

OPTIMAL RISK EXCHANGES *

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The determination of optimal rules for sharing risks and constructing reinsurance treaties has important practical and theoretical interest. Medolaghi, de Finetti, and Ottaviani developed the first linear reciprocal reinsurance treaties based upon minimizing individual and aggregate variance of risk. Borch then used the economic concept of utility to justify choosing Pareto-optimal forms of risk exchange; in many cases, this leads to familiar linear quota-sharing of total pooled losses, or to stop-loss arrangements. However, this approach does not give a unique, risk-sharing agreement, and may lead to substantial fixed side payments. Gerber showed how to constrain a Pareto-optimal risk exchange to avoid invasion of reserves.

To these ideas, the authors have added the actuarial concept of long-run fairness to each participant in the risk exchange; the result is a unique, Pareto-optimal risk pool, with "quota-sharing-by-layers" of the total losses. There are many interesting special cases, especially when all individual utility functions are of exponential form, giving linear quota-sharing-by-layers. Algorithms and numerical examples are given.

1. INTRODUCTION

Insurance companies and other financial risk-bearing entities may enter into formal risk-sharing agreements for a variety of reasons, the most important of which is the simultaneous reduction of risk for all participants. For example, it is well known that two companies can both reduce the variance of their risk portfolios by agreeing to cover fixed quotas (β , $1-\beta$) of their pooled losses (and perhaps making a side payment to keep the pool "on fair terms"); this joint improvement occurs for some interval of values of β in $(0, 1)$, so the actual quota must be negotiated by other considerations.

Other corporate objectives, such as market penetration or financial stability may lead to different, non-linear forms of exchange, in which extreme, catastrophic losses are reallocated to the treaty members in different ways to "spread the risk". In fact, there is no difficulty in including under risk exchanges such "one-sided" arrangements as a reinsurance treaty, in which one of the participants brings no risks to the pool, but agrees to take a portion of the excess losses above some retention limit, in exchange for a fixed fee. Clearly it would be desirable to develop a theory which would explain the variety of actual risk-sharing agreements observed in the real world.

Given that a group of insurance companies has agreed to enter into a risk pool, this paper explores the general forms of exchange that result in simultaneous improvement of risk for all parties, under the following assumptions:

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1. All companies share the same information about the statistical nature of the individual, possibly dependent, risks to be shared. We formalize this by assuming that all companies work with the same probability distributions of the risks concerned.
2. Each company measures the risk of its portfolio by an individual utility function that is unaffected by the negotiations, i.e. there is no effort to change attitudes.
3. The companies may also agree, through mechanisms not considered here, on certain individual or joint side conditions, such as limits on losses paid, or on side payments.

In a series of important papers (1960a, 1960b, 1962), Borch showed that the use of utility functions leads to the economic concept of Pareto-optimal risk exchanges, in which the form of the agreements is determined by the individual utility functions (however *not* by the probability distribution!). Solution parameters are still open to competitive negotiations. In this paper we shall add the assumption that

4. all companies wish the exchange to be fair, in the sense that, according to a commonly accepted premium principle, all companies agree that, over the long run, no company in the pool should profit at the expense of the others.

We shall see that adding this insurance concept of fairness will lead to a unique Pareto optimal risk-sharing agreement. Many familiar forms of exchanges then follow under special assumptions about utility functions, volume of business, individual participation constraints, etc.

It is interesting that so far all authors have tried to arrive at a specific element within the set of Pareto Optima by game theoretic considerations (BORCH (1960b), LEMAIRE (1977)). Our paper achieves unicity by introducing the actuarial concept of fairness.

2. THE MODEL

2.1. *General Considerations*

Consider n insurance companies, indexed $i = 1, 2, \dots, n$, each with a risk portfolio characterized by

- a fixed premium income, $\Pi_i \geq 0$;
- a random loss (possible claims), $S_i \geq 0$;

over some common exposure period. The set of all losses $[S_i]$ is defined over some probability space Ω with known joint distribution $P(\omega)$ of possible outcomes ω . Set $\Pi = \Sigma \Pi_i$ and $S = \Sigma S_i$.

