CALCULATION OF PRICE EQUILIBRIA FOR UTILITY FUNCTIONS OF THE HARA CLASS

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Abstract

We explicitly calculate price equilibria for power and logarithmic utility functions which—together with the exponential utility functions—form the so-called HARA (Hyperbolic Absolute Risk Aversion) class.

A price equilibrium is economically admissible in the market which is a closed system. Furthermore it is on the one side individually optimal for each participant of the market (in the sense of maximal expected utility), on the other side it is a Pareto optimum and thus collectively optimal for the market as a whole.

Key words

Risk exchange, Price equilibrium, Utility function, Pareto optimality, Risk aversion.

1. Introduction

This paper follows up Buhlmann (1980). First we repeat some of the notions and results of Bühlmann (1980).

The *market* consists of *n participants* ("decision makers"), numbered from 1 to n, typically buyers of direct insurance, insurers, reinsurers.

Each participant i ($1 \leq i \leq n$) is characterized by

- w_i: Initial wealth
- $u_i(x)$: Utility function (as usual $u_i' > 0$, $u_i'' \leq 0$)
- $X_i(\omega)$: Initial portfolio (random variable)
- $Y_i(\omega)$: Payment (reimbursement) (random variable) bought for the premium $E[\phi Y_i]$.

$\phi(\omega)$ is a *price function.*

Remark. All random variables are given on a probability space $(\Omega, \mathfrak{A}, P)$ and are expected to be sufficiently regular in all cases. Particularly the appearing expected values should always exist.

The "exchange variable" Y_i, bought by i, represents the payment which i obtains from the other participants. So the $Y_1, \ldots, Y_n$ must satisfy the economic admissibility condition of the market

$$(1) \qquad \sum_{i=1}^{n} Y_i(\omega) = 0 \quad \forall \omega \in \Omega.$$

After having bought Y_i and paid for that the amount $E[\phi Y_i]$, i has a modified portfolio

$$X_i - Y_i + E[\phi Y_i].$$

We call each such vector of random variables

$$\boldsymbol{Y} = (Y_1, \ldots, Y_n),$$

satisfying (1), a *risk exchange* in the market. Furthermore a *price equilibrium* of the market is a pair

$$(\tilde{\phi}, \tilde{\boldsymbol{Y}} = (\tilde{Y}_1, \ldots, \tilde{Y}_n))$$

satisfying the two conditions
(I) $\sum_{i=1}^{n} \tilde{Y}_i = 0$, i.e., $\tilde{\boldsymbol{Y}}$ is a risk exchange
(II) $\forall i$: $\tilde{Y}_i$ maximizes the function

$$Y_i \mapsto E[u_i(w_i - X_i + Y_i - E[\tilde{\phi} Y_i])].$$

$\tilde{\boldsymbol{Y}}$ is called an *equilibrium risk exchange* whereas $\tilde{\phi}$ is an *equilibrium price*.

Condition (II) implies that each participant evaluates his "situation" according to the principle of expected utility. In BUHLMANN (1980) is shown that (II) is equivalent to the condition (III)

$$\text{(III)}\quad \forall i: \quad u_i'(w_i - X_i(\omega) + \tilde{Y}_i(\omega) - E[\tilde{\phi}\tilde{Y}_i]) = \tilde{\phi}(\omega) \underbrace{E[u_i'(w_i - X_i + \tilde{Y}_i - E[\tilde{\phi}\tilde{Y}_i])]}_{=: C_i}.$$
$$\text{for almost all } \omega \in \Omega.$$

Hence an equilibrium $(\tilde{\phi}, \tilde{\boldsymbol{Y}})$ is also characterized by (I) and (III).

