A PORTFOLIO OF ENDOWMENT POLICIES
AND ITS LIMITING DISTRIBUTION

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Abstract

Two methods for approximating the limiting distribution of the present value of the benefits of a portfolio of identical endowment insurance contracts are suggested. The model used assumes that both future lifetimes and interest rates are random The first method is similar to the one presented in PARKER (1994b). The second method is based on the relationship between temporary and endowment insurance contracts.

1. INTRODUCTION

In PARKER (1994b) we present a method for approximating the distribution of the present value of the average cost per policy of a portfolio of temporary insurance contracts In this paper, based on chapter 7 of PARKER (1992), we first show how this method can be applied to portfolios of endowment contracts. The relationship between temporary and endowment policies is also used to suggest an alternative way of approximating this distribution Some justifications for the suggested approximation are given. Finally, we indicate how to extend these results to the case of contracts payable by annual premiums The references section lists some recent papers on the subject of actuarial functions with random interest and mortality

2. A PORTFOLIO

Consider a portfolio of c identical n-year endowment insurance contracts, all with sum insured 1, being sold to lives insured at age x at time of issue Let $\mathcal{L}(c)$ be the random variable representing the present value of the benefits of the portfolio. This is relevant for contracts payable by single premiums

Here, summing over the n policy-years of the contract and using the notation of PARKER (1994b), one can express $\mathcal{L}(c)$ as

$$(2.1) \qquad \mathcal{L}(c) = \sum_{i=0}^{n-1} c_i \ e^{-y(i+1)} + c_n \ e^{-y(n)}$$

When studying $\mathscr{L}(c)$, we will again assume that $(c_i)_{i=1}^n$ is multinomial. We also assume that the future lifetimes and the forces of interest, $\{\delta_s\}_{s \geq 0}$, are independent.

For the portfolio under consideration the average cost per policy, $\mathscr{L}(c)/c$, tends almost surely, hence also in distribution to (see, for example, FREES (1990)).

$$(2\ 2) \qquad \beta_n = \sum_{i=0}^{n-1} {}_i|q_x \; e^{-y(i+1)} + {}_np_x \cdot e^{-y(n)}$$

3. A RECURSIVE APPROXIMATION

Let z_i and y_j be possible realizations of β_i and $y(j)$ respectively. And let the function $h_n(z_n, y_n)$ be defined as

$$(3.1) \qquad h_n(z_n, y_n) = P(\beta_n \leq z_n) \cdot f_{y(n)}(y_n \mid \beta_n \leq z_n)$$

or equivalently,

$$(3\ 2) \qquad h_n(z_n, y_n) = f_{y(n)}(y_n) \cdot P(\beta_n \leq z_n \mid y(n) = y_n)$$

where $f_{y(n)}$ is the probability density function (pdf) of $y(n)$

It is an immediate consequence of this definition that the pdf of β_n is

$$(3\ 3) \qquad F_{\beta_n}(z_n) = \int_{-\infty}^{\infty} h_n(z_n, y_n) \; dy_n$$

This function $h_n(z_n, y_n)$ is similar to the function $g_n(z_n, y_n)$ defined in section 4 of PARKER (1994b). It can also be calculated recursively with a high degree of accuracy by using

$$(3.4) \qquad h_n(z_n, y_n) \cong \int_{-\infty}^{\infty} f_{y(n)}(y_n \mid y(n-1) = y_{n-1})$$

$$h_{n-1}(z_n - {}_{n-1}p_x \cdot (e^{-y_n} - e^{-y_{n-1}}), y_{n-1}) \, dy_{n-1}.$$

with the starting value

$$(3\ 5) \qquad h_1(z_1, y_1) = \begin{cases} \Phi\left(\dfrac{y_1 - E[y(1)]}{V[y(1)]^{0\,5}}\right) & \text{if } z_1 \geq e^{-y_1}, \\[2ex] 0 & \text{otherwise,} \end{cases}$$

where $\Phi(\ )$ denotes the pdf of a standardized normal random variable.

