretoria.

WILLMOT, G.E. (1987) The Poisson-Inverse Gaussian distribution as an alternative to the negative binomial. *Scandinavian Actuarial Journal*, 113-127.

Address for correspondence:

F.J. VÁZQUEZ-POLO
Department of Quantitative Methods
University of Las Palmas de Gran Canaria
35017-Las Palmas de Gran Canaria
Spain
E-mail: fjvpolo@dmc.ulpgc.es

J.M. SARABIA
Department of Economics
University of Cantabria
39005-Santander
Spain
E-mail: sarabiaj@unican.es

E. GÓMEZ
Department of Quantitative Methods
University of Las Palmas de Gran Canaria
35017-Las Palmas de Gran Canaria
Spain
E-mail: egomez@dmc.ulpgc.es