# MODELLING OF DISCRETIZED CLAIM NUMBERS IN LOSS RESERVING

BY OLE HESSELAGER

*University of Copenhagen*
*Laboratory of Actuarial Mathematics*

ABSTRACT

We investigate the usual method of discretizing loss reserving data by calendar year and show how this procedure may introduce fluctuations in the delay probabilities. These fluctuations, when treated as random fluctuations, possess a special correlation structure and we present a simple credibility method accounting for these fluctuations. The results are illustrated by a numerical example.

KEYWORDS

Loss reserving; Claim numbers; Poisson process; Discretizing by calendar year; Variations in delay probabilities; Credibility adjustments.

## 1. INTRODUCTION

Starting from a continuous time model, with claims occurring according to an inhomogeneous Poisson process and with the waiting times until notification being iid real-valued variables, it is shown how the traditional way of discretizing the observations according to calendar year can introduce a time dependence in the delay probabilities. Thus, variations between occurrence years in the delay probabilities may occur simply as a consequence of the way the data is discretized, even when the distribution of the actual delays is independent of time.

In recent years a number of papers on loss reserving have appeared where the delay probabilities for observations in discrete time are assumed to vary between occurrence years. In these papers (HESSELAGER and WITTING, 1988; NEUHAUS, 1992; LAWLESS, 1994; HESS and SCHMIDT, 1994a, b) the variations are treated as random fluctuations between occurrence years and are modelled by a Dirichlet distribution which is a mathematically very convenient construct. In the present paper it is shown that fluctuations induced in the discretizing process possess a special structure and that these fluctuations can not be described by a Dirichlet distribution in a reasonable manner. In fact, in typical cases the probability that a claim is reported in the year of occurrence is negatively correlated with all the other delay probabilities, and these are positively correlated. In a Dirichlet distribution, all probabilities are negatively correlated.

It should be noted that this paper deals only with the fluctuations which are introduced in the discretizing process, and we are of course aware that there may be other sources of variation.

## 2. FROM CONTINUOUS TO DISCRETE TIME MODELS

We start by considering the continuous time process $(T_n, W_n)$, $n = 1, 2, ...,$ where $0 < T_1 < T_2 < ...$ denote the occurrence times, and $W_1, W_2, ...$ are the corresponding real-valued waiting times from occurrence until notification. Most methods for loss reserving assume that the observations have been discretized according to calendar year as illustrated in Figure 1.

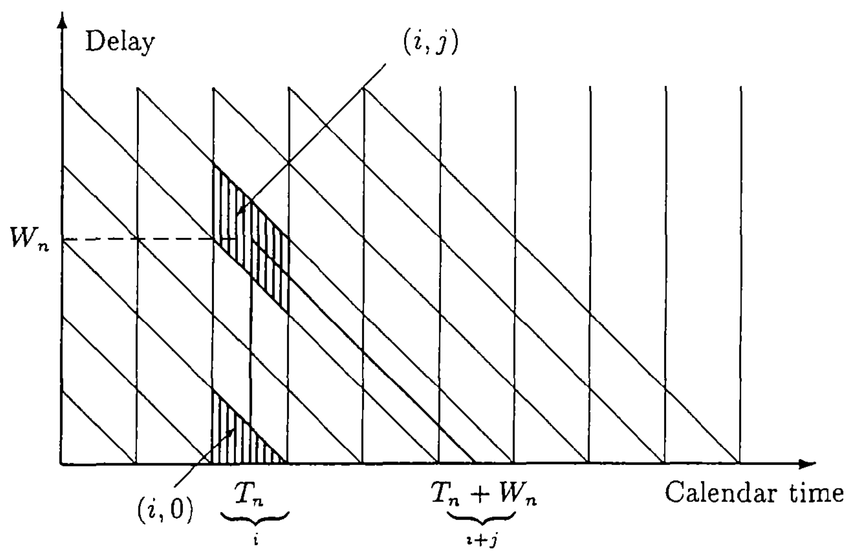


FIGURE 1. Diagram for discretizing observations by calendar year.

The time axis is divided into intervals $(i-1, i]$, $i = 1, 2, ...,$ and the time interval $(i-1, i]$ is called year $i$. A claim incurred in year $i$ is according to this method reported with a delay of $j$ years if it is reported in year $i+j$, and Figure 1 shows the combinations of values $(T_n, W_n)$ which fall into cell $(i, j)$. It should be noted that the cell $(i, 0)$ is differently shaped and only half the size of the cells $(i, j)$, $j \geq 1$. Note also that the actual waiting time corresponding to observations in $(i, j)$ is in the interval $[j-1, j+1]$, $j \geq 1$.

For $i = 1, 2, ...,$ and $j = 0, 1, ...,$ we denote by

$$N_{ij} = \sum_{n \geq 1} I\{T_n \in (i-1, i], T_n + W_n \in (i+j-1, i+j]\},$$

the number of claims incurred in year $i$ and reported with a delay of $j$ years. The total number of claims incurred in year $i$ is

$$N_i = \sum_{j=0}^{\infty} N_{ij} = \sum_{n \geq 1} I\{T_n \in (i-1, i]\}.$$

We now make the following model assumptions pertaining to the continuous time setting

(a) The claims occurrences $\{T_n\}_n$ are generated by a Poisson process with parameter $\Lambda(\cdot)$ on $[0, \infty)$, where $\Lambda(t)$ is a non-negative, non-decreasing and right-continuous function representing the risk exposure for the interval $[0, t]$.
(b) The delays $W_1$, $W_2$, ... are independent of $\{T_n\}_n$ and iid with common distribution $F$.