By a risk exchange or risk pool we mean any formal mutual agreement among the n companies that, operating as an entity

- (2.1) $\left\{ \begin{array}{l} 1. \text{ accepts the responsibility for paying for an input } X_i = f_i(S_i, \Pi_i) \\ \text{ from company } i, \text{ where } f_i \text{ is a fixed but arbitrary function;} \\ 2. \text{ charges company } i \text{ an output } Y_i \text{ for accepting the input, according} \\ \text{ to the agreed-upon rule for sharing risks;} \\ 3. \text{ Operates on a zero-balance conservation principle} \\ \qquad \qquad \qquad \Sigma Y_i(\omega) = \Sigma X_i(\omega) = X(\omega) \\ \text{ for all outcomes } \omega \in \Omega. \end{array} \right.$

Since risk pools are intended to redistribute only actual losses (and possibly the associated premiums), but not the individual wealth of the company, one would typically include in the exchange protocol side constraints of the form

$$(2.2) \qquad Y_i \geq A_i \quad (i = 1, 2, \dots, n)$$

where A_i is a constant (or random variable $A_i(\omega)$), designed to limit negative charges (payouts) to company i .

The risk agreements might also include constraints of the type

$$(2.3) \qquad Y_i \leq A_i + B_i, \quad (i = 1, 2, \dots, n),$$

designed to directly protect the liquidity of the individual companies. It is clear that we must have $B_i \geq 0$. We shall not consider pools in which coalition constraints, relating to subsets of the Y_i , are possible. The importance of side constraints is mentioned by Borch (1968), and first incorporated in an exchange model by Gerber (1978).

2.2 *The Claims Pool; Linear and Quota Exchanges*

An important special case is the claims pool in which companies keep their premiums and share all their losses:

$$(2.4) \qquad X_i = S_i; Y_i \geq 0, \quad (i = 1, 2, \dots, n).$$

One possible risk-sharing rule is a linear exchange, in which n^2 constants θ_{ij} are given, so that

$$(2.5) \qquad Y_i = \sum_{j=1}^n \theta_{ij} X_j, \quad (i = 1, 2, \dots, n).$$

To satisfy the clearing condition $\Sigma Y_i = \Sigma X_i$, we must have

$$(2.6) \qquad \sum_{i=1}^n \theta_{ij} = 1, \quad (j = 1, 2, \dots, n),$$

so that a feasible linear exchange has only $n^2 - n$ free constants, $0 \leq \theta_{ij} \leq 1$.

If the losses X_i are uncorrelated, it is easy to show (for $n=2$, see BEARD (1969)) that the variance of all output losses $V\{Y_i\}$ can be reduced by adjusting the constrained θ_{ij} until a Pareto-optimal frontier is reached at which a variance reduction for one company must be traded-off against a variance increase for another. On this frontier, there are only n exchange constants, $\theta_{ij} = \theta_i$, and

$$(2.7) \quad Y_i = \theta_i X_i; \quad 0 \leq \theta_i \leq 1; \quad \sum \theta_i = 1.$$

This arrangement is called a quota claims pool; the quota fraction θ_i , taken by the i th company of the total losses S , is often fixed on the basis of "volume", e.g.,

$$(2.8) \quad \theta_i = \Pi_i / \Pi.$$

Note that one could add conservative side payments to (2.5) to make $E\{Y_i\} = E\{X_i\}$ without changing the variances of the quota claims pool. The first models of linear quota pools were developed by P. Medolaghi, B. de Finetti, and G. Ottaviani; references are given in SEAL (1969).

2.3. *The Business Pool*

Another important special case is when companies agree to share both premiums and claims:

$$(2.9) \quad X_i = S_i - \Pi_i; \quad Y_i \geq -\Pi, \quad (i = 1, 2, \dots, n).$$

Notice the lower limit to prevent the invasion of reserves.

The traditional *quota business pool* is a linear exchange of this type, in which

$$(2.10) \quad Y_i = \theta_i(S - \Pi); \quad 0 \leq \theta_i \leq 1; \quad \sum \theta_i = 1.$$

Observe that if these quotas are set on the basis of "volume" (2.8), then the quota claims and quota business pools coincide, in the sense that the net charge to company i , $Y_i - X_i$, is the same in both cases.