Each vector of random variables

$$\boldsymbol{Z} = (Z_1, \ldots, Z_n),$$

satisfying

$$\text{(2)} \qquad \sum_{i=1}^{n} Z_i(\omega) = \sum_{i=1}^{n} X_i(\omega) \quad \forall \omega \in \Omega,$$

is an *admissible modification* of the initial portfolios $(X_1, \ldots, X_n)$. An admissible modification $\boldsymbol{Z}$ is *Pareto optimal* if there exists *no* admissible modification $\boldsymbol{Z}^*$ with

$$\forall i: \quad E[u_i(w_i - Z_i^*)] > E[u_i(w_i - Z_i)].$$

The admissible modification $\tilde{\boldsymbol{Z}}$,

$$\text{(3)} \qquad \tilde{Z}_i := X_i - \tilde{Y}_i + E[\tilde{\phi}\tilde{Y}_i] \qquad (1 \leq i \leq n),$$

induced by the equilibrium $(\tilde{\phi}, \tilde{\boldsymbol{Y}})$, is Pareto optimal. For this see BUHLMANN (1980) ((III) is just Borch's condition, necessary and sufficient for Pareto optimality).

A further and immediate consequence of the equivalence (II)$\Leftrightarrow$(III) is that the equilibrium price is automatically "normalized":

$$\text{(4)} \qquad E[\tilde{\phi}] = 1.$$

It follows from (4) that $\tilde{Y}$ is determined only up to an additive constant: $(\tilde{Y}_1 + c_1, \ldots, \tilde{Y}_n + c_n)$ is an equilibrium exchange if $(\tilde{Y}_1, \ldots, \tilde{Y}_n)$ is one and if $\sum_{i=1}^{n} c_i = 0$.

Hence we can put without loss of generality

$$(5) \qquad E[\tilde{\phi}\tilde{Y}_i] = 0 \qquad (1 \leq i \leq n) \qquad (c_i := -E[\tilde{\phi}\tilde{Y}_i]).$$

Then (3) becomes

$$(6) \qquad \tilde{Z}_i = X_i - \tilde{Y}_i.$$

REMARK. Observe that (2) (as well as (1)) is a "clearing condition" in the market which is a closed system.

REMARK. The *existence* of an equilibrium is dealt with in BÜHLMANN (1984) (which follows up BUHLMANN (1980)).

2. EXPLICIT CALCULATIONS

We explicitly calculate equilibria where the utility functions u_i are

(a) power functions

(b) logarithmic functions

(c) exponential functions.

Observe in the following that our considered utility functions u_i are indeed increasing ($u_i' > 0$) and concave ($u_i'' \leq 0$).

Hence: *Given*: w_i, X_i, u_i $(1 \leq i \leq n)$

Wanted: Pair $(\tilde{\phi}, \tilde{Y})$ $(\tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_n))$ satisfying (I), (III) and (5).

We will see that the $\tilde{Y}_i$ have the form

$$\tilde{Y}_i = X_i - \gamma_i \sum_{i=1}^{n} X_i - \delta_i$$

where the γ_i's and δ_i's are constants with

$$\gamma_i \geq 0, \qquad \sum_{i=1}^{n} \gamma_i = 1; \qquad \sum_{i=1}^{n} \delta_i = 0.$$

We can also write in the "Z-language" (as in (6))

$$\tilde{Z}_i = X_i - \tilde{Y}_i = \gamma_i \sum X_i + \delta_i.$$

Hence according to the modified portfolio $\tilde{Z}_i$ participant i pays the quota γ_i of the total claims $\sum X_i$ as well as a deterministic side payment δ_i. This result does not surprise, see e.g., BUHLMANN and JEWELL (1979).

(a) *Power Case*

We consider

$$(7) \qquad u_i(x) := \pm(a_i \mp x)^c \quad \text{for} \begin{cases} c > 1 \\ 0 < c < 1 \\ c < 0 \end{cases} \qquad (1 \leq i \leq n).$$

Observe that the u_i's are concave indeed ($u_i'' \leq 0$). The domain of u_i is given by the requirement that $u_i' > 0$. The a_i's are constants.