For a Poisson process with parameter $\Lambda(\cdot)$ the number of events occurring in disjoint time intervals are stochastically independent, and the number of events occurring in $(s, t]$ is Poisson distributed with parameter $\lambda(s, t) = \Lambda(t) - \Lambda(s)$. A Poisson process with intensity $\lambda(\cdot)$ has a parameter $\Lambda(t) = \int_0^t \lambda(x)\,dx$, such that $\Lambda(\cdot)$ in this case is the measure with density $\lambda(\cdot)$.

While it is customary in collective risk theory to assume the existence of an intensity, we choose here to work with the more general setting primarily because we shall consider mixed Poisson models where $\Lambda(\cdot)$ is considered random, and where $t \to \Lambda(t)$ is not necessarily continuous (with probability one). Apart from this technical reason, the reader should note that the case where $\Lambda(\cdot)$ is a non-random measure having a mass point at $t_0$, say, represents the situation where some event is known to take place at time $t_0$ generating a Poisson distributed number of claims with parameter $\Lambda(t_0) - \Lambda(t_0-)$. When in addition $\Lambda(\cdot)$ is taken to be random (a stochastic process), one will also cover the cases where the time epochs for such multiple claims (or catastrophes) are not known in advance.

For the moment we consider $\Lambda(\cdot)$ as non-random and introduce for $i = 1$, $2, \ldots,$

(2.1) $$\Lambda_i = \Lambda(i-1, i),\ U_i(x) = \Lambda(i-1, i-1+x)/\Lambda_i,\ x \in (0, 1],$$

which is the total exposure for year $i$ and the relative distribution of the exposure over year $i$, respectively. It then follows from assumptions (a), (b) that the claim numbers $N_{ij}$ are mutually independent and

(2.2) $$N_{ij} \sim \text{Poisson}(\Lambda_i p_{ij}),$$

where

$$p_{ij} = \int_0^1 [F(j+1-x) - F(j-x)]\, U_i(dx),\ j \geq 0,$$

and it is understood that $F(z) = 0$ for $z < 0$. The result (2.2), (2.3) is easily verified by standard calculations making use of the fact that the occurrence times within year $i$, conditionally given $N_i$, are distributed as the order statistics of $N_i$ iid variables with distribution $U_i(\cdot)$ on $(0, 1]$. Alternatively, the result can be shown using arguments similar to those given in the proof of Theorem 2 in NORBERG (1993).

The delay probabilities $p_{ij}$ will according to (2.3) in general depend on time $i$ through the distribution $U_i$ of exposure over year $i$. This is very unfortunate from a practical point of view, since it makes statistical estimation of the delay probabilities difficult. However, if the Poisson process has an intensity $\lambda(\cdot)$ which is piecewise constant over the occurrence years,

$$\lambda(t) = \lambda_{[t]+1},$$

where $[t]$ denotes the integer part of $t$, then

$$U_i(x) = \frac{\int_0^x \lambda(i-1+s)\,ds}{\int_0^1 \lambda(i-1+s)\,ds} = x,\ x \in (0,\ 1], \tag{2.4}$$

is the uniform distribution and is independent of time $i$. The delay probabilities $p_{ij}$ are therefore also independent of $i$, and can now be estimated by statistical methods in a straightforward manner. A more general situation where $p_{ij}$ becomes independent of $i$ is that where

$$\Lambda(i-1,\ i-1+x) = \lambda_i \Lambda_0(x),\ x \in (0,\ 1],\ i = 1,\ 2,\ \ldots$$

This covers the situation where there exists a fixed measure $\Lambda_0(x)$ on $(0,1]$ representing seasonal variation in the exposure within occurrence years, whereas the factors $\lambda_i$ represent the variation between the occurrence years. We obtain in this case from (2.1) that

$$U_i(x) = \Lambda_0(x)/\Lambda_0(1),\ x \in (0,\ 1], \tag{2.5}$$

which is again independent of $i$.

*Remark 1.* In the case (2.4) with a uniform distribution of exposure we may express the delay probabilities (2.3) in terms of the stop-loss transform

$$\Pi(x) = \int_x^\infty (1-F(y))\ dy \tag{2.6}$$

for the continuous time delay distribution. This yields the expression

$$p_j = \begin{cases} 1+\Pi(1)-\Pi(0), & j=0 \\ \Pi(j-1)+\Pi(j+1)-2\Pi(j), & j \geq 1 \end{cases}. \tag{2.7}$$

□

*Remark 2.* Had the occurrence times $T_n$ and the waiting times $W_n$ been discretized separately, we would have that

$$N_{ij} = \sum_{n=1}^{\infty} I\{T_n \in (i-1,\ i],\ W_n \in (j,\ j+1]\},$$

and it would then hold that the claim numbers $N_{ij}$ are mutually independent and Poisson distributed as

$$N_{ij} \sim \text{Poisson}(\Lambda_i p_j),$$

with $\Lambda_i$ as in (2.1) and now with delay probabilities

$$p_j = F(j+1) - F(j), j = 0, 1, \ldots,$$

which are always time-independent. The problem with this approach is of course that the statistics $N_{ij}$ with $i+j = \tau$ can not be constructed (observed) at the end of year $\tau$ (at time $\tau$), since they involve reportings in the interval $(\tau - 1, \tau + 1]$.

From these remarks we are now able to conclude that when discretizing the observations according to calendar year, or some other time unit, it is important to choose the time unit in such a way that the risk can be regarded as constant over these time periods, except possibly for the same seasonal variation within the time periods. If this is not the case, one will introduce fluctuations in the delay probabilities between the occurrence years. In particular, when seasonal variations may occur within calendar years, one should be cautious about discretizing on a quarterly basis, as is often done in practice.

