

ON THE DIFFERENCE BETWEEN THE CONCEPTS "COMPOUND" AND "COMPOSED" POISSON PROCESSES

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In the discussion in the ASTIN Colloquium 1962, which followed my lecture on the numerical evaluation of the distribution functions defining some *compound* Poisson processes, a remark was made, which drew my attention to the paper quoted here below under reference number (1), On *composed* Poisson distributions I.

As I have the impression that this remark may induce some confusion of the terms "compound" and "composed", the more as the same French word (*composé*) is used for the two terms, a comparison between the two kinds of processes shall be made.

1. The most general propositions of (1) are a theorem which concerns a general *homogeneous* Markoff process (l.c. § 2) and a theorem for the family $\{P(k, \phi)\}$ of distribution functions of positive integer-valued variables with mean ϕ , where ϕ runs over all non-negative numbers, and where the convolution of $P(k, \phi_1)$ and $P(k, \phi_2)$ is equal to $P(k, \phi_1 + \phi_2)$ (l.c. § 3). By these propositions the characteristic functions corresponding in the 1st case to the distribution functions defining the homogeneous Markoff process, and in the 2nd case to the distribution functions belonging to the family $\{P(k, \phi)\}$ can all be written in the same form, namely

$$\exp \left[\phi \sum_{k=1}^{\infty} c_k (e^{uk} - 1) \right], \quad (\text{a})$$

where u is an entirely imaginary variable, ϕ is the parameter of the process, respectively of $P(k, \phi)$ and c_1, c_2, \dots non-negative constants such that $\sum_{k=1}^{\infty} c_k$ converges. If, in the 1st case, $\sum_{k=1}^{\infty} k c_k$ converges, and as, in the 2nd case, this series converges, the distributions defined by these characteristic functions are called *composed Poisson distributions*, which define *homogeneous composed Poisson processes*.

2. The *compound Poisson processes*, defined in my lecture, were

defined by probability distributions of the number m of changes in the time-interval $(0, t)$ in the form of the Laplace-Stieltjes integral

$$\int_0^{\infty} e^{-vt} (vt)^m dU(v,t)/m!, \quad (b)$$

where $U(v,t)$ is a distribution function, which may depend on t (compound Poisson processes in the wide sense) or be independent of t (compound Poisson processes in the narrow sense). These processes are in the general case heterogeneous in time, though the inhomogeneity in particular cases can be eliminated by a change of the time scale. These particular cases are the Poisson process and the Polya process, for which by the introduction of the new time parameter p for the natural time τ leads to homogeneous processes, where p and τ are connected by the relations $p = \int_0^{\tau} c(v)dv$ for the Poisson process and $p = \frac{1}{\delta} \log(1 + \tau\delta)$ for the Polya process. In (1) it has also been proved that the Poisson process is a *composed* Poisson process, which fulfils one of the alternative conditions, either that $c_n = 0$ for $n \geq 2$ in the expression (a), or that the variance for a fixed value of p of $P(k,p)$ is minimal as compared with other members of the family $\{P(k,p)\}$, and that the Polya process has a *composed* Poisson distribution, where $c_n = \frac{1}{p\delta^n} \left[\frac{p\delta}{v+p} \right]^n$, $n = 1, 2, \dots$. This transformation of the characteristic functions defining a Polya process is equivalent to Ammeter's transformation of such a process for the case where the conditioned distribution of the size of one change is the unity distribution and to Lundberg's transformation of the same process (cf. (5), p. 183, (6), pp. 67-69). Other *compound* Poisson processes are, however, "genuinely" heterogeneous in time taken to mean (cf. Rényi, (2), p. 87) that their heterogeneity cannot be removed by the transformation of the time scale (Lundberg, (6), p. 58). Further, *compound* Poisson processes, except the Poisson process, are processes with *dependent* increments, i.e. their distribution functions do not generally fulfil the condition that the convolution of $P(k,p_1)$ and $P(k,p_2)$ is equal to $P(k,p_1 + p_2)$ (cf. Lundberg, l.c. pp. 89-91). According to my opinion the results of (1) cannot, generally, be applied to the compound processes, as defined in my lecture. — In

(1) is also given a system of equations for the calculation of c_k , $k = 1, 2, 3, 4$.

3. In another paper, Rényi (2) has extended the class of *composed* Poisson processes to include also time-heterogeneous Markoff processes. In Theorem 1 of (2) it is asserted that the characteristic function defining a process of *independent increments*, which fulfils a certain postulate of “rarity”, can be written in the form (a) with

$p = 1$ and with the substitution of $\int_s^t c_k(\tau) d\tau$ for c_k , $k = 1, 2, \dots$,

where $c_k(\tau)$ are non-negative, integrable functions of time and $\sum_{k=1}^{\infty} c_k(\tau)$ converges almost everywhere. The “rarity” postulate of

this theorem is given in the following form. A $\delta > 0$ can be found for an arbitrary small $\epsilon > 0$ and for an arbitrary $T > 0$ such that the probability of non-occurrence of an event in the interval (s_1, t_r)

will exceed $1 - \epsilon$, if for an arbitrary positive integer r , $\sum_{j=1}^r (t_j - s_j)$

