

VALUE AND PRICES IN A REINSURANCE MARKET *

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This paper concerns the Borch model of a reinsurance market seen as a model of an economy under uncertainty.

In a market of this type the goods traded are unit coverings contingent to a particular state of nature (n -tuple of claims).

Our idea is to regard the probability of a state of nature as a sort of intrinsic value of the related contingent covering. From this point of view we examine the role of the reinsurance market in modifying values in market equilibrium prices and other questions, related to this classical economic problem, in the particular case of a quadratic utility function for all companies.

1. INTRODUCTION

A classical problem in economic theory is the relation between intrinsic value and exchange value of goods ¹.

Before to leave the concept of value for a utilitarian approach to the study of economic problems, much work was done to analyse the role of the market in modifying values in market prices.

In this paper we propose to apply these "old" ideas to the study of a reinsurance market seen, as suggested by BORCH (1960, 1962), as a particular economy under uncertainty, working along the lines proposed by ARROW (1953) on which the modern theory of risk is based ².

2. BORCH'S MODEL

Let us briefly recall Borch's model of a reinsurance market. There are n insurance companies, indexed 1, . . . , n , whose risk situation is described by

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¹ Perhaps the earliest question about this theme was, in the Middle Age, that of fair interest in connection with the usury problem.

Later the idea of value can be found in Adam Smith's treatment of a natural price of the goods as distinct from market prices determined by the balancing of demand and supply

The labour theory of value of Ricardo is a refinement of this idea and was in turn the basis for Marxian treatment of surplus. Since that time only the Marxian economist accepted the notion of intrinsic value of goods. On the other side the so-called Austrian (utilitarian) school developed his theory of a purely subjective value in strong opposition with any idea of objective or intrinsic value. For these classical economic questions see DOBB (1973).

² Recently another interesting model for the study of insurance markets was proposed by BÜHLMANN and GERBER (1978).

This model is based on ARROW (1974), but taking the approach of markets for securities rather than of contingent markets.

a vector $\mathbf{T} = (T_1, \dots, T_n)$ of available funds to pay claims, and a random vector \mathbf{X} of independent random variables X_1, \dots, X_n , claims for portfolio's risks of the n companies.

A state of nature \mathbf{x} of our economy is a particular n -tuple (x_1, \dots, x_n) of claims. For convenience we assume here that the number S of the set $\{\mathbf{x}\}$ is finite (we will use the notation $\mathbf{x}^s, s = 1, \dots, S$) and that all companies agree on a common evaluation $\pi(\mathbf{x}^s) = \pi_s$ of their probabilities.

The reinsurance market works as a particular economy under uncertainty in the following way: before trading on the market a company is obliged to pay $x_i^s = x_{is}$ for claims if state s happens.

But there is fixed a vector $\mathbf{p} = (p(\mathbf{x}^1), \dots, p(\mathbf{x}^S))$ or briefly (p_1, \dots, p_S) of market prices, such that the company can, by payment of a sum $p(\mathbf{x}^s)y_i(\mathbf{x}^s) = p_s y_{is}$, buy the right to receive a sum y_{is} if state of nature \mathbf{x}^s happens (and nothing else); that is the right for a covering y_{is} against that risk (briefly a contingent covering) ³.

Then if state s is observed the company actually loses for payments of claims a sum equal to $x_{is} - y_{is}$.

The total sum spent by the company acting in this way over all contingent markets, that is buying (or selling) contingent coverings for y_{i1} against $\mathbf{x}^1, \dots, y_{iS}$ against \mathbf{x}^S is then equal to $\sum p_s y_{is}$.

So its utility, U_i , after these transactions, is given according to the Bernoulli expected utility approach, with $u_i(x)$ as the utility of money, by the formula:

$$(1) \quad U_i(y_{i1}, \dots, y_{iS}) = \sum \pi_s u_i(T_i - \sum p_s y_{is} - x_{is} + y_{is}).$$

The company seeks to maximize his own utility by a suitable choice of y_i , that is by conveniently buying or selling contingent coverings. Maximization of (1) for every company offers a set $\{\mathbf{y}_i\}$ of vectors; the resulting situation is not an equilibrium one unless $\sum y_{is} = 0$ for every s ; that is unless the price vector is fixed in such a way that optimal independent decisions of the companies clear all the contingent markets.

If this happens the $(n+1)$ -tuple of S vectors $\mathbf{p}, \dots, \mathbf{y}_n$ is called a competitive equilibrium of the reinsurance market seen as a particular economy under uncertainty.

