

# A NOTE ON THE NET PREMIUM FOR A GENERALIZED LARGEST CLAIMS REINSURANCE COVER

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## ABSTRACT

In the present paper the author gives net premium formulae for a generalized largest claims reinsurance cover. If the claim sizes are mutually independent and identically 3-parametric Pareto distributed and the number of claims has a Poisson, binomial or negative binomial distribution, formulae are given from which numerical values can easily be obtained. The results are based on identities for compounded order statistics.

## KEYWORDS

Net premium; Reinsurance, LCR; ECOMOR, Compounded order statistics

## 1 INTRODUCTION

An expression for the pure premium for the largest claim reinsurance cover was already introduced by AMMETER (1964a) and for the  $p$  largest claims reinsurance cover by AMMETER (1964b). Simple formulae were presented under the assumptions that the claim sizes obeyed a one parametric Pareto distribution and the number of claims was Poisson distributed. For the same claim size distribution KUPPER (1971) gave a formula for the largest claim reinsurance when the number of claims was geometrically distributed and CIMINELLI (1976) considered a negative binomial distribution. BERLINGER (1972) extended the results by AMMETER and deduced the variance for the  $p$  largest claims reinsurance cover. Net premium for a general claim size and claim number distribution was given by KREMER (1985) and for some generalized claim number distributions and a general claim size distribution by KREMER (1988a). The results in the latter were, however, not so practical for a specific claim size distribution. The author of this paper gives net premium formulae for a generalized largest claims reinsurance cover, assuming that the claim sizes are mutually independent and identically 3-parametric Pareto distributed and when the number of claims has a Poisson, binomial or negative binomial distribution. The formulae presented in this paper are simple and easily calculated.

2 PRELIMINARIES

From now on, let  $X_1, X_2, \dots, X_N$  denote non-negative, mutually independent and identically distributed claim sizes, which are independent of the number of claims  $N$  that occur in a given time period. Denote by

$$X_{N:1} \geq X_{N:2} \geq \dots \geq X_{N:N}$$

the claims ordered in a decreasing size. The  $i$ -th largest claim is called the  $i$ -th ordered claim or more generally the  $i$ -th compounded order statistic. Let

$$f_i: [0, \infty) \rightarrow [0, \infty)$$

( $i \geq 1$ ) be measurable functions, that satisfy

$$f_i(0) = 0 \text{ and } \sum_{i=1}^n f_i(y_i) \in \left[ 0, \sum_{i=1}^n y_i \right]$$

for all  $0 \leq y_n \leq \dots \leq y_2 \leq y_1$ . This representation was first made by KREMER (1982) and the following main definition by KREMER (1984):

**Definition.** The reinsurance treaty defined by

$$R_N(X_{N:1}, X_{N:2}, \dots, X_{N:N}) = R_N = \sum_{i=1}^N f_i(X_{N:i}),$$

which determines the reinsurers share of the total loss  $\sum_{i=1}^N X_i$ , is called a reinsurance treaty based on ordered claims

We are especially interested in the case

$$f_i(x) = a_i x,$$

where  $a_i, i \geq 1$ , are real constants. This reinsurance treaty is defined as the generalized largest claims cover (KREMER 1988b). We get for

$$a_1 = a_2 = \dots = a_p = 1, a_i = 0 \quad \forall i > p$$

the so called LCR( $p$ ) treaty covering the  $p$  largest claims and for

$$a_1 = a_2 = \dots = a_{p-1} = 1, a_p = 1 - p, a_i = 0 \quad \forall i > p$$

the so called ECOMOR( $p$ ) treaty covering all claims in excess of the  $p$ -th largest claim

We will subsequently use some special functions. The incomplete gamma function is defined as

$$\Gamma(a, x) = \int_0^x e^{-u} u^{a-1} du \quad , \quad a > 0, \quad x \geq 0$$

and the complete gamma function as  $\lim_{x \rightarrow \infty} \Gamma(a, x) = \Gamma(a)$  The incomplete beta function is defined as

$$B(a, b, x) = \int_0^x u^{a-1}(1-u)^{b-1} du = \int_0^{\frac{x}{1-x}} u^{a-1}(1+u)^{-(a+b)} du \quad , a, b > 0, 0 \leq x < 1$$

and the complete beta function as  $\lim_{x \rightarrow 1} B(a, b, x) = B(a, b)$ . The complete beta function and the complete gamma function are related by

