A SEMIMARTINGALE APPROACH TO SOME PROBLEMS IN RISK THEORY

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Abstract

The purpose of this note is to draw attention to a semimartingale method which can be applied to very general types of risk models to obtain local martingales or martingales, which can then be used in the now classical way to evaluate ruin probabilities. Relations to the theory of exponential families of stochastic processes are also pointed out and utilized.

1 INTRODUCTION

Since Gerber (1973) introduced the use of martingale methods in risk theory, these methods have become a standard technique, see also Gerber (1979). Several papers have appeared, including Dassios and Embrechts (1989), Delbaen and Haezendonck (1989) and Schmidli (1995), where martingale methods have been used to analyse increasingly complicated risk models. A more comprehensive review of the literature can be found in Grandell (1991) and Schmidli (1994). In this note we use results from the general theory of semimartingales to derive martingales or local martingales which can be used in the now classical way to asses the probability of ruin in very general risk models.

2 THE RUIN PROBABILITY FOR A GENERAL RISK MODEL

In this section we will consider risk processes of the following type

$$X_t = u + B_t + Z_t + S_t, \tag{2.1}$$

where $B_0 = Z_0 = S_0 = 0$ such that the constant u is the initial capital. The process B represents the total premium payments between time 0 and time t and the accumulation of other regular and predictable streams of income or payment. It is assumed to be a continuous process of finite variation. The process S is a jump process representing the accumulated claims, while Z is a random perturbation which is assumed to be a continuous local martingale. Thus S_t is the sum of the jumps of X in the time interval $[0, t]$. We assume that there exists a predictable process $\tilde{S}$ of finite

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variation such that $S_t - \tilde{S}_t$ is a local martingale. This is a very weak condition which usually follows from the Doob-Meyer decomposition theorem. It implies that the risk process is a semimartingale (It is, in fact, a particularly nice kind of semimartingale, which in the stochastic calculus is referred to as a special semimartingale) We will, moreover, assume that the times at which claims occur cannot be predicted. The technical way of expressing this is that we assume the process X to be quasi-left-continuous. A more precise definition of this concept can, for instance, be found in Jacod and Shiryaev (1987), but for the discussion here the slightly loose definition just given is sufficient. Note that a quasi-left-continuous process is by no means required to have continuous sample paths To state these conditions precisely, it is necessary that all processes are defined on a probability space $(\Omega, \mathcal{F}, P)$ with a right-continuous filtration $\{\mathcal{F}_t\}$ It is not necessary to be very precise about this here, but it may certainly be so in some applications

The process Z need not simply be some unspecific perturbation. It could, for instance, be due to the randomly varying value of a portfolio of stocks If A_t is the value of the portfolio at time t, a simple classical model for the variation of A is the geometric Brownian motion $dA_t = \alpha A_t dt + \sigma A_t dW_t$, where W is a Wiener process In this case the accumulated income in $[0, t]$ from the stock portfolio $\alpha \int_0^t A_s ds$ is included in B_t, while $Z_t = \sigma \int_0^t A_s dW_s$.

We have assumed that the sum of the jumps of X in $[0, t]$ is convergent This is not the case for all semimartingales, but we make the assumption because it simplifies the exposition considerably and is satisfied for most risk models of practical interest. Note, however, that there exist results without this assumption which are similar to, but more complicated than, the following. Since S represents the claims, all jumps are downwards The assumptions imposed imply the existence of a predictable random measure $\nu(\omega, dt, dx)$ on $(0,\infty) \times (-\infty, 0)$ satisfying $\nu(\{t\} \times (-\infty, 0)) = 0$ almost surely for all $t > 0$ and

$$-\int_0^t \int_{-\infty}^0 x\nu(ds, dx) < \infty \tag{2 2}$$

almost surely for all $t > 0$, such that

$$S_t - \int_0^t \int_{-\infty}^0 x\nu(ds, dx)$$

is a local martingale, see Jacod and Shiryaev (1987, Section II.1) In the context of risk theory, ν could be called the claim intensity measure. The *net-profit* condition for the model (2.1) can be expressed as

$$B_t > -\int_0^t \int_{\infty}^0 x\nu(ds, dx) \quad \text{for all } t > 0 \tag{2 3}$$

This means that the insurance company adopts the sensible policy to let the premiums follow the claim intensity. Whether this can exactly be done in practice depends on how the claim intensity varies with time.

