ON THE STABILITY OF RECURSIVE FORMULAS

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ABSTRACT

Based on recurrence equation theory and relative error (rather than absolute error) analysis, the concept and criterion for the stability of a recurrence equation are clarified. A family of recursions, called congruent recursions, is proved to be strongly stable in evaluating its non-negative solutions. A type of strongly unstable recursion is identified. The recursive formula discussed by PANJER (1981) is proved to be strongly stable in evaluating the compound Poisson and the compound Negative Binomial (including Geometric) distributions. For the compound Binomial distribution, the recursion is shown to be unstable. A simple method to cope with this instability is proposed. Many other recursions are reviewed. Illustrative numerical examples are given.

KEYWORDS

Recursive formula; compound distribution; probability of ruin; dominant solution; subordinate solution; congruent recursion; index of error propagation; stable; strongly stable; strongly unstable; relative error analysis; empirical inflation factor.

1. INTRODUCTION

Compound distributions are used extensively in modeling the total claims for insurance portfolios. Consider the family of claim frequency distributions satisfying the recursion:

$$p_n = p_{n-1}\left(a + \frac{b}{n}\right), \qquad n = 1, 2, 3, \ldots \tag{1}$$

where p_n denotes the probability that exactly n claims occur in a fixed time interval such as one year and p_0 is an initial value. If the claim severity has a probability function (p.f.) $f(x)$, $x > 0$, the total claims has a compound distribution with a p.f.:

$$g(x) = \sum_{n=0}^{\infty} p_n f^{*n}(x), \qquad x \geq 0. \tag{2}$$

PANJER [12] has shown that, if the claim severity distribution is defined on the positive integers with a p.f. $f(x)$, $x > 0$, the compound distribution in (2) can be evaluated recursively as:

$$g(x) = \sum_{j=1}^{x} \left(a+b\frac{j}{x}\right) f(j)\, g(x-j), \qquad x = 1, 2, 3, \ldots \tag{3}$$

$$g(0) = p_0. \tag{4}$$

This recursive formula is very useful for computer programming and significantly reduces the computing time comparing with the brute-force method directly using formula (2).

As with any algorithm, round-off errors are inevitable since computers only represent a finite number of digits. Practical observations show that algorithm (3) works well in evaluating compound distributions. However, in the actuarial literature, there are also some comments which diverge from the above observations and make the picture somewhat fuzzy. There is an obvious need for a clearer picture of the stability of recursive computation.

To convey some impression that round-off errors are not necessarily small, we start with a numerical example.

Example 1: In a compund Poisson model, the claim frequency has a Poisson distribution with mean $\lambda = 10$, the claim severity has a two points distribution:

$$f(1) = .95, \qquad f(2) = .05.$$

By directly applying recursion (3) in the usual *forward* direction:

$$g(x) = \frac{\lambda}{x}\,[f(1)\, g(x-1) + 2f(2)\, g(x-2)], \tag{5}$$

$$= \frac{10}{x}\,[.95\, g(x-1) + .1\, g(x-2)], \tag{6}$$

with initial values

$$g(-1) = 0, \qquad g(0) = \exp(-\lambda) = \exp(-10), \tag{7}$$

one can obtain the compound distribution easily.

Values at $x = 9$ and $x = 10$ are

$$g(9) = .1140989798, \qquad g(10) = .1183785348.$$

Equation (6) can be used in the *backward* direction as:

$$g(x-2) = x\, g(x) - 9.5\, g(x-1). \tag{8}$$

With $g(10)$ and $g(9)$ as starting points, we obtained the surprising results in Table 1 when 6 digits of floating points are used. One can see that round-off errors blow up rapidly!

TABLE 1

AN EXAMPLE USING ALGORITHM (3) IN THE BACKWARD DIRECTION

points	probability
8	.099850
7	.078315
6	.054807
5	.027538
4	.067231
3	−.501005
2	5.02847
1	−49.2735
0	478.155

The catastrophic instability in the backward direction can indicate strong stability in the forward direction. The well-known Miller's algorithm (see [16], p. 153) is based on this principle. Thus, the stability of a recursion depends on the direction in which it is used. In this paper, unless otherwise stated, the direction of recursive evaluation is the forward direction.

2. RELATIVE ERROR *VS* ABSOLUTE ERROR

GOOVAERTS and DE VYLDER [9] (p. 57) have discussed the propagation of absolute errors of the recursion (3). Based on their analysis about the inflation of absolute errors, they concluded that the recursion (3) seems to be unstable.

There is nothing wrong in their error analysis, but the conclusion they drew is inappropriate because the absolute error has little bearing on the behavior of errors relative to the required solution. We want to stress one basic point in standard numerical analysis: "*as a measure of accuracy, the absolute error may be misleading and the relative error more meaningful*" – BURDEN and FAIRES [1] (p. 13). The criterion for the stability of an algorithm should be relative error, rather than absolute error.

Example 2: For a Poisson distribution with a large mean λ, say $\lambda = 1000$, assume ideal computing which gives exact solutions using the recursion:

$$p_n = \frac{\lambda}{n} p_{n-1}, \qquad n \geq 1. \tag{9}$$

Thus, in the above ideal computing process there is no error propagation. Rounding errors only occur when the computer outputs the exact solution. Only a finite number r (r can be any desired number) digits can be represented in the output. In this way, both the first point, p_0, and the mean point, p_{1000}, are obtained. When $r = 10$, one has

$$p_0 = .5075958897 \times 10^{-434}, \qquad \text{and} \qquad p_{1000} = 0.1261461134,$$

with absolute errors of about

$$10^{-444}, \quad \text{and} \quad 10^{-12},$$

respectively.

For any value of r, the absolute error is inflated 10^{432} times when the recursive evaluation moves from p_0 to p_{1000}. Obviously one *cannot* conclude that the algorithm (9) is unstable.

On the other hand, one can see that the algorithm (9) is stable by observing a constant relative error in the evaluation process (the relative errors for p_0 and p_{1000} are about the same at 10^{-r}).

To conclude this section, we cite Oliver's ([11], p. 324) argument about the criterion of stabilities of recursions:

> "*If we should wish to determine the number of significant figures in the computed values, then the absolute stability of the relation is quite irrelevant; what matters is the behavior of the propagated errors relative, not to unity, but to the required solution.*"

3. LINEAR RECURSIONS OF FINITE ORDER

Consider the linear homogeneous recurrence equation in the forward direction

$$g(x) = \sum_{j=1}^{m} A_j(x)\, g(x-j), \quad x > k, \quad A_m(x) \neq 0, \tag{10}$$

where m is called the *order* of the recurrence equation. The point k is the *starting point* of the recursion and $g(k-m+1), \ldots. g(k)$ are the initial values.

For any given initial conditions

$$\{g(j) = \alpha_j;\ j = k-m+1, \ldots, k\};\ (\alpha_{k-m+1}, \ldots, \alpha_k) = \vec{\alpha}, \tag{11}$$

the linear recurrence equation (10) has one and only one solution, $g_{\vec{\alpha},k}(x)$. Any solution of (10) can be represented by its initial values. Also, the solution $g_{\vec{\alpha},k}(x)$ linearly depends upon the initial vector $\vec{\alpha}$:

$$g_{c_1\vec{\alpha}+c_2\vec{\beta},k}(x) = c_1 g_{\vec{\alpha},k}(x) + c_2 g_{\vec{\beta},k}(x) \tag{12}$$

The homogeneous linear recurrence equation (10) possesses a linearly independent set of solutions $\{g^{(h)}(x),\ 1 \leq h \leq m\}$, called a **fundamental set,** and any solution of (10) can be expressed as a linear combination of these functions.

Definition 1: A solution $g(x)$ of equation (10) is called a **dominant** solution, if for any solution $h(x)$ of equation (10) there exists a constant $C > 0$, such that

$$|g(x)| \geq C|h(x)|, \quad x > K \quad \text{for some} \quad K \geq k. \tag{13}$$

A solution $h(x)$ of equation (10) is called a **subordinate** solution, if there exists a solution $g(x)$ of equation (10) such that

$$\lim_{x\to\infty} \left|\frac{g(x)}{h(x)}\right| = \infty; \tag{14}$$

in this case, we say that $g(x)$ dominates $h(x)$.

It should be noted that some solutions may be neither dominant nor subordinate. However, for most recurrence equations that are encountered in practical applications, their coefficients $A_j(x)$ satisfy some regularity conditions and there exists a fundamental set $\{g^{(h)}(x),\ 1 \le h \le m\}$ such that

- $g^{(1)}(x)$ is a dominant solution and free from zero for x sufficiently large;
- $\lim_{x\to\infty} g^{(1)}(x)/g^{(h)}(x) = \infty$, for $2 \le h \le m$.

(See CASH [2], p. 2; WIMP [23], p. 19 and p. 272-9).