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

### 3 FORMULAE FOR THE NET PREMIUM

The two most common risk loaded premium principles, the variance principle and the standard deviation principle, are based on the expectation and the variance of a certain risk For a generalized largest claims reinsurance cover the expectation is given by

$$E[R_N] = \sum_{i=1}^{\infty} a_i E[X_{N_i}]$$

and the variance by

$$Var[R_N] = \sum_{i=1}^{\infty} a_i^2 E[X_{N_i}^2] + 2 \sum_{j=2}^{\infty} \sum_{i=1}^{j-1} a_i a_j E[X_{N_i} X_{N_j}] - \left( \sum_{i=1}^{\infty} a_i E[X_{N_i}] \right)^2$$

The following theorem is due to CIMINELLI (1976) and KREMER (1985), where

$$\phi(s) = \sum_{n=0}^{\infty} P(N = n) s^n$$

denotes the probability generating function of  $N$ , which is assumed to have derivatives  $\phi^{(i)}$  on  $(0,1)$  of each order  $i \geq 1$

**Theorem 1** If the claim sizes  $X_1, X_2, \dots, X_N$  have a continuous distribution function  $F$  the density function of the  $i$ -th ordered claim is given by

$$P(X_{N_i} = x) = \frac{1}{\Gamma(i)} f(x) [1 - F(x)]^{i-1} \phi^{(i)}(F(x))$$

and the joint density function of the  $i$ -th and  $j$ -th ordered claims  $(0 < i < j)$  is given by

$$P(X_{N_i} = x_i, X_{N_j} = x_j) = \frac{1}{\Gamma(i)\Gamma(j-i)} [1 - F(x_i)]^{i-1} [F(x_i) - F(x_j)]^{j-i-1} \phi^{(j)}(F(x_j)) f(x_i) f(x_j)$$

**Theorem 2** If the claim sizes  $X_1, X_2, \dots, X_N$  have a continuous distribution function  $F$  the  $k$ -th moment around the origin of the  $i$ -th ordered claim is given by

$$E[X_{N,i}^k] = \frac{1}{\Gamma(i)} \int_0^1 F^{-1}(u)^k [1 - u]^{i-1} \phi^{(i)}(u) du$$

and the expectation of the cross product of the  $i$ -th and  $j$ -th ordered claims ( $0 < i < j$ ) is given by

$$E[X_{N,i} X_{N,j}] = \frac{1}{\Gamma(i)\Gamma(j-i)} \int_0^1 F^{-1}(v)(1-v)^{j-i} \phi^{(j)}(v) \int_0^1 F^{-1}(1-u(1-v))u^{i-1}(1-u)^{j-i-1} dudv.$$

**Proof** The first part of the statement follows from theorem 1 after the substitution  $u = F(x)$ . For the second part we have for  $0 < i < j$  and  $0 \leq X_{N,j} \leq X_{N,i}$  that

$$E[X_{N,i} X_{N,j}] = C \int_0^\infty \int_{x_j}^\infty x_i x_j [1 - F(x_i)]^{i-1} [F(x_i) - F(x_j)]^{j-i-1} \phi^{(j)}(F(x_j)) f(x_i) f(x_j) dx_i dx_j,$$

where

$$C = \frac{1}{\Gamma(i)\Gamma(j-i)}$$

After substituting  $u = \frac{1-F(x_i)}{1-F(x_j)}$  and  $v = F(x_j)$  we obtain

$$E[X_{N,i} X_{N,j}] = C \int_0^1 \int_0^1 F^{-1}(1-u(1-v))F^{-1}(v)u^{i-1}[1-v]^{j-i-1}[1-u]^{j-i-1}\phi^{(j)}(v)dudv = C \int_0^1 F^{-1}(v)[1-v]^{j-i-1}\phi^{(j)}(v) \int_0^1 F^{-1}(1-u(1-v))u^{i-1}[1-u]^{j-i-1} dudv$$

