THE STANDARD ERROR OF CHAIN LADDER RESERVE ESTIMATES: RECURSIVE CALCULATION AND INCLUSION OF A TAIL FACTOR

BY

Thomas MACK

Munich Re, Munich

ABSTRACT

In Mack (1993), a formula for the standard error or chain ladder reserve estimates has been derived. In the present communication, a very intuitive and easily programmable recursive way of calculating the formula is given. Moreover, this recursive way shows how a tail factor can be implemented in the calculation of the standard error.

KEYWORDS

Chain Ladder, Standard Error, Recursive Calculation, Tail Factor

INTRODUCTION

Let C_{ik} denote the cumulative loss amount of accident year $i = 1, ..., n$ at the end of development year (age) $k = 1, ..., n$. The amounts C_{ik} have been observed for $k \leq n + 1 - i$ whereas the other amounts have to be predicted. The chain ladder algorithm consists of the stepwise prediction rule

$$\hat{C}_{i,k+1} = \hat{C}_{ik}\hat{f}_k$$

starting with $\hat{C}_{i,n+1-i} = C_{i,n+1-i}$. Here, the age-to-age factor $\hat{f}_k$ is defined by

$$\hat{f}_k = \sum_{i=1}^{n-k} w_{ik}C_{ik}^{\alpha}F_{ik} \bigg/ \sum_{i=1}^{n-k} w_{ik}C_{ik}^{\alpha}, \qquad \alpha \in \{0; 1; 2\},$$

where

$$F_{ik} = C_{i,k+1}/C_{ik}, 1 \leq i \leq n, 1 \leq k \leq n - 1,$$

are the individual development factors and where

$$w_{ik} \in [0; 1]$$

are arbitrary weights which can be used by the actuary to downweight any outlying F_{ik}. Normally, $w_{ik} = 1$ for all i, k. Then, $\alpha = 1$ gives the historical chain ladder age-to-age factors, $\alpha = 0$ gives the straight average of the observed individual development factors and $\alpha = 2$ is the result of an ordinary regression of $C_{i,k+1}$ against C_{ik} with intercept 0. Note that in case $C_{ik} = 0$, the corresponding two summands should be omitted when calculating $\hat{f}_k$.

The above stepwise rule finally leads to the prediction

$$\hat{C}_{in} = C_{i,n+1-i}\hat{f}_{n+1-i} \cdot \; \dots \; \cdot \hat{f}_{n-1}$$

of C_{in} but – because of limited data – the loss development of accident year i does not need to be finished at age n. Therefore, the actuary often uses a tail factor $\hat{f}_{ult} > 1$ in order to estimate the ultimate loss amount $C_{i,ult}$ by

$$\hat{C}_{i,ult} = \hat{C}_{in}\hat{f}_{ult} \; .$$

A possible way to arrive at an estimate for the tail factor is a linear extrapolation of $\ln(\hat{f}_k - 1)$ by a straight line $a \cdot k + b$, $a < 0$, together with

$$\hat{f}_{ult} = \prod_{k=n}^{\infty}\hat{f}_k \; .$$

However, the tail factor used must be plausible and, therefore, the final tail factor is the result of the personal assessment of the future development by the actuary.

In Mack (1993), a formula for the standard error of the predictor $\hat{C}_{in}$ was derived for $\alpha = 1$ and all $w_{ik} = 1$. In the next section, this formula is generalized for the cases $\alpha = 0$ or $\alpha = 2$ and $w_{ik} < 1$. Furthermore, a recursive way of calculating the standard error is given. In the last section it is shown how a tail factor can be implemented in the calculation of the standard error.

RECURSIVE CALCULATION OF THE STANDARD ERROR

In order to calculate the standard error of the prediction $\hat{C}_{in}$ as compared to the true loss amount C_{in}, Mack (1993) introduced an underlying stochastic model (for $\alpha = 1$ and $w_{ik} = 1$) which is given here in its more general form without the restriction on α and w_{ik}:

(CL1) $E(F_{ik}|C_{i1}, \dots, C_{ik}) = f_k$, $1 \leq i \leq n$, $1 \leq k \leq n - 1$,

(CL2) $\mathrm{Var}(F_{ik}|C_{i1}, \dots, C_{ik}) = \dfrac{\sigma_k^2}{w_{ik}C_{ik}^\alpha}$, $1 \leq i \leq n$, $1 \leq k \leq n - 1$,

(CL3)   The accident years $(C_{i1}, \dots, C_{in})$, $1 \leq i \leq n$, are independent.