Remarks:

For positive arithmetic severities with finite support, by a simple rescaling, one can assume that $f(x)$ is defined on positive integers with finite support $\{x_1, x_2, \ldots, x_r\}$ such that

$$1 \le x_1 < x_2 < \ldots < x_r < \infty, \tag{15}$$

$$gcd(x_1, x_2, \ldots, x_r) = 1, \tag{16}$$

where *gcd* stands for *greatest common divisor*. In this case, formula (3) becomes a special case of (10) with $m = x_r$ and $k = 0$:

$$g(x) = \sum_{j=1}^{m} \left(a + b\frac{j}{x}\right) f(j)\, g(x-j) \tag{17}$$

with initial values:

$$\{g(x) = 0;\ x = -m+1, \ldots, -1\};\ g(0) = p_0 > 0. \tag{18}$$

4. RELATIVE STABILITY THEORY

For the general linear recurrence equation (10), OLIVER[1] [11] proposed a theory of relative stability. Oliver's relative stability theory is presented with modifications and refinement.

[1] J. Oliver, wrote his Ph.D. dissertation partly on the relative stability theory of linear recurrence algorithms under J.C.P. Miller at Cambridge.

4.1. Concepts and definitions

Definition 2: The desired solution of recursion (10) is a special solution to be computed, which can be represented by the initial values

$$\{g(j) = \alpha_j;\ j = k-m+1, \ldots, k\};\ (\alpha_{k-m+1}, \ldots, \alpha_k) = \vec{\alpha}. \tag{19}$$

We denote this desired solution as $g_{\vec{\alpha},k}(x)$.

Notation: We use ε to denote absolute errors and η to denote relative errors.

Two possible ways to generate round-off errors are: (i) rounding, and (ii) chopping. Most computers use rounding; however, some computers do use chopping.

As indicated in Example 2, when the desired solution is a rapidly varying solution, the absolute round-off errors also vary rapidly. However, OLIVER [11] (p. 326-7) pointed out that, for a rapidly varying solution, floating point arithmetic would be used. If floating point arithmetic is used then the actual relative round-off errors η_i are fairly evenly distributed within a small range

$$\begin{cases} [-\bar{\eta}, \bar{\eta}] & \text{if rounding is used,} \\ [-\bar{\eta}, 0] & \text{if chopping is used.} \end{cases}$$

If r digits are assigned by a user to the computer, $r+1$ digits would be actually used by the computer to leave some room for rounding or chopping [2]. Then every real number in the floating-point range of the computer can be represented with a relative error bounded by

$$\bar{\eta} = \begin{cases} .5 \times 10^{-r} & \text{if rounding is used,} \\ 10^{-r} & \text{if chopping is used.} \end{cases} \tag{20}$$

(See DAHLQUIST and BJORCK [4], p. 45).

To symbolize this fact, we give the following definition.

Definition 3: The basis relative error generator $\vec{\eta}_{gen}$ is a random variable uniformly distributed on

$$\begin{cases} [-\bar{\eta}, \bar{\eta}] & \text{if rounding is used,} \\ [-\bar{\eta}, 0] & \text{if chopping is used.} \end{cases} \tag{21}$$

During the recursive evaluation by computers, each of the initial values $\{g(j);\ j = k-m+1, \ldots, k\}$ has only initial round-off error. After the starting point k, there are two sources of errors in each step of the evaluation of $g(x)$:

[2] To be consistent, 'the number of digits' will refer to the number of digits assigned to the computer.

(i) the propagation of earlier errors, and (ii) the newly generated round-off error when the computer outputs its 'exact' result assuming that all inputs are exact. We assume that the newly generated round-off errors are independent and identically distributed random variable $\vec{\eta}_{gen}$. Obviously, for any newly generated error, it will be propagated in the same way as the true 'value' and thus satisfies the recursion (10).

Definition 4: The relative error for the initial value $g(j) = \alpha_j$ is η_j (a value of $\vec{\eta}_{gen}$). The propagation of the initial value errors is a solution $\varepsilon_k(x)$ of (10) which satisfies the initial condition:

$$\varepsilon_k(j) = \eta_j \alpha_j, \quad j = k-m+1, \ldots, k. \tag{22}$$

We shall adopt the following **convention:** if one of the initial values α_j is zero, then the actual value used will be correct. This is equivalent to assuming that the computer can represent zero exactly, i.e. all bits set to zero. For example, in the initial conditions (18) of the recursion (17), the first $m-1$ initial values are zero, and in actual computing they are used as zero without error. OLIVER [11] (p. 330) also supports this convention.

Definition 5: The (newly generated) round-off relative error at point τ ($\tau > k$) is η_τ (a value of $\vec{\eta}_{gen}$). The propagation of the round-off error at τ is a solution $\varepsilon_\tau(x)$ which satisfies the initial condition at τ:

$$\{\varepsilon_\tau(\tau-m+j) = 0;\ j = 1, \ldots, m-1\};\ \varepsilon_\tau(\tau) = \eta_\tau\, g_{\bar{\alpha},k}(\tau). \tag{23}$$

4.2. The basic error propagation

Consider the first order homogeneous linear recursion:

$$g(x) = cg(x-1), \quad x \geq 1. \tag{24}$$

For recursion (24), it is easy to see that the propagated value of any generated error remains constant relative to the solution $g(x)$:

$$\frac{\varepsilon_i(x)}{g(x)} = \eta_i, \quad i = 0, 1, 2, \ldots \tag{25}$$

An upper bound for the accumulated relative error is

$$\frac{|\sum_{i=0}^{x} \varepsilon_i(x)|}{|g(x)|} \leq \frac{\sum_{i=0}^{x} |\varepsilon_i(x)|}{|g(x)|} \leq (x+1)\bar{\eta}. \tag{26}$$

Note that at worst the accumulated relative error increases linearly with the number of points that have been evaluated. "*This is an acceptable form of error accumulation, since if floating point arithmetic is used then doubling the range of evaluation corresponds to the loss of a single binary digit (in terms of error bounds rather than actual errors).*" – OLIVER [11] (p. 325).

We define the basic error propagation for which (26) holds, i.e. relative error bound grows linearly with a slope no greather than 1, and we judge the acceptability of error behavior in the general case by comparing it with the above basic error propagation.

4.3. Index of error propagation

Definition 6: The range of interest for recursion (10) is the interval $[k, R]$ over which the values of $g(x)$ are to be computed.

Definition 7: The index of error propagation for the recursion (10) in evaluating the desired solution $g_{\bar{\alpha},k}(x)$ over the range $[k, R]$ is defined by

$$I(k, R) := \sup_{x \in [k, R]} \left\{ \frac{1}{(x-k+1)\bar{\eta}} \frac{|\sum_{i=k}^{x} \varepsilon_i(x)|}{|g_{\bar{\alpha},k}(x)|} \right\}. \tag{27}$$

In evaluating the desired solution,

1. if $I(k, R)$ is bounded, we say that the recursion (10) is **stable** over the range $[k, R)$.
2. if $I(k, R) \leq 1$, we say that the recursion (10) is **strongly stable** over the range $[k, R)$.
3. if $I(k, R) = \infty$, we say that the recursion (10) is **unstable** over the range $[k, R)$.

In other words, a recursive evaluation is stable if the round-off error grows linearly, and being strongly stable if the linear slope is bounded by 1; a recursive evaluation is unstable if the round-off error grows more rapidly than linear; for example, exponentially.

Theorem 1: The linear recursion (10) is stable for evaluating its dominant solutions, and unstable for evaluating its subordinate solutions.

This result can be found in WIMP [23] (p. 10) and CASH [2] (p. 3). Here we just give an intuitive interpretation.

Let $g^{(h)}(x)$, $(h = 1, 2, \ldots, m)$, be a fundamental set of (10) such that $g^{(1)}(x)$ is a dominant solution and

$$\lim_{x \to \infty} \frac{g^{(h)}(x)}{g^{(1)}(x)} = 0, \quad \text{for} \quad h = 2, \ldots, m. \tag{28}$$

The solution $g_{\bar{\alpha},k}(x)$ to be computed can be written as a linear combination of this fundamental set:

$$g_{\bar{\alpha},k}(x) = d_1 g^{(1)}(x) + \ldots + d_m g^{(m)}(x), \tag{29}$$

where

$$d_1 \begin{cases} = 0 & \text{if } \; g_{\bar{\alpha},k}(x) \text{ is subordinate} \\ \neq 0 & \text{if } \; g_{\bar{\alpha},k}(x) \text{ is dominant} \end{cases}$$

On the other hand, the round-off error propagation $\varepsilon_\tau(x)$, as a disturbance solution, can be written as a linear combination of the fundamental set:

$$\varepsilon_\tau(x) = c_1 g^{(1)}(x) + \ldots + c_m g^{(m)}(x), \tag{30}$$

where even though c_1 is small, but with probability 1 that $c_1 \neq 0$.

Since $c_1 \neq 0$, one has

$$\lim_{x\to\infty} \frac{\varepsilon_\tau(x)}{g_{\bar{\alpha},k}(x)} = \begin{cases} \infty & \text{if } \; g_{\bar{\alpha},k}(x) \text{ is subordinate} \\ \dfrac{c_1}{d_1} & \text{if } \; g_{\bar{\alpha},k}(x) \text{ is dominant} \end{cases} \tag{31}$$

where $\dfrac{c_1}{d_1}$ can be made arbitrarily small by using sufficient number of digits.