□

From now on we will focus on the case where the claim sizes are distributed according to the 3-parametric Pareto distribution

$$F(x) = 1 - \left(\frac{d + \beta}{x + \beta}\right)^\alpha \quad x \geq d > 0. \tag{3.1}$$

where the parameters  $\alpha$ ,  $\beta$  and  $d$  satisfy  $\alpha > 0$  and  $b > -d$ . The distribution (3.1) is the most used claim size distribution, especially if there is a possibility of large claims. In the literature the 3-parametric Pareto distribution is sometimes also called the "shifted" Pareto distribution (RYTGAARD 1990) or the complete Pareto distribution (DAYKIN et al. 1994). Since

$$F^{-1}(x) = \frac{d + \beta}{(1 - x)^{\frac{1}{\alpha}}} - \beta$$

the expectations of theorem 2 becomes after binomial expansion and simplifications

$$E[X_N^k | J] = \frac{1}{\Gamma(t)} \sum_{h=0}^k \binom{k}{h} (d + \beta)^{k-h} (-\beta)^h \int_0^1 (1 - u)^{t - \frac{k-h}{\alpha} - 1} \phi^{(h)}(u) du$$

and for  $\alpha > \frac{1}{t}$

$$E[X_N | X_N | J] = \frac{1}{\Gamma(t)} \left[ A_1 \int_0^1 (1 - v)^{t - \frac{1}{\alpha} - 1} \phi^{(1)}(v) dv - A_2 \int_0^1 (1 - v)^{t - \frac{1}{\alpha} - 1} \phi^{(2)}(v) dv + A_3 \int_0^1 (1 - v)^{t - 1} \phi^{(3)}(v) dv \right]$$

where

$$A_1 = (d + \beta)^2 \frac{\Gamma(t - \frac{1}{\alpha})}{\Gamma(t - \frac{1}{\alpha})}$$

$$A_2 = \beta(d + \beta) \left[ \frac{\Gamma(t)}{\Gamma(t)} + \frac{\Gamma(t - \frac{1}{\alpha})}{\Gamma(t - \frac{1}{\alpha})} \right]$$

$$A_3 = \beta^2 \frac{\Gamma(t)}{\Gamma(t)}$$

The restriction on the parameter  $\alpha$  is needed to get a finite expression. Assuming further that the number of claims  $N$  is Poisson distributed

$$P(N = n) = \frac{\lambda^n}{n!} e^{-\lambda} \quad \lambda > 0, n \geq 0, \quad (3.2)$$

negative binomially distributed

$$P(N = n) = \frac{\Gamma(r + n)}{\Gamma(r)n!} \left( \frac{1}{1 + \lambda} \right)^r \left( \frac{\lambda}{1 + \lambda} \right)^n \quad r, \lambda > 0, n \geq 0 \quad (3.3)$$

or binomially distributed

$$P(N = n) = \binom{m}{n} q^n (1 - q)^{m-n} \quad 0 \leq q \leq 1, n = 0, 1, \dots, m, \tag{3.4}$$

where  $m$  is a non-negative integer, we have the following corollaries

**Corollary 3.** Assume that the claim sizes  $X_1, X_2, \dots, X_N$  are Pareto distributed (3.1) and that the claim number  $N$  is Poisson distributed (3.2). Then the  $k$ -th moment around the origin of the  $i$ -th ordered claim is, for  $\alpha > \frac{k}{i}$  given by

$$E[X_N^k] = \frac{1}{\Gamma(i)} \sum_{h=0}^k \binom{k}{h} (d + \beta)^{k-h} (-\beta)^h \lambda^{\frac{k-h}{\alpha}} \Gamma(i - \frac{k-h}{\alpha}, \lambda)$$

and the expectation of the cross product of the  $i$ -th and  $j$ -th ordered claims ( $0 < i < j$ ) is, for  $\alpha > \max\{\frac{i}{j}, \frac{2}{j}\}$ , given by

$$E[X_{N,i} X_{N,j}] = \frac{1}{\Gamma(i)} \left[ A_1 \lambda^{\frac{2}{\alpha}} \Gamma(j - \frac{2}{\alpha}, \lambda) - A_2 \lambda^{\frac{1}{\alpha}} \Gamma(j - \frac{1}{\alpha}, \lambda) + A_3 \Gamma(j, \lambda) \right]$$

**Proof** Since the  $j$ -th derivative of the probability generating function  $\phi$  for a Poisson distributed random variable (3.2) is given by

$$\phi^{(j)}(s) = \lambda^j e^{\lambda(s-1)}$$

we have, for  $\gamma > 0$ , that

$$\int_0^1 (1-u)^{\gamma-1} \phi^{(j)}(u) du = \lambda^j \int_0^1 (1-u)^{\gamma-1} e^{\lambda(u-1)} du$$