Within this model, the following statements hold (see Mack (1993)):

$E(C_{i,k+1}|C_{i1}, ..., C_{ik}) = C_{ik}f_k,$

$E(C_{in}|C_{i1}, ..., C_{i,n+1-i}) = C_{i,n+1-i}f_{n+1-i} \cdot \ ... \ \cdot f_{n-1},$

$\hat{f}_k$ is the minimum variance unbiased linear estimator of f_k (for w_{ik} and α given),

$\hat{f}_{n+1-i} \cdot \ ... \ \cdot \hat{f}_{n-1}$ is an unbiased estimator of $f_{n+1-i} \cdot \ ... \ \cdot f_{n-1}$.

Therefore, the model CL1-3 can be called underlying the chain ladder algorithm. Furthermore,

$$\hat{\sigma}_k^2 = \frac{1}{n-k-1} \sum_{i=1}^{n-k} w_{ik} C_{ik}^\alpha (F_{ik} - \hat{f}_k)^2, \qquad 1 \le k \le n-2,$$

is an unbiased estimator for $\hat{\sigma}_k^2$ which can be supplemented by

$$\hat{\sigma}_{n-1}^2 = \min(\hat{\sigma}_{n-2}^4/\hat{\sigma}_{n-3}^2, \ \min(\hat{\sigma}_{n-3}^2, \hat{\sigma}_{n-2}^2)) \ .$$

Based on this model for $\alpha = 1$ and all $w_{ik} = 1$, Mack (1993) derived the following formula for the standard error of $\hat{C}_{in}$, which at the same time is the standard error of the estimate $\hat{R}_i = \hat{C}_{in} - C_{i,n+1-i}$ for the claims reserve $R_i = C_{in} - C_{i,n+1-i}$:

$$\left(\text{s.e.}(\hat{C}_{in})\right)^2 = \hat{C}_{in}^2 \sum_{k=n+1-i}^{n-1} \frac{\hat{\sigma}_k^2}{\hat{f}_k^2} \left(\frac{1}{\hat{C}_{ik}} + \frac{1}{\sum_{j=1}^{n-k} C_{jk}} \right) \ .$$

This formula can be rewritten as

$$(*) \qquad (\text{s.e.}(\hat{C}_{in}))^2 = \hat{C}_{in}^2 \sum_{k=n+1-i}^{n-1} \left((\text{s.e.}(F_{ik}))^2 + (\text{s.e.}(\hat{f}_k))^2 \right) \Big/ \hat{f}_k^2$$

where $(\text{s.e.}(F_{ik}))^2$ is an estimate of $\text{Var}(F_{ik}|C_{i1}, ..., C_{ik})$ and $(\text{s.e.}(\hat{f}_k))^2$ is an estimate of

$$\text{Var}(\hat{f}_k) = \sigma_k^2 \Big/ \sum_{j=1}^{n-k} w_{jk} C_{jk}^\alpha \ .$$

In this last form, formula (*) also holds for $\alpha = 0$ and $\alpha = 2$ and any $w_{ik} \in [0; 1]$ as can be seen by applying the proof for $\alpha = 0$ and $w_{ik} = 1$ analogously. Moreover, from this proof the following easily programmable recursion can be gathered:

$$(\text{s.e.}(\hat{C}_{i,k+1}))^2 = \hat{C}_{ik}^2 \left((\text{s.e.}(F_{ik}))^2 + (\text{s.e.}(\hat{f}_k))^2 \right) + (\text{s.e.}(\hat{C}_{ik}))^2 \hat{f}_k^2$$