If we make the further assumption that there exists an $r_0 > 0$ such that

$$\int_0^t \int_{x<-1} e^{-rx}\nu(ds, dx) < \infty \tag{2 4}$$

almost surely for all $t > 0$ when $0 < r \leq r_0$, then it follows from the general semimartingale theory that the stochastic process

$$M_t(r) = \exp[-r(X_t - u) - G_t(r)] \tag{2 5}$$

is a local martingale for every r in $[0, r_0]$, see Liptser and Shiryaev (1989, Chapter 4). Here

$$G_t(r) = -rB_t + \frac{1}{2}r^2 < Z >_t + \int_0^t \int_{-\infty}^0 (e^{-rx} - 1)\nu(ds, dx) \tag{2 6}$$

and $< Z >$ denotes the predictable quadratic variation of Z.

The local martingale $M(r)$ can be used to evaluate the *ruin probability* in the way that is now standard in risk theory. Let

$$\tau = \inf\{t > 0 \quad X_t < 0\}$$

be the time of ruin. Then because a non-negative local martingale is a supermartingale and because $M_0(r) = 1$, it follows that

$$1 \geq E(M_{\tau \wedge t}(r)) \geq E(M_\tau(r) | \tau \leq t)P(\tau \leq t)$$

for every r in $[0, r_0]$ and $t > 0$ Hence

$$P(\tau \leq t) \leq \frac{e^{-ru}}{E(\exp[-G_\tau(r)] | \tau \leq t)} \quad \text{for all } r \in [0, r_0], \tag{2 7}$$

where we have used that $X_\tau \leq 0$ on $\{\tau \leq t\}$, see also Gerber (1979, p 133)

By Jensen's inequality we see that

$$E(\exp[-G_\tau(r)] | \tau \leq t)^{-1} \leq E(\exp[G_\tau(r)] | \tau \leq t) \tag{2 8}$$

Note that $G_\tau(r)$ is a strictly convex function of r satisfying that $G_\tau(0) = 0$ and

$$-G'_\tau(0) = B_\tau + \int_0^\tau \int_{-\infty}^0 x\nu(ds, dx) > 0 \tag{2 9}$$

by the net-profit condition (2 3) That we can differentiate the integral with respect to ν under the integral follows by a standard argument because zero is an interior point in the range of r-values for which the integral exists From these observations it follows that

$$E(\exp[-ru + G_\tau(r)]|\tau \leq t)$$

is a strictly convex function of r which decreases from the value 1 at $r = 0$ When (2.4) holds for all $r > 0$, it increases to plus infinity as $r \to \infty$ Hence there exists a unique $r^* \in [0, r_0]$ for which this function attains its minimum, and by (2 8)

$$P(\tau \leq t) \leq \frac{e^{-r^* u}}{E(\exp[-G_\tau(r^*)]|\tau \leq t)} \tag{2 10}$$

is probably often close to the best evaluation of the ruin probability obtainable from (2.7).

A simpler evaluation of the ruin probability is obtained if an r-value $R_t > 0$ exists for which the denominator of (2 7) equals one. This r-value need not be unique, and it typically depends on t We see that

$$P(\tau \leq t) \leq e^{-R_t u} \tag{2 11}$$

If R_t exists for all $t > 0$ and if $R = \lim_{t \to \infty} R_t$ exists, then

$$P(\tau < \infty) \leq e^{-uR} \tag{2 12}$$

Example 2.1 Consider the classical risk model perturbed by a Wiener process W

$$X_t = u + ct + \sigma W_t - \sum_{i=1}^{N_t} Y_i \tag{2 13}$$

Here c is the premium rate, N is a Poisson process with intensity λ, and the Y_i's are positive independent identically distributed random variables with distribution function F and mean value μ. We assume that W, N and $\{Y_i\}$ are independent This model has been studied by Gerber (1970), Dufresne and Gerber (1991), Furrer and Schmidli (1994) and Schmidli (1995).