THEOREM. The equilibrium $(\tilde{\phi}, \tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_n))$ can be explicitly calculated and has the form

$$\tilde{\phi}(\omega) = \frac{(\sum a_i \mp \sum w_i \pm \sum X_i(\omega))^{c-1}}{E[(\sum a_i \mp \sum w_i \pm \sum X_i)^{c-1}]} \begin{cases} c > 1 \\ 0 \neq c < 1 \end{cases} \tag{8}$$

$$X_i(\omega) - \tilde{Y}_i(\omega) = \gamma_i \cdot \sum X_i(\omega) + \delta_i \qquad (1 \leq i \leq n)$$

where

$$\begin{cases} \gamma_i = \dfrac{E[(\sum a_i \mp \sum w_i \pm \sum X_i)^{c-1}(a_i \mp w_i \pm X_i)]}{E[(\sum a_i \mp \sum w_i \pm \sum X_i)^c]} \\ \delta_i = \pm(\sum a_i \mp \sum w_i) \cdot \gamma_i \mp (a_i \mp w_i). \end{cases} \tag{9}$$

Observe that it is indeed

$$\sum \gamma_i = 1, \qquad \sum \delta_i = 0.$$

REMARK. The case of three signs is for $\begin{cases} c > 1 \\ 0 < c < 1 \\ c < 0 \end{cases}$, whereas we put two signs for $\begin{cases} c > 1 \\ 0 \neq c < 1 \end{cases}$.

PROOF. With (7) we have

$$u_i'(x) = (\overset{+}{\pm} c)(a_i \overset{-}{\mp} x)^{c-1} \tag{10}$$

and it follows from (III) and (5)

$$(\overset{+}{\pm} c)(a_i \overset{-}{\mp} [w_i - X_i + \tilde{Y}_i])^{c-1} = \tilde{\phi} C_i$$

or

$$a_i \overset{-}{\mp} w_i \overset{+}{\pm} X_i \overset{-}{\mp} \tilde{Y}_i = \tilde{\phi}^{1/(c-1)} \left(\frac{C_i}{\overset{+}{\pm} c} \right)^{1/(c-1)}, \tag{11}$$

and after summation over all i it follows with (I)

$$\sum a_i \overset{-}{\mp} \sum w_i \overset{+}{\pm} \sum X_i = \tilde{\phi}^{1/(c-1)} \sum \left(\frac{C_i}{\overset{+}{\pm} c} \right)^{1/(c-1)}.$$

When we solve the last equation for $\tilde{\phi}$, we obtain with the "norming condition" (4) the formula (8).

We see that in the sequel it is enough to distinguish only two cases for the signs, concerning $\tilde{\phi}$.

We determine now $\tilde{Y}_i$. For this we multiply both sides of (11) with $\tilde{\phi}$ and then we take the expected value (so we take on both sides $E[\tilde{\phi}\cdot]$). We obtain with (5)

$$E[\tilde{\phi}(a_i \mp w_i \pm X_i)] = E[\tilde{\phi}\tilde{\phi}^{1/(c-1)}]\left(\frac{C_i}{\pm c}\right)^{1/(c-1)}. \tag{12}$$

(11) divided by (12) gives

$$\frac{a_i \mp w_i \pm X_i \mp \tilde{Y}_i}{E[\tilde{\phi}(a_i \mp w_i \pm X_i)]} = \frac{\tilde{\phi}^{1/(c-1)}}{E[\tilde{\phi}\tilde{\phi}^{1/(c-1)}]}$$

or

$$\frac{a_i \mp w_i \pm X_i \mp \tilde{Y}_i}{\tilde{\phi}^{1/(c-1)}} = \frac{E[\tilde{\phi}(a_i \mp w_i \pm X_i)]}{E[\tilde{\phi}\tilde{\phi}^{1/(c-1)}]}. \tag{13}$$

We insert $\tilde{\phi}$ of (8) in (13) and obtain after simplication (the denominator of (8) disappears)

$$\frac{a_i \mp w_i \pm X_i \mp \tilde{Y}_i}{\sum a_i \mp \sum w_i \pm \sum X_i} = \frac{E[(\sum a_i \mp \sum w_i \pm \sum X_i)^{c-1}(a_i \mp w_i \pm X_i)]}{E[(\sum a_i \mp \sum w_i \pm \sum X_i)^c]}.$$

We see from this that

$$\tilde{Z}_i = X_i - \tilde{Y}_i$$

is of the type

$$\gamma_i \sum X_i + \delta_i$$

where γ_i and δ_i have indeed the form (9).