with starting value s.e.$(\hat{C}_{i,n+1-i}) = 0$. This recursion, which leads to formula (*), is very intuitive: $(\text{s.e.}(F_{ik}))^2$ estimates the (squared) random error $\text{Var}(F_{ik}) = E(F_{ik} - f_k)^2$, i.e. the mean squared deviation of an individual F_{ik} from its true mean f_k, and $(\text{s.e.}(\hat{f}_k))^2$ estimates the (squared) estimation error $\text{Var}(\hat{f}_k) = E(\hat{f}_k - f_k)^2$, i.e. the mean squared deviation of the estimated average $\hat{f}_k$ of the F_{ik}, $1 \leq i \leq n$, from the true f_k. From this interpretation it is clear that we have $\text{Var}(\hat{f}_k) < \text{Var}(F_{ik})$ if $\hat{f}_k$ is unbiased and accident year i belongs to those years over which $\hat{f}_k$ is the average.

INCLUSION OF A TAIL FACTOR

The recursion can immediately be extended to include a tail factor $\hat{f}_{ult}$:

$$\left(\text{s.e.}(\hat{C}_{i,ult})\right)^2 = \hat{C}_{in}^2 \left((\text{s.e.}(F_{i,ult}))^2 + (\text{s.e.}(\hat{f}_{ult}))^2\right) + (\text{s.e.}(\hat{C}_{in}))^2 \hat{f}_{ult}^2$$

and an actuary who develops an estimate for f_{ult} should also be able to develop an estimate s.e.$(\hat{f}_{ult})$ for its estimation error $\sqrt{\text{Var}(\hat{f}_{ult})}$ (How far will $\hat{f}_{ult}$ deviate from f_{ult}?) and an estimate s.e.$(F_{i,ult})$ for the corresponding random error $\sqrt{\text{Var}(F_{i,ult})}$ (How far will any individual $F_{i,ult}$ deviate from f_{ult} on average?). Note that at F_{ik}, f_k and σ_k, index $k = ult$ is the same as $k = n$ whereas at C_{ik} we have $ult = n + 1$.

As a plausibility consideration, we will usually be able to find an index $k < n$ with

$$\hat{f}_{k-1} > \hat{f}_{ult} > \hat{f}_k .$$

Then we can check whether it is reasonable to assume that the inequalities

$$\text{s.e.}(\hat{f}_{k-1}) > \text{s.e.}(\hat{f}_{ult}) > \text{s.e.}(\hat{f}_k)$$

and

$$\text{s.e.}(F_{i,k-1}) > \text{s.e.}(F_{i,ult}) > \text{s.e.}(F_{ik})$$

hold, too, or whether there are reasons to fix s.e.$(\hat{f}_{ult})$ and/or s.e.$(F_{i,ult})$ outside these inequalities.

As an example, we take the data of Table 4 from Mack (1993). From these (using $\alpha = 1$ and all $w_{ik} = 1$, we get the results given in Table 1 for $k = 1, ..., 8$:

TABLE 1

PARAMETER ESTIMATES FOR THE DATA OF TABLE 4 OF MACK (1993)

k	1	2	3	4	5	6	7	8	ult
$\hat{f}_k$	11.10	4.092	1.708	1.276	1.139	1.069	1.026	1.023	1.05
s.e.$(\hat{f}_k)$	2.24	0.517	0.122	0.051	0.042	0.023	0.015	0.012	0.02
s.e.(F_{3k})	7.38	1.89	0.357	0.116	0.078	0.033	0.015	0.007	0.03
$\hat{\sigma}_k$	1337	988.5	440.1	207.0	164.2	74.60	35.49	16.89	71.0

