NOTE ON THE BACKGROUND TO THE SUBJECT: THEORY OF RISK, FUNDAMENTAL MATHEMATICS AND APPLICATIONS

based on Report I by

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1. By a general theorem the necessary and sufficient condition for a function $\varphi_0(\sigma)$ being completely monotonic for σ lying in the right semi-plane, i.e. that the n^{th} derivative with respect to σ has the sign of $(-1)^n$, is that the function may be represented by the Laplace-Stieltjes integral $\int_0^\infty e^{-(\sigma-s)v}\, d\, U(v)$, where $U(v)$ is a non-decreasing function of v, independent of σ and bounded in every finite interval, σ a real or complex variable represented in the right semi-plane, s a real constant $\leq$ the real part of σ. By the notation $\varphi_n(\sigma)$ we designate $\int_0^\infty e^{-(\sigma-s)v}\, v^n\, dU(v)$, which for $U(v)$ being independent of σ, as assumed above, is equal to $(-1)^n\, \varphi_0^{(n)}(\sigma)$.

Definition 1. *A compound Poisson process* (*in the narrow sense*) is a process for which the *probability distribution of the number of changes* in the random function $Y(t)$, constituting the process, occurring while the parameter, which is represented on the positive real axis, is in the interval $(0, t)$ for every value of t, is defined by the following relation

$$\pi_n(t) = t^n\, \varphi_n(s + t)\, /\, n!, \tag{1}$$

the function $\varphi_n(\sigma)$ being defined by the integral given above and subject to the condition that $\varphi_0(\sigma)$ tends to unity, when σ tends to s. The function $U(v)$ in the integrand of $\varphi_n(\sigma)$ is, then, a distribution function which defines the *risk distribution*, in this case said to be *t-independent*.

It shall be remarked here that in this definition we have—as in the most part of this report and of my second report—restricted our considerations to processes with a parameter represented on the positive axis, such a process being here called *univariate*. In some references, however, we shall talk of *bivariate* and *multivariate processes*.

Definition 2. A process is said to be *elementary*, if the size of one change is a constant, and *non-elementary*, if it is a variable distributed in a conditioned distribution relative to the hypothesis that one change has occurred, this distribution (or the distribution function) will in this and the following report be called the *distribution of the size of one change.*

For the terminology used here is referred to Ove Lundberg (1940), to Feller (1943, 1950; though Feller 1957 has changed his terminology) and to a previous paper by myself (1961b).*

2. A particular case of the processes defined in the previous section is the Poisson process for which $dU(v)$ assumes the value 1 for a particular value of v and is equal to zero for other values of v, i.e. $\varphi_0(s+t) = e^{-qt}$, where q is a positive constant. The collective risk theory in its classical form is introduced by Filip Lundberg (1903, 1909, 1919, 1926-1928, 1930, 1932, 1934) and developed by Cramér (1919, 1926, 1928, 1930, 1946, 1954 and other papers), Esscher (1932) and many others. In these papers the parameter t is often treated as a time parameter and transformed into a scale depending on the mean of $\pi_n(t)$, called the operational scale, and the distribution of the size of one change being defined as the distribution of the amount of one claim and called the risk sum distribution, probably due to the fact that it is—for life insurance—chiefly dependent on the distribution of the risk sum insured. In this and the following report the parameter t and the distribution of the size of one change represent, not necessarily, the time respectively the distribution of the amount of one claim.—Cramér (1955) gave an excellent general exposition of the collective risk theory, where i.a. reference is made to generalizations such as those made by Esscher with respect to the variation of the distribution of the size of one claim (with t or any other argument) and to a modification of the expansion of the distribution function of the

*) Literature references in a separate list, common to Report I and II, here given below.

standardized $Y(t)$. Further, Cramér refers to Ove Lundberg (1940) and to Ammeter (1946, 1948, 1949, 1951, 1954) with respect to generalization of the probability distribution of the number of changes in $Y(t)$. A short and eloquent survey with particular consideration of applications to non-life insurance was given by Wilhelmsen (mimeograph, 1955). Segerdahl (1959) gave a more general survey of some results reached in this theory, where also some results for the expansion of the distribution function of a standardized $Y(t)$ attached to a Polya process with a t-independent distribution of the size of one change as given by Ammeter and Ove Lundberg, were mentioned. A summary with particular respect to motor insurance has recently been given by Mehring (1960), to which is referred for literature in the field.