This approximation was obtained by assuming that

$$(3\ 6) \qquad f_{\beta_{n-1}}(z - {}_{n-1}p_x \; (e^{-y_n} - e^{-y_{n-1}}) \mid y(n) = y_n, \, y(n-1) = y_{n-1})$$

$$\cong f_{\beta_{n-1}}(z - {}_{n-1}p_x \; (e^{-y_n} - e^{-y_{n-1}}) \mid y(n-1) = y_{n-1})$$

Recall that $\{y(t)\}_{t=1}^{n}$ is multivariate normal, therefore, $y(n)$ given $y(n-1)$ is normally distributed with mean and variance given by (4 7)[1] and (4.8) of PARKER (1994b)

Note that (3 4) is exact if $y(t)$ is a Markov process. However, when modelling the force of interest by a Wiener process or an Ornstein-Uhlenbeck process, the resulting $y(t)$ is not Markov and (3.4) is an approximation

4 JUSTIFICATIONS

Our justifications of the approximation are based on two correlation coefficients which we would like to be high From Table 1 of PARKER (1994b), we know that $y(n)$ and $y(n-1)$ are highly correlated and this suggests that the approximation is acceptable. From MARDIA, KENT and BIBBY (1979, section 6.5) the approximation would also be acceptable if the correlation coefficient between $e^{-y(n)}$ and β_n is high This correlation is given by

$$(4\ 1)\quad \rho(e^{-y(n)}, \beta_n) = \frac{\text{cov}\left(e^{-y(n)}, \sum_{i=0}^{n-1}{}_{i}|q_i \cdot e^{-y(i+1)} + {}_{n}p_i \quad e^{-y(n)}\right)}{\left\{V[e^{-y(n)}]\quad V\left[\sum_{i=0}^{n-1}{}_{i}|q_i \cdot e^{-y(i+1)} + {}_{n}p_i \cdot e^{-y(n)}\right]\right\}^{1/2}}$$

Using the male ultimate rates of the CA 1980-82 mortality table (COWARD, 1988, pp 227-231), Table 1 shows that the correlation coefficients are high especially for n small

Note that $\rho(e^{-y(1)}, \beta_1)$ is exactly 1, this implies that the first recursion in (3 4) is always exact

TABLE I

CORRELATION COEFFICIENT BETWEEN $e^{-y(n)}$ AND β_n
FORCE OF INTEREST AS WHITE NOISE AND ORNSTEIN-UHLLNBECK PROCESSES

n	White Noise * $\Delta = 0\ 06,\ \sigma_w = 0\ 01$ $\lambda = 30$	Ornstein-Uhlenbeck $\delta = 0\ 06,\ \delta_0 = 0\ 1,\ \alpha = 0\ 1$		
		$\sigma = 0\ 01\ \lambda = 30$	$\sigma = 0\ 02\ \lambda = 30$	$\sigma = 0\ 01\ \lambda = 50$
1	1 000000	1 000000	1 000000	1 000000
5	0 999998	1 000000	1 000000	0 999998
10	0 999983	0 999996	0 999996	0 999848
20	0 999742	0 999915	0 999924	0 996258
40	0 982191	0 990842	0 992647	0 887285
60	0 859629	0 884964	0 871195	—

* In the case of the White Noise process, we know that (3 4) is exact Nevertheless, this case is of interest since it provides a lower bound for the correlation coefficients resulting from the use of other Gaussian processes for the force of interest

[1] Note that the denominator $V[y(n)]$ should be $V[y(n-1)]$

5 RELATIONSHIP BETWEEN g_n AND h_n

The function g_n introduced in PARKER (1994b) and the function h_n are of very similar nature, but are based on different random variables (ζ_n for the former and β_n for the latter) These random variables are the limiting average cost per policy of a portfolio of temporary and endowment insurance contracts respectively

As the difference between endowment insurance and temporary insurance is the pure endowment benefit, it is possible to link the random variables, ζ_n and β_n We have,

$$(5\ 1) \qquad\qquad \beta_n = \zeta_n + {}_np_\lambda \cdot e^{-\lambda(n)}.$$

This leads to the following relationship between g_n and h_n.