2.4. *The Canonical Risk Exchange (REX)*

It is easy to see that if both X_i and Y_i are changed by the same (possibly random) amount, the net charges of a risk pool remain the same. To simplify exposition in the sequel, we shall subtract out any A_i appearing in (2.2) from the definitions of X_i and Y_i in (2.1), giving $X_i = f_i(S_i, \Pi_i) - A_i$ and $Y_i \geq 0$. This will also affect any other side conditions, such as (2.3).

Our canonical definition will then be:

A risk exchange (REX) (X, Y) is a formal rule for changing a random vector $X = (X_1, X_2, \dots, X_n)$ into a random vector $Y = (Y_1, Y_2, \dots, Y_n)$ so that

$$(2.11a) \quad 1. \quad \sum Y_i = \sum X_i.$$

$$(2.11b) \quad 2. Y_i \geq 0, \quad (i = 1, 2, \dots, n).$$

$$(2.11c) \quad 3. Y_i \leq B_i, \quad (i = 1, 2, \dots, n).$$

3. THE FAIR RISK EXCHANGE (FAIRREX)

Most insurance business is carried out using a measure of risk called premium. If Z is any random variable defined on Ω , we may define a premium principle for Z as follows:

$$(3.1) \quad \text{Premium}[Z] = \int_{\Omega} Z(\omega)G(\omega)dP(\omega).$$

$G(\omega) > 0$ is a loading factor that weighs Z in a predetermined manner over the possible outcomes. If G is the random variable with values $G(\omega)$, we can also write $\text{Premium}[Z] = E\{ZG\}$. Typically, $E\{G\} \geq 1$; for later purposes, we require $G > 0$ on the support of Z . Finally, if $G = 1$ for all ω , then (3.1) gives the usual fair premium $E\{Z\}$.

In establishing a risk pool, particularly a mutual agreement among similar companies, we argue that no company should profit from any other in the long run, no matter what form of agreement is mutually best for the outcomes in each exposure period. This implies that each company, using a commonly accepted premium principle, would judge that the premium of its input to the pool should be identical with the premium of the output it actually pays, in order that the pool is perceived to be equitable in the long run.

Therefore we define.

A REX (X, Y) is a FAIRREX if it satisfies the fairness condition

$$(3.2) \quad \text{Premium}[Y_i] = \text{Premium}[X_i]$$

for each company ($i = 1, 2, \dots, n$), using a common premium principle (3.1). Note that for all values of θ_i the quota business pool (2.9) is a FAIRREX if Π_i is the premium calculated on a fair premium basis. Unfortunately the linear form of exchange is not usually optimal in the sense described in the next section.

4. THE PARETO-OPTIMAL RISK EXCHANGE (POREX)

4.1. *The Unconstrained Case*

BORCH (1960a, 1960b, 1962) observed that the form of treaty acceptable to all parties in a REX should depend upon the individual attitudes towards risk. Assuming that each company is rational in ordering its preferences (i.e., satisfies the Bernoulli hypothesis), then it is well-known that under weak technical assumptions this implies the existence of a non-decreasing utility function u_i for each company ($i = 1, 2, \dots, n$), and the ranking of risky outcomes according to its expected utility U_i . For example, suppose that, prior to

joining a REX, company i held capital wealth W_i against a random loss X_i . Its prior expected utility would be

$$(4.1) \quad U_i^0 = E\{u_i(W_i - X_i)\}.$$

Posterior to the REX (2.11), its expected utility will be

$$(4.2) \quad U_i = E\{u_i[W_i - Y_i(X_1, X_2, \dots, X_n)]\},$$

and so it will perceive the REX as advantageous for company i , if $U_i > U_i^0$.

Borch argued that if the companies acted rationally and cooperatively, they would not agree on a REX (X, Y) if there existed another REX (X, \tilde{Y}) such that the expected utility of at least one company was improved without decreasing the expected utility of all other participants. This leads naturally to the idea that the all interesting treaties are Pareto-optimal, defined as

A REX (X, Y) is a Pareto-Optimal Risk Exchange if there is no other REX (X, \tilde{Y}) with

$$(4.3) \quad E\{u_i(W_i - \tilde{Y}_i)\} \geq E\{u_i(W_i - Y_i)\}$$

for all i , with strict inequality for at least one i .

