AN APPLICATION OF CREDIBILITY THEORY TO SOLVENCY MARGINS SOME COMMENTS ON A PAPER BY G. W. DE WIT AND W. M. KASTELIJN

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ABSTRACT

Some comments are given on a recent paper by DE WIT and KASTELIJN (1980) and alternative methods for analysing loss ratios are proposed in connection with the determination of the necessary solvency margins of non-life insurance companies. The methods are illustrated by a numerical example.

1. INTRODUCTION

DE WIT and KASTELIJN (1980) present an analysis of the solvency requirements in non-life insurance companies in the Netherlands. The purpose of their analysis is to update the results in an OECD report by CAMPAGNE (1961), which formed a basis for the present EEC rules for the solvency requirements in non-life insurance companies. In the OECD report data from the period 1952–1957 were analysed for a number of countries. For the Netherlands it contained 53 loss ratios from ten companies. The loss ratios were assumed to follow a beta distribution, whose two parameters were estimated by the method of moments. With an average expense ratio equal to 53% it followed that a solvency margin equal to 31% of the premium income was sufficient to ensure survival for the next year with a probability of 0.9997.

DE WIT and KASTELIJN (1980) analysed loss ratios for the years 1976, 1977, and 1978 from 71 companies along the same lines as in the previous OECD report. For the new data the average expense ratio had dropped to 30%, but due to a higher average loss ratio and much greater variation in the loss ratios it appeared that only a solvency margin equal to 60% of the premium was sufficient to ensure survival with the same probability as above.

The report by DE WIT and KASTELIJN (1980) is an element in a strong current interest in the problem of determining solvency margins for non-life insurance companies. There is much other testimony of the great attention which these questions have attracted in recent years. Under the leadership of T. Pentikainen, the system of solvency margins used in Finland, and described by HOVINEN (1969), has just been revised, and the new results may be found in PENTIKAINEN (1982) and RANTALA (1982). LØVIK (1981) presented an interesting example, and so has AMSLER (1978, 1979, 1980). In the 1980 Cambridge Seminar, the problems concerning fluctuation reserves were studied in detail, OAKES (1980)

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and KARSTEN (1980) described the current systems in Germany and Finland, where rules for the calculation of solvency margins have existed for several years, and the same method as in DE WIT and KASTELIJN (1980) was applied to data from ten large U.K. companies for the period 1971–1978. Given all this interest, it must be worthwhile looking into the details of the procedure used and to discuss whether it is possible to extend the analysis. The purpose of the present note is to give such a critical evaluation of current methodology and to present an alternative analysis. Our comments are as follows.

1. It is hard to see why the analysis should be based on the beta distribution. It is difficult to estimate the parameters of this particular distribution efficiently, and it fails to allow for loss ratios larger than 1, as has been noted by DE WIT and KASTELIJN (1980) already. Since one only needs an approximate description of the loss ratios anyway, it is more natural to choose some other more convenient distribution instead. We propose an alternative below.

2. In previous work the parameters of the beta distribution have been estimated by the method of moments. If one were to stick to this distribution it would be interesting to see what results could be obtained with a more efficient estimation procedure, such as maximum likelihood.

3. The most critical point in the previous analyses is the assumption that all observed loss ratios are stochastically independent and identically distributed with the same (beta) distribution throughout. This assumption can hardly be appropriate, not even approximately. It must be more realistic to assume that different companies may have independent loss ratios, but with differing distributions. (Perhaps the loss ratios for different years for a fixed company are independent and identically distributed.) When this possibility is ignored and all the observations are assumed independent and identically distributed, even across companies, a greater variance is introduced into the data, which may explain why the solvency margin computed is rather high. The solvency margin ought not to be a figure common to all companies. It should depend on the policy of the particular company, on its portfolio mix, on its premium level, on its reinsurance arrangments, as well as on many other factors which may vary from company to company.

4. In the OECD report figures from ten different companies were analysed, while DE WIT and KASTELIJN (1980) used loss ratios from 71 companies. One can fear that the variation between companies increases with the number of companies involved, and this may help explain why the solvency margins in DE WIT and KASTELIJN (1980) exceed those in the OECD report.

2. AN ALTERNATIVE APPROACH.

Using the ideas noted above we now describe how one may analyse the figures of DE WIT and KASTELIJN (1980) in a different way.