Theorem: The functions g_n and h_n are linked by the following equation

$$(5\ 2) \qquad\qquad h_n(z_n, y_n) = g_n(z_n - {}_np_\lambda\ e^{-\lambda_n}, y_n)$$

Proof: From definition (3 2), conditioning on $y(n) = y_n$ and using (5.1), we have

$$(5.3)\quad h_n(z_n, y_n) = f_{\lambda(n)}(y_n)\ P(\beta_n - {}_np_\lambda\ e^{-y(n)} \leq z_n - {}_np_\lambda\cdot e^{-\lambda(n)} \mid y(n) = v_n)$$
$$= f_{\lambda(n)}(y_n)\cdot P(\zeta_n \leq z_n - {}_np_\lambda\cdot e^{-v_n} \mid y(n) = y_n)$$

Finally, from (4 3) of PARKER (1994b), the right-hand side of (5 3) is simply $g_n(z_n - {}_np_\lambda\ e^{-\lambda_n}, y_n)$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

An immediate use of this result is to offer an alternative way of approximating the distribution of β_n From (3.3) and (5 2), we have

$$(5\ 4) \qquad\qquad F_{\beta_n}(z_n) = \int_{-\infty}^{\infty} g_n(z_n - {}_np_\lambda\ e^{-\lambda_n}, y_n)\ dy_n$$

So the distribution function of β_n can be obtained by the numerical integration of (3.3) or (5.4). It should be noted that using (5 4) involves one additional source of inaccuracy in that the value of $g_n(z_n - {}_np_\lambda\ e^{-\lambda_n}, y_n)$ has to be approximated (say by linear interpolation) because it is generally not evaluated at the specific values that $z_n - {}_np_\lambda\ e^{-\lambda_n}$ may take in (5.4)

6. ILLUSTRATIONS

An arbitrary discritization method (details are provided in PARKER (1992)) using the relationship between g_n and h_n has been used to obtain the distribution of β_n for different values of n The results are illustrated in Figure 1

FIGURE 1 Cumulative distribution function of β_n
Endowment insurance policies issued at age 30
Ornstein-Uhlenbeck $\delta = 0.06$ $\delta_0 = 0.1$, $\alpha = 0.1$, $\sigma = 0.01$


```
———    5 years
- -   10 years
      15 years
- -   20 years
- -   25 years
```
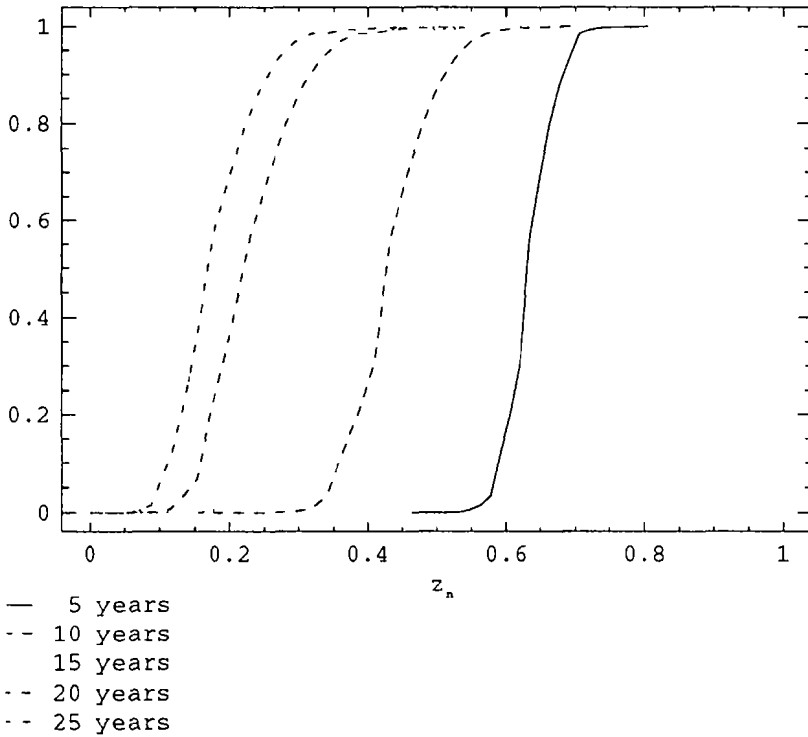
The range of possible values for β_5 is shorter than the one for β_{25} This is due to the fact that with a limiting portfolio, there is no fluctuation due to mortality, and therefore, all the possible variations in the random variable β_n are caused by the force of interest When there are only five years of fluctuating force of interest involved, it is clear that the results will be less spread than when there are 25 years of fluctuating force of interest Due to the later time of payment of the survival benefit, β_{25} takes smaller values than β_5