Therefore, a recursive evaluation by (10) is stable if the desired solution $g_{\bar{\alpha},k}(x)$ is dominant; and is unstable if the desired solution $g_{\bar{\alpha},k}(x)$ is subordinate.

Also, we can see that, regardless of our desired solution, the computation always generate a dominant solution $g_{\bar{\alpha},k}(x) + \varepsilon_\tau(x)$.

Example 3: Consider the following linear recursion:

$$g(x) = g(x-1) - \frac{3}{16} g(x-2), \qquad x > 2. \tag{32}$$

Equation (32) has a fundamental set of solutions

$$g^{(1)}(x) = (.75)^x, \qquad g^{(2)}(x) = (.25)^x. \tag{33}$$

Where $g^{(1)}(x)$ is a dominant solution, and $g^{(2)}(x)$ is a subordinate solution. A combination $c_1 g^{(1)}(x) + c_2 g^{(2)}(x)$ is a dominant solution if and only if $c_1 \neq 0$. Also, a solution $g(x)$ is a dominant solution if and only if

$$\lim_{x\to\infty} \frac{g(x)}{g(x-1)} = .75.$$

(I). Evaluate the desired solution $g^{(1)}(x)$ by recursion (32) with initial values:

$$g^{(1)}(1) = .75, \qquad g^{(1)}(2) = .75^2.$$

The computed results for some selected points are listed in Table 2 (5 digits are used in the evaluation).

TABLE 2

EVALUATION OF THE DOMINANT SOLUTION $g^{(1)}(x)$

x	calculated $g^{(1)}(x)$	relative error	x	calculated $g^{(1)}(x)$	relative error
5	.23730	$-.000019753$	75	$.42622 \times 10^{-9}$	.000089971
10	.056318	.000079649	100	$.32074 \times 10^{-12}$	.000061680
20	.0031710	$-.000066832$	200	$.10291 \times 10^{-24}$	.00047191
30	.00017858	$-.000011704$	300	$.33020 \times 10^{-37}$	.00091698
40	.000010057	.000041251	400	$.10592 \times 10^{-49}$	.0010888
50	$.56639 \times 10^{-6}$	.00012068	500	$.33970 \times 10^{-62}$	.0010686

From Table 2, one can observe that the relative error grows very slowly. The accumulated error at x is bounded by $(x-1)\bar{\eta}$, (i.e. the evaluation of $g^{(1)}(x)$ is stable).

(II). Evaluate the desired solution $g^{(2)}(x)$ by recursion (32) with initial values:

$$g^{(2)}(1) = .25, \qquad g^{(2)}(2) = .25^2.$$

The computed results for some selected points are listed in Table 3 (5 digits are used in the evaluation).

TABLE 3

EVALUATION OF THE SUBORDINATE SOLUTION $g^{(2)}(x)$

x	calculated $g^{(2)}(x)$	relative error	$g^{(2)}(x)/g^{(2)}(x-1)$
3	.015625	0	.25000
5	.0009763	$-.00026880$	.24995
10	$.8781 \times 10^{-6}$	$-.079245$	.23643
20	$-.42568 \times 10^{-8}$	-4681.4	.75032
30	$-.23977 \times 10^{-9}$	$-.27644 \times 10^{9}$	.75001
50	$-.76036 \times 10^{-12}$	$-.96387 \times 10^{18}$	.75000

From Table 3, one can observe that the round-off errors blow up rapidly. The recursive evaluation is very unstable. By checking the ratio $g^{(2)}(x)/g^{(2)}(x-1)$ of the computed results, one can see that the computed solution (eventually) follows a pattern of a dominant solution.

Remarks:

1. In general, every (non-trivial) linear recursion is stable for some solution and unstable for other solutions. Thus it is meaningless to merely talk about the stability of a recursion without mentioning the desired solution.

However, for simplicity, when we talk about the stability of a recursion without specifying which solution it refers to, we assume that the desired solution is implicitly known.

2. It is the rate of growth of the desired solution with respect to other solutions of the recursive equation that determines whether or not the recursive computation is successful. In terms of initial value representations, the family of subordinate solutions form an $m-1$ dimensional surface in the m dimensional space of all solutions of (10). As a result of round-off errors and higher-order round-off errors, the disturbance solution can be in any direction in the space of all solutions of (10). Therefore, in general, no matter whether the desired solution is dominant or not, the computed result follows a pattern of a dominant solution. When the desired solution is a subordinate solution, round-off errors will blow up and make the recursive evaluation ineffective.

We have clarified the stability concept of linear recursions. In the next section, we shall give a family of recursions whose non-negative solutions are dominant solutions.

5. CONGRUENT RECURSIONS OF FINITE ORDER AND THEIR DOMINANT SOLUTIONS

Definition 8: A linear recurrence equation of the form:

$$g(x) = \sum_{j=1}^{m} B_j(x) f(j)\, g(x-j), \tag{34}$$

with the following restrictions:

- $f(x)$ is non-negative with finite support on $\{x_1, x_2, \ldots, x_r\}$ which satisfies (15) and (16). Note that $f(x)$ does not have to be a probability function
- $B_j(x)$, $j = 1, 2, \ldots, m$, are strictly positive functions of $x > 0$

is called a **congruent** recursion of **finite order** m.

In this section, we are going to give the dominant solutions of congruent recursions.

We first discuss a set of solutions $g^{(h)}(x)$ of (34) with starting point k (≥ 0) and initial values

$$g^{(h)}(k-m+j) = \delta_{h,j} = \begin{cases} 1 & \text{if } j = h, \\ 0 & \text{if } j \neq h, \end{cases} \qquad 1 \leq h,\ j \leq m. \tag{35}$$

Proposition 1: For a positive number n, if it is a linear combination of $x_1, x_2, \ldots, x_r$ with coefficients in $\mathbb{Z}^0 = \{0, 1, 2, \ldots\}$, then

$$g^{(h)}(k+h+n) > 0. \tag{36}$$

Proof: Since

$$g^{(h)}(k-m+h) = 1, \qquad B_j(\cdot) f(j) \geq 0,$$

from equation (34), we have

$$g^{(h)}(k+h) \geq B_m(k+h) f(m) > 0 .$$

Now from point $k+h$, apply the recurson (34) again:

$$g^{(h)}(k+h+x_1) \geq B_{x_1}(\cdot) f(x_1)\, g^{(h)}(k+h) > 0$$

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By induction, for any n which is a linear combination of $x_1, x_2, \ldots, x_r$ with coefficients in $\mathbb{Z}^0$, we have

$$g^{(h)}(k+h+n) > 0 . \qquad \square$$

Lemma 1: Let $x_1, x_2, \ldots, x_r \in \mathbb{Z}$ with x_i not all zero. The following statements are equivalent:

- $x_1 z_1 + x_2 z_2 + \ldots + x_r z_r = 1$ has a solution in integers z_i;
- $gcd(x_1, x_2, \ldots, x_r) = 1$.

Proof: See FLATH ([7], p. 13).

Proposition 2: Let $x_1, x_2, \ldots, x_r$ be positive integers with

$$gcd(x_1, x_2, \ldots, x_r) = 1 .$$

There exists a constant N_0, such that for any integer $k > N_0$, k can be expresssed as a linear combination of $x_1, x_2, \ldots, x_r$ with coefficients in $\mathbb{Z}^0 = \{0, 1, 2, \ldots\}$.

Proof: From Lemma 1, there exist $z_1, z_2, \ldots, z_r \in \mathbb{Z}$ such that

$$x_1 z_1 + x_2 z_2 + \ldots + x_r z_r = 1 .$$

Let

$$N_0 = x_1 (|z_1| x_1 + |z_2| x_2 + \ldots + |z_r| x_r) .$$

For any $n > N_0$, apply the division algorithm to positive integers $n - N_0$ and x_1, we obtain

$$n = N_0 + u x_1 + v, \qquad u \geq 0, \qquad 0 \leq v < x_1 .$$

By replacing v with

$$v(x_1 z_1 + x_2 z_2 + \ldots + x_r z_r),$$

we obtain

$$n = N_0 + v(x_1 z_1 + x_2 z_2 + \dots + x_r z_r) + u\, x_1,$$

which is a linear combination of $x_1, x_2, \dots, x_r$ with coefficients in $\mathbb{Z}^0$. □

Theorem 2: For the congruent recursion (34), the solution $g^{(h)}(x)$ is non-negative and free from zero when x gets large:

(37) $\quad g^{(h)}(x) \geq 0, \quad \text{for} \quad x \geq k; \qquad g^{(h)}(x) > 0, \quad \text{for} \quad x > N_0;$

where N_0 is some constant.

