THE REINSURER'S MONOPOLY AND THE BOWLEY SOLUTION

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ABSTRACT

The reinsurer has a monopoly in the following sense: He will select a random variable P that determines the reinsurance premiums. The first insurer can purchase a payment of R (a random variable) for a premium of $\pi = E[PR]$. For known P, the first insurer chooses R to maximize his expected utility. Knowing this, i.e., the demand for reinsurance as a function of P, the reinsurer chooses P to maximize his utility. The resulting pair (P, R) is called the Bowley solution. Assuming exponential, quadratic and/or linear utility functions, some explicit results are obtained.

KEY WORDS

Utility function, reinsurance, Bowley solution.

1. INTRODUCTION

We consider a model with two economic agents: the first insurer (who buys reinsurance) and the reinsurer (who sells reinsurance).† The reinsurance premium is determined according to a certain principle. Knowing this principle, the first insurer will buy the optimal coverage; thus the *demand* for reinsurance is a function of the principle. It is assumed that the reinsurer knows this function and has the monopoly; thus he will choose the principle that is optimal for him. In economics the resulting solution is called the *Bowley solution.*

BUHLMANN (1968) and GERBER (1984) have discussed the Bowley solution in the case, where only proportional reinsurance is considered. Here we shall drop this restriction, and we shall try to find the solution in a global sense.

2. THE MODEL

The first insurer has a utility function $u(\cdot)$, and the reinsurer has a utility function $v(\cdot)$. Both functions are strictly increasing; u is assumed to be strictly concave, and v is assumed to be concave (possibly linear). In the absence of reinsurance

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† The labels "first insurer" and "reinsurer" will be used to emphasize the asymmetric roles of the two agents, for the same reason we shall avoid the term "risk exchange"

the first insurer's fortune is X, and the reinsurer's fortune is Y; X and Y are random variables.

The first insurer can be buy a payment of R (a random variable) for a premium of

$$\pi = E[PR]. \tag{1}$$

Here the "price intensity" P is a random variable satisfying $P > 0$ and $E[P] = 1$, and will be determined by the reinsurer. To fix ideas, the reader might assume that P is a function of X and Y.

We have chosen the particular form (1) to define reinsurance premiums, because it is a simple principle on the one hand, but allows the premium to depend not only on the distribution of the payment, but also on its relationship to other relevant random variables (such as X and Y) on the other.

Knowing P, the first insurer chooses R in order to maximize

$$E[u(X - \pi + R)], \tag{2}$$

his expected utility. We shall now derive a condition for the optimality of R.

Suppose that R maximizes (2). Let Q be an arbitrary random variable, and t a number. We consider $R_t = R + tQ$ and the corresponding premium $\pi_t = E[PR_t]$. Then the function

$$g(t) = E[u(X - \pi_t + R_t)] \tag{3}$$

has a maximum at $t = 0$. It follows that $g'(0) = 0$, a condition that can be written as

$$E[Q\{-PE[u'(X - \pi + R)] + u'(X - \pi + R)\}] = 0. \tag{4}$$

Since Q is arbitrary, the expression inside the braces must vanish, i.e.,

$$\frac{u'(X - \pi + R)}{E[u'(X - \pi + R)]} = P. \tag{5}$$

This condition has been derived by BUHLMANN (1980, Theorem A'), who showed that it is also sufficient. A generalization to more general principles of premium calculations than (1) can be found in DEPREZ and GERBER (1985, Theorem 9).

It should be noted that the optimal choice of R is not unique: if a constant is added to R, the same constant is added to the premium (since $E[P] = 1$), and the value of (2) remains unchanged.

Condition (5) allows us to verify what is intuitively obvious: the first insurer buys full coverage ($R = -X$), if and only if P is the constant 1.

To summarize, knowing P, the first insurer chooses R to maximize (2), i.e., according to condition (5).

Having this information, the reinsurer determines P in order to maximize

$$E[v(Y + \pi - R)], \tag{6}$$

his expected utility; in this expression R and π depend on P as indicated by formulas (5) and (1).