Useful information for pricing or solvency purposes is contained in the right tail of the distribution of β_n Some illustrative right-tail probabilities of the distributions of β_5 and β_{25} are contained in Table 2.

From Table 2, we know, for example, that a company charging a single premium of 0 734243 to each life insured in a very large portfolio of 5-year endowment contracts will meet its future liabilities with a probability of about 0.9968. Of course, the company may choose to charge a lower premium In this case, the difference should be put aside if the future liabilities are to be met with the same probability

TABLE 2

RIGHT TAIL OF THE APPROXIMATE DISTRIBUTION OF β_n
5 AND 25 YEARS ENDOWMENT INSURANCE ISSUED AT AGE 30
ORNSTEIN-UHLENBECK $\delta = 0\ 06,\ \delta_0 = 0\ 1,\ \alpha = 0\ 1,\ \sigma = 0\ 01$

5 years endowment		25 years endowment	
z_5	$F_{\beta_5}(z_5)$	z_{25}	$F_{\beta_{25}}(z_{25})$
0 691584	0 931803	0 270676	0 933177
0 705804	0 983629	0 294392	0 966609
0 720023	0 992748	0 365540	0 993782
0 734243	0 996828	0 412971	0 997693
0 776901	0 999589	0 460403	0 999229

7. VALIDATIONS

The method has been validated by comparing the exact first three moments of β_n with the first three moments estimated from the distribution illustrated in section 6 (see Table 3). The expressions for the estimated moments can be found in section 7 of PARKER (1994b).

TABLE 3

COMPARISON OF EXACT AND APPROXIMATE MOMENTS OF β_n
n-YEAR ENDOWMENT INSURANCE ISSUED AT AGE 30
ORNSTEIN-UHLENBECK $\delta = 0\ 06,\ \delta_0 = 0\ 1,\ \alpha = 0\ 1,\ \sigma = 0\ 01$

n	$E[\beta_n^m]$			$E[\beta_n^m] - \hat{E}[\beta_n^m]$		
	$m = 1$	$m = 2$	$m = 3$	$m = 1$	$m = 2$	$m = 3$
1	0 90660	0 82196	0 74523	0 00000	0 00000	0 00000
5	0 63471	0 40402	0 25792	0 00025	0 00027	0 00022
10	0 43263	0 19026	0 08505	− 0 00054	− 0 00069	− 0 00061
15	0 30965	0 09980	0 03348	− 0 00056	− 0 00052	− 0 00034
20	0 22975	0 05658	0 01494	0 00005	− 0 00027	− 0 00020
25	0 17581	0 03415	0 00734	− 0 00263	− 0 00100	− 0 00029

The exact moments of β_n about the origin may be obtained by using the definition of β_n given by (2 2) Its mth moment about the origin is then given by

$$(7.1) \qquad E[\beta_n^m] = E\left[\left(\sum_{t=0}^{n-1} {}_t|q_x \cdot e^{-\delta(t+1)} + {}_np_x \cdot e^{-\delta(n)}\right)^m\right]$$