The existence of competitive equilibria in this version of the reinsurance market is guaranteed by mild conditions ⁴ on the utility functions, $u_i(x)$, of the n companies.

³ Note that a company can also sell contingent coverings; in this case the sign of y_{is} is obviously negative.

⁴ For existence of equilibria in economies under uncertainty and related questions see ARROW and HAHN (1971, p. 122) and DEBREU (1959, p. 102).

3. THE CASE OF A QUADRATIC UTILITY FUNCTION

The above conditions are satisfied for example in case of a quadratic utility function $u_i(x) = x - a_i x^2$, $a_i > 0$, for every company. In this particular case BORCH (1962, p. 440) has given explicit expressions for equilibrium prices and quantities.

To this purpose we need the following notations:

$$A_i = 1/2a_i - T_i; \sum_i A_i = A; Z(\mathbf{x}^s) = z_s = \sum_i x_{is};$$

$$D = E(Z) = \sum_s \pi_s z_s;$$

i.e. Z is the random amount of accumulated claims over all companies, D is the expectation of Z , $1/2a_i$ is the "saturation level" that is the wealth point of greatest utility, and A_i then the residual capacity of company i .

The unique vector of competitive equilibrium prices is then given by the relations:

$$(2) \quad p_s = \pi_s \frac{A + z_s}{A + D} \quad s = 1, \dots, S$$

while for the quantities it turns out

$$(3) \quad y_{is} = x_{is} - q_i z_s$$

or

$$(4) \quad x_{is} - y_{is} = q_i z_s \quad s = 1, \dots, S; i = 1, \dots, n$$

where

$$(5) \quad q_i = \frac{A_i + \sum_s p_s x_{is}}{A + \sum_s p_s z_s} \quad i = 1, \dots, n.$$

The meaning of (3) is that the market transactions produce for any state of nature an allocation of risks between the companies in fixed proportions, according to the q_i , of the accumulated claims z_s (as described in (4)).

4. PRICES AND PERSONAL PROBABILITIES: THE CORRECTION FACTOR

Now let us return to the key relation (2). We claim that the number π_s , probability of the state of nature \mathbf{x}^s is a sort of "inner" value of the particular contingent good the right to receive one monetary unit if \mathbf{x}^s is observed and nothing else (shortly the value of a unit of a contingent covering).

In fact the subjective approach to the concept of probability defines in an operational way, the probability of an event as the "fair" price of the contingent right to receive a monetary unit ⁵.

⁵ See DE FINETTI (1974) and SAVAGE (1971).

Can a subjective fair price be given an objective meaning as implied by the word value?

The matter is questionable, but we think the difficulty can be overcome, at least in cases where all subjective evaluations of a probability are based on some general well accepted criteria, and thus agree on a common number.

Indeed this is our case, as we assumed all companies evaluations π_{is} of the probability of state of nature (event) \mathbf{x}^s agree on a common value π_s .

In this optic then we are able to give an interesting meaning to (2): it shows that the market price p_s of a good of our reinsurance market (that is of a unit contingent covering) is obtained by multiplying the "value" of the good (his probability) π_s by a market correction factor:

$$(6) \quad \beta_s = \frac{A + z_s}{A + D} \quad s = 1, \dots, S.$$

It is worth to observe that the factor β_s , keeps account both of the companies risk aversion⁶ embodied in A_i , then in A , and of the specific risk connected with the state s (remember the meaning of z_s).

We think this possibility, not restricted to the quadratic case as we shall see later, to factorize the price of a contingent covering in terms of the related probability has an outstanding importance in the analysis of the reinsurance market.

Indeed the existence of a logical connection between prices of contingent goods and probabilities of the related states of nature was recognized by the economists as a leading feature of economies under uncertainty.

Concerning a general model of an economy of this type, for example DRÈZE (1971) shows that in equilibrium the following relation must hold:

$$(7) \quad p_{s1} = \frac{\lambda_{is}}{\lambda_i} \pi_{is} \quad s = 1, \dots, S$$

which links the equilibrium price p_{s1} of the numéraire (physical good labelled 1 of the multigood economy) and the personal evaluation of subject i of the probability of state of nature s .