3. SPECIAL ASSUMPTIONS

In order to arrive at concrete results, we shall make some further assumptions. First, let us assume that the first insurer has an exponential utility function:

$$u(x)=\frac{1}{a}(1-e^{-ax}), \tag{7}$$

where $a>0$ is his constant risk aversion. Then (5) has a simple solution; we get

$$R=-X-\frac{1}{a}\ln P+k, \tag{8}$$

where k is an arbitrary constant. This formula, which appears as formula (3) in BUHLMANN (1980), shows that the demand for reinsurance is determined by two factors: the *need* for reinsurance, and the *price* for reinsurance.

Next we assume that $v(y)=y$, i.e., that the reinsurer is indifferent to risk. Thus he chooses P in order to maximize his expected gain, i.e.,

$$\pi-E[R]=-E[PX]-\frac{1}{a}E[P\ln P]+E[X]+\frac{1}{a}E[\ln P]. \tag{9}$$

This problem can be solved as follows Suppose that P maximizes (9). Let Q be a random variable with $E[Q]=0$, and let $g(t)$ denote the value of the right-hand side of (9), if "P" is replaced by "$P+tQ$". Thus the function $g(t)$ has a maximum at $t=0$, and $g'(0)=0$, or

$$E\left[Q\left\{-X-\frac{1}{a}\ln P+\frac{1}{a}\frac{1}{P}\right\}\right]=0. \tag{10}$$

If we denote the expression inside the braces by V, and set $Q=V-E[V]$, we see that

$$\operatorname{Var}[V]=E[(V-E[V])V]=0. \tag{11}$$

Thus V is a constant:

$$-X-\frac{1}{a}\ln P+\frac{1}{a}\frac{1}{P}=c \tag{12}$$

for some constant c; finally, the value of c follows from the condition that $E[P]=1$.

Since $1/p-\ln p$ is a decreasing function of p for $0<p<\infty$, formula (12) shows that P is a decreasing function of X; this corresponds to our intuition: if, for example, the value of X is small for a particular point of the sample space, the need for reinsurance is high, and consequently its price, i.e., the value of P, is high for such a point.

From (8) and (12) we obtain

$$R=-\frac{1}{a}\frac{1}{P}+k \tag{13}$$

(where k is an arbitrary constant). Thus R is a decreasing function of X.

Furthermore, the reinsurer's expected gain is

$$\pi - E[R] = \frac{1}{a}\left(E\left[\frac{1}{P}\right] - 1\right). \tag{14}$$

To summarize, the Bowley solution is given by formulas (12) and (8) or (13).

For the further discussion of the Bowley solution we assume that a is small and expand P in terms of powers of a. Since $P \equiv 1$ for $a = 0$, P must be of the form

$$P = 1 + aF_1 + a^2F_2 + \cdots, \tag{15}$$

where the F_k's are functions of X with $E[F_k] = 0$. Substituting

$$\begin{aligned} \ln P &= (P-1) - \tfrac{1}{2}(P-1)^2 + \cdots \\ &= aF_1 + a^2F_2 - \tfrac{1}{2}a^2F_1^2 + \cdots \end{aligned} \tag{16}$$

and

$$\begin{aligned} \frac{1}{P} &= 1 - (P-1) + (P-1)^2 - (P-1)^3 + \cdots \\ &= 1 - aF_1 - a^2F_2 + a^2F_1^2 \\ &\quad - a^3F_3 + 2a^3F_1F_2 - a^3F_1^3 + \cdots \end{aligned} \tag{17}$$

in (12), and comparing the coefficients of a and a^2, we find that

$$\begin{aligned} F_1 &= -\tfrac{1}{2}(X-\mu) \\ F_2 &= \tfrac{3}{16}[(X-\mu)^2 - \sigma^2], \end{aligned} \tag{18}$$

where $\mu = E[X]$ and $\sigma^2 = \operatorname{Var}(X)$. Thus for small values of a we obtain the following approximation:

$$P \approx 1 - \frac{a}{2}(X-\mu) + \tfrac{3}{16}a^2[(X-\mu)^2 - \sigma^2]. \tag{19}$$

From (13) and (17) we get

$$\begin{aligned} R &\approx F_1 + aF_2 - aF_1^2 \\ &= -\tfrac{1}{2}(X-\mu) - \frac{a}{16}(X-\mu)^2 + k. \end{aligned} \tag{20}$$

