SHORT CONTRIBUTIONS

MARTINGALES AND TAIL PROBABILITIES

BY HANS U. GERBER

At the twenty-eighth Actuarial Research Conference of the Society of Actuaries, WILLMOT and LIN (1993) presented a paper whose central result is a bound on the tail probability of a random sum. In the subsequent discussion, Professor Bühlmann raised the question, if this bound could be derived by martingale methods. The purpose of this note is to show how it can be done

We consider a random variable of the form

$$S = X_1 + \dots + X_N .$$

Here the random variables $N, X_1, X_2, \dots$ are independent, and the X_k's are assumed to be positive and identically distributed; their common distribution function is denoted by $F(x)$.

Let

$$p_k = \Pr(N = k), \qquad k = 0, 1, \dots$$

We assume the existence of a number ϕ, $0 < \phi < 1$, with

$$\Pr(N > k \mid N \geq k) \leq \phi \qquad \text{for} \qquad k = 1, 2, \dots \tag{1}$$

and a positive number r with

$$\phi \cdot \int_0^\infty e^{rx}\, dF(x) \leq 1 \tag{2}$$

(if $F(x)$ is sufficiently regular, we might choose the value of r for which equality holds). Then the result of Willmot and Lin is that

$$\Pr(S \geq x) \leq \frac{1 - p_0}{\phi} \cdot e^{-rx}$$

for any $x > 0$.

For the following proof we introduce

$$S_k = X_1 + \dots + X_k$$

and

$$Y_k = \begin{cases} e^{rS_k} & \text{if} \quad N \geq k \\ 0 & \text{if} \quad N < k . \end{cases}$$

We note the recursive relationship

$$Y_k = Z_k \cdot Y_{k-1}, \qquad k = 1, 2, \ldots$$

with

$$Z_k = \begin{cases} e^{rX_k} & \text{if} \quad N \geq k \\ 0 & \text{if} \quad N < k. \end{cases}$$

According to (1) and (2), the conditional expectation of Z_{k+1} (given $N \geq k$) is less than or equal to 1, which shows that the sequence $Y_1, Y_2, \ldots$ is a supermartingale

If we stop it at time

$$T = \min\{k : S_k \geq x \quad \text{or} \quad N < k\}$$

it follows that, given $N \geq 1$ and X_1,

$$Y_1 \geq E[Y_T | N \geq 1, X_1]$$

or

$$e^{rX_1} \geq E[e^{rS_T} 1_{\{S \geq x\}} | N \geq 1, X_1] \geq e^{rx} \Pr(S \geq x | N \geq 1, X_1).$$

Then we get

$$\begin{aligned} \Pr(S \geq x) &= (1-p_0) \cdot E[\Pr(S \geq x | N \geq 1, X_1)] \\ &\leq (1-p_0) \cdot E[e^{rX_1} \cdot e^{-rx}] \\ &\leq \frac{1-p_0}{\phi} e^{-rx}, \end{aligned}$$

which completes the proof.

REFERENCE

WILLMOT, G E and LIN X (1993) *Lundberg bounds on the tails of compound distributions* Research Report 93-14, Institute of Insurance and Pension Research University of Waterloo Forthcoming in the Journal of Applied Probability

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