After the substitution  $t = \lambda(1-u)$  we obtain

$$\begin{aligned} \int_0^1 (1-u)^{\gamma-1} \phi^{(j)}(u) du &= \lambda^{-\gamma} \int_0^\lambda t^{\gamma-1} e^{-t} dt \\ &= \lambda^{-\gamma} \Gamma(\gamma, \lambda). \end{aligned}$$

which gives the result. □

**Corollary 4** Assume that the claim sizes  $X_1, X_2, \dots, X_N$  are Pareto distributed (3.1) and that the claim number  $N$  is negative binomially distributed (3.3). Then the  $k$ -th moment around the origin of the  $i$ -th ordered claim is, for  $\alpha > \frac{k}{i}$ , given by

$$E[X_{N,i}^k] = \frac{l}{B(i,r)} \sum_{h=0}^k \binom{k}{h} (d + \beta)^{k-h} (-\beta)^h \lambda^{\frac{k-h}{\alpha}} B(i - \frac{k-h}{\alpha}, r + \frac{k-h}{\alpha}; \frac{\lambda}{l+\lambda})$$

and the expectation of the cross product of the  $i$ -th and  $j$ -th ordered claims ( $0 < i < j$ ) is, for  $\alpha > \max\{\frac{l}{i}, \frac{2}{j}\}$ , given by

$$E[X_{N,i} X_{N,j}] = \frac{l}{B(j,r)} \frac{\Gamma(j)}{\Gamma(i)} [A_1 \lambda^{\frac{2}{\alpha}} B(j - \frac{2}{\alpha}, r + \frac{2}{\alpha}, \frac{\lambda}{l+\lambda}) - A_2 \lambda^{\frac{1}{\alpha}} B(j - \frac{1}{\alpha}, r + \frac{1}{\alpha}, \frac{\lambda}{l+\lambda}) + A_3 B(j; i, \frac{\lambda}{l+\lambda})]$$

**Proof** Since the  $j$ -th derivative of the probability generating function  $\phi$  for a negative binomially distributed random variable (3.3) is given by

$$\phi^{(j)}(s) = \frac{\Gamma(r+j)}{\Gamma(r)} \lambda^j [l - \lambda(s - l)]^{-(r+j)}$$

we have, for  $\gamma > 0$ , that

$$\int_0^l (l - u)^{\gamma-l} \phi^{(j)}(u) du = \frac{\Gamma(r+j)}{\Gamma(r)} \lambda^j \int_0^l (l - u)^{\gamma-l} [l - \lambda(u - l)]^{-(r+j)} du$$

After the substitution  $t = \lambda(l - u)$  we obtain

$$\begin{aligned} \int_0^l (l - u)^{\gamma-l} \phi^{(j)}(u) du &= \frac{\Gamma(r+j)}{\Gamma(r)} \lambda^{\gamma-j} \int_0^{\lambda} t^{\gamma-l} (l + t)^{-(r+j)} dt \\ &= \frac{\Gamma(r+j)}{\Gamma(r)} \lambda^{\gamma-j} B(\gamma; l + j - \gamma, \frac{\lambda}{l+\lambda}), \end{aligned}$$

from which the result follows after simplification □

**Corollary 5.** Assume that the claim sizes  $X_1, X_2, \dots, X_N$  are Pareto distributed (3.1) and that the claim number  $N$  is binomially distributed (3.4). Then the  $k$ -th moment around the origin of the  $i$ -th ordered claim is, for  $\alpha > \frac{k}{i}$ , given by

$$E[X_{N,i}^k] = i \binom{m}{i} \sum_{h=0}^k \binom{k}{h} (d + \beta)^{k-h} (-\beta)^h q^{\frac{k-h}{\alpha}} B(i - \frac{k-h}{\alpha}, m - i + l, q)$$

and the expectation of the cross product of the  $i$ -th and  $j$ -th ordered claims ( $0 < i < j$ ) is, for  $\alpha > \max\{\frac{l}{i}, \frac{2}{j}\}$ , given by

$$E[X_{N,i} X_{N,j}] = \binom{m}{j} \frac{\Gamma(j+l)}{\Gamma(i)} [A_1 q^{\frac{2}{\alpha}} B(j - \frac{2}{\alpha}, m - j + l, q) - A_2 q^{\frac{1}{\alpha}} B(j - \frac{1}{\alpha}, m - j + l, q) + A_3 B(j; m - j + l, q)]$$