In case of agreement on the above probabilities with π_s as the common probability (7) becomes $\left(\text{posing } \frac{\lambda_{is}}{\lambda_i} = \lambda_s \right)$

$$(8) \quad p_{s1} = \lambda_s \cdot \pi_s \quad s = 1, \dots, S$$

⁶ A measure of the risk aversion expressed by a utility function $u_i(x)$, is the so called risk absolute aversion function $R_i(x) = -u''_i(x)/u'_i(x)$ (PRATT (1964)). It is easy to see that in our quadratic case the value assumed by $R_i(x)$ for $x = T_i$ is nothing but the reciprocal $1/A_i$ of the residual capacity.

where the meaning of λ_s is that of relative marginal utility of the numéraire under state s , so that it really plays the role of a "correction factor".

Moreover it could be seen that the above marginal utilities reflect relative scarcities of the numéraire in the various states of nature.

After this enlightening walk in the economic field let us now return to our reinsurance market. Here there is only one physical good: money (traded either as covering or in the form of side payments). It of course plays the role of numéraire in our model.

The correction factor β_s of (6) then, could be seen as the marginal utility of wealth given state s , and it is obviously greater (*ceteris paribus*) when accumulated claims are relatively high, so that there is relative scarcity of money to honour coverings (to pay claims).

One remark more; it follows from (6) that β_s is linked to s only via the accumulated amount of claims z_s . This fact implies an interesting relation between prices and values of unit contingent coverings for states s' , s'' different but with the same amount of accumulated claims $z_{s'} = z_{s''}$. It is precisely ⁷:

$$(9) \quad p_{s'}/p_{s''} = \pi_{s'}/\pi_{s''}$$

which follows immediately being $\beta_{s'} = \beta_{s''}$ as implied from (6) and the above assumptions on z_s .

5. MARKET VALUES AND INNER VALUES

The role played by the market prices as a mechanism for rearrangement of values is clearly described in another relation:

$$(10) \quad \sum_s p_s = \sum_s \pi_s$$

A relation of this type too, is characteristic of a general economy under uncertainty. Indeed once fixed the numéraire (good 1) the sum $\sum_s p_{s1}$ of the equilibrium prices of the numéraire turns out to be 1. Should this fact imply that the prices of the numéraire can be considered as a sort of "market evaluations" of the probabilities of state s ? While this interpretation is in general ill-founded ⁸, it could be accepted in our particular model of a one-good economy with unanimous agreement on the probabilities of the states of nature.

The proof of (10) is immediate; (2) and summation over s gives:

$$\sum_s p_s = \sum_s \pi_s \cdot \frac{A + z_s}{A + D}$$

⁷ In fact relation (9) holds, more generally, under mild convenient assumptions on the utility functions $u_i(x)$ of the companies. See CASPI (1975).

⁸ See DRÈZE, 1971, 150-152.

but $\sum_s \pi_s = 1$, and $\sum_s \pi_s z_s = E(Z) = D$, and so

$$\sum_s p_s = \frac{A + D}{A + D} = 1 = \sum_s \pi_s.$$

Another interesting relation is:

$$(11) \quad \sum_s \sum_t p_s y_{ts} = \sum_s \sum_t \pi_s y_{ts} = 0.$$

It resumes the fact that neither monetary sums nor inner values can be created or destroyed in the market transactions, and it is then an obvious tautology. For a formal proof, change the summation order, keep p_s (or respectively π_s) independent from i , sum over t and remember that $\sum_t y_{ts} = 0$ for all s .

Obviously (11) does not imply that for any company:

$$(12) \quad \Delta(i) = \sum_t (p_s - \pi_s) y_{ts} = 0 \quad i = 1, \dots, n.$$

It is worth to analyse the left hand side of equation (12) as it gives us for a company the difference between the total cost (market value) and the total inner value of the covering received from the market transactions.

Indeed if all transactions could take place at a price equal to the ideal fair price π_s , the total market value of the covering received must be equal to the inner value.

So if the difference is negative we can say the company has a propitious (strong) market position, on the contrary positive sign means weak position (obviously this reasoning is meaningless in a utilitarian sense, as all companies have a utility advantage from their operations, otherwise they would not engage in market transactions).

An interesting result concerning this point is that, as we shall see later, after some convenient calculations exploiting relations (2) (5), we succeed in writing down the left hand side of (12) without any reference to price or quantities of equilibrium, but in terms of nothing but the risk situation of the companies prior to the market transactions; more precisely in terms of the residual capacity and the variance of the distributions of the single and accumulated claims.

6. THE MARKET POSITION OF A COMPANY

Concerning the question raised at the end of the last chapter, we shall prove in appendix A that the following relation holds for any company:

$$(13) \quad \Delta(i) = \sum_t (p_s - \pi_s) y_{ts} = \frac{A \text{ var}(X_i) - A_t \text{ var}(Z)}{(A + D)^2 + \text{var}(Z)} \quad i = 1, \dots, n.$$

The sign of the above expression is the one of the numerator, so that (13) is positive if and only if $A \text{ var } (X_i) - A_i \text{ var } (Z) > 0$.