In this particular case, $B_t = ct$, $< Z >_t = \sigma^2 t$ and $\nu(\omega; dt, dx) = \lambda F^* (dx)dt$, with $F^*(x) = 1 - F(-x)$, so

$$G_t(r) = g(r)t = \left(-rc + \frac{1}{2}\sigma^2 r^2 + \lambda[\varphi_F(-r) - 1]\right)t, \tag{2 14}$$

where $\varphi_F(s) = \int e^{-sx} dF(x)$ is the Laplace transform of F Since X in this case is a process with independent increments, it is well-known that $M_t(r)$ is a martingale for every r in the domain of φ_F We see that $R_t = R$ is the positive solution of $g(r) = 0$ When $\sigma^2 = 0$, R is the classical adjustment (or Lundberg) coefficient

A bound on finite time ruin probabilities, which is more precise than $\exp(-Ru)$, can in some cases be obtained as follows For $r \in [R, r_0]$ we have that $g(r) \geq 0$, so by (2 7)

$$P(\tau \leq t) \leq \exp[-ru + g(r)t] \quad \text{for all } r \in [R, r_0] \tag{2 15}$$

The right hand side of (2 15) attains its minimum at r^*, which is given as the solution of $g'(r^*) = u/t$, provided there is a solution in $[0, r_0]$. Otherwise the minimum is attained at $r^* = r_0$, in which case $g'(r^*) < u/t$. If $t \leq u/g'(R)$, the convexity of g implies that $r^* \geq R$, so

$$P(\tau \leq t) \leq \exp[-r^* u + g(r^*)t] \text{ for } t \leq u/g'(r) \tag{2.16}$$

Since $g(r^*) < (r^* - R)u/t$ for $r^* > R$ (using again the convexity of g and the fact that $g(R) = 0$), we see that the right hand side of (2.16) is strictly smaller than $\exp(-uR)$ when $t < u/g'(R)$. Concerning (2.16), see also Gerber (1979, p 139).

3 A PARTICULAR TYPE OF RISK MODELS

In this section we consider a particular type of the general risk models studied in Section 2 Specifically, we assume that

$$X_t = u + B_t + \int_0^t \sigma_s dW_s - \sum_{i=1}^{N_t} Y_i, \tag{3.1}$$

where B is as in the previous section, W is a standard Wiener process, and σ is a predictable process The value of σ_s at time s only depends on things which are known at that time The process N is a counting process with predictable intensity λ, i e. λ is a predictable stochastic process such that $N_t - \int_0^t \lambda_s ds$ is a local martingale The positive random variables $Y_1, Y_2, \ldots$, the claims, are assumed to be mutually independent The distribution of the claim Y_i may depend on the time at which the i'th jump occurs, but is otherwise non-random and independent of the N–process. Thus the Y-s can depend on the N–process only through the time-dependence of the distributions of the Y_i-s. An example of time-dependence is when the claims are subject to inflation

Under these simplifying assumptions $\nu(dt,dx) = \lambda_t F_t^*(dx)dt$, where F_t is the claim-size distribution at time t, and $F_t^*(x) = 1 - F_t(-x)$ is the distribution of $-Y_t$ at time t. Hence

$$G_t(r) = -rB_t + \frac{1}{2}r^2 \int_0^t \sigma_s^2 ds + \int_0^t [\varphi_s(-r) - 1]\lambda_s ds, \tag{3.2}$$