3. The general theory of compound Poisson processes (in the narrow sense) has been given by Ove Lundberg (l.c.). Hofmann (1955) introduced a sub-class of such processes, containing Poisson and Polya processes as particular cases. He discusses, also, a process constituted by two random functions, each with a probability distribution of the number of changes in the form of such a distribution in a Polya process. Bivariate Polya processes were, tentatively, applied by me (1955) to the growth of plants. Univariate Polya processes were treated by Ove Lundberg (l.c.), by Ammeter (l.c. and 1957a, 1957b) and by Bichsel (1959). The connections between such processes and different urn schemes were demonstrated by Ove Lundberg (l.c.), by Feller (1943), by Ammeter (1948) (cf. also Thompson (1954)), the same topic being treated by me (1956); in the last-mentioned paper a comparison was made with a process introduced by Arfwedson (1955). The normal expansion of the distribution function of the standardized $Y(t)$ was extended to a Polya process with t-dependent distribution of the size of one change and to other processes of the Hofmann sub-class with t-independent such a distribution by me (1957, 1961b). The Polya process was by Ove Lundberg and Hofmann applied to sickness and accident statistics, by the latter in the extended form referred to above. A slightly modified form of a Polya process (with a risk distribution having three parameters) was applied by Delaporte (1959, 1960) to the bonus problem in motor insurance. Thyrion (1959, 1960) used another particular process of the Hofmann sub-class to the same problem. For the same purpose Grenander (1957), Fréchet (1959), Franckx (1959, 1960) used Markov processes, which are related to the process discussed here above. Depoid (1959) used several Poisson distributions as bonus model. Hofmann applied two particular processes of his sub-class to the accident statistics studied by him. In a recent paper the applications mentioned here above have been discussed by me (1961a). In two earlier papers I have applied the normal expansion of the distribution function of the standardized $Y(t)$ attached to a stationary and ergodic process to insurance against loss of profit (1959a) and to loss excess reinsurance (1959b). Ammeter (1946, 1951, 1957b, 1957c, 1960) has applied the Polya process to reinsurance problems and the Poisson process to stop loss reinsurance and experience rating. Benktander and Segerdahl (1960) have studied claim distributions with particular reference to excess of loss reinsurance. Sparre Andersen (1957) has developed a risk theory for a case of contagion.

4. Almer (1957, and in papers under preparation kindly given to me in manuscript) has from a very general starting point developed a risk theory using mathematical tools different from those generally applied in mathematical statistics. For some results reached by Almer is referred to my second report and to two of my recent papers (1961b, 1961c). With respect to the application to motor insurance, see also Philipson (1960).

5. In a recent thesis, Matérn (1960) has studied different bivariate and even multivariate processes as models for spatial variation mainly by the study of the correlation functions, based on covariance functions and their relation to the spectral and, the so-called radial densities of the process. Inter alia he defines the multivariate stationary (in the weak sense) compound Poisson process, obtained by substituting a random function attached to a multivariate integrable, stationary and positive process (called the primary process) for the constant intensity of a Poisson process. Even in the univariate case this process is more general than those defined in Definition 1. These processes were constructed in a way mentioned by Quenouille (1949) (cf. Bartlett (1954) and Thompson (1955)). Matérn gives expressions for the covariance function of such processes in the form of a sum of two terms, one being the covariance function of a Poisson process and the other a double integral over a function, generally called product density, in particular cases being equal to the correlation function of the primary process. These relations show that the stationarity of the primary process is necessary for the compound process being stationary. The multivariate processes have in Matérn's excellent thesis been applied to sampling problems arising in estimating the distribution of plants over an area and the timber production by forest trees in an area. Especially, I want to draw attention to the demonstrations of three particularly chosen models of randomly located points with normal, super-normal and sub-normal dispersion. Each model has been explained by a particular expression for the covariance function of the compound process and by a numerical example, as recorded in two graphs, in one of which a map over the actual distribution has been drawn in a plane representing the spatial coordinates. A study of the chapter referred to (l.c. pp. 48-51) seems to be of outstanding value for the comprehensibility of the concept stationary compound Poisson process and of the processes dealt with in the papers referred to in the next section.