Proof: It is an immediate application of Propositions 1 and 2. □

Now we are ready to generalize our results to the solutions of equation (34) with an arbitrary non-negative initial vector $\vec{\alpha}$ with at least one positive element:

(38) $\quad g_{\vec{\alpha},k}(k-m+j) = \alpha_{k-m+j} \geq 0, \quad (j = 1, 2, \dots, m); \quad \sum_{j=1}^{m} \alpha_{k-m+j} > 0.$

Theorem 3: For the congruent recursion (34) with initial conditions (38), the solution $g_{\vec{\alpha},k}(x)$ is non-negative and free from zero when x gets large:

(39) $\quad g_{\vec{\alpha},k}(x) \geq 0, \quad \text{for} \quad x \geq k; \qquad g_{\vec{\alpha},k}(x) > 0, \quad \text{for} \quad x > N_0$

where N_0 is some constant.

We say that $\vec{\alpha} \geq \vec{\beta}$ if an only if $\alpha_{k-m+j} \geq \beta_{k-m+j}$ for $j = 1, 2, \dots, m$.

Theorem 4 (Comparison): For the congruent recursion (34), if $\vec{\alpha} \geq \vec{\beta}$, then

$$g_{\vec{\alpha},k}(x) \geq g_{\vec{\beta},k}(x), \qquad \text{for} \quad x \geq k.$$

Proof: Since $\vec{\alpha} \geq \vec{\beta}$, we have $\vec{\alpha} - \vec{\beta} \geq 0$ and from equation (34) we have

$$g_{\vec{\alpha},k}(x) - g_{\vec{\beta},k}(x) = g_{\vec{\alpha}-\vec{\beta},k}(x) \geq 0. \qquad □$$

Theorem 5: For the congruent recursion (34) with initial conditions (38), the solution $g_{\vec{\alpha},k}(x)$ is a dominant solution.

Proof: From Theorem 3, we can move the starting point from k to a new point K such that

(40) $\quad \vec{\gamma} = (\gamma_{K-m+1}, \dots, \gamma_K) = \{g_{\vec{\alpha},k}(K-m+1), \dots, g_{\vec{\alpha},k}(K)\}$

has all its m components strictly positive.

Obviously $g_{\vec{\alpha},k}(x)$ and $g_{\vec{\gamma},K}(x)$ are the same for $x > K$.

Let $h(x)$ be any solution of the congruent recursion (34), and

(41) $$\vec{\beta} = (\beta_{K-m+1}, \ldots, \beta_K) = \{h(K-m+1), \ldots, h(K)\}.$$

Since a finite number of values are always bounded, there is a positive constant ξ $(0 < \xi < \infty)$ such that

$$\xi \, \gamma_{K-m+j} \geq |\beta_{K-m+j}|, \qquad j = 1, 2, \ldots, m.$$

Therefore

$$g_{\vec{\alpha}, k}(x) = g_{\vec{\gamma}, K}(x) \geq \xi^{-1} |h(x)|, \quad \text{for} \quad x > K. \qquad \square$$

6. NON-HOMOGENEOUS RECURSIONS OF INFINITE ORDER

Now we extend our discussions to a general family of non-homogeneous recursions of infinite order.

Definition 9: A recurrence equation of the form

(42) $$g(x) = \sum_{j=1}^{x} A_j(x)\, g(x-j) + H(x), \quad x > k \geq 0,$$

is called a **non-homogeneous** recursion of **infinite** order.

Definition 10: The recurrence equation of infinite order

(43) $$g(x) = \sum_{j=1}^{x} A_j(x)\, g(x-j), \quad x > k \geq 0,$$

is called the **homogeneous counterpart** of recursion (42).

The homogeneous counterpart (43) is also a special case of (42) with $H(x) = 0$. When $H(x) = 0$, the homogeneous counterpart of equation (43) is itself.

For an example, when claim severity has a infinite support, the recursion (3) is homogeneous recurrence equation of infinite order.

Definition 11: The desired solution is a special solution of the non-homogeneous recursion (42) to be computed, which can be represented by the initial values:

(44) $$g(j) = \alpha_j, \quad j = 0, 1, \ldots, k; \quad (\alpha_0, \alpha_1, \ldots, \alpha_k) = \vec{\alpha}.$$

We denote this desired solution as $g_{\vec{\alpha}, k}(x)$.

In the above definition, without loss of generality, we assumed that the initial points are $\{0, 1, \ldots, k\}$. If initial points are $\{r, r+1, \ldots, r+k\}$, one can always introduce a new variable $x' = x - r$ and get a new equation in terms of x'. Of course, the stabilities for these two recursions are equivalent. Note that, by a transformation $x' = x - (k - m + 1)$ the recursion (10) of finite order is a special case of (42) with $H(x) = 0$ and $A_j(x) = 0$ for $j > m$.

Since both the desired solution $g_{\bar{\alpha},k}(x)$ and the computed values $\hat{g}_{\bar{\alpha},k}(x)$ satisfy the non-homogeneous recursion (42), the accumulated absolute error $\hat{g}_{\bar{\alpha},k}(x) - g_{\bar{\alpha},k}(x)$ satisfies the homogeneous counterpart (43).

Definition 12: The relative error for the initial value $g(j) = \alpha_j$ is η_j. The propagation of initial value errors is a solution $\varepsilon_k(x)$ of the homogeneous counterpart (43) with the initial condition

$$\varepsilon_k(j) = \eta_j \alpha_j, \quad j = 0, 1, \ldots, k. \tag{45}$$

Definition 13: The (newly generated) round-off relative error at point τ ($\tau > k$) is η_τ. The propagation of the round-off error at τ is a solution $\varepsilon_\tau(x)$ of the homogeneous counterpart (43) with the initial condition

$$\varepsilon_\tau(j) = 0, \quad j = 0, 1, \ldots, \tau - 1; \quad \varepsilon_\tau(\tau) = \eta_\tau g_{\bar{\alpha},k}(\tau). \tag{46}$$

Other definitions (e.g. index of error propagation and strongly stable, etc.) can be similarly defined as in the finite homogeneous case.

Definition 14: A non-homogeneous **congruent** recursion of infinite order is defined by:

$$g(x) = \sum_{j=1}^{x} B_j(x)\, g(x-j) + H(x), \quad x > k, \tag{47}$$

with $B_j(x) \geq$, and $H(x) \geq 0$.

The dominance ranking between the desired solution and the error solution determines whether the recursive evaluation is successful or not.

Unlike its homogeneous counterpart, a non-homogeneous first order recursion is not necessarily stable. This is because that, for a non-homogeneous recursion, the desired solution and the error solution satisfy two different equations.

Example 4: Consider the first order forward recursion:

$$g(x) = g(x-1) - .5^x, \qquad x \geq 1, \tag{48}$$

with an initial value $g(0) = 1$. The desired solution is $g(x) = .5^x$. A fundamental set of the homogeneous counterpart is given by $g^{(1)}(x) = 1$. Since $g^{(1)}(x)$ dominates $g(x)$, the recursive evaluation is unstable in evaluating $g(x)$. This instability can be easily verified on a computer. If 5 digits are used, the computed results for the points after $x = 40$ become a constant $.79228 \times 10^{-7}$, which again follows a pattern of a dominant solution.

Similarly, we have a comparison theorem.

Theorem 6: Let $g_{\vec{\alpha},k}(x)$ and $g_{\vec{\beta},k}(x)$ be two solutions of the non-homogeneous congruent recursion (47), $\varepsilon_{\vec{\beta},k}(x)$ be a solution of the homogeneous counterpart of (47). If $\vec{\alpha} \geq \vec{\beta}$, then

$$g_{\vec{\alpha},k}(x) \geq g_{\vec{\beta},k}(x) \geq \varepsilon_{\vec{\beta},k}(x), \qquad \text{for} \quad x > k.$$

From the above theorem, or by mathematical deduction, for non-negative initial vector $\vec{\alpha}$, the solution $g_{\vec{\alpha},k}(x)$ of (47) is non-negative.

Theorem 7 (Strongly Stable): A non-homogeneous congruent recursion of infinite order (47) is strongly stable in evaluating $g_{\vec{\alpha},k}(x)$ provided that $\vec{\alpha}$ is non-negative.

Proof: After the initial points, any vanishing of $g_{\vec{\alpha},k}(x)$ results solely from zeros in the initial values and does not depend on previous non-zero $g_{\vec{\alpha},k}(x)$ values. There is no error in this case.

We need only to be concerned with positive values of $g_{\vec{\alpha},k}(x)$.
For the propagation $\varepsilon_k(x)$ of initial value errors, since

$$|\varepsilon_k(j)| \leq \bar{\eta} g(j) = \bar{\eta}\alpha_j, \qquad j = 0, 1, \ldots, k,$$

from Theorem 6, we have

$$\frac{|\varepsilon_k(x)|}{g_{\vec{\alpha},k}(x)} \leq \bar{\eta}, \quad x > k. \tag{49}$$

For the propagation $\varepsilon_\tau(x)$ of the newly generated round-off error at point τ, since

$$\varepsilon_\tau(j) = 0, \quad (j = 0, 1, \ldots, \tau - 1); \quad |\varepsilon_\tau(\tau)| \leq \bar{\eta} g_{\vec{\alpha},k}(\tau),$$

we have

$$\frac{|\varepsilon_\tau(x)|}{g_{\vec{\alpha},k}(x)} \leq \bar{\eta}, \quad x > k. \tag{50}$$

Therefore,

$$\frac{1}{(x-k+1)\bar{\eta}} \frac{|\sum_{i=k}^{x} \varepsilon_i(x)|}{|g_{\vec{\alpha},k}(x)|} \leq 1, \quad x > k.$$

The strongly stable condition (27) holds. □

In the proof, the inequalities (49) and (50) can be very loose. Thus, 1 is only a gross upper bound for $I(k, \infty)$. It can be much less than 1 in actual error propagation. Another important factor is the offset of positive and negative relative errors when rounding is used by the computer.