For example, we obtain

$$
(7.2) \quad E[\mathfrak{Z}_n^3] = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} {}_i|q_x \cdot {}_j|q_x \; {}_k|q_x \; E[e^{-\nu(i+1)-\nu(j+1)-\nu(k+1)}]
$$

$$
+ 3 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} {}_i|q_x \; {}_j|q_x \; {}_np_x \; E[e^{-\nu(i+1)-\nu(j+1)-\nu(n)}]
$$

$$
+ 3 \sum_{i=0}^{n-1} {}_i|q_x \cdot ({}_np_x)^2 \; E[e^{-\nu(i+1)-2\nu(n)}] + ({}_np_x)^3 \; E[e^{-3\nu(n)}]
$$

The moments of $\mathfrak{Z}_n$ are exactly the limiting moments of the average cost per policy obtained by PARKER (1994d).

For $n \le 25$, the absolute difference between the exact and the approximate first three moments of $\mathfrak{Z}_n$ is always less than 0.003 and the relative error is less than about 4%.

A discretization of (3 3) and (3.4) was also tried. It generally gave very similar results to those presented here, but on some occasions, the results were worse due to round-off errors

8 ANNUAL PREMIUMS

The ideas presented here are easily extended to more general types of insurance or annuity contracts Note that for contracts where the benefits depend on the interest process, the application of these results may not be so straightforward

As a simple illustration of how to extend the approach presented here, consider a portfolio of n-year endowment contracts with a benefit of 1 payable by annual premiums π_i, $i = 0, 1, ., n - 1$ From proposition 5 of FREES (1990), we know that the distribution of the present value of the cash flows generated by this portfolio tends in distribution to

$$
(8.1) \qquad \qquad \mathfrak{Z}_n = \sum_{i=0}^{n} CF_i \; e^{-\nu(i)},
$$

where

$$
(8.2) \qquad CF_i = \begin{cases} -\pi_0 & i = 0 \\ {}_{i-1}|q_x - \pi_i \; {}_ip_x & i = 1, 2, \quad , n-1 \\ {}_{n-1}p_x & i = n \end{cases}
$$

One can then show, with appropriate modifications, that the cumulative distribution function of $\mathfrak{Z}_n$ is given by (3.3) and (3 4) but with the first argument of h_{n-1} in (3 4) replaced by $z_n - CF_n \cdot e^{-\nu_n}$. The condition for the non-zero starting value in (3 5) should also be changed to $z_n \ge CF_0 + CF_1 \cdot e^{-\nu_1}$.

Note that the intermediary values for h_i, $i = 1, 2, ..., n - 1$ are specific to portfolios of n-year endowment contracts. They relate to the first i annual cash flows generated by that portfolio and should not be used for portfolios of i-year endowment contracts This is in contrast with the results of sections 3 and 5 where intermediary h_i can be used to obtain the results for portfolios of endowment contracts of duration up to n.

9. CONCLUSION

In this paper, we have suggested two ways of approximating the distribution of limiting portfolios which can be used as long as the conditional probability density function of $y(n)$ given $y(n - 1)$ and some covariances are known. Using a Gaussian process for the interest rates simplifies things considerably.

From the very nature of the problem at hand, approximation (3 4) appears to be highly acceptable The worst possible case for Gaussian interest rates is when they are independent, i.e a White-Noise process. Even in this case, the correlation resulting between consecutive present value functions is fairly high

Due to the round-off errors frequently encountered when using the method involving h_n, it is preferable to use the relationship between temporary and endowment insurance contracts, i.e. (5.4) which uses the equation linking g_n and h_n

The distribution of the average cost per policy is useful in pricing, valuation, solvency and reinsurance The justified and validated approximation we suggest in this paper is certainly accurate enough for these purposes.

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