6. The papers mentioned in the sections 3 and 4 (1961a, 1961b, 1961c) are or will be published by me in 1961.

In the papers a general class of compound Poisson processes (in the narrow sense) was first constructed, which class implies a wide generalization of the Hofmann sub-class, which was used as a starting point. Then, this class has been extended to a class of processes, which have been called compound Poisson processes in the wide sense and defined by allowing also for t-dependent risk distributions. For the construction of these processes a general class of distribution functions, connected with different functions commonly used, has been defined. The mathematical developments have been based on a recent book by Slater (1960) and on a paper by Odhnoff (1946), the former for certain hypergeometric functions, the latter for the Fourier-Laplace transforms of Pearson's density functions. The deductions lead inter alia to comparatively simple expressions for certain probabilities and for the characteristic functions of $Y(t)$ attached to the processes concerned. These compound Poisson processes have, however, been studied for the univariate case. Matérn's stationary compound Poisson processes are, in fact, particular cases of the compound Poisson processes in the wide sense, if the latter were to be generalized to multivariate such processes. Report II to this colloquium is a summary of the three papers referred to and contains a generalization of some theorems given in the papers

7. For a more general outlook on the impact of stochastic process theory on statistics is referred to an interesting paper by Bartlett (1959). From this paper the following concluding remarks will be quoted:

"We have seen that the statistican's greater breadth of outlook" (by the wider approach based on the theory of stochastic processes) "can lead to greater power in handling any analysis. At the same time it will warn him to be rather wary of empirical analysis, at least on non-experimental material, not based on a complete, and sometimes necessarily extensive, theoretical appraisal."

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* Abreviations of journals and books see below

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Abbreviations

AB	= *The Astin Bulletin.*
AIP	= *Annales de l'Institut Henri Poincaré.*
AMS	= *The Annals of Mathematical Statistics.*
ASFL	= *Actuarial Studies dedicated to Filip Lundberg,* Stockholm 1946.
B	= *Biometrika.*
Baf	= *Bulletin trimestriel de l'Instutit des Actuaires Francais.*
Bas	= *Bulletin des actuaires suisses.*
BDGV	= *Blätter der Deutschen Gesellschaft für Versicherungsmathematik.*
BFR	= *Bulletin of the Swedish State Institute for Forestry Research.*
CM	= *Cambridge Mathematical Tract.*
HCV	= *Probability and Statistics, The Harald Cramér Volume,* Stockholm, New York, 1959.
NTA	= *Actuaries of the Nordic Tariff Associations.*
SA	= *Skandinavisk Aktuarietidskrift.*
Trans	= *Transactions of the International Congress of Actuaries,* XIII in Scheveningen, XIV in Madrid, XV in New York, XVI in Brussels.
UCP	= *University of California Publications in Statistics.*

Addition

After completing my reports I and II, I have obtained the last issue of Bull. des actuaires belges, where Thyrion publishes Note sur les distributions "par grappes". In this paper he assumes that the changes in $Y(t)$ attached to a random process are generated from centres attached to a primary process. In one case it is assumed that the primary process is a compound Poisson process in the narrow sense. This leads to $Y(t)$ being attached to a compound Poisson process in the wide sense with the risk distribution defined by the frequency function $k(t) \exp(-\lambda k(t))$, where

$$k(t) = t[1 - b(t)]/b(t), 0 < b(t) < 1$$

for each value of $t > 0$. The deduction is similar to that given by me in report II to this colloquium.