From (14) and (17) we see that the reinsurer's expected gain is approximately

$$aE[F_1^2] + 2a^2E[F_1F_2] - a^2E[F_1^3] = \frac{a}{4}\sigma^2 - \frac{a^2}{16}E[(X-\mu)^3]. \tag{21}$$

Finally, a little calculation shows that the first insurer's surplus after reinsurance is

$$\begin{aligned} X - \pi + R &= X + F_1 + aF_2 - aF_1^2 + \cdots \\ &\approx \tfrac{1}{2}(X+\mu) - \frac{a}{16}(X-\mu)^2 - \frac{3a}{16}\sigma^2. \end{aligned} \tag{22}$$

This result can be interpreted as follows. According to the leading term the original surplus, X, has been replaced by the average of X and its mean, as if 50% of X had been reinsured for a net premium. The price of this advantageous arrangement is reflected by the two linear terms; it is interesting that this price is itself a function of X.

4. ALTERNATIVE ASSUMPTIONS

We still assume (7), but assume now that the reinsurer has also an exponential utility function:

$$v(y)=\frac{1}{b}(1-e^{-by}). \tag{23}$$

Thus he chooses P in order to minimize

$$E[e^{-b(Y+\pi-R)}], \tag{24}$$

where π and R are given by (1) and (8). Hence the problem is to minimize the expression

$$e^{b\{E[PX]+(1/a)E[P\ln P]\}}E[e^{-b\{Y+X+(1/a)\ln P\}}]. \tag{25}$$

Using variational calculus as before, we find that the optimal P is obtained from the equation

$$-\left(X+\frac{1}{a}\ln P\right)E[e^{-b\{Y+X+(1/a)\ln P\}}]+\frac{1}{a}\frac{1}{P}e^{-b\{Y+X+(1/a)\ln P\}}=c, \tag{26}$$

where the value of c follows from the condition that $E[P]=1$. Note that (12) is the limiting case $b=0$. From (8) and (26) we obtain

$$R=-\frac{1}{a}\frac{1}{P}\frac{e^{-b\{Y+X+(1/a)\ln P\}}}{E[e^{-b\{Y+X+(1/a)\ln P\}}]}+k \tag{27}$$

which generalizes (13).

At least in principle equation (26) can be solved by a series expansion. Assuming that $b/a=q$ is a fixed ratio, we expand P in powers of a,

$$P=1+aF_1+a^2F_2+\cdots \tag{28}$$

where the F_k's are functions of X and Y with $E[F_k]=0$. Substituting (16), (17) and

$$P^{-q}=1-aqF_1+\cdots \tag{29}$$

in (26), and comparing the coefficients of "1", a, $a^2,\ldots,$ we can determine $F_1, F_2, F_3, \ldots$. We find that

$$F_1=-\frac{1+q}{2+q}(X-\mu)-\frac{q}{2+q}(Y-\nu), \tag{30}$$

where $\nu=E[Y]$. Thus the following first order approximation holds for small values of a:

$$P \approx 1 + aF_1 = 1 - a\frac{1+q}{2+q}(X-\mu) - a\frac{q}{2+q}(Y-\nu) \tag{31}$$

$$R = -X - \frac{1}{a}\ln P \approx -X - F_1 \tag{32}$$

$$= -\frac{1}{2+q}(X-\mu) + \frac{q}{2+q}(Y-\nu) + k.$$

As a consequence, the first insurer's surplus with reinsurance is approximately

$$X - \pi + R \approx \frac{1}{2+q}\mu + \frac{1+q}{2+q}X + \frac{q}{2+q}(Y-\nu). \tag{33}$$

5. QUADRATIC UTILITY FUNCTIONS

Let us return to the situation of Section 3 and replace assumption (7) by the assumption of a quadratic utility function:

$$u(x) = x - \frac{c}{2}x^2, \qquad x < \frac{1}{c}, \tag{34}$$

where $s = 1/c$ is the level of saturation of the first insurer. From (5) we see that the first insurer's demand for coverage will be

$$R = -X - fP + k \tag{35}$$

with

$$f = \frac{s - E[PX]}{1 + \mathrm{Var}(P)}. \tag{36}$$

If we set $k = 0$, we obtain

$$\pi = E[PR] = -E[PX] - fE[P^2] = -s. \tag{37}$$

Hence the reinsurer's expected gain is

$$\pi - E[R] = -s + \mu + f. \tag{38}$$

Thus the reinsurer chooses P in order to maximize f. Since

$$f = \frac{s - \mu - \rho\sigma_X\sigma_P}{1 + \sigma_P^2} \tag{39}$$

where ρ is the correlation coefficient of X and P, the problem boils down to the optimal choice of ρ and σ_P. Obviously $\rho = -1$, and consequently