where $\varphi_s(u) = \int e^{-ux} dF_s(x)$ is the Laplace transform of F_s

For the models considered here, the net-profit condition (2.3) is

$$B_t > \int_0^t \mu_s \lambda_s ds \tag{3.3}$$

for all $t > 0$, where μ_s denotes the mean claim size at time s

We shall now describe simple situations where the ruin probability can easily be evaluated. We suppose that for each $t > 0$ there exists a distribution function $\tilde{F}_t$ such that $F_s(x) \geq \tilde{F}_t(x)$ for all $x > 0$ and all $s \leq t$, i e the claim-size distribution at time s is stochastically dominated by a single distribution $\tilde{F}_t$ for all $s \leq t$ This is, for instance, the case if the claims are subject to deterministic inflation Under the condition just imposed, $\mu_s \leq \tilde{\mu}_t$ for $s \leq t$ and

$$\int_0^s [\varphi_u(-r) - 1]\lambda_u du \leq [\tilde{\varphi}_t(-r) - 1]\Lambda_s \quad \text{for } s \leq t, \tag{3 4}$$

where $\tilde{\mu}_t$ denotes the mean value of $\tilde{F}_t$, $\tilde{\varphi}_t(u) = \int e^{-ux} d\tilde{F}_t(x)$, and $\Lambda_s = \int_0^t \lambda_s ds$ is the integrated intensity of N

Now we make the further assumption that the insurance company adopts the prudent policy that for some constant $c > 1$

$$B_s \geq c\tilde{\mu}_t \int_0^s \lambda_u du \tag{3 5}$$

for $s \leq t$ If, moreover, σ_s^2 is bounded by a constant ζ_t^2 for $s \leq t$, (3.4) and (3.5) implies that

$$\begin{aligned} G_s(r) &\leq [-rc\tilde{\mu}_t + \tilde{\varphi}_t(-r) - 1]\Lambda_s + \frac{1}{2}r^2\zeta_t^2 s \\ &= g_t(r)\Lambda_s + \frac{1}{2}r^2\zeta_t^2 s \end{aligned} \tag{3 6}$$

for all $s \leq t$ The function $g_t(r)$ is well-known from classical risk theory Under the conditions imposed it is convex, $g_t(0) = 0$ and $g_t'(0) < 0$, so there is a range $[0,R_t]$ of r-values for which $g_t(r) \leq 0$ Note that R_t is an analogue of the classical adjustment coefficient. For $r \in [0, r_0]$ it follows from (2 7) that

$$P(\tau \leq t) \leq \frac{e^{-ru+\frac{1}{2}r^2\zeta_t^2 t}}{E(\exp[-g_t(r)\Lambda_\tau] \mid \tau \leq t)} \tag{3 7}$$

The Laplace transform of Λ_τ is rarely known, but when the Laplace transform of Λ_t is known, it is sometimes possible to proceed in a way analogous to the derivation of the upper bound (2.16). Quite generally we can use that $-ru + \frac{1}{2}r^2\zeta_t^2 t$ has a minimum at $r = u/(t\zeta_t^2)$, which implies the inequality

$$P(\tau \leq t) \leq e^{-\frac{1}{2}u^2/(t\zeta_t^2)}$$

provided $u/(t\zeta_t^2) \leq R_t$. In general, we have the result

$$P(\tau \leq t) \leq \exp\left(-R_t u + \frac{1}{2}R_t^2\zeta_t^2 t\right)$$

This evaluation is, of course, most precise when we can choose R_t such that $g_t(R_t) =$

0. Note that we could, in a similar way, treat the case where ζ_t^2 is a random variable independent of Λ_T provided the Laplace transform of ζ_t^2 is known Another possible and manageable assumption is that the process σ^2 is bounded by a constant times the intensity λ

