ON THE EXACT COMPUTATION OF THE AGGREGATE CLAIMS DISTRIBUTION IN THE INDIVIDUAL LIFE MODEL

By Nelson De Pril

Instituut voor Actuariële Wetenschappen, K.U. Leuven, Belgium

ABSTRACT

A recursive expression is derived for computing exactly the distribution of aggregate claims of a portfolio of life insurance policies. The recursion generalizes a formula of White and Greville for the claim numbers distribution and improves Kornya's approximation method for the aggregate claims distribution. It can be seen as the counterpart in the individual model of Panjer's recursion formula for the collective model.

KEYWORDS

Aggregate claims distribution, recursion formula, individual life model, individual risk theory.

1. INTRODUCTION

Consider a portfolio of independent life insurance policies. Each policy has a face amount (or an amount at risk) which is an integral multiple of some convenient monetary unit such as \$1.000 or \$5.000. The benefit is payable if death occurs within a certain exposure period. The portfolio is classified into amount and mortality classes, as displayed in Table 1.

Let i = face amount of a policy in column i, $i = 1, 2, \dots, a$; q_j = mortality rate in the exposure period for policies in row j, $j = 1, 2, \dots, b$; c_{ij} = number of policies in column i and row j.

Further we set $p_j = 1 - q_j$ = survival probability for policies in row j, $c_j = \sum_{i=1}^{a} c_{ij}$ = number of policies with mortality rate q_j, $c = \sum_{j=1}^{b} c_j$ = total number of policies, $m = \sum_{i=1}^{a} \sum_{j=1}^{b} ic_{ij}$ = maximum possible amount of aggregate claims.

TABLE 1
CLASSIFICATION OF THE PORTFOLIO

		Face amount in units 1	2	i	a
Mortality rate	q_1				
	q_2				
	q_j			c_{ij}	
	q_b				

Denote by S the total amount of claims in the exposure period and by $p_S(s)$ the probability that S will be precisely s units.

In this paper a recursion formula for $p_S(s)$ is derived, giving an optimal way to calculate exactly the aggregate claims distribution. Our development is inspired on the algorithm of KORNYA (1983) and PANJER's recursion (1981) for compound distibutions in the collective model.

2. THE RECURSION

THEOREM 1. *Let*

$$a(j,k)=\left(\frac{q_j}{p_j}\right)^k \tag{1}$$

and

$$A(i,k)=(-1)^{k+1}i\sum_{j=1}^{b}c_{ij}a(j,k) \tag{2}$$

then the following recursion holds

$$p_S(0)=\prod_{j=1}^{b}(p_j)^{c_j} \tag{3a}$$

$$sp_S(s)=\sum_{i=1}^{\min(a,s)}\sum_{k=1}^{[s/i]}A(i,k)p_S(s-ki), \qquad s=1,2,\ldots,m \tag{3b}$$

where $[x]$ denotes the greatest integer less than or equal to x.

PROOF. The probability generating function of S is

$$G_S(u)=\prod_{i=1}^{a}\prod_{j=1}^{b}(p_j+q_ju^i)^{c_{ij}} \tag{4}$$

and putting $u=0$ gives immediately formula (3a).

To prove (3b), we take the derivative of (4)

$$G'_S(u)=G_S(u)\sum_{i=1}^{a}\sum_{j=1}^{b}c_{ij}iu^{i-1}\frac{q_j}{p_j+q_ju^i}. \tag{5}$$

By expanding we get

$$G'_S(u)=G_S(u)\sum_{i=1}^{a}\sum_{j=1}^{b}c_{ij}iu^{i-1}\frac{q_j}{p_j}\sum_{k=0}^{\infty}(-1)^k\left(\frac{q_j}{p_j}u^i\right)^k$$

$$=G_S(u)\sum_{i=1}^{a}\sum_{k=1}^{\infty}A(i,k)u^{ki-1}. \tag{6}$$

Taking, according to Leibnitz's formula, the derivative of order $s-1$ of both sides of (6) and setting $u=0$ leads to the recursion (3b).