Theorem 8: In evaluating non-negative solutions of the congruent recursion (47), if rounding is used by the computer, the accumulated relative error at point x is a random variable $\vec{u}(x)$ with values in

$$[-(x-k+1)\bar{\eta},\ (x-k+1)\bar{\eta}]. \tag{51}$$

$\vec{u}(x)$ has a mean of zero and variance $(x-k+1)\bar{\eta}^2/3$.

Proof: It is a direct result from a sum of $x-k+1$ *i.i.d.* random variables which are uniformly distributed on $[-\bar{\eta}, \bar{\eta}]$. □

From Theorem 8, even though the upper bound for accumulated relative errors at x grows linearly with x, the standard deviation is only a constant multiple of $\sqrt{x-k+1}$. A 99% confidence interval for $\vec{u}(x)$ is approximately

$$[-1.5\sqrt{x-k+1}\ \bar{\eta},\ 1.5\sqrt{x-k+1}\ \bar{\eta}]. \tag{52}$$

7. FORWARD DIRECTION *VS* BACKWARD DIRECTION

The earlier discussions can also be easily extended to recursions in the backward direction. For simplicity, we only discuss recursions of finite order.

Definition 15: A recurrence equation of the form

$$g(y) = \sum_{j=1}^{m} A_j(y)\, g(y+j) + H(y), \quad y < k, \tag{53}$$

with

$$g(j) = \alpha_j, \quad (j = k, \ldots, k+m-1), \quad \bar{\alpha} = (\alpha_k, \ldots, \alpha_{k+m-1}) \tag{54}$$

is called a non-homogeneous recursion in the **backward** direction with starting point k and initial vector $\bar{\alpha}$. We denote this solution as $g_{\bar{\alpha},k}(y)$.

Definition 16: A non-homogeneous **congruent** recursion in the **backward** direction is defined by:

$$g(y) = \sum_{j=1}^{m} B_j(y)\, g(y+j) + H(y), \quad y < k, \tag{55}$$

with $B_j(y) \geq 0$, and $H(y) \geq 0$.

Similarly, we have a strongly stable theorem.

Theorem 9: The non-homogeneous congruent recursion (55) is strongly stable in evaluating its non-negative solution $g_{\bar{\alpha},k}(y)$ in the backward direction.

When a congruent recursion in the forward direction is rewritten as a recursion in the backward direction, it is *no longer* a congruent recursion in the backward direction. Thus, **'congruent' is direction dependent!**

The links between the two directions are important.

For a first order homogeneous recursion, since there is no dominance ranking among the solutions, the recursion is strongly stable in both directions.

For a second order homogeneous recursion, there are only two solutions in a fundamental set. If the recursion is unstable in one direction, which means the undesired error solution grows unboundedly with respect to the desired solution, then this undesired error solution will decrease rapidly in the reverse direction, and thus the recursion is stable in the reverse direction.

For a recursion of order $m \geq 2$, its solutions are ranked by their dominance relationship. There may be solutions which are subordinate in both directions; for these solutions, the recursion is unstable in both directions. Nevertheless, if the desired solution dominates all other solutions (in a fundamental set) in one direction, then the same desired solution will be dominated by other solutions in the reverse direction. Thus, if a recursion is stable in one direction, it is unstable in the reverse direction. In general, the more stable a recursoin is in one direction, the more unstable when it is used in the reverse direction.

Definition 17: Asssuming that $m \geq 2$, a recursion is called **strongly unstable** in one direction for a desired solution if it is strongly stable in the reverse direction for the same desired solution.

The next two theorems follow directly from this definition.

Theorem 10: A recurrence equation ($m \geq 2$)

$$g(x) = B_m(x)\, g(x-m) - \sum_{j=1}^{m-1} B_j(x)\, g(x-j) - H(x), \quad x > k, \tag{56}$$

with $B_j(x) \geq 0$ and $H(x) \geq 0$ is strongly unstable in the forward direction in evaluating its non-negative solutions.

Theorem 11: A recurrence equation ($m \geq 2$)

$$g(y) = B_m(y)\, g(y+m) - \sum_{j=1}^{m-1} B_j(y)\, g(y+j) - H(y), \quad y < k, \tag{57}$$

with $B_j(y) \geq 0$ and $H(y) \geq 0$ is strongly unstable in the backward direction in evaluating its non-negative solutions.

Example 5: Reconsider Example 1. From Theorem 7, the forward recursion (6) is strongly stable in evaluating its non-negative solutions. From Theorem 11, the backward recursion (8) is strongly unstable in evaluating its non-negative solutions.

Example 6: Consider the recurrence equations for modified Bessel functions (see Press, et al. [16], p. 192):

$$I_{n+1}(x) = -(2n/x)\ I_n(x) + I_{n-1}(x), \tag{58}$$

$$K_{n+1}(x) = +(2n/x)\ K_n(x) + K_{n-1}(x). \tag{59}$$

Since $I_n(x)$ and $K_n(x)$ are non-negative solutions for $x \geq 0$, the recursion (58) is strongly unstable in the forward direction, and the recursion (59) is strongly stable in the forward direction.

8. EMPIRICAL INFLATION FACTOR

In this section, based on the signs of the coefficients $A_j(x)$ and the term $H(x)$ in (42), we investigate the growth of the relative errors in each step of the recursive evaluation.

Lemma 2: Let a and b be two positive real values, with their estimates $\hat{a}$ and $\hat{b}$ having relative errors η_1, η_2, respectively. Then, $\hat{a}+\hat{b}$, as an estimate of $a+b$, has a relative error

$$\frac{a}{a+b}\eta_1 + \frac{b}{a+b}\eta_2 \tag{60}$$

which is bounded by $[-\eta, \eta]$ where

$$\eta = \max(|\eta_1|, |\eta_2|). \tag{61}$$

As a special case, if $\eta_2 = 0$ ($\hat{b}$ is exact), then, $\hat{a}+b$, as an estimate of $a+b$, has a relative error which is less than η_1. We say that the relative error is **damped**.

Lemma 3: Let a and b be two positive real values, with their estimates $\hat{a}$ and $\hat{b}$ having relative errors η_1, η_2, respectively. Then, $\hat{a}\hat{b}$, as an estimate of ab, has a relative error $\eta_1+\eta_2$, provided that η_i is small relative to 1 ($\eta_i \ll 1$, $i = 1, 2$). As a special case, $\hat{a}b$, as an estimate of ab, has a relative error η_1.

Lemma 4: Let a and b be two positive real values, with their estimates $\hat{a}$ and $\hat{b}$ having relative errors of any value in the range $(-\bar{\eta}, \bar{\eta})$. Then, $\hat{a}-\hat{b}$, as an estimate of $a-b$, can have a relative error of any value in the range $(-\gamma\bar{\eta}, \gamma\bar{\eta})$, where

$$\gamma = \frac{a+b}{|a-b|}, \tag{62}$$

is called the **error inflation factor**.

In Lemma 4, one can see that, when $a \approx b$, γ can be infinitely large, which causes extraordinary unstable result. This should be avoided in any computing schemes.

Consider the non-homogeneous recursion (42) of infinite order. The value $g(x)$ at point x depends upon all previous values $g(x-j), j = 1, \ldots, x$. In each step of recursive evaluation, there are $x+1$ terms involved:

$$H(x) \quad \text{and} \quad A_j(x)\, g(x-j), \quad (j = 1, \ldots, x).$$

Some of them may be positive, and some may be negative. To indicate clearly the sign of each term, we re-write the equation (42) into the following form:

$$g(x) = \sum_{j=1}^{x} s_j(x)\, B_j(x)\, g(x-j) + H^+(x) - H^-(x), \tag{63}$$

such that

$$B_j(x)\, g(x-j) = |A_j(x)\, g(x-j)| \geq 0, \tag{64}$$

and

$$s_j(x) = \begin{cases} 1 & \text{if} \quad A_j(x)\, g(x-j) > 0, \\ 0 & \text{if} \quad A_j(x)\, g(x-j) = 0, \\ -1 & \text{if} \quad A_j(x)\, g(x-j) < 0, \end{cases} \tag{65}$$

and

$$H^+(x) = \frac{|H(x)| + H(x)}{2}, \qquad H^-(x) = \frac{|H(x)| - H(x)}{2}. \tag{66}$$

Definition 18: Associated with the computed solution $g(x)$, we define a positive part $g_+(x)$ and a negative part $g_-(x)$ at each point x such that

$$g_+(x) = \sum_{s_j = 1} B_j(x)\, g(x-j) + H^+(x) \geq 0 \tag{67}$$

$$g_-(x) = \sum_{s_j = -1} B_j(x)\, g(x-j) + H^-(x) \geq 0 \tag{68}$$

$$g(x) = g_+(x) - g_-(x) \tag{69}$$

Definition 19: An empirical inflation factor at x is defined by

$$\hat{\gamma}(x) = \frac{g_+(x) + g_-(x)}{|g_+(x) - g_-(x)|}, \quad \text{if} \quad g_+(x) \neq g_-(x); \tag{70}$$

and

$$\hat{\gamma}(x) = \infty, \quad \text{if} \quad g_+(x) = g_-(x). \tag{71}$$

Definition 20: If k is the starting point of the recursion (63), then we define an empirical accumulated relative error bound recursively:

$$\hat{u}(x) = \hat{u}(x-1)\,\hat{\gamma}(x) + \bar{\eta}, \qquad x > k, \tag{72}$$

with initial value $\hat{u}(k) = \bar{\eta}$.