Let Y_{ij} be the loss ratio for Company Number i in year j, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. (The method may easily be extended to cover the case where data for different companies are available for different number of years.) We will assume that for Company No. i, $Y_{i1}, \ldots, Y_{in}$ are independent and identically distributed according to a lognormal distribution with parameters θ_i and σ^2, where σ^2 is the same for all companies. If $X_{ij} = \log Y_{ij}$, then $X_{i1}, \ldots, X_{in}$ are independent and identically normally distributed with mean θ_i and variance σ^2. It is of course necessary to check whether it is possible to describe the loss ratios by a lognormal distribution, but it suffices here to assert that the shape of the lognormal curve is appealing in this context and that it has been applied before, HUNTER (1980), to model loss ratio data. One way to model this situation and to take into account that we analyse companies operating in the same market in the same country is to assume that $\theta_1, \ldots, \theta_m$ are themselves independent and identically distributed according to a normal distribution with a mean θ_0 and a variance τ^2. This is the conjugate prior distribution of the family of univariate normal distributions $N(\theta, \sigma^2)$, parametrised by θ, with σ^2 fixed.

When we do not condition on θ_i, the observations $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{in})'$ of the ith company are normally distributed with mean $(\theta_0, \ldots, \theta_0)'$ and a covariance matrix $\boldsymbol{\Sigma}$, where

$$\sigma_{kl} = \operatorname{Cov}(X_{ik}, X_{il}) = \delta_{kl}\sigma^2 + \tau^2,$$

where δ_{kl} is the Kronecker symbol. In the present situation any statement about the solvency of Company No. i should be based on the conditional distribution of $Y_{i,n+1}$, given the past $Y_{i1}, \ldots, Y_{in}$. Well known results from credibility theory or from the analysis of variance show that the conditional distribution of $X_{i,n+1}$, given $X_{i1}, \ldots, X_{in}$, is normal with mean

$$\tilde{X}_{i,n+1} = (n/(n+\kappa))\bar{X}_{i\cdot} + (\kappa/(n+\kappa))\theta_0 \tag{2.1}$$

and variance

$$\nu_{n+1}^2 = \sigma^2 + \tau^2\sigma^2/(\sigma^2 + n\tau^2), \tag{2.2}$$

where $\bar{X}_{i} = \sum_{j=1}^{n} X_{ij}/n$ and

$$\kappa = E\operatorname{Var}(X_{ij}|\theta_i)/\operatorname{Var}E(X_{ij}|\theta_i) = \sigma^2/\tau^2.$$

One may note that $\nu_{n+1}^2 = \sigma^2 + (1-Z)\tau^2$, where $Z = n/(n+\kappa)$ is the usual credibility factor. Note also that the variance ν_{n+1}^2 is independent of $\boldsymbol{X}_i$, which in one sense is a weakness of the model. It is seen that the conditional variance ν_{n+1}^2 tends to σ^2 as $n \to \infty$, which is the relevant variance figure for known θ_i.

Summarizing, we note that the conditional distribution of $Y_{i,n+1}$, given $Y_{i1}, \ldots, Y_{in}$, is a lognormal distribution with the parameters $\tilde{X}_{i,n+1}$ and ν_{n+1}^2. The upper limit of the loss ratio, $y_{i,1-\varepsilon}$, for Company No. i at the probability level $1-\varepsilon$ may then be calculated from

$$\begin{aligned}1-\varepsilon &= P\{Y_{i,n+1} \leq y_{i,1-\varepsilon} | Y_{i1}, \ldots, Y_{in}\}\\ &= P\{X_{i,n+1} \leq \log y_{i,1-\varepsilon} | X_{i1}, \ldots, X_{in}\}\\ &= \Phi[(\log y_{i,1-\varepsilon} - \tilde{X}_{i,n+1})/\nu_{n+1}],\end{aligned}$$

and $y_{i,1-\varepsilon} = \exp\{\tilde{X}_{i,n+1} + \Phi^{-1}(1-\varepsilon)\nu_{n+1}\}$.

For practical applications one must estimate the unknown parameters θ_0, σ^2, and τ^2. From the analysis of variance the maximum likelihood estimators are known to be

$$\hat{\theta}_0 = \bar{X} = (1/mn) \sum_{i,j} X_{ij}, \tag{2.3}$$

$$\hat{\sigma}^2 = \sum_{ij} (X_{ij} - \bar{X}_{i\cdot})^2 / [m(n-1)],$$

and

$$\hat{\tau}^2 = (1/(m-1)) \sum_{i=1}^{m} (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 - \hat{\sigma}^2/n.$$

From these estimators one can easily estimate the upper limit of the loss ratio for Company No. i.