## 3. THE STRUCTURE OF THE INDUCED VARIATION

We investigate here the structure of the fluctuations in the delay probabilities $p_{ij}$ induced by variations in the distribution $U_i$ of exposure over year $i$. Considering only a fixed year $i$, we shall drop the subscript $i$ in this section. For reasons of simplicity we also assume that the continuous time delay distribution $F$ has a density $f$.

With $P_j = p_0 + \ldots + p_j$ being the cdf. for the discretized delay distribution, we obtain from (2.3) that

$$\begin{aligned} P_j &= \int_0^1 F(j+1-x)\, U(dx) \\ &= F(j) + \int_0^1 f(j+1-x)\, U(x)\, dx, \end{aligned} \tag{3.1}$$

where the latter equality follows by partial integration.

A pair of probability distributions with cumulative distribution functions $G$ and $G^*$ are ordered in stochastic order, written as $G \leq_{st} G^*$, if $(1-G)(x) \leq (1-G^*)(x)$ for all $x$. In the actuarial literature one also says that $G^*$ is more dangerous than $G$, written as $G \leq_d G^*$, if there exists a $c$ such that

$$\begin{aligned} G(x) &\leq G^*(x),\ x < c \\ G(x) &\geq G^*(x),\ x \geq c \end{aligned}, \tag{3.2}$$

and $\int xG(dx) \leq \int xG^*(dx)$. In addition to the sign change condition (3.2) for the cumulative distribution function we shall also work with a sign change condition for the discretized delay probabilities. In this case we write $P \leq_0 P^*$ if

$$
p_0 \geq p_0^*,
\qquad (3.3)
$$
$$
p_j \leq p_j^*, \quad j > 0
$$

It is well-known (e.g. KAAS *et al.*, 1994, Th. III.1.3) that $G \leq_d G^*$ implies that $G$ is smaller than $G^*$ in stop-loss order, and also that $P \leq_0 P^*$ implies that $P \leq_{st} P^*$ (e.g. KAAS *et al.*, 1994, Th. II.1.3).

**Lemma 1**

*(a)* $U \leq_{st} U^* \Rightarrow P \leq_{st} P^*$

*(b) If the density f is decreasing, then* $U \leq_{st} U^* \Rightarrow P \leq_0 P^*$

*(c) If the density f is decreasing and convex, then* $U \leq_d U^* \Rightarrow P \leq_0 P^*$

*Proof.* Assertion (a) follows immediately from (3.1) since

$$
P_j^* - P_j = \int_0^1 f(j+1-x)\underbrace{\left(U^*(x) - U(x)\right)}_{\leq 0} dx \leq 0.
$$

Since $P_0 = p_0$, this also proves (3.3) for $j = 0$. For $j \geq 1$, when $f$ is decreasing, we similarly find that

$$
p_j^* - p_j = \int_0^1 \underbrace{\left(f(j+1-x) - f(j-x)\right)}_{\leq 0} \underbrace{\left(U^*(x) - U(x)\right)}_{\leq 0} dx \geq 0.
$$

which verifies (3.3) and hence assertion (b).

To verify (c) we write

$$
p_0 - p_0^* = \int_0^1 f(1-x)\left(U(x) - U^*(x)\right) dx
$$
$$
= \int_0^1 [f(1-x) - f(1-c)]\,[U(x) - U^*(x)] + f(1-c)(m^* - m),
$$

where $m = \int (1 - U(x))\,dx$ is the mean of the distribution $U$, and $m^*$ is the mean of $U^*$. Since $f(1-x)$ is increasing it holds that $f(1-x) - f(1-c)$, and by assumption also $U(x) - U^*(x)$, is positive for $x > c$ and negative for $x < c$, and therefore

$$
p_0 - p_0^* \geq f(1-c)(m^* - m) \geq 0.
$$

For $j \leq 1$ we note that $f$ being convex implies that $h(x) = f(j+1-x) - f(j-x)$ is decreasing. It then holds that $h(x) - h(c)$ and $U(x) - U^*(x)$ have opposite signs, and

$$\begin{aligned} p_j - p_j^* &= \int_0^1 (h(x) - h(c))(U(x) - U^*(x))\,dx + h(c)\int_0^1 [U(x) - U^*(x)]\,dx \\ &\leq h(c)\int_0^1 [U(x) - U^*(x)]\,dx \\ &= h(c)(m^* - m) \leq 0, \end{aligned}$$

since $h(c) \leq 0$. QED

Assertion (a) allows us to obtain lower and upper bounds for the discrete time delay distribution when the distribution $U$ of the exposure is completely unknown. It follows that the lower and upper bounds are obtained by putting all the exposure at the beginning and at the end of the year, respectively, which gives

$$P^- = (F(1), F(2), \ldots), \tag{3.4}$$

$$P^+ = (0, F(1), F(2), \ldots). \tag{3.5}$$

(Note that the $-$ signifies the smallest distribution in stochastic order, which is the largest cdf, and similarly for the $+$).

If the continuous time delay distribution has a decreasing density it can be concluded that a stochastic increase in the distribution of exposure results in a special type of stochastic increase in the discrete time delay distribution; namely where mass is taken from the reportings with delay $j = 0$ and transferred to the reportings with delay $j \geq 1$. The same type of stochastic increase occurs if the distribution of exposure becomes "more dangerous" (which is a weaker requirement), provided that the delay density is also convex.

We have seen that at least certain types of changes in the exposure distribution lead to the special type of change in the delay probabilities where the probability of immediate reporting varies inversely with the remaining delay probabilities. Thus, if variations in the exposure distribution are considered as random, and only variations satisfying the conditions of Lemma 1 are followed for, we will have that the random probability of immediate reporting is negatively correlated with the rest of the delay probabilities, and that these are positively correlated.