Theorem 12: Assuming that r digits are used. For the computed value $g(x)$ by recursion (42), an empirical upper bound of the relative error is given by

$$\hat{u}(x) \le \begin{cases} (x-k+1)\prod_{i=k}^{x}\hat{\gamma}(i) \times .5 \times 10^{-r} & \text{if rounding is used,} \\ (x-k+1)\prod_{i=k}^{x}\hat{\gamma}(i) \times 10^{-r} & \text{if chopping is used.} \end{cases} \tag{73}$$

Proof: It can be easily verified by mathematical induction.

Definition 21: We say that the number of significance digits in the computed value $g(x)$ is $\nu(x)$ if the relative error is less than $10^{-\nu(x)}$.

One can empirically estimate the number of significant digits $\nu(x)$ in the computed value $g(x)$ by the following inequality:

$$\nu(x) \ge \hat{\nu}(x) = [-\log_{10}\hat{u}(x)] = \left[-\frac{\ln \hat{u}(x)}{\ln 10}\right] \tag{74}$$

where $\ln x$ denotes the natural logarithm of x, $[x]$ denote the largest integer which is no greater than x. For example, $[2.317] = 2$, and $[-2.317] = -3$.

Example 7: Reconsider the backward recursion (8) in Example 1. Now we calculate the estimated $\hat{\nu}(x)$ and compare it with the actual $\nu(x)$ at each point. The results are listed in Table 4.

TABLE 4

EMPIRICAL ESTIMATION OF THE NUMBER OF SIGNIFICANT DIGITS FOR THE BACKWARD RECURSION (8) WHEN 6 DIGITS ARE USED

x	computed $g(x)$	$\hat{\nu}(x)$	$\hat{\gamma}(x)$	exact $g(x)$	actual $\nu(x)$
8	.099850	4	22.71	.0998450	4
7	.078315	3	25.22	.0783629	3
6	.054807	2	28.15	.0543124	2
5	.027538	0	38.81	.0325723	0

The catastrophic instability of the backward recursion (8) can be seen from the large inflation factors $\hat{\gamma}(x)$ in Table 4.

Remarks:

If all the terms are of the same sign, (i.e. either $g_+(x) = 0$ or $g_-(x) = 0$ for all $x \geq k$), then $\hat{u}(x) = (x-k+1)\bar{\eta}$ and

$$(75) \qquad \nu(x) \geq \begin{cases} r+[\log_{10} 2 - \log_{10}(x-k+1)] & \text{if rounding is used,} \\ r+[-\log_{10}(x-k+1)] & \text{if chopping is used.} \end{cases}$$

Our earlier results about the non-negative solutions of congruent recursions are 'recovered'.

One should interpret the inflation factors with care. For an example, in evaluating the dominant solution $g^{(1)}(x)$ in Example 3, the inflation factors are a constant $\hat{\gamma} = 1.6667$, but error inflations seldom occur and the evaluation is stable.

9. APPLICATIONS

Note that the recursion (3) is a special case of (47) with $H(x) = 0$ and starting point $k = 0$. The initial value $g(0)$ is positive and the desired compound distribution is non-negative. If the claim frequency is in the family of Poisson, Negative Binomial or Geometric distributions, we have, from PANJER [12],

$$(76) \qquad B_j(x) = a + b\frac{j}{x} > 0, \qquad j = 1, \ldots, x.$$

As an immediate application of Theorem 7, the recursion (3) is strongly stable in evaluating compound Poisson, compound Negative Binomial and compound Geometric distributions.

In using recursion (3) to evaluate compound Poisson, compound Negative Binomial and compound Geometric distributions, the accumulated relative error bound grows linearly with a slope no greater than 1. If the evaluation starts at point $x = 0$ and r digits are used, a guaranteed number of significance digits in the computed $g(x)$ can be estimated by the following simple inequality:

$$(77) \qquad \nu(x) \geq \begin{cases} r+[\log_{10} 2 - \log_{10}(x+1)] & \text{if rounding is used,} \\ r+[-\log_{10}(x+1)] & \text{if chopping is used.} \end{cases}$$

If rounding is used by the computer, with a probability of 99%,

$$(78) \qquad \nu(x) \geq r + \left[\log_{10}\frac{4}{3} - \frac{1}{2}\log_{10}(x+1)\right].$$

For example, if both claim frequency and claim size have a mean 1000, one wishes to get an accuracy with relative errors less than 10^{-7} over the interested range $[0, 10^7]$. One can achieve this accuracy by using 14 digits. Also, with (at least) 99% confidence, one can achieve this accuracy by using only 11 digits. This strongly stable property has practical significance in applications of discretization method (see GERBER [8], PANJER [13], PANJER and LUTEK [14]). If one increases the number of points by a factor of 100 in the discretization of severity distribution, simply adding 2 digits can keep the same level of accuracy.

As an application of Lemma 2 and Lemma 3, the effet of round-off coefficients can be considered. For any finite number of positive values, their summation has the same level of relative error, and their product has a relative error bound which is the summation of individual relative error bounds. For any non-negative solution of the recursion (47), if the relative round-off errors of $B_j(x)$ and $f(j)$ are *i.i.d.* random variable $\vec{\eta}_{gen}$, then the index of relative error propagation enlarges only by a constant multiple of 3. One additional digit is sufficient to protect the solution from round-off errors in the coefficients.

The condition (76) does not hold for the family of compound Binomial distributions. Compound Binomial distributions share a special feature that it has only finite support when claim size has finite support. Since the desired solution eventually becomes zero in the forward direction, it can not be a dominant solutoin. From Theorem 1, recursion (3) is unstable in evaluating compound Binomial distributions. This instability can be encountered at the right tail of the compound distribution in the forward direction. A special treatment for compound Binomial distributions is given in the next section.

10. THE CASE OF COMPOUND BINOMIAL

In this section, we investigate in more detail about the instability of compound Binomial distribution. Based on some special features of compound Binomial distribution, a simple method to cope with this instability is given.

Consider the case that the claim frequency has a Binomial distribution:

$$p_n = \frac{N!}{n!\,(N-n)!}\,\theta^n (1-\theta)^{N-n}, \qquad 0 \le n \le N. \tag{79}$$

Then, in recursion (3),

$$a = -\frac{\theta}{1-\theta}, \qquad b = (N+1)\frac{\theta}{1-\theta}. \tag{80}$$

Example 8 [3]: Consider compound Binomial distribution with parameters

$$\theta = .95, \qquad N = 100,$$

and with claim severity distribution as in Table 5.

In order to investigate how unstable the recursion (3) is in evaluating this compound Binomial distribution, we use 200 digits in the calculation. The

[3] All the numerical examples in this paper are done on Maple V [6], on which one can freely assign the number of digits. Rounding is used by Maple V.

TABLE 5

THE DISTRIBUTION OF CLAIM SEVERITY A

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	.150	.200	.250	.125	.075	.050	:050	.050	.025	.025

TABLE 6

THE EMPIRICAL INFLATION FACTOR AND ESTIMATED SIGNIFICANT DIGITS IN THE COMPUTED VALUE WHEN 200 DIGITS ARE USED IN THE COMPUTATION

x	$\hat{\gamma}(x)$	$\hat{v}(x)$	x	$\hat{\gamma}(x)$	$\hat{v}(x)$	x	$\hat{\gamma}(x)$	$\hat{v}(x)$
50	1	199	350	9.5099	61	650	50.818	−345
100	1	198	400	12.905	8	700	68.394	−434
150	1.9574	191	450	17.447	−51	750	99.559	−529
200	2.8849	171	500	22.699	−116	800	157.89	−634
250	4.6382	143	550	29.683	−186	850	328.04	−752
300	6.4586	106	600	38.372	−263	900	—	—

empirical inflation factors are calculated along with the recursive evaluation. The results for some selected points are listed in Table 6.