Finally, we make the stronger assumption that for $s \leq t$ the intensity λ_s is bounded by a constant $d_t > 0$ By (2 7)

$$P(\tau \leq t) \leq \frac{e^{-ru}}{E(\exp[-h_t(t)\tau] \mid \tau \leq t)} \tag{3 8}$$

with $h_t(t) = g_t(r)d_t + \frac{1}{2}r^2\zeta_t^2$. The right-hand side of (3 8) is of the type considered in Example 2.1, and many ideas that have been used to study that example can be applied here too Obviously,

$$P(\tau \leq t) \leq \exp(-R_t u), \tag{3 9}$$

where R_t is the unique strictly positive solution to $h_t(r) = 0$ Here we use that $h_t(r)$ is strictly convex with $h_t(0) = 0$ and $h_t'(0) < 0$ provided the strong net-profit condition (3 5) is satisfied

4 THE MARTINGALE CASE

Sometimes it can be proved that $E(M_t(r)) = 1$ for all $t \geq 0$. Then it follows that the supermartingale $M_t(r)$ is a martingale We shall briefly consider this situation, where more accurate results can be obtained, see e.g. Gerber (1979) and Schmidli (1995). A nice way of seeing this, which also shows how the theory is related to the theory of exponential families of processes, is to define for each $r \in [0, r_0]$ a new probability measure Q_r by

$$Q_r(A) = \int_A M_t(r)dP \tag{4 1}$$

for $A \in \mathcal{F}_t$ and for all $t > 0$ By the fundamental identity of sequential analysis (see e g Kuchler and Sørensen, 1994)

$$P(\tau \in B) = E_{Q_r}(\exp[rX_\tau + G_\tau(r)]1_{\{\tau \in B\}})e^{-ru}, \tag{4 2}$$

where $B \subseteq \mathbb{R}$ The right-hand side can be evaluated as discussed earlier.

The family of probability measures $\{Q_r \quad 0 < r \leq r_0\}$ was also studied in Sørensen (1993) Under an additional assumption on ν it is the *exponential family of processes* generated by the semimartingale X. This is, for instance, the case for the general type of models considered in Section 3 when the claim-size distribution is constant. We will now concentrate on such models

Under Q_r the process X is of the form

$$X_t = u + \widetilde{B}_t + \int_0^t \sigma_s d\widetilde{W}_s - \sum_{i=1}^{N_t} Y_i, \tag{4.3}$$

where $\widetilde{W}$ is a standard Wiener process, N has intensity $\varphi(-r)\lambda$, the claim-size distribution is $\exp[rx - \log\varphi(-r)]dF(x)$ (i.e. it belongs to the exponential family generated by F), and

$$\widetilde{B}_t = B_t - r < Z >_t$$

All independence assumptions made under P hold under Q_r too These results follow e.g from Jacod and Mémin (1976), see also Kuchler and Sørensen (1989).

We shall now consider the event $\{\tau < \infty\}$. The probability of this event under Q_r is determined by the predictable drift of X under Q_r, given by

$$\widetilde{B}_t - \int_0^t \int_0^\infty x \exp[rx - \log\varphi(-r)]dF(x)\varphi(-r)\lambda_s ds = B_t - r < Z >_t + \varphi'(-r)\Lambda_t$$
$$= -G_t'(r),$$

where we have used that, by standard exponential family theory, the mean value of the claim-size distribution under Q_r is $-\varphi'(-r)/\varphi(-r)$ We saw in Section 2 that under the net-profit condition $-G_t(r)$ is a strictly concave function of r satisfying $G_t(0) = 0$ and $-G_t'(0) > 0$. Now suppose the model is sufficiently ergodic that $t^{-1}G_t(r)$ converges almost surely to a non-random limit $g(r)$, which will then be convex. Assume further that we can find $R > 0$ such that $g(R) = 0$. Then $-g'(R) < 0$, so $-G_t'(R)$ will tend to minus infinity as $t \to \infty$. Hence $Q_r(\tau < \infty) = 1$ for $r \geq R$, so it follows from (4.2) that

$$P(\tau < \infty) = E_Q(\exp[rX_\tau + G_\tau(r)])e^{-ru} \tag{4.4}$$

for $r \geq R$

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