As all factors here are positive, this is equivalent to say that positivity holds in case:

$$(14) \quad \frac{\text{var } (X_i)}{\text{var } (Z)} > \frac{A_i}{A} \quad i = 1, \dots, n$$

$$(15) \quad \frac{\text{var } (X_i)}{A_i} > \frac{\text{var } (Z)}{A} \quad i = 1, \dots, n.$$

Now positivity of $\Delta(i)$ means weak position on the market for company i ; then (14) and (15) say explicitly that in an ideal market for contingent coverings a weak position of a company follows from relatively high values of the ratio $\text{var } (X_i)/A_i$ in comparison with the "market ratio" $\text{var } (Z)/A$.

In turn high values of the above ratio could derive either from (relatively) high variance of the company's portfolio, or from (relatively) small residual capacity, that is high risk aversion (given the company's wealth, see footnote 6).

These results are not surprising and match very well with intuitive considerations.

7. FINAL COMMENTS

Let us spend in this final chapter some words about the relation connecting the theoretical questions examined in this paper and the practical reinsurance world.

The results we found in the last chapter could be seen as describing pretty well the essential features of a real reinsurance market. In the real world indeed a small company (likely with relatively high risk aversion) which wants to reinsure big risks in his portfolio, is obliged to accept "unfair" conditions to find covering by the big companies. This is exactly the rational anticipation of our theoretical model.

So even if managers of our companies do not think in terms of states of nature, contingent coverings and equilibria prices, an economic model under uncertainty of the type introduced captures quite likely some of the essentials of the real reinsurance world even if the prices do not seem to conform to the "holy" principle of actuarial equivalence.

Another point is of great practical interest. We found in para 3 that the competitive equilibrium gives rise to a situation that corresponds to a quota treaty, described by quotas q_i as in (5), and unconditional side payments $W(i) = \sum p_{is} y_{is} = q_i A - A_i$ (see appendix B).

It is not surprising that both quotas and side payments could be expressed explicitly as functions of the risk situation of the companies without reference

to the equilibrium prices and quantities (as was for expression (12)). Precisely it is (as proved in Appendix C):

$$(16) \quad q_i = \frac{[A_i + E(X_i)][A + E(Z)] + \text{var}(X_i)}{[A + E(Z)]^2 + \text{var}(Z)}$$

$$(17) \quad W(i) = \frac{A \text{var}(X_i) - A_i \text{var}(Z) + [AE(X_i) - A_iE(Z)][A + E(Z)]}{[A + E(Z)]^2 + \text{var}(Z)}$$

The relevance of this fact goes exceedingly over this formal property. With formulas (16) and (17) at disposal it is no more necessary to think that an equilibrium situation must be generated by a competitive market of contingent coverings; indeed the companies could well accept to sign a treaty (or to accept an arbitral proposal for that) like the one described by (16) and (17) without any recourse to market operations⁹.

This way they could reach the same situation generated by an ideal market for contingent coverings, but avoiding operational expenses and any deviations from the equilibrium point caused by errors or any other "noise" factors.

APPENDIX A

$$(18) \quad \sum_i (\phi_s - \pi_s) y_{is} = \sum_i \left(\frac{\pi_s(z_s + A)}{A + D} - \pi_s \right) y_{is} = \sum_i y_{is} \frac{(\pi_s z_s - \pi_s D)}{A + D} = \sum_i [x_{is} - q_i z_s] \cdot \frac{(\pi_s z_s - \pi_s D)}{A + D}$$

For the quantity in square brackets of (18) is by (5)

$$(x_{is} - q_i z_s) = x_{is} - z_s \cdot \frac{(A_i + \sum_j \phi_j x_{is})}{(A + \sum_j \phi_j z_s)} = \frac{A x_{is} - A_i z_s}{A + \sum_j \phi_j z_s}$$

then (6) becomes

$$(19) \quad \sum_i \frac{(A x_{is} - A_i z_s)}{(A + \sum_j \phi_j z_s)} \cdot \frac{(\pi_s z_s - \pi_s D)}{(A + D)}$$