3. ANOTHER APPROACH

The model in the previous section may be described by

$$X_{ij} = \theta_i + Z_{ij},$$

where $\theta_1, \ldots, \theta_m, Z_{11}, \ldots, Z_{mn}$ are independent normally distributed and where each θ_i has mean θ_0 and variance τ^2, while each Z_{ij} has mean 0 and variance σ^2. Therefore, for given θ_i, $X_{i1}, \ldots, X_{in}$ are i.i.d. The assumption that the distribution remains the same from year to year is often unrealistic since the company may for example revise its tariff and change its premium level, which influences the loss ratios Y_{ij} and therefore also the X_{ij}. To take care of effects of this nature we introduce variables $\eta_1, \ldots, \eta_n$, which are assumed to be independent of the θ_i and the Z_{ij} and to be independent normally distributed with mean 0 and variance ω^2, and we change our model to

$$X_{ij} = \theta_i + \eta_j + Z_{ij}.$$

This model may be treated as a two-way analysis of variance with stochastic row and column effects. It may also be viewed as a parametric credibility model with seasonal random factors, a model which has been described in detail in SUNDT (1979). The credibility estimator $\tilde{X}_{i,n+1}$ of $X_{i,n+1}$ is given by

$$\begin{aligned} \tilde{X}_{i,n+1} = {} & (n/(n+\kappa-\rho))\{\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot}\} + (n/(n+\kappa+(m-1)\rho))\bar{X}_{\cdot\cdot} \\ & + ((\kappa+(m-1)\rho)/(n+\kappa+(m-1)\rho))\theta_0, \end{aligned} \tag{3.1}$$

where now $\kappa = (\omega^2+\sigma^2)/\tau^2$, $\rho = \omega^2/\tau^2$.

Correspondingly, the conditional variance becomes

$$\begin{aligned} \nu_{n+1}^2 = {} & \tau^2 + \omega^2 + \sigma^2 - [n(m-1)\tau^4/m(n\tau^2+\sigma^2)] \\ & - [n\tau^4/m(n\tau^2+\sigma^2+m\omega^2)]. \end{aligned} \tag{3.2}$$

If $\omega^2 = 0$, these two formulas reduce to (2.1) and (2.2). The structural parameters θ_0, σ^2, τ^2, and ω^2 can be estimated by

(3.3) $$\hat{\theta}_0 = \bar{X}_{..}$$

$$\hat{\sigma}^2 = [(n-1)(m-1)]^{-1} \sum_{i,j} (X_{ij} - \bar{X}_{i\cdot} - \bar{X}_{\cdot j} + \bar{X}_{..})^2,$$

$$\hat{\tau}^2 = (m-1)^{-1} \sum_i (\bar{X}_{i\cdot} - \bar{X}_{..})^2 - \hat{\sigma}^2/n,$$

and

$$\hat{\omega}^2 = (n-1)^{-1} \sum_j (\bar{X}_{\cdot j} - \bar{X}_{..})^2 - \hat{\sigma}^2/m,$$

which are unbiased estimators of the parameters. The corresponding upper limit becomes

$$y_{i,1-\varepsilon} = \exp\{\tilde{X}_{i,n+1} + \Phi^{-1}(1-\varepsilon)\nu_{n+1}\}$$

as in the previous section.

4. A NUMERICAL EXAMPLE

In this section we illustrate how the figures used by DE WIT and KASTELIJN (1980) may be analysed along the ideas above. In the Appendix we have stated the loss ratios for the 71 Dutch companies for the years 1976, 1977, and 1978, and we will analyse the figures by the method described in Section 3. By (3.3) the structural parameters are estimated by

$$\hat{\theta}_0 = 4.222, \qquad \hat{\sigma}^2 = 0.04147, \qquad \hat{\tau}^2 = 0.08623, \quad \text{and} \quad \hat{\omega}^2 = 0.0003786,$$

which implies that $\hat{\nu}_4^2 = 0.05385$. When we substitute $\hat{\theta}_0 = \bar{X}_{..}$ (= 4.222) in the expression for the credibility estimator $\tilde{X}_{i,4}$, we get

$$\tilde{X}_{i,4} = Z\bar{X}_{i\cdot} + (1-Z)\bar{X}_{..},$$

where $Z = 3/(3+\kappa-\rho)$, the credibility factor, is estimated by $\hat{Z} = 0.8618$. In Table 1 we have shown the upper limits $y_{i,1-\varepsilon}$ with $\varepsilon = 1‰$ for some selected companies together with the credibility adjusted means $\exp\{\tilde{X}_{i,4} + \frac{1}{2}\nu_4^2\}$. The right column in Table 1 shows the corresponding solvency margins and is calculated as the positive part of $y_{i,0.999} + 30 - 100$ indicating an expense ratio equal to 30% as in DE WIT and KASTELIJN (1980).