## 4. INDUCED RANDOM FLUCTUATIONS

In this section we consider the situation where the parameter $\Lambda(\cdot)$ itself is a stochastic process. Note that there is a one to one correspondence between $\Lambda(\cdot)$ and the pairs $(\Lambda_i, U_i)$, $i = 1, 2, \ldots$, and in the stochastic models considered here these pairs are iid for $i = 1, 2, \ldots$, and $\Lambda_i$ is independent of $U_i$. The delay probabilities (see (2.3))

$$p_{ij} = \int_0^1 h_j(x)\, U_i(dx), \tag{4.1}$$

$$h_j(x) = F(j+1-x) - F(j-x),\ j = 0,\ 1,\ \ldots, \tag{4.2}$$

are in this case random variables with

$$\pi_j = \mathsf{E} p_{ij} = \int_0^1 h_j(x)\, u(dx), \tag{4.3}$$

$$c_{jl} = \mathsf{Cov}(p_{ij},\ p_{il}) = \int_0^1 \int_0^1 h_j(x)\, h_l(y)\, \varrho(dx,\ dy), \tag{4.4}$$

where

$$u(x) = \mathsf{E}\, U_i(x), \quad \varrho(dx,\ dy) = \mathsf{Cov}(U_i(dx),\ U_i(dy)). \tag{4.5}$$

Conditionally given $(\Lambda_i,\ U_i)$, the claim numbers $N_{ij}$, $j = 0, 1, \ldots$, are mutually independent and Poisson distributed with parameters $\Lambda_i p_{ij}$ (see (2.2)), which gives the (unconditional) moments

$$\mathsf{E} N_{ij} = \nu \pi_j, \tag{4.6}$$

$$\mathsf{Cov}(N_{ij},\ N_{il}) = \delta_{jl} \nu \pi_j + \pi_j \pi_l \tau^2 + c_{jl}(\tau^2 + \nu^2), \tag{4.7}$$

where

$$\nu = \mathsf{E}\Lambda_i, \quad \tau^2 = \mathsf{Var}\Lambda_i. \tag{4.8}$$

At the end of calendar year $I$, the observed claim numbers in respect of occurrence year $i$ are $N_{i\le} = (N_{i0}, \ldots, N_{i,\, I-i})'$, and for $j > I-i$ we may predict $N_{ij}$ using credibility estimation as (see e.g. SUNDT, 1993, p. 34)

$$\bar{N}_{ij} = \mathsf{E} N_{ij} + \mathsf{Cov}(N_{ij},\ N'_{i\le})(\mathsf{Var} N_{i\le})^{-1}(N_{i\le} - \mathsf{E} N_{i\le}). \tag{4.9}$$

The second order moments $c_{jl}$ may be written as $c_{jl} = \sqrt{c_{jj} c_{ll}}\, \varkappa_{jl}$, where $\varkappa_{jl}$ denotes the coefficient of correlation. We shall see that these for all practical purposes can be a approximated by

$$\{\varkappa_{jl}\}_{jl} = \begin{pmatrix} 1 & -1 & -1 & -1 & \ldots \\ -1 & 1 & 1 & 1 & \ldots \\ -1 & 1 & 1 & 1 & \ldots \\ -1 & 1 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{4.10}$$

and for this situation we have the following

**Lemma 2**

*With a moment structure (4.6), (4.7), and a correlation matrix (4.10), the credibility estimator (4.9) is given by*

$$\bar{N}_{ij} = \pi_j\{(1-z_i)\nu + z_i\hat{\Lambda}_i + \eta_i(\hat{\Omega}_i - \mathsf{E}\,\hat{\Omega}_i)(v_j - z_i\bar{v})\},$$

*where*

$$z_i = \frac{\tau^2\pi_\bullet}{\nu+\tau^2\pi_\bullet}, \quad \hat{\Lambda}_i = \sum_{l\le I-i} N_{il}/\pi_\bullet,$$

$$\hat{\Omega}_i = \frac{1}{m_i\,\pi_\bullet}\sum_{l\le I-i} v_l(N_{il} - z_i\hat{\Lambda}_i\pi_l), \quad \mathsf{E}\,\hat{\Omega}_i = \nu(1-z_i)\,\bar{v}/m_i,$$

$$\eta_i = \frac{(\tau^2+\nu^2)\;m_i\pi_\bullet}{\nu+(\tau^2+\nu^2)m_i\pi_\bullet}, \quad m_i = \frac{1}{\pi_\bullet}\sum_{l\le I-i}\pi_l v_l^2 - z_i\bar{v}^2,$$

*and*

$$\pi_\bullet = \sum_{l\le I-i}\pi_l, \quad \bar{v} = \frac{1}{\pi_\bullet}\sum_{l\le I-i}\pi_l v_l,$$

*with* $v_0 = -\sqrt{c_{00}}/\pi_0$ *and* $v_j = \sqrt{c_{jj}}/\pi_j$ *for* $j \ge 1$.