Some special cases of formula (8)

$c := 2$ gives the quadratic utility functions

$$u_i(x) = -(a_i - x)^2 \qquad (x \leq a_i) \qquad (1 \leq i \leq n).$$

We then have for the "quadratic price"

$$\tilde{\phi}(\omega) = (\sum a_i - \sum w_i + \sum X_i(\omega))/E[\sum a_i - \sum w_i + \sum X_i].$$

$c := \frac{1}{2}$ fournishes utility functions of square root type:

$$u_i(x) = \sqrt{a_i + x} \qquad (x \geq -a_i) \qquad (1 \leq i \leq n).$$

In this case we have

$$\tilde{\phi}(\omega) = \frac{1}{\sqrt{\sum a_i + \sum w_i - \sum X_i(\omega)}} \Big/ E\left[\frac{1}{\sqrt{\sum a_i + \sum w_i - \sum X_i}}\right].$$

$c := -1$ finally leads to hyperbolic utility functions

$$u_i(x) = -\frac{1}{a_i + x} \qquad (x > -a_i) \qquad (1 \leq i \leq n).$$

Then the equilibrium price becomes

$$\tilde{\phi}(\omega)=\frac{1}{(\sum a_i+\sum w_i-\sum X_i(\omega))^2}\Big/E\left[\frac{1}{(\sum a_i+\sum w_i-\sum X_i)^2}\right].$$

(b) *Logarithmic Case*

$$u_i(x):=\log(a_i+x) \qquad (x>-a_i) \qquad (1\leq i\leq n).$$

The derivatives u_i'—because of the property (III) essential for the calculation of the equilibrium—which appear here,

$$u_i'(x)=(a_i+x)^{-1}, \tag{14}$$

belong to the same type of functions like those in the power case. If we compare (14) with (10) we obtain from (8) and (9) directly the formulas

$$\tilde{\phi}(\omega)=\frac{(\sum a_i+\sum w_i-\sum X_i(\omega))^{-1}}{E[(\sum a_i+\sum w_i-\sum X_i)^{-1}]}$$

$$X_i(\omega)-\tilde{Y}_i(\omega)=\gamma_i\sum X_i(\omega)+\delta_i \qquad (1\leq i\leq n)$$

where

$$\gamma_i=E\left[\frac{a_i+w_i-X_i}{\sum a_i+\sum w_i-\sum X_i}\right]$$

$$\delta_i=-(\sum a_i+\sum w_i)\gamma_i+(a_i+w_i).$$

Hence the logarithmic case is a special case of the power case.

Observe again

$$\sum\gamma_i=1, \qquad \sum\delta_i=0.$$

(c) *Exponential Case*

Let's finally consider the well known exponential utility functions

$$u_i(x)=\frac{1}{a_i}(1-e^{-a_ix}) \qquad (x\in\mathbb{R}) \qquad (1\leq i\leq n)$$

and for completeness cite BUHLMANN's (1980) formulas. The parameter $a_i>0$ stands here for the risk aversion of i. With the abbreviation

$$\frac{1}{a}:=\sum\frac{1}{a_i}$$

it is

$$\tilde{\phi}(\omega)=\exp(a\sum X_i(\omega))/E[\exp(a\sum X_i)]$$

$$\gamma_i = \frac{1/a_i}{1/a}$$

$$\delta_i = E[\exp(a \sum X_i)(X_i - \gamma_i \sum X_i)]/E[\exp(a \sum X_i)]$$

(and $\sum \gamma_i = 1$, $\sum \delta_i = 0$).

REFERENCES

BÜHLMANN, H (1980) An Economic Premium Principle *ASTIN Bulletin* **11** (1), 52-60.
BUHLMANN, H (1984) The General Economic Premium Principle *ASTIN Bulletin* **14** (1)
BUHLMANN, H and JEWELL, W S (1979) Optimal Risk Exchanges. *ASTIN Bulletin* **10** (3), 249

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