It follows that the solvency margin depends very much on the individual company and cannot be described by a single figure. However, the margins in Table 1 are much greater than the 56% DE WIT and KASTELIJN (1980) found, and the main reason is that we have used a lognormal distribution instead of a beta distribution. The lognormal distribution is a far more "dangerous" distribution

TABLE 1

UPPER LIMITS, PREDICTED LOSS RATIOS, AND CORRESPONDING SOLVENCY MARGINS FOR SOME SELECTED COMPANIES

Company No	$y_{i\,0\,999}$	$\exp\{\tilde{X}_{i4}+\frac{1}{2}\nu_4^2\}$	Solvency margin
12	178 00	89 06	108 00
31	190.71	95 42	120 71
36	51.64	25 84	0.00
40	138 04	69 07	68 04
60	151 64	75 87	81 64
68	130.41	65 25	60.41

than the beta distribution with an upper limit equal to 150%, and it is also more dangerous than the Weibull distribution quoted at the end of DE WIT and KASTELIJN (1980). However, one has to be aware of the fact that among the 71 companies there seem to be some with a peculiar loss pattern, and it is mainly these companies which contribute to the rather greater variations in the loss ratios. Since it is questionable whether the model assumptions are valid for these companies, and since one should only analyse companies which are similar in some way, we have deleted Company No. 10, 32, 33, 34, and 38 and have performed a similar analysis for the remaining part of the figures.

For the remaining part we get a much lower conditional variance and a much greater credibility, since $\hat{\nu}_4^2 = 0.01429$ and $\hat{Z} = 0.9526$, and in Table 2 we have shown results similar to the results in the previous Table 1. We see that we now get solvency margins which are approximately 40–50% lower than in the previous case. This illustrates that in an analysis of this kind one should be very careful about including companies with a diverging loss experience, since they may influence the solvency margins of the other companies in an unreasonable way. It also raises the question whether it is possible to model all the figures in the Appendix by the model in our Section 2 or the model in DE WIT and KASTELIJN (1980). Therefore, one could try to only model the loss ratios for those companies, which are of the same size and have similar types of business. That would also

TABLE 2

UPPER LIMITS, PREDICTED LOSS RATIOS, AND CORRESPONDING SOLVENCY MARGINS FOR SELECTED COMPANIES, WHEN COMPANY NO 10, 12, 33, 34, AND 38 ARE DELETED

Company No	$y_{i,0\,999}$	$\exp\{\tilde{X}_{i4}+\frac{1}{2}\nu_4^2\}$	Solvency margin
12	128 94	89.65	58 94
31	139 15	96 75	69 15
36	32 83	22 83	0 00
40	97.35	67 69	27.35
60	108 00	75 09	38 00
68	91 42	63 56	21 42

take into account that the size of the variation in the loss ratios depends on the size of the company.

5. FINAL REMARKS

The above proposed analyses of loss ratio figures for several years for various companies in the same country have been carried out by straightforward credibility methods. For a review of credibility theory we refer to NORBERG (1979).

DE WIT and KASTELIJN (1980) state that the solvency margin should be the same for all non-life insurance companies. They reason that one should not upset the relations between competitors in the market. As has been illustrated in our numerical example it is hard to agree, however, that the maximum loss ratio $y_{1-\varepsilon}$ should be independent of the loss experience of the individual company. In addition the average solvency margin of 56% of the gross premium earned, calculated by DE WIT and KASTELIJN (1980) according to a one year ruin probability of 1 per mille seems very high. One would be curious to know whether Dutch companies are able to satisfy solvency requirements at such a high level.

This paper follows the current trend in the literature and in practice by focussing on loss ratios. This trend is unfortunate. In general, solvency margins and related problems should not be discussed on the basis of loss ratio figures alone, but should be investigated in much greater detail. One may easily understand the current reliance on loss ratios, for they are readily calculated. One should realize, however, that premiums and loss ratios reflect the risk profile of the portfolio very imperfectly. Therefore an analysis of the solvency margin should be more complete, and should contain a detailed description of the various classes of business, the portfolio mixture, the claim occurrence, claim distributions, reinsurance arrangements and retention limits, inflation rate and interest earned on the premium income, as well as possible other relevant factors, and the solvency requirement should be based on the distribution of the total claim amount minus the premium income. These notions open vistas for future research.