$< \delta$, $s_1 < t_1 \leq s_2 < t_2 \leq \dots \leq s_r < t_r < T$. — In (1), § 1, appears another “rarity” postulate, which implies the exclusion of multiple events and which is given in the form that the fraction $P_1(t) / [1 - P_0(t)]$ tends to 1 for $t \rightarrow 0$, where $P_1(t)$, $P_0(t)$ are the probabilities for the occurrence of one event, respectively no events in the interval $(0, t)$; in this case the process reduces to an ordinary Poisson process. In a continuation of (2), Aczél (3) has given a theorem, which is more general than Theorem 1 of (2), as the “rarity” condition has been left out. Aczél’s theorem has the following form: “Let the number of random events during the time-interval (t_1, t_2) be independent of the number of random events during the time-interval (t_3, t_4) , provided that $t_1 < t_2 \leq t_3 < t_4$ and let $w_k(t_1, t_2)$ denote the probability of exactly k events occurring in the time-interval (t_1, t_2) . Then,

$$w_0(t_1, t_2) = \exp [L(t_2) - L(t_1)]$$

$$w_k(t_1, t_2) = \exp [L(t_2) - L(t_1)]$$

$$r_1 + 2r_2 + \dots + kr_k = r \quad \prod_{k=1}^r \frac{[C(t_2) - C(t_1)]^{r_k}}{r_k!} \quad (c)$$

where $r_j \geq 0$; $C_j(t)$ ($j = 1, 2, \dots$) are arbitrary functions such that $\sum_{j=1}^{\infty} [C_j(t_2) - C_j(t_1)]$ exists and is equal to $L(t_1) - L(t_2)$ (General inhomogeneous composed Poisson distribution)". Aczél asserts, further, that, if and only if $\lim_{t_2 \rightarrow t_1} \frac{1 - w_0(t_1, t_2)}{w_1(t_1, t_2)} = 0$, which condition corresponds to the "rarity" condition in (1) § 1, the process is an ordinary Poisson process in an inhomogeneous form, the inhomogeneity can be eliminated by transformation of the time-scale. This cannot be done in the most general inhomogeneous Markoff process, defined by (c), which can be considered a sum of an infinity of independent ordinary inhomogeneous Poisson processes, the j^{th} process with the mean $C_j(t_2) - C_j(t_1)$ consisting of a t -tuple of events—this is consistent with a similar remark for homogeneous Markoff processes made by Kolmogoroff (in (1) p. 211).—In this connection I want to remark that Thyron (1960) used a postulate of an infinite sequence of random processes consisting of a k -tuple of events as a starting-point, when constructing a very general class of processes in which the increments may be dependent or independent.

4. The time-heterogeneous composed Poisson process with independent increments, defined by Rényi in Theorem 1 of (2) according to the previous paragraph, will in this note in accordance with notations in (2) be called a ζ_t -process. In Theorem 2 of (2) Rényi introduces a transform of the ζ_t -process, which, here, will be called an η_t -process. In this theorem Rényi still uses the term composed Poisson distribution, though the decrements of η_t are not explicitly stated to be independent. The theorem has, namely, the following content. Supposing that each event of a η_t -process, for which the mean of ζ_t exists for every $t > 0$, is the starting-point of some happening, the duration of which is a variable distributed with the distribution function $1 - \Phi(\tau, t)$, where t is the starting-point, and where $\Phi(\tau, t)$ is a continuous positive function for all (τ, t) ; η_t denotes the number of happenings going on at time t . The characteristic function of η_t is, then, of the same form as that ζ_t with

$$d_k(t) = \int_0^t \sum_{n=k}^{\infty} c_k(\tau) \binom{n}{k} \Phi^k(\tau, t - \tau) [1 - \Phi(\tau, t - \tau)]^{n-k} d\tau \quad (d)$$

substituted for $\int_0^t c_k(\tau) d\tau$. $d_k(t)$ is, evidently, non-negative for all k, t .—If particularly ζ_t is attached to a Poisson process the characteristic function of η_t reduces to

$\exp [(e^\mu - 1) \int_0^t c_1(\tau) \Phi(\tau, t - \tau) d\tau]$ i.e. η_t is also attached to a Poisson process.

5. In Theorem 3 of (2) it is asserted that, if $\xi_{n_1}, \xi_{n_2} \dots \xi_{n_{k_n}}$ are non-negative independent, integer-valued random variables, and if the probability for ξ_{nk} being equal to s , $p_{nk}(s)$ say, fulfils the relation $\lim_{n \rightarrow \infty} \max_{1 \leq k \leq k_n} p_{nk}(0) = 0$, then the sufficient and necessary condition for the convergence of the distribution function of the sum of ξ_{nk} ($k = 1, 2 \dots k_n$) for n tending to ∞ is the existence of a sequence non-negative numbers $c_1, c_2 \dots c_n$ such that $\sum_{n=1}^{\infty} c_n$ converges to a non-negative value and that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} p_{nk}(s) = c_s = 0.$$

Under these conditions the distribution function of the sum of ξ_{nk} converges to a *composed* Poisson distribution with a characteristic function in the form of (a) with $\phi=1$.