Now it is:

$$\begin{aligned} A + \sum_i \phi_s z_s &= A + \sum_i z_s \pi_s \frac{(A + z_s)}{(A + D)} = \frac{A^2 + AD + A \sum_i \pi_s z_s + \sum_i \pi_s z_s^2}{A + D} = \\ &= \frac{A^2 + 2AD + E(Z^2)}{A + D} = \frac{(A + D)^2 + \text{var}(Z)}{A + D} \end{aligned}$$

⁹ Moreover it could be seen that the above treaty is Pareto efficient, so that it represents a point of collective rationality. More details on Pareto efficient treaties in BORCH (1960)

Substitution in (19) gives:

$$(20) \quad \sum_i \frac{(Ax_{is} - Aiz_s)(\pi_s z_s - \pi_s D)}{(A + D)^2 + \text{var}(Z)}$$

Let us consider the numerator of (20):

$$\begin{aligned} \sum_i (Ax_{is} - Aiz_s)(\pi_s z_s - \pi_s D) &= AE(X_i Z) - A_i E(Z^2) - AE(Z)E(X_i) + A_i D^2 = \\ &= A[E(X_i Z) - E(X_i)E(Z)] - A_i \text{var}(Z). \end{aligned}$$

Now the quantity in square brackets is $\text{cov}(X_i, Z)$ and it is easy to see that, provided $Z = \sum X_i$ and independence of the X_i , $\text{cov}(X_i, Z) = \text{var}(X_i)$ holds.

Then we have:

$$\sum_i (p_s - \pi_s)y_{is} = \frac{A \text{var}(X_i) - A_i \text{var}(Z)}{(A + D)^2 + \text{var}(Z)} \quad Q.E.D.$$

APPENDIX B

It is

$$(21) \quad \begin{aligned} W(i) &= \sum_i p_s y_{is} = \sum_i p_s (x_{is} - q_i z_s) = \\ &= \sum_i p_s \cdot \frac{Ax_{is} - Aiz_s}{A + \sum p_s z_s} \quad (\text{see Appendix A}), \end{aligned}$$

sum and subtract the same quantity A_i in (21):

$$\begin{aligned} \frac{A \sum p_s x_{is} - A_i \sum p_s z_s}{A + \sum p_s z_s} + A_i - A_i &= \frac{A \sum p_s x_{is} + A A_i}{A + \sum p_s z_s} - A_i = \\ &= A \cdot \frac{A_i + \sum p_s x_{is}}{A + \sum p_s z_s} - A_i = q_i A - A_i. \quad Q.E.D. \end{aligned}$$

APPENDIX C

It is

$$(22) \quad q_i = \frac{A_i + \sum p_s x_{is}}{A + \sum p_s z_s} = \frac{A_i + \sum p_s x_{is}}{\frac{(A + D)^2 + \text{var}(Z)}{A + D}} \quad (\text{see Appendix A}).$$

As for the numerator of (22) we have:

$$(23) \quad A_i + \sum_s \left(\pi_s \cdot \frac{A + z_s}{A + D} \right) x_{is} = A_i + \frac{A \sum \pi_s x_{is} + \sum \pi_s x_{is} z_s}{A + D} = \frac{A_i A + A_i D + \frac{AE(X_i) + E(X_i)E(Z)}{A + D}}{A + D}$$

In case of independence of X_i, Z , we have:

$$E(X_i, Z) = \text{cov}(X_i, Z) + E(X_i)E(Z) = \text{var}(X_i) + E(X_i)E(Z),$$

so that (23) becomes:

$$\frac{A_i A + A_i D + AE(X_i) + E(X_i)E(Z) + \text{var}(X_i)}{A + D},$$

and (22) then:

$$\frac{[A_i + E(X_i)] [A + E(Z)] + \text{var}(X_i)}{(A + D)^2 + \text{var}(Z)}$$

and (16) is proved.

To derive $W(i)$ we recall appendix B and the formula for q_i , just proved and obtain:

$$W(i) = \frac{\{[A_i + E(X_i)] [A + E(Z)] + \text{var}(X_i)\}}{(A + D)^2 + \text{var}(Z)} A - A_i = \frac{A_i A^2 - A_i A E(Z) + A^2 E(X_i) + A E(X_i) E(Z) + A \text{var}(X_i) - A_i A^2 - A_i D^2 - 2A_i A E(Z) - A_i \text{var}(Z)}{(A + D)^2 + \text{var}(Z)}$$

After some elementary manipulations on the numerator we obtain:

$$W_i = \frac{A \text{var}(X_i) - A_i \text{var}(Z) + [A + D] [A E(X_i) - A_i E(Z)]}{(A + D)^2 + \text{var}(Z)} \quad Q.E.D.$$

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