APPENDIX

LOSS RATIOS FOR THE YEARS 1976, 1977, AND 1978 FOR 71 DUTCH COMPANIES

Company	Loss ratios (%) 1976	1977	1978	Company	Loss ratios (%) 1976	1977	1978
1	73.54	49 68	54.24	37	79 21	87 33	66 48
2	68.54	65 20	63.13	38	134 83	39.58	45 84
3	93.95	110 35	112.15	39	64 01	57 86	55 03
4	57 74	58 66	50 60	40	67 57	74.30	60 18
5	83 52	90.32	107 75	41	82.61	88 83	96 27
6	95 13	105.24	98 62	42	79.15	65.25	61 82
7	68 92	71.32	67 57	43	77,20	98 17	64 31
8	83 30	85 85	86 03	44	49 30	24.39	26 32
9	91 72	94.23	96 05	45	95.25	85 07	83 49

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	Loss ratios (%)				Loss ratios (%)		
Company	1976	1977	1978	Company	1976	1977	1978
10	75 49	120 64	79 49	46	86 99	80 15	62 42
11	97.85	97.48	92.54	47	80.44	55 67	59 25
12	82 55	87.65	101 07	48	73.62	75 67	75.70
13	78 27	75.64	74.71	49	78 27	74 82	62 01
14	70 80	68 02	61 97	50	79 44	76 22	69 98
15	72 37	71.20	74 13	51	69 03	63 75	56 26
16	82 53	82 09	81 34	52	63 79	59 01	62.41
17	67 35	72 51	67 20	53	99 34	95 22	99 39
18	48 98	50 07	50 50	54	67 40	63.76	45.01
19	76.00	73.71	75 33	55	80 44	84 54	38 89
20	43 99	46.15	57 68	56	57 76	71 92	56 17
21	65 21	63 22	58 67	57	71 21	63.64	56 64
22	69 29	68 60	62 51	58	69 10	67 31	57 22
23	33 50	37 52	34.08	59	64 13	64 15	61 85
24	75 57	76.06	69 45	60	79 52	77.17	68.22
25	63 88	56 96	61 75	61	76 21	71 93	64 41
26	65 58	69 53	68.80	62	64 25	67 66	61 40
27	50 35	53.36	52 03	63	79 38	73 03	61 54
28	90.05	90 71	92 38	64	66 51	67 35	65 10
29	68 75	68 12	63.15	65	90 78	94 01	94 68
30	83.41	60 26	82 09	66	63 73	52 97	58 59
31	100.15	97.27	95.45	67	84 91	92.04	98.78
32	36 78	9 31	14 53	68	64 68	60 66	63 05
33	11 88	84.66	74.06	69	66 64	73 57	60 80
34	108 61	101 60	73 48	70	76 41	94 89	83 92
35	90 59	84.51	95 14	71	80.14	85 03	80 12
36	18.75	22 14	23.75				

REFERENCES

AMSLER, M (1978) L'équation générale d'équilibre d'un risque collectif *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, 221–245

AMSLER, M (1979) Ruine et provision de fluctuation *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, 85–105

AMSLER, M (1980). Ruine et fusion de portefeuilles d'assurance. *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, 41–57.

CAMPAGNE (1961) Standard minimum de solvabilité, applicable aux entreprises d'assurances, OECD report dated 11 3 1961

HOVINEN, E (1969) Procedures and basic statistics to be used in magnitude control of equalisation reserves in Finland *Astin Bulletin*, **5**, 227–238

HUNTER, R (1980). Fluctuation reserves—simulation of German method for calculating fluctuation reserves. *General Insurance Conference, Cambridge*, available by the Institute of Actuaries' Office

KARSTEN, H (1980). Fluctuation reserves—the Finnish system *General Insurance Conference, Cambridge*, available by the Institute of Actuaries' Office

LØVIK, J (1981) En modell for styring av sikringskapitalen i skadesforsikring, og en vurdering av enkeltransjenes kapitalbehov *Nordisk forsikringstidsskrift*, 1981 (3), 250–254

NORBERG, R (1979) The credibility approach to experience rating *Scandinavian Actuarial Journal*, 181–221

OAKES, M W (1980) Fluctuation reserves—the German system. *General Insurance Conference, Cambridge*, available by the Institute of Actuaries' Office.
PENTIKAINEN, T (1982). *Solvency of Insurers and Equalization Reserves, Volume I, General Aspects.* Insurance Publishing Company Helsinki
RANTALA, J (1982) *Solvency of Insurers and Equalization Reserves, Volume II, Risk Theoretical Model* Insurance Publishing Company Helsinki
SUNDT, B (1979) An insurance model with collective seasonal random factors *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, 57–64
WIT, B W DE, and KASTELIJN, W. M (1980) The solvency margin in non-life insurance companies *Astin Bulletin*, **11**, 136–144