**Proof** Since the  $j$ -th derivative of the probability generating function  $\phi$  for a binomially distributed random variable (3.4) is given by

$$\phi^{(j)}(s) = \frac{\Gamma(m+1)}{\Gamma(m-j+1)} q^j [qs + 1 - q]^{m-j} \quad j \leq m,$$

we have, for  $\gamma > 0$ , that

$$\int_0^1 (1-u)^{\gamma-1} \phi^{(j)}(u) du = \frac{\Gamma(m+1)}{\Gamma(m-j+1)} q^j \int_0^1 (1-u)^{\gamma-1} (qu + 1 - q)^{m-j} du$$

After the substitution  $t = q(1-u)$  we obtain

$$\begin{aligned} \int_0^1 (1-u)^{\gamma-1} \phi^{(j)}(u) du &= \frac{\Gamma(m+1)}{\Gamma(m-j+1)} q^{j-\gamma} \int_0^q t^{\gamma-1} (1-t)^{m-j} dt \\ &= \frac{\Gamma(m+1)}{\Gamma(m-j+1)} q^{j-\gamma} B(\gamma, m-j+1, q), \end{aligned}$$

from which the result follows after simplification □

If  $0 < \alpha < 1$ , which indicates a very heavy tailed distribution, we have according to the results above that the first moment around origin of a certain number of the largest ordered claims does not exist. We could therefore consider the number of ordered claims, for which the first moment around the origin does not exist, as a measure for how dangerous a Pareto distribution is. Since many computer programs have built-in routines for computing the complete gamma, incomplete gamma and the incomplete beta function, the expectations in results above can be calculated easily.

If the claim sizes obey an exponential distribution

$$F(x) = 1 - e^{-\beta(x-a)} \quad \beta > 0, x \geq a,$$

we cannot get useful expressions for the moments around the origin and the cross product by applying theorem 2. Using well known results from order statistics for a deterministic number of claims (DAVID 1970) and then the iterativity of the expectation operator, expression for the pure premium can be constructed. Exponentially distributed claim sizes have been studied by KUPPER (1971) and KREMER (1985 and 1986).

#### 4 A NUMERICAL EXAMPLE

Let the distribution for the claim sizes be Pareto distributed (3.1) with  $d = 0$ . For the insurance line under consideration the method of moments gives the following parameter estimates  $\hat{\alpha} = 2.3401$  and  $\hat{\beta} = 13692$ . Since the most important claim number distributions are the Poisson and the negative binomial, we will restrict



the example to them. Using the same estimation method we have the following parameter estimates. Poisson  $\hat{\lambda} = 79\ 667$ , negative binomial  $\hat{\lambda} = 1\ 0865$  and  $\hat{r} = 73\ 326$ . We have the following numerical results

Expectation of LCR( $p$ ) and ECOMOR( $p$ ) treaties

$p$	Poisson	negative binomial	$p$	Poisson	negative binomial
1	124 597	124 368	1	0	0
2	190 099	189 738	2	59 095	58 997
3	238 679	238 215	3	92 937	92 783
4	278 390	277 837	4	119 548	119 350
5	312 395	311 763	5	142 369	142 133
LCR( $p$ )-treaty			ECOMOR ( $p$ )-treaty		

Standard deviation of LCR( $p$ ) and ECOMOR( $p$ ) treaties

$p$	Poisson	negative binomial	$p$	Poisson	negative binomial
1	178 069	178 129	1	0	0
2	191 632	191 860	2	134 587	134 549
3	198 847	199 254	3	182 222	182 206
4	203 797	204 389	4	188 799	188 815
5	207 581	208 363	5	193 255	193 405
LCR( $p$ )-treaty			ECOMOR ( $p$ )-treaty		

The difference between the numerical values for Poisson and the negative binomial cases is quite small. If we assume that in the incomplete beta function  $b$  is large and  $a$  is bounded we have the following asymptotic representation (ABRAMOWITZ and STEGUN 1972)

$$\frac{B(a; b; x)}{B(a; b)} = \frac{\Gamma\left(a, \frac{x(2b + a - 1)}{2 - x}\right)}{\Gamma(a)} + O(b^{-2}).$$

This explains the similarity in the numerical results above. This suggests, that the Poisson distribution might be the right claim number model if the parameter value  $r$  is large and  $\lambda$  is small in the negative binomial distribution

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