*Proof.* Letting $a_j = v_j\pi_j$, and assuming a correlation matrix (4.10), we may according to (4.7) write the covariance matrix $\mathsf{Var}\,\boldsymbol{N}_{i\le}$ as

$$\mathsf{Var}\,\boldsymbol{N}_{i\le} = \nu\,\mathrm{Diag}(\boldsymbol{\pi}) + \tau^2\boldsymbol{\pi}\boldsymbol{\pi}' + (\tau^2+\nu^2)\,\mathbf{aa}',$$

where $\boldsymbol{\pi} = (\pi_0, \ldots, \pi_{I-i})'$, and Diag $(\boldsymbol{\pi})$ is the diagonal matrix with the elements of $\boldsymbol{\pi}$ placed in the diagonal. Using the inversion lemma (e.g. SUNDT, 1984, Lemma 6.1)

$$(\boldsymbol{Q}+\boldsymbol{f}\boldsymbol{g}')^{-1} = \boldsymbol{Q}^{-1} - \boldsymbol{Q}^{-1}\boldsymbol{f}\boldsymbol{g}'\boldsymbol{Q}^{-1}\frac{1}{1+\boldsymbol{g}'\boldsymbol{Q}^{-1}\boldsymbol{f}},$$

one first calculates the inverse of $\boldsymbol{P} = \nu\,\mathrm{Diag}(\boldsymbol{\pi}) + \tau^2\boldsymbol{\pi}\boldsymbol{\pi}'$ and subsequently uses the same lemma to invert the matrix $\boldsymbol{P} + (\tau^2+\nu^2)\mathbf{aa}'$. The result then follows by straightforward calculations from (4.9). □

*Remark 3.* The term $\pi_j[(1-z_i)\nu + z_i\hat{\Lambda}_i]$ appearing in Lemma 2 is the credibility estimator for $N_{ij}$ in the discrete time mixed Poisson model

$$\text{(4.11)} \qquad \begin{aligned} &N_{ij}\,|\,\Lambda_i \overset{indep.}{\sim} \mathrm{Poisson}(\Lambda_i\pi_j),\\ &\mathsf{E}\,\Lambda_i = \nu, \quad \mathsf{Var}\,\Lambda_i = \tau^2, \end{aligned}$$

with fixed delay probabilities $\pi_j$ and a random parameter $\Lambda_i$ as considered by NORBERG (1986). The second term can therefore be viewed as a correction term, taking account of the random fluctuations in the delay probabilities induced in the discretizing process. □

### 4.1. The exponential delay distribution

Consider the exponential continuous time delay distribution

$$F(x) = 1 - e^{-x/\mu},\ x \geq 0.$$

From (2.3) we obtain that

$$p_{i0} = 1 - e^{-1/\mu}\Psi_i \tag{4.12}$$

$$p_{ij} = e^{-j/\mu}(1 - e^{-1/\mu})\Psi_i,\ j = 1, 2, ..., \tag{4.13}$$

where

$$\Psi_i = \int_0^1 e^{x/\mu}\, U_i(dx). \tag{4.14}$$

It is seen from (4.12), (4.13) that the probabilities $p_{ij}$ are linearly dependent, and that the correlation structure (4.10) is exact in this case.

Because (4.10) holds true, we may then use the credibility approach described in Lemma 2, and for the constants $v_j$ introduced in Lemma 2 we furthermore observe from (4.12), (4.13) that

$$v_j = \frac{\sqrt{\mathrm{Var}\,\Psi_i}}{\mathrm{E}\,\Psi_i},\ j \geq 1, \quad v_0 = -\frac{\sqrt{\mathrm{Var}\,\Psi_i}}{e^{1/\mu} - \mathrm{E}\,\Psi_i}. \tag{4.15}$$

Thus, if the parameters $\pi_j$ and $v$, $\tau^2$ of the mixed Poisson model (4.11) are known, it is also possible to calculate the credibility adjustment accounting for random fluctuations in the delay probabilities $p_{ij}$, when the mean $\mathrm{E}\,\Psi_i$ and the coefficient of variation $\sqrt{\mathrm{Var}\,\Psi_i}/\mathrm{E}\,\Psi_i$ are known.

Consider the simple model where

$$\frac{dU_i(x)}{dx} = (1 - b_i)1_{[0,\frac{1}{2}]}(x) + (1 + b_i)1_{(\frac{1}{2},1]}(x), \tag{4.16}$$

and $b_i$ is a random variable with mean zero. This means that the exposure is (a priori) expected to be constant over the year, but that variations occur such that the actual exposure is $(1 - b_i) \times 100\%$ during the first half of the year and $(1 + b_i) \times 100\%$ during the second half. In this case we find that

$$\Psi_i = \int_0^1 e^{x/\mu}\,dx + b_i\left[\int_{1/2}^1 e^{x/\mu}\,dx - \int_0^{1/2} e^{x/\mu}\,dx\right],$$

and

$$\mathrm{E}\,\Psi_i = \mu\,(e^{1/\mu} - 1), \tag{4.17}$$

$$\frac{\sqrt{\mathrm{Var}\,\Psi_i}}{\mathrm{E}\,\Psi_i} = \sqrt{\mathrm{Var}\,b_i}\,\frac{e^{1/2\mu} - 1}{e^{1/2\mu} + 1} \tag{4.18}$$

*Remark 4.* We may view the above as a quick and dirty method of performing the credibility adjustment in Lemma 2, based on the assumption of an exponential delay distribution. Note that we do not necessarily suggest that the parameters $\pi_j$ are obtained from (4.3) with an exponential delay distribution, since these can be estimated in a straightforward manner from the observed claim numbers $N_{ij}$ in the run-off triangle. The exponential distribution is only used to generate the correlation structure (4.10) as assumed in Lemma 2 and to reduce the number of second order parameters $c_{jj}$ (or $v_j$) via (4.15). In Section 5 it will be demonstrated by example that the correlation structure (4.10) can safely be assumed even when the delay distribution is not exponential.