The parameter estimates $\hat{f}_k$ and $\hat{\sigma}_k$ for $1 \leq k \leq 8$ are the same as in Mack (1993). From these, the estimates s.e.$(\hat{f}_k) = \hat{\sigma}_k \bigg/ \sqrt{\sum_{j=1}^{n-k} C_{jk}}$ and s.e. $(F_{ik}) = \hat{\sigma}_k / \sqrt{C_{ik}}$ for $k \leq n+1-i$ or s.e.$(F_{ik}) = \hat{\sigma}_k \bigg/ \sqrt{\hat{C}_{ik}}$ for $k > n+1-i$ are calculated which give the estimation error and the random error, respectively. Note that the random error s.e.(F_{ik}) varies also over the accident years because model assumption CL2 states that for $\alpha = 1$ the variance of the individual development factor F_{ik} is the smaller the greater the previous claims amount (volume) C_{ik} is. Therefore, only the values of s.e.(F_{ik}) for accident year $i = 3$ of average volume are given. The last column of Table 1 shows a possible tail estimation by the actuary: He expects a tail factor of 1.05 with an estimation error of ±0.02 and a random error of ±0.03 for accident year $i = 3$. From this, the estimate $\hat{\sigma}_{ult} = \text{s.e.}(F_{3,ult})\sqrt{\hat{C}_{3,n}} = 71.0$ has been deduced and is used to calculate s.e.$(F_{i,ult})$ for the other accident years. These tail estimates fit well between the columns $k = 6$ and $k = 7$. (Note that the extrapolated estimate for σ_8 leads to a rather small s.e.$(F_{3,8})$ as compared to s.e.$(\hat{f}_8)$. This is due to the fact that $\hat{f}_8$ does not follow a loglinear decay as it was assumed for the calculation of σ_8. Therefore, an estimate $\hat{\sigma}_8 \approx 30$ would have been more reasonable.)

Table 2 shows the resulting estimates for the ultimate claims amounts. The rows $\hat{C}_{i9}$ and s.e.$(\hat{C}_{i9})$ are identical to the results given in Mack (1993). Row $\hat{C}_{i,ult}$ is 5% higher than row $\hat{C}_{i,9}$ and the last row s.e.$(\hat{C}_{i,ult})$ shows the standard errors which result from the formula given above.

i	1	2	3	4	5	6	7	8	9
$\hat{C}_{i9}$	1950	4219	5608	7698	7216	9563	5442	3241	1660
$\hat{C}_{i,ult}$	2048	4420	5888	8073	7577	10041	5714	3403	1743
s.e.$(\hat{C}_{i9})$	0	61	140	319	596	1038	1298	1806	2182
s.e.$(\hat{C}_{i,ult})$	107	180	250	418	670	1128	1377	1902	2293

Finally, we give a recursive formula for the total reserve of all accident years together:

$$\left(\text{s.e.}\left(\sum_{i=n+1-k}^{n}\hat{C}_{i,k+1}\right)\right)^2 = \left(\text{s.e.}\left(\sum_{i=n+2-k}^{n}\hat{C}_{ik}\right)\right)^2 \cdot \hat{f}_k^2 +$$

$$+ \sum_{i=n+1-k}^{n}\hat{C}_{ik}^2 \cdot \left(\text{s.e.}(F_{ik})\right)^2 + \left(\sum_{i=n+1-k}^{n}\hat{C}_{ik}\right)^2 \cdot \left(\text{s.e.}(\hat{f}_k)\right)^2$$

starting at $k = 1$. This formula can also be gathered from the proof of the corollary to Theorem 3 in Mack (1993). In the above example, this formula yields

$$\text{s.e.}\left(\sum_{i=1}^{9}\hat{C}_{i,ult}\right) = 4054$$

as standard error of the ultimate total claims amount $\sum_{i=1}^{9}\hat{C}_{i,ult} = 48906$ (amounts in 1000s).

REFERENCE

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THOMAS MACK
Münchener Rückversicherungs-Gesellschaft
Königinstrasse 107
D-80791 München
e-mail:tmack@munichre.com

NELSON DE PRIL
1950-99

Only 49 years old, Nelson De Pril died on April 9, 1999. It seems so unreal that Nelson is no longer with us. He was so full of life, not abstaining from practical jokes or dancing on the table.

Nelson obtained his doctoral degree in actuarial sciences from the Katholieke Universiteit Leuven in 1979 with a dissertation on bonus-malus systems. The problem of a "fair" segmentation of the market kept his attention, and he had intended to present a paper on that subject at the ASTIN Colloquium in Tokyo.

In the early eighties, recursive methods for aggregate claims distributions became a hot subject. At that time most people in the area were concerned with collective models, that is, methods for compound distributions. However, Nelson turned his mind to individual models. It is interesting to follow the development of his research. First he presented an algorithm for evaluation of the n-fold convolution of a distribution. This algorithm is