$$P = 1 - \frac{\sigma_P}{\sigma_X}(X - \mu). \tag{40}$$

It remains to choose σ_P in order to maximize

$$f=\frac{s-\mu+\sigma_X\sigma_P}{1+\sigma_P^2}. \tag{41}$$

The solution of the resulting quadratic equation is

$$\sigma_P=\frac{\sqrt{(s-\mu)^2+\sigma_X^2}-(s-\mu)}{\sigma_X}. \tag{42}$$

To summarize, the Bowley solution is given by formulas (35) and (40), together with (41) and (42).

For small values of σ_X we may expand the square root in (42) to obtain the approximation

$$\sigma_P\approx\frac{1}{2}\frac{\sigma_X}{s-\mu}. \tag{43}$$

The corresponding approximation for P is

$$P\approx 1-\frac{1}{2}\frac{1}{s-\mu}(X-\mu)=1-\tfrac{1}{2}r(\mu)(X-\mu), \tag{44}$$

where $r(x)=-u''(x)/u'(x)$ is the risk aversion function corresponding to (34). We note the close analogy with the first order part in formula (19). For small values of σ_X, f is approximately $s-\mu$. Substituting this and (44) in (35) we get

$$R\approx-\tfrac{1}{2}(X-\mu) \tag{45}$$

(modulo an additive constant). Note that this corresponds to the leading term of the approximation (20).

Let us now combine (34) with the assumption that the reinsurer has the exponential utility function (23). Thus the reinsurer chooses P in order to minimize (24), where π and R are defined in formulas (35)-(37). Hence the problem is to minimize the expression

$$E[e^{-b(Y+X+fP)}] \tag{46}$$

where f is given by formula (36). Using variational calculus we find that the optimal P must satisfy the condition that

$$(X+2fP)E[Pe^{-b(Y+X+fP)}]-(s-E[PX])\,e^{-b(Y+X+fP)}=c \tag{47}$$

where the value of c follows from the condition that $E[P]=1$.

For small values of $\alpha=(s-\mu)^{-1}$ and b an approximate solution of (47) can be derived as follows: For a fixed value of the ratio $q=b/\alpha$ we consider the expansion

$$P=1+\alpha F_1+\alpha^2F_2+\cdots \tag{48}$$

where the F_k's are functions of X and Y with $E[F_k]=0$. Substituting (48) in (36) we get

$$f=\frac{1}{\alpha}\{1-\alpha^2 E[F_1 X]-\alpha^2 \operatorname{Var}[F_1]+\cdots\}. \tag{49}$$

Substituting these expansions in (47), and comparing the coefficients of α^0, we find that

$$F_1=-\frac{1+q}{2+q}(X-\mu)-\frac{q}{2+q}(Y-\nu), \tag{50}$$

which is the same as (30), with the proviso that a has now been replaced by $\alpha=r(\mu)$ in the definition of q. The same analogies then also hold for the first order approximations of P and R:

$$P\approx 1+\alpha F_1=1-\alpha\frac{1+q}{2+q}(X-\mu)-\alpha\frac{q}{2+q}(Y-\nu) \tag{51}$$

$$R=-X-fP\approx-\frac{1}{2+q}(X-\mu)+\frac{q}{2+q}(Y-\nu)+k. \tag{52}$$

This leads us to the conjecture that for small risk aversions the Bowley solution is independent of the specific form of the utility functions.

REFERENCES

BUHLMANN, H (1968) Marktverhalten unter Risiko—das Deckungsmonopol *Transactions of the 18th International Congress of Actuaries* 379-392

BÜHLMANN, H. (1980) An economic premium principle. *Astin Bulletin* **11**, 52-60

DEPREZ, O and GERBER, H. U (1985) On Convex Principles of Premium Calculation. *Insurance Mathematics & Economics* **4**, 179-189

GERBER, H U. (1984) Chains of Reinsurance *Insurance Mathematics & Economics* **3**, 43-48