The model (4.16) is not essential to the simple approach presented here, and the coefficient of variation $\sqrt{\mathrm{Var}\,\Psi_i}/\mathrm{E}\,\Psi_i$ can easily be obtained using other models for $U_i$. Adopting (4.16) we suggest to fix the standard deviation $\sqrt{\mathrm{Var}\,b_i}$ on a purely subjective basis. □

### 4.2. The time continuous Poisson/Gamma model

Recall that the claim occurrences in the continuous time setting are assumed to be generated by a Poisson process with parameter $\Lambda(\cdot)$. In this section we consider the time continuous Poisson/Gamma model, where $\Lambda(\cdot)$ is viewed as the realization of a gamma process with parameters $(\gamma(\cdot), \beta)$.

The gamma process with parameters $(\gamma(\cdot), \beta)$ has independent increments and $\Lambda(s, t)$ is gamma distributed with shape parameter $\gamma(s, t) = \gamma(t) - \gamma(s)$ and scale parameter $\beta$. Since the sample paths of a gamma process are not continuous, the (conditional) Poisson with parameter $\Lambda(\cdot)$ does not have an intensity.

*Remark 5.* The Poisson/Gamma process has independent increments since the Poisson process as well as the mixing gamma process have independent increments. With

$$N(s, t) = \#\{n \mid T_n \in (s, t]\}$$

denoting the number of occurrences in $(s, t]$, it follows that the distribution of $N(s, t)$ is a gamma mixture of Poisson distributions, which yields a negative binomial distribution,

$$\mathrm{P}(N(s, t) = k) = \binom{\gamma(s, t) + k - 1}{k} q^k (1 - q)^{\gamma(s, t)},$$

with $q = (1 + \beta)^{-1}$. □

From the distribution of $\Lambda(\cdot)$ we want to derive the distribution of $(\Lambda_i, U_i)$ for $i = 1, 2, \ldots$, and use this to investigate the distribution of the (random) delay probabilities.

Since $\Lambda_i$ and $U_i$ depend only on the increments of $\Lambda(\cdot)$ over year $i$ according to (2.1), we immediately conclude that $(\Lambda_i, U_i)$, $i = 1, 2, \ldots$, are stochastically independent, and since $\Lambda_i$ is the increment of $\Lambda(\cdot)$ over year $i$ we also have that $\Lambda_i \sim \mathrm{Gamma}(\gamma(i-1, i), \beta)$. The distribution $U_i$ is obtained from (2.1) by normalizing the increments of $\Lambda(\cdot)$ over year $i$, and so it follows from FERGUSON (1973, Section 4) that $U_i(\cdot)$ is a Dirichlet process on (0,1] with parameter $\gamma(i-1, i-1+x)$, $x \in [0, 1]$, and furthermore that $U_i(\cdot)$ is stochastically independent of $\Lambda_i$. In particular, $U_i(x)$ has a beta distribution with mean

$$u(x) = \mathrm{E}\, U_i(x) = \gamma(i-1, i-1+x)/\gamma(i-1, i),$$

and (FERGUSON, 1973, Th. 4)

$$\text{(4.19)} \quad \mathrm{Cov}(U_i(dx), U_i(dy)) = \frac{1}{\gamma(i-1, i)+1} \{\delta_{x,y} u(dx) - u(dx)u(dy)\},$$

where $\delta_{x,y}$ equals 1 if $x = y$ and zero otherwise.

In order for the pairs $(\Lambda_i, U_i)$ to be iid (which was previously assumed) we then need to require that $\gamma(i-1, i-1+x)$ and $\gamma(i-1, i)$ are independent of $i$, and expressed in terms of the parameters (4.8) we have that $\gamma(i-1, i) = \nu^2/\tau^2$ and $\beta = \nu/\tau^2$. From (4.3)-(4.5) together with (4.19) we then find the first and second order moments of the discrete time delay probabilities, which become

$$\text{(4.20)} \quad \pi_j = p_{ij} = \int_0^1 h_j(x)\, u(dx),$$

$$\text{(4.21)} \quad c_{jl}^2 = \frac{\tau^2}{\nu^2+\tau^2} \left\{ \underbrace{\int_0^1 h_j(x) h_l(x)\, u(dx)}_{=:\pi_{jl}} - \pi_j \pi_l \right\},$$

where $h_j(x) = F(j+1-x) - F(j-x)$ was introduced in (4.2).

Notice that the bracket in (4.21) can be written as $\mathrm{Cov}(h_j(Y), h_l(Y))$, where $Y$ is a random variable with distribution $u(x)$ on (0, 1]. The delay probabilities $p_{ij}$ and $p_{il}$ are therefore positively correlated if both $h_j$ and $h_l$ are either increasing or decreasing, and are negatively correlated if one is increasing while the other is decreasing. It is seen from (4.2) that $h_0(x)$ is always decreasing, and $h_j(x)$ for $j \geq 1$ is certainly increasing if $F$ has a decreasing density. This result matches assertion (b) in Lemma 1.

*Example 1.* The (American) Pareto distribution with parameters $(\eta, \alpha)$ has cdf (see e.g. HOGG and KLUGMAN, 1984)

$$F(x) = 1 - \left(\frac{\eta}{\eta+x}\right)^\alpha, \quad x \geq 0,$$

a (decreasing) density

$$f(x) = \frac{\alpha \eta^{\alpha}}{(\eta + x)^{\alpha+1}}, \quad x \geq 0,$$

and a mean $\mu = \dfrac{\eta}{\alpha - 1}$. It is a noteworthy property of this distribution that when $X$ is Pareto distributed with parameters $(\eta, \alpha)$, then the conditional distribution of $X - y$ given that $X > y$ is again Pareto with parameters $(\eta + y, \alpha)$. The mean residual lifetime (MRL) then becomes

$$m(y) = \mathrm{E}(X - y \mid X > y) = \frac{\eta + y}{\alpha - 1} = \mu + \frac{1}{\alpha - 1} y. \tag{4.22}$$

BENKTANDER and SEGERDAHL (1960) suggested to use the MRL as a useful tool for investigating the tail of a severity distribution.