From Table 6, one can see that the error inflation factor remains flat at 1 when $x \leq 100$, and accelerates after $x > 100$. The acclerating growth in the error inflation factors indicates that the recursive evaluation becomes more and more unstable when it proceeds to the right half of the compound Binomial distribution. Even 200 digits can not protect the desired solution from the disturbance of rounding-off errors! In the computed values of $g(x)$, we obtained the following absurd results:

$$g(898) := -.19502 \times 10^{-93}, \quad \text{and} \quad g(1000) := -.59052 \times 10^{-70}.$$

The computed $g(x)$ becomes negative at $x = 898$, which tells us that the empirical estimates $\hat{\gamma}(x)$ and $\hat{v}(x)$ after point $x = 898$ are no longer reliable.

10.1. A combined usage of two directions

This method involves two recursions: (i) the forward direction, and (ii) the reverse recursion in the backward direction staring at the end point mN.

When the claim severity has a finite support $\{x_1, x_2, \ldots, x_r\}$, recursion (3) can be written into a recursion (17) of finite order $m = x_r$. The recursion (17) can be easily turned into a backward recursion:

$$(81) \qquad g(y) = \frac{1}{P(y)} \left\{ g(y+m) - \sum_{j=1}^{m-1} \left(a + b \frac{m-j}{y+m} \right) f(m-j)\, g(y+j) \right\}$$

where

$$(82) \qquad P(y) = \left(a + b \frac{m}{y+m} \right) f(m).$$

For compound Binomial distributions, the boundary condition at the end point mN is known and can be used as an initial value for the backward recursion:

(83) $$g(mN) = \theta^N f(m)^N, \quad g(mN+j) = 0, \quad \text{for} \quad j = 1, 2, \ldots$$

Theorem 13: In evaluating compound Binomial distributions, we have the following results:

1. The forward recursion (3) is locally strongly stable over the range $[0, N+1]$.
2. The forward recursion (17) is locally strongly stable over the range $[0, N+1]$, and becomes strongly unstable when it proceeds to the range $[mN-N-1, mN]$.
3. The backward recursion (81) is locally strongly stable over the range $[mN-N-1, mN]$, and becomes strongly unstable when it retreats to the range $[0, N+1]$.
4. As a special case, when $m = 2$, a combination of recursions in both directions gives a locally strongly stable evaluation over the interested range $[0, mN]$.

Proof: When $0 < x \leq N+1$, we have

$$a+b\frac{j}{x} \geq 0, \qquad j = 1, 2, \ldots$$

Therefore, the coefficients of the forward recursion (3) are all non-negative over the range $[0, N+1]$.

When $mN-(N+1) \leq y \leq Nm$, we have

$$P(y) > 0, \quad \text{and} \quad -\left(a+b\frac{m-j}{y+m}\right) \geq 0, \quad j = 1, \ldots, m-1.$$

Thus the coefficients of the backward recursion (81) are all non-negative over the range $[mN-(N+1), mN]$.

From earlier results, the theorem is proved. □

One can see the connection between compound Binomial and compound Poisson distributions. When $N \to \infty$ and $\theta = \lambda/N \to 0$, the limiting distribution of compound Binomial is nothing but a compound Poisson distribution, which is strongly stable over the range $[0, \infty)$.

Example 9: Reconsider the compound Binomial distribution discussed in Example 8. We use both the forward recursion (3) and the backward recursion (81) to evaluate the compound Binomial distribution. This time, instead of using 200 digits, we use only 20 digits in the evaluation. The results are displayed in Table 7.

TABLE 7

EVALUATE COMPOUND BINOMIAL RECURSIVELY IN BOTH DIRECTIONS

x	$g(x)$ (forward)	$g(x)$ (backward)
0	$.7888609052 \times 10^{-130}$	$.1025580868 \times 10^{11}$
1	$.2248253579 \times 10^{-127}$	$-.8857199512 \times 10^{10}$
...		
200	$.1254727678 \times 10^{-22}$	$-.5706507331 \times 10^{-6}$
...		
305	$.2472423462 \times 10^{-2}$	$.2472423462 \times 10^{-2}$
306	$.2694072242 \times 10^{-2}$	$.2694072242 \times 10^{-2}$
...		
378	$.8779196867 \times 10^{-2}$	$.8779196867 \times 10^{-2}$
379	$.8381164919 \times 10^{-2}$	$.8381164919 \times 10^{-2}$
...		
600	$.2300721278 \times 10^{25}$	$.1099653604 \times 10^{-20}$
...		
999	$-.2066050091 \times 10^{111}$	$.3684354379 \times 10^{-160}$
1000	$.3516174897 \times 10^{111}$	$.3684354379 \times 10^{-162}$

From Table 7, one can observe that the computed results by recursions in both directions meet each other over the middle range [305, 379] in their first 10 non-zero digits. If we use the results of forward recursion for points before 379, and the results of backward recursion for points after 305, then we have confidence in that there are at least 10 significant digits in the combined results.

Note that in (79), $a = -\dfrac{\theta}{1-\theta}$, thus $a \to \infty$ as $\theta \to 1$.

In this numerical example, $\theta = .95$, which gives a large negative value $a = -19$ and thus causes rapid round-off error blow-ups. The effectiveness of both forward and backward recursions are compared in Table 8, for different values of θ, and in Table 9, for different values of N.

TABLE 8

EVALUATE COMPOUND BINOMIAL WITH SEVERITY A AND $N = 100$ IN TWO DIRECTIONS (20 DIGITS ARE USED TO ENSURE 10 SIGNIFICANCE DIGITS IN THE COMPUTED RESULT)

θ	forward range	forward mass	backward range	backward mass
.01	$0 \to 965$	$1-.67415 \times 10^{-326}$	$443 \leftarrow 1000$	$.90480 \times 10^{-114}$
.05	$0 \to 926$	$1-.90647 \times 10^{-234}$	$465 \leftarrow 1000$	$.75752 \times 10^{-82}$
.1	$0 \to 889$	$1-.24936 \times 10^{-187}$	$462 \leftarrow 1000$	$.18494 \times 10^{-63}$
.2	$0 \to 841$	$1-.74181 \times 10^{-140}$	$442 \leftarrow 1000$	$.11263 \times 10^{-41}$
.3	$0 \to 801$	$1-.49640 \times 10^{-110}$	$432 \leftarrow 1000$	$.88769 \times 10^{-30}$
.4	$0 \to 772$	$1-.78696 \times 10^{-90}$	$415 \leftarrow 1000$	$.22285 \times 10^{-20}$
.5	$0 \to 747$	$1-.36413 \times 10^{-74}$	$396 \leftarrow 1000$	$.43928 \times 10^{-13}$
.6	$0 \to 732$	$1-.39255 \times 10^{-63}$	$375 \leftarrow 1000$	$.20499 \times 10^{-7}$
.7	$0 \to 692$	$1-.44574 \times 10^{-48}$	$335 \leftarrow 1000$	.0023744
.8	$0 \to 624$	$1-.35322 \times 10^{-30}$	$299 \leftarrow 1000$	.45509
.9	$0 \to 523$	$1-.44492 \times 10^{-12}$	$278 \leftarrow 1000$	.98987
.95	$0 \to 379$	.87753	$305 \leftarrow 1000$	.97773
.99	$0 \to 214$	$.15973 \times 10^{-13}$	$187 \leftarrow 1000$	$1-.245 \times 10^{-20}$

TABLE 9

EVALUATE COMPOUND BINOMIAL WITH SEVERITY A AND $\theta = .5$ IN TWO DIRECTIONS
(20 DIGITS ARE USED TO ENSURE 10 SIGFICANCE DIGITS IN THE COMPUTED RESULT)

N	forward range	forward mass	backward range	backward mass
10	$0 \rightarrow 100$	1	$0 \leftarrow 100$	1
20	$0 \rightarrow 192$	$1-.20096 \times 10^{-31}$	$12 \leftarrow 200$	.99630
50	$0 \rightarrow 422$	$1-.14619 \times 10^{-51}$	$142 \leftarrow 500$	.0043459
100	$0 \rightarrow 747$	$1-.36413 \times 10^{-74}$	$396 \leftarrow 1000$	$.43928 \times 10^{-13}$
200	$0 \rightarrow 1337$	$1-.81694 \times 10^{-111}$	$962 \leftarrow 2000$	$.67602 \times 10^{-46}$
500	$0 \rightarrow 2996$	$1-.52880 \times 10^{-207}$	$2754 \leftarrow 5000$	$.31541 \times 10^{-165}$
1000	$0 \rightarrow 5763$	$1-.87859 \times 10^{-372}$	$5763 \leftarrow 10000$	$.87859 \times 10^{-372}$

Remarks:

1. In terms of probability mass (not number of points) covered by the valid range in which the accuracy meets a specified level, the effectiveness of the forward direction increases when N increases, and increases when θ decreases. This can be seen from Table 8 and Table 9, which is also consistent with the result in Theorem 13.
   However, the forward direction can be very unstable when θ gets close to 1 or the claim distribution is highly negative skewed. In such cases, the backward recursion can play a major part in evaluating the compound distribution.
2. From Table 8, we can see that, when $\theta \leq .5$, the backward direction can give accurate results for more than one third of the points over the whole range; however, their total probability mass is very small. Thus, when $\theta \leq .5$, the actual usefulness of the backward direction can be used to check the accuracy of the forward direction.
3. In most insurance applications, $\theta \leq .5$ and N is large and the claim size distribution $f(x)$ is positively skewed, if additional digits are used in the evaluation, one should not be bothered by seeing negative probabilities in the extreme far right tail, since almost all of the compound distribution except the very extreme right tail has been evaluated with desired accuracy. A check of accuracy can be done by a recursive evaluation in the backward direction. If two directions do not meet over the middle range, increasing the number of digits in the evaluation can make them so.
4. As mentioned by CHAN ([3], p. 175) and SHIU ([19], p. 181), the famous J. C. P. Miller formula has been used to evaluate the power of polynominals and the N-fold convolution of arithmetic distributions. Essentially, J. C. P. Miller formula is a variant of recursion (3) for the compound Binomial case. Assume that a discrete distribution $f(x)$ is defined on integers $\{x_0, x_1, \ldots, x_r\}$. With a transformation $x' = x - N x_0$, the N-fold convolution of $f(x)$ is equivalent to a compound Binomial with $\theta = 1 - f(x_0)$. In such situations, the Binomial parameter θ can be very close to 1, or $f(x)$

itself can have a high negative skewness, which may cause difficulties when using the recursion in the forward direction. This instability can be easily handled by using two recursive evaluations in both directions.