In (4) Prékopa gives a continuation of (2) (in a paper presented by Rényi) which contains an interesting modification of Rényi’s developments. He proves the following theorem. Let J denote a finite interval on the t -axis and $J_1, J_2 \dots J_n$ a subdivision of the interval I , so that $\sum_{r=1}^n J_r = I$, and ξ_J the increment of a random function in the interval J . If the variables ξ_{J_k} are mutually independent for $k = 1, 2 \dots n$, if ξ_J only assume the values of a countable set of real numbers $\lambda_0 = 0, \lambda_1, \lambda_2 \dots$, which set is independent of the special selection of J , and if, further, $1 - W_0(j) \rightarrow 0$, when J contracts to a fixed point, where $W_\lambda(j)$ is the probability of $\xi_J = \lambda$, then the characteristic function of the accumulated sum of ξ_J can be written

$$\exp \left[\sum_{k=1}^{\infty} C_{\lambda_k}(I) (e^{\lambda_k \mu} - 1) \right] \tag{e}$$

where $C_{\lambda_k}(I) = \int_I W_{\lambda_k}(J)$, $\lambda_k \neq 0$; $\sum_{k=1}^{\infty} C_{\lambda_k}(I) = \int_I [1 - W_0(J)] < \infty$,

the integrals being taken in the sense of Burkill. The theorem is proved by an indirect proof which is a generalization of the proof of Theorem 4 in (2) and by a direct proof which is a generalization of the deduction in the first section of (2), where λ_k is identical with the set of non-negative integers. For the last-mentioned case, Prékopa (§3, (4)) gives an explicit form of $W_k(I)$ in the terms of Bessel functions.

6. By Theorem 4 of (2) the class of *composed* Poisson distributions can be characterized as the class of infinitely divisible distributions of non-negative integer-valued variables, which assume the value zero with a positive probability. The proof of this theorem is given only in terms of c_n being independent of time, i.e. the proof relates, directly only to the homogeneous composed Poisson distributions. By the developments in (4), the theorem holds also for heterogeneous composed Poisson processes, if the increments are independent. This means that also the distribution functions of the ζ_t -process are infinitely divisible. If the functions $\Phi(\tau, t)$ appearing in the Theorem 2 of (2) are defined such as to ensure the independency of the decrements of η_t , the assertion holds also for the η_t -process. The η_t -process is a particular transform of the ζ_t -process. Even if the increments of the η_t -process should be mutually dependent, one cannot expect the *compound* Poisson processes to be contained in the class of the η_t -processes.

7. For the elementary *compound* Poisson process in the narrow sense the probability of non-occurrence of an event in the time interval $(0, t)$ can be written

$$P_0(t) = \int_0^{\infty} e^{-vt} dU(v) \tag{f}$$

and the characteristic function of the number of events occurring during the same time-interval

$$P_0 [t(1 - e^u)], \tag{g}$$

where u is an entirely imaginary variable (cf. (6), pp. 71 and 103). By definition the characteristic function corresponding to $U(v)$ in (f) can be written

$$P_0(-u) = \int_0^\infty e^{uv} dU(v) = \exp \left[\sum_{v=1}^\infty \chi_v u^v/v! \right], \quad (h)$$

where χ_v is the v^{th} semi-invariant of v , and u , as before, entirely imaginary. Consequently, by comparing (g) and (h) and by writing z for e^u , we may, provided that the sum in the exponent converges, write

$$P_0[t(I-z)] = \exp \left[\sum_{v=1}^\infty (-t)^v (I-z)^v \chi_v/v! \right] \quad (i)$$

Supposing that the series converges absolutely for $t < T$, the expression can be transformed, after expansion of $(I-z)^v$ according to Newton’s binomial theorem, by the reversion of the order of summations, and by some easy computation be written in the form.

$$\exp \left[\sum_{\mu=1}^\infty (e^{u\mu}-I) \sum_{v=\mu}^\infty (-I)^{u-v} \binom{v}{\mu} \chi_v t^v/v! \right] \text{ for } t < T \quad (j)$$

which is of similar form as the characteristic function corresponding to a *composed* Poisson distribution (cf. (a)). As, however, the *compound* Poisson process, in the general case, are processes of *dependent increments*, (j) is *not*, generally, to be considered a particular case of the characteristic functions corresponding to the class of infinitely divisible distributions. In this case Rényi’s theorem 4 (2) and Prékopa’s theorem (4) are only applicable to the Poisson process or to the Polya process after Ammeter’s transformation of the time scale.

The deduction of the last paragraph can straight forwardly be generalized, by the introduction of the semi-invariant functions $\chi_v(t)$, say, of $U(v,t)$ for χ_v , to include also the compound Poisson processes in the wide sense.

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