With a Pareto$(\eta, \alpha)$ delay distribution, and $u(x) = x$, $x \in [0, 1]$, the average delay distribution is calculated from (2.7) with the stop-loss transform given by

$$\Pi(x) = (1 - F(x))\, m(x) = \mu \left( \frac{\eta}{\eta + x} \right)^{\alpha - 1}, \tag{4.23}$$

whereas the parameters $\pi_{jl}$ appearing in (4.21) have to be calculated numerically.

□

## 5. NUMERICAL RESULTS

For a portfolio of accident policies we have observed all claim occurrences between 1/1/82 and 31/12/90, which have been reported before 3/3/92. The data have been discretized according to calendar period, using an interval length of 3 month. Table 1 shows the run-off triangle by 31/12/90 containing the numbers of reported disability claims, and since the portfolio has been observed until 3/3/92 we are also able to construct the claim numbers which were eventually reported in this case.

TABLE 1

DISABILITY CLAIMS REPORTED BY 31/12/90, AND LATER REPORTINGS

| | Run-off triangle | | | | | | Actual reportings | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | delay $j$ | | | | | | delay $j$ | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 72 | 35 | 7 | 4 | 3 | 0 | — | — | — | — | — | — |
| 2 | 71 | 35 | 6 | 3 | 2 | — | — | — | — | — | — | 2 |
| 3 | 69 | 42 | 4 | 4 | — | — | — | — | — | — | 3 | 3 |
| 4 | 70 | 31 | 9 | — | — | — | — | — | — | 2 | 1 | 2 |
| 5 | 67 | 31 | — | — | — | — | — | — | 4 | 2 | 1 | 1 |
| 6 | 55 | — | — | — | — | — | — | 48 | 7 | 4 | 2 | 2 |

These are shown in Table 1 as well. For the period 1/1/82-31/12/90 there has been reported a total of 4015 disability claims, and the average reporting delay for these claims was $\mu = 0.91$ (with a time unit of 3 month).

In the mixed Poisson model (4.11) we have estimated the delay probabilities $\pi_j$ on the basis of all the observations for the period 1/1/82-31/12/90, and the estimated probabilities $\pi_j$ are shown for $j = 0, \ldots, 5$ in Table 2. Finally, using the method of moments, we have estimated the parameters (4.8) obtaining the results

$$\nu = 110.5, \quad \tau^2 = 164.0$$

In this mixed Poisson model, the credibility predictions for the outstanding claim numbers are calculated in accordance with Remark 4, and the result is shown in Table 3 as the first set of predictions.

TABLE 2

DISCRETE TIME DELAY PROBABILITIES IN %

| $j$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\pi_j$ | 58.03 | 29.27 | 4.72 | 2.38 | 1.57 | 0.69 |

TABLE 3

CREDIBILITY PREDICTIONS

| | "Ordinary" credibility predictions | | | | | | Credibility predictions based on Lemma 2 | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | delay $j$ | | | | | | delay $j$ | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | — | — | — | — | — | — | — | — | — | — | — | — |
| 2 | — | — | — | — | — | 1 | — | — | — | — | — | 1 |
| 3 | — | — | — | — | 2 | 1 | — | — | — | — | 2 | 1 |
| 4 | — | — | — | 3 | 2 | 1 | — | — | — | 3 | 2 | 1 |
| 5 | — | — | 5 | 3 | 2 | 1 | — | — | 5 | 3 | 2 | 1 |
| 6 | — | 30 | 5 | 2 | 2 | 1 | — | 34 | 6 | 3 | 2 | 1 |

By inspection of the data in Table 1 it is seen that the observed number of immediate reportings in respect of period 6 is significantly below average. In the mixed Poisson model this will be interpreted as a result of a low-risk period, and the predictions will be correspondingly low. An alternative explanation would be that the claims in this period have occurred later than expected by the model, and that a smaller proportion (than expected) of the incurred claims have therefore been reported already in the occurrence period. Comparing with the actual reportings in Table 1 it is seen that this was probably the case, since the number of reported claims with delay $j = 1$ is above average for that period.

In order to account for fluctuations due to the discretization we apply the credibility method in Lemma 2 using the simple approach described in Section 4.1. With a standard deviation $\sqrt{\mathrm{Var}\, b_i} = 0.2$ we obtain the predictions as shown in Table 3. Comparing with Table 1 it is seen that the credibility method based on Lemma 2 performs better in this case, even though both methods underestimate the number $N_{6,1}$.

Lemma 2 is based on the assumption of a correlation structure (4.10), which from Section 4.2 is known to be correct in the case of an exponential delay distribution. For the data considered here we have access to the continuous time delays $W_n$, $n = 1, \ldots, 4015$, and in Figure 2 we have shown the empirical MRL for these observations, the MRL for the Pareto distribution considered in Example 1, and we have also fitted the curve