11. REVIEW OF OTHER RECURSIONS

11.1. The generalized (a, b) class

SUNDT and JEWELL [2] extended recursion (3) to a larger family of claim frequencies

$$\frac{p_n}{p_{n-1}} = a + \frac{b}{n}, \qquad n = r+1, \ldots \tag{84}$$

The compound distribution for this family of claim frequency satisfies:

$$g(x) = \sum_{j=1}^{x} \left(a + b\frac{j}{x}\right) f(j)\, g(x-j) + \sum_{i=1}^{r} \left(p_i - \left(a + \frac{b}{i}\right) p_{i-1}\right) f^{*i}(x). \tag{85}$$

Among the generalized claim frequencies (84), the class with $r = 1$ is of special interest and is given a name (a, b) class. In the (a, b) class, p_0 can be any value in the interval $[0, 1]$. The family of frequencies in (1) given by PANJER [12] is a subclass of the (a, b) class with $r = 0$ and is called the $(a, b, 0)$ subclass. The family of frequencies in the (a, b) class with $r = 1$ and $p_0 = 0$, is called the $(a, b, 1)$ subclass.

As counterparts of the $(a, b, 0)$ subclass, truncated Poisson, truncated Negative Binomial, truncated Geometric and truncated Binomial are members in the $(a, b, 1)$ subclass. Another member in the $(a, b, 1)$ subclass is the logarithmic distribution. SUNDT and JEWELL [20] and WILLMOT [21] completed the enumeration of members in the $(a, b, 1)$ subclass by adding in the extended truncated Negative Binomial (ETNB) distribution.

For members in the $(a, b, 1)$ subclass, we can modify the probabilities at zero arbitrarily. We name the members of the (a, b) class as: zero-modified Poisson, zero-modified Negative Binomial, zero-modified Geometric, zero-modified Binomial, zero-modified extended Negative Binomial, and log-zero distribution.

For claim frequencies in the (a, b) class, the non-homogeneous recursion (85) becomes

$$g(x) = \sum_{j=1}^{x} \left(a + b\frac{j}{x}\right) f(j)\, g(x-j) + (p_1 - (a+b)p_0) f(x). \tag{86}$$

If we decompose the recursion (86) into the following form:

$$g(x) = \sum_{j=1}^{x-1} \left(a + b\frac{j}{x}\right) f(j)\, g(x-j) + p_1 f(x) + (a+b)\,(g(0) - p_0) f(x), \tag{87}$$

and utilize the initial condition $g(0) = p_0$, recursion (86) can be reduced to:

$$g(x) = \sum_{j=1}^{x-1} \left(a+b\frac{j}{x}\right) f(j)\, g(x-j) + p_1 f(x), \tag{88}$$

with two initial values

$$g(0) = p_0, \qquad g(1) = p_1 f(1). \tag{89}$$

From Theorem 7, one can easily see that the recursion (88) is strongly stable in evaluating compound zero-modified Poisson, compound zero-modified Negative Binomial, compound zero-modified Geometric and compound log-zero distributions.

The recursion (88) is unstable in evaluating compound zero-modified Binomial. The method developed for compound Binomial in the last section can be applied to this case without any difficulty.

For the compound zero-modified extended Negative Binomial distribution, we have

$$0 < a < 1, \qquad b = (r-1)a, \qquad -1 < r < 0. \tag{90}$$

For positive claim severities with a finite support $\{x_1, \ldots, x_r = m\}$, we have

$$a+b\frac{j}{x} > 0, \qquad j = 1, \ldots, m, \quad \text{for} \quad x > (1+|r|)m. \tag{91}$$

Therefore, for compound zero-modified extended Negative Binomial distribution, the recursion (86) is stable. Also, once recursive evaluation has reached at a point $k > (1+|r|)m$, the recursive evaluation for future points are strongly stable.

In all the previous recursions for aggregate claims, it was assumed that claims were positive valued. For non-negative claim severities including zero claims, PANJER and WILLMOT [15] proposed a simple method, by which the spike at zero can be easily removed and the previous recursions for positive claims can be used.

11.2. Compound Poisson (*a*, *b*) (CPAB) class

WILLMOT and PANJER [22] discussed various contagious counting distributions which involve a sequential usage recursion (3). For example, a compound Poisson Inverse Gaussian (P-IG) distribution can be evaluated by a two-stage usage of recursion (3): (i) a compound ETNB over the claim severity distribution; (ii) a compound Poisson with the compound ETNB distribution obtained in the first stage as its severity distribution.

If each recursive evaluation is stable, their combined usage is also stable. ISLAM and CONSUL ([10], p. 93) commented that the use of CPAB frequency model may cause serious numerical instabilities. Clearly their comment was wrong.

11.3. Improved recursions aren't improved

When the claim frequency has a Poisson distribution, and the claim severity has a special pattern of piecewise constant or piecewise linear, DE PRIL [17] gave some simplified recursions in terms of numbers of calculations required. However, since both positive and negative signs evenly appeared in the coefficients of the recursions, they are unstable and thus not really improved.

11.4. Probability of ultimate ruin

PANJER [13] proposed a method of direct evaluation of the probability of ruin. Since the desired probability is a compound Geometric distribution, the recursive evaluation is strongly stable.

GOOVAERTS and DE VYLDER [9] proposed a different approach to approximate the probability of ruin. The upper bounds are evaluated by a recurrence equation:

$$(92)\quad \hat{\Psi}_u(x\,h) = \frac{1}{1+\theta}\left\{K(x\,h) - \sum_{i=1}^{x} \Delta K((i-1)\,h)\,\hat{\Psi}_u((x-i)\,h)\right\},\ x = 1, 2, \ldots$$

The lower bounds are evaluated by a recurrence equation:

$$(93)\quad \hat{\Psi}_l(x\,h) = \frac{1}{1+\theta+\Delta K(0)}\left\{K(x\,h) - \sum_{i=1}^{x-1} \Delta K(i\,h)\,\hat{\Psi}_u((x-i)\,h)\right\}, x = 1, 2, .$$

Since

$$(94)\qquad K(s) = \int_s^{\infty} \frac{1-F(y)}{p_1}\,dy, \qquad s \geq 0,$$

we have

$$(95)\qquad -\Delta K(i\,h) = \int_{i\,h}^{(i+1)\,h} \frac{1-F(y)}{p_1}\,dy > 0.$$

Therefore, the recursions (92) and (93) are indeed strongly stable in evaluating the desired ruin probability.

RAMSAY [18] recently commented that his numerical result did not agree with that of GOOVAERTS and DE VYLDER [9] and was unable to explain the difference ([18], p. 58). Now it becomes clear that, the instability that RAMSAY [18] discussed about was not from inherent rounding error accumulations by using recursions (92) and (93), but from the unstable evaluation of the coefficients $\Delta K(i\,h) = K((i+1)\,h) - K(i\,h)$ by subtracting two nearly equal numbers. Also, the inaccuracy in the numerical results of GOOVAERTS and DE VYLDER [9] can be explained by the slow convergence (as proved by Ramsay) of the approximation scheme of Goovaerts and De Vylder, and not because of the instability of the recursions.

11.5. Probability of finite time ruin

In their paper [5], Dickson and Waters suggested a method of recursive evaluation of finite time ruin probabilities. DICKSON and WATERS [5] (p. 211) commented that they experienced some numerical instabilities when using a combination of two recursions. One (see (4.2) of DICKSON and WATERS [5], p. 208) is now known as strongly stable; the other recursion (see (3.2) of DICKSON and WATERS [5], p. 206) involves many differencing terms. It can be verified that, (3.2) of DICKSON and WATERS is unstable in evaluating the desired probabilities.

The basic ideas and results in this paper can be extended and applied to other recursions (not necessarily in actuarial field). For unstable recursions, alternative methods of evaluation merit further research.

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