3. COMPARISON WITH KORNYA'S ALGORITHM

Kornya's rth-order approximation to $p_S(0)$ can be written as

$$p_S^{(r)}(0) = \exp\left\{\sum_{i=1}^{a}\sum_{j=1}^{b} c_{ij} \sum_{k=1}^{r} \frac{(-1)^k}{k}\left(\frac{q_j}{p_j}\right)^k\right\}. \tag{7}$$

It is easily seen that—theoretically—this formula reduces only in the limit for $r \to \infty$ to the exact expression of $p_S(0)$. Since all other rth-order approximations $p_S^{(r)}(s)$ are computed recursively and depend on the value of $p_S^{(r)}(0)$, they only become exact in the limit for $r \to \infty$.

In the recursion formula (3) we start from the exact value of $p_S(0)$ and the correct value of a given $p_S(s)$ can be computed directly. Even if we put $A(i, k) = 0$ for $k > r$, as in Kornya's approximation, we get exact values for $p_S(s)$ for each $s \leqslant r$.

4. A SPECIAL CASE

A recursion for the number of claims N can be obtained by putting $a = 1$ in (3).

COROLLARY 1. *Let*

$$A(k) = (-1)^{k+1} \sum_{j=1}^{b} c_j \left(\frac{q_j}{p_j}\right)^k \tag{8}$$

then

$$p_N(0) = \prod_{j=1}^{b} (p_j)^{c_j} \tag{9a}$$

$$np_N(n) = \sum_{k=1}^{n} A(k) p_N(n-k), \qquad n = 1, 2, \ldots, c. \tag{9b}$$

This recursion was derived by WHITE and GREVILLE (1959) and applied to problems involving multiple life contingencies.

5. A BACKWARD RECURSION

So far we formulated the model for the case of death benefits. However, the model can also be applied when a benefit is payable if the insured is alive at the end of the exposure period. Using the same notation as before, the q_j will now play the role of survival probabilities and, since $q_j > \frac{1}{2}$, the quantities $A(i, k)$ will grow very fast in magnitude with increasing k. Noting that the $A(i, k)$ also alternate it is clear that, in this case, the recursion (3) is not appropriate for actual computations. This problem can be removed by replacing the forward recursion (3), which starts from $p_S(0)$, by a backward recursion starting from the other end of the distribution $p_S(m)$. The maximum possible claim amount m can be quite large, but is nevertheless finite.

THEOREM 2. *Let*

$$b(j,k)=\frac{1}{a(j,k)}=\left(\frac{p_j}{q_j}\right)^k \tag{10}$$

and

$$B(i,k)=(-1)^{k+1}i\sum_{j=1}^{b}c_{ij}b(j,k) \tag{11}$$

then

$$p_S(m)=\prod_{j=1}^{b}(q_j)^{c_j} \tag{12a}$$

$$sp_S(m-s)=\sum_{i=1}^{\min(a,s)}\sum_{k=1}^{[s/i]}B(i,k)p_S(m-s+ki),\qquad s=1,2,\ldots,m. \tag{12b}$$

PROOF. Consider the random variable T defined by

$$T=m-S. \tag{13}$$

The probability generating function of T is

$$G_T(u)=u^mG_S\left(\frac{1}{u}\right)=\prod_{i=1}^{a}\prod_{j=1}^{b}(q_j+p_ju^i)^{c_{ij}} \tag{14}$$

and (12a) is obtained by putting $u=0$.

Proceeding as in theorem 1 we get

$$sp_T(s)=\sum_{i=1}^{\min(a,s)}\sum_{k=1}^{[s/i]}B(i,k)p_T(s-ki) \tag{15}$$

from which the recursion (12b) follows.

We remark that, as in corollary 1, a backward recursion for the distribution of the number of survivors can be obtained by putting $a=1$ in (12).

REFERENCES

KORNYA, P. S. (1983) Distribution of aggregate claims in the individual risk theory model. *Transactions of the Society of Actuaries* **35**, 823-836. Discussion 837-858.

PANJER, H. H. (1981) Recursive evaluation of a family of compound distributions. *ASTIN Bulletin* **12**, 22-26.

WHITE, R. P. and GREVILLE, T. N. E. (1959) On computing the probability that exactly k of n independent events will occur. *Transactions of the Society of Actuaries* **11**, 88-95. Discussion 96-99.

NELSON DE PRIL

Instituut voor Actuariële, Wetenschappen, Katholieke Universiteit Leuven, Dekenstraat 2, B-3000 Leuven (Belgium).