$$m_1(x) = \mu(1 + x/a)^{1-c},$$

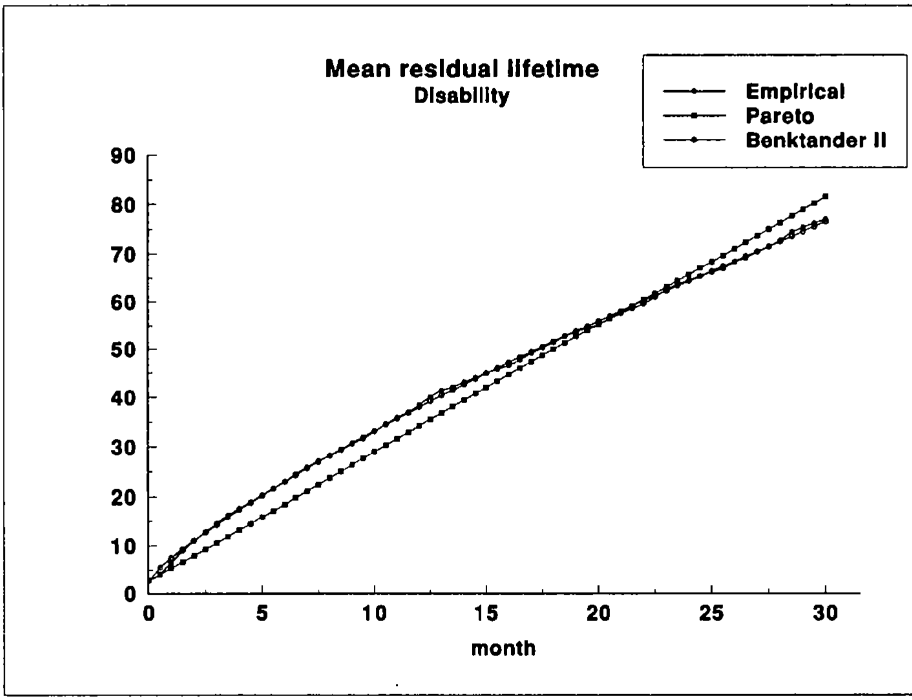


FIGURE 2. The MRL for the continuous time delay distribution, disability claims.

obtaining the values $\hat{a} = 0.409$ and $\hat{c} = 0.226$. It is seen that this gives an almost perfect fit, and the corresponding distribution, a shifted version of Benktander's type II distribution (see BEARD *et al.*, p. 82), has cdf

$$F(x) = 1 - \left(\frac{a}{a+x}\right)^{1-c} e^{-\frac{a}{c\mu}[(1+x/a)^{-c}-1]}.$$

In Figure 3 we have also plotted the empirical MRL for reporting delays for dental claims from the same portfolio. It is seen that the Pareto distribution from Example 1 gives a adequate description in this case, and the estimated parameters are $\mu = 1.14$ (with a time unit of 1 month in this case) and $\hat{\alpha} = 1.343$.
For these two delay distributions we have calculated the coefficients of correlation assuming the time continuous Poisson/Gamma model treated in Section 4.2. This

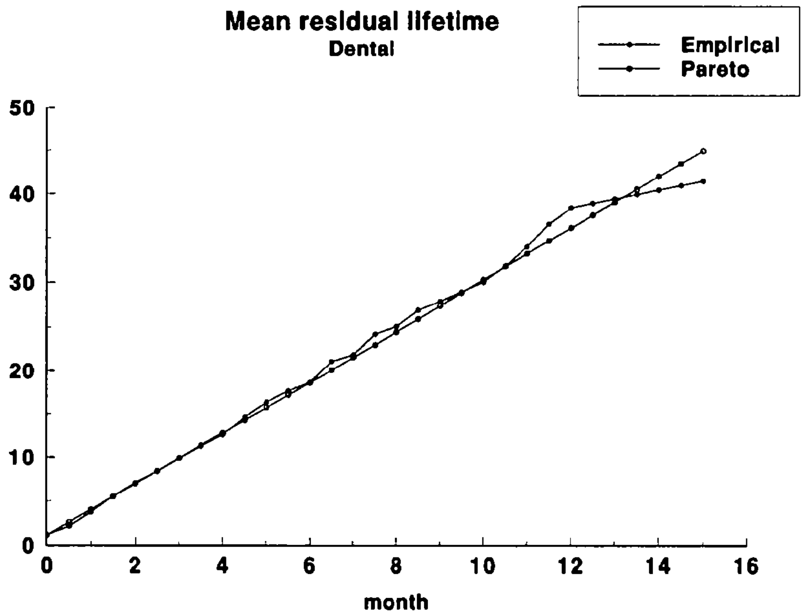


FIGURE 3. The MRL for the continuous time delay distribution, dental claims.

gives the results shown in Table 4, and it is seen that even though the continuous time delay distributions in these cases are far from exponential, the correlation matrix is well approximated by (4.10) as assumed in Lemma 2.

TABLE 4

CORRELATION MATRIX. BENKTANDER TYPE II DELAY DISTRIBUTION FOR DISABILITY CLAIMS (UPPER TABLE) AND PARETO DELAY DISTRIBUTION FOR DENTAL CLAIMS (LOWER TABLE)

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | −0.999 | −0.991 | −0.982 | −0.977 | −0.973 |
| | 1 | 0.984 | 0.974 | 0.967 | 0.963 |
| | | 1 | 0.999 | 0.997 | 0.995 |
| | | | 1 | 0.999 | 0.999 |
| | | | | 1 | 0.999 |
| | | | | | 1 |

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | −0.999 | −0.982 | −0.969 | −0.960 | −0.955 |
| | 1 | 0.976 | 0.960 | 0.951 | 0.945 |
| | | 1 | 0.998 | 0.995 | 0.993 |
| | | | 1 | 0.999 | 0.999 |
| | | | | 1 | 0.999 |
| | | | | | 1 |

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OLE HESSELAGER
*University of Copenhagen*
*Laboratory of Actuarial Mathematics*
*Universitetsparken 5*
*DK-2100 Copenhagen Ø*
*Denmark*