Since we will be dealing only with non-negative losses, we can simplify our formulae by changing to disutility functions v_i and expected disutilities V_i , measured about the current wealth:

$$(4.4) \quad v_i(x) = -u_i(W_i - x).$$

(4.3) then reads $E\{v_i(\tilde{Y}_i)\} \leq E\{v_i(Y_i)\}$. Expected disutility for company i (which one wants to keep small!) is denoted by V_i . We shall assume $v_i' > 0$ and $v_i'' > 0$ for all i (risk aversion). As convenient, we shall make transformations of the form $a + bv_i(x)$, with $b > 0$, which do not affect preference orderings.

Borch (1960a) observed that Pareto-optimality could be obtained for every outcome, and therefore, did not depend upon the distribution of the X_i . Working with unconstrained REXs (i.e. without (2.11b)), he characterized these solutions as follows:

Theorem 1

REX (X, Y) is an unconstrained POREX if and only if there exist positive constants k_1, k_2, \dots, k_n for which

$$(4.5) \quad k_i v_i'(Y_i) = k_1 v_1'(Y_1), \quad (i = 1, 2, \dots, n).$$

A proof is given by DuMouchel (1968). Clearly, there are only $n-1$ effective constants that parametrize the possible POREXs. It also follows that the POREX treaties $Y_i = y_i(X_1, X_2, \dots, X_n)$ are scalar functions of the total losses X only. The values x of X can replace ω as the "state of the world", and the POREX is hence described by the functions $y_i(x)$, ($i = 1, 2, \dots, n$).

POREXs are pools, in which all losses are first merged, and then divided up. Borch (1968) refers to this result as "non olet" ("l'argent n'a pas d'odeur") with reference to the Vespasian anecdote

The form of the unconstrained treaties follows easily from (4.5). For example, if the companies all have utility functions that are in one of the following classes (and their positive linear transformations)

$$(4.6a) \quad u_i(x) = \begin{cases} -e^{-ax} \\ \pm (b_i \pm x)^c \\ \ln(x + d_i) \end{cases} \begin{cases} (-, -) \text{ for } c > 1 \\ (+, +) \text{ for } 0 < c < 1 \\ (-, +) \text{ for } c < 0 \end{cases}$$

then the POREX sharing rules are *linear* in the total losses Borch (1968), Wilson (1968)

$$(4.6b) \quad y_i(x) = \beta_i x + \gamma_i, \quad \sum \beta_i = 1, \quad \sum \gamma_i = 0.$$

The quota share fractions β_i and the side payments γ_i are determined from the individual utility parameters in (4.6a) as well as the Pareto multipliers from (4.5). Only in the exponential utility case are the quota-sharing fractions independent of the k_i , and hence the same for all POREXs.

(4.6b) shows a disadvantage of the Borch formulation: because the side payments must sum to zero, there are some companies making payments to others even when all losses are zero. This phenomenon, which one wants to eliminate in certain cases, leads immediately to constrained optimization as considered in the next section.

4.2. The Constrained Case

Gerber (1978) was the first to incorporate a non-invasion side constraint and suggested $Y_i \leq X_i + 1_i$, where the 1_i are given non-negative constants; this is a special case of our definition of a REX. Gerber also generalized Borch's theorem (4.5) to the constrained case; for our formulation (2.11) it takes the following form:

Theorem 2

REX (X, Y) is a constrained POREX if and only if there exist positive constants k_1, k_2, \dots, k_n and a positive random variable $\Lambda(\omega)$ such that for almost all outcomes ω and all companies $i = 1, 2, \dots, n$

$$(4.7) \quad \begin{aligned} k_i v'_i(Y_i(\omega)) &= \Lambda(\omega) & \text{if } 0 < Y_i(\omega) < B_i(\omega) \\ k_i v'_i(Y_i(\omega)) &\geq \Lambda(\omega) & \text{if } Y_i(\omega) = 0 \\ k_i v'_i(Y_i(\omega)) &\leq \Lambda(\omega) & \text{if } Y_i(\omega) = B_i(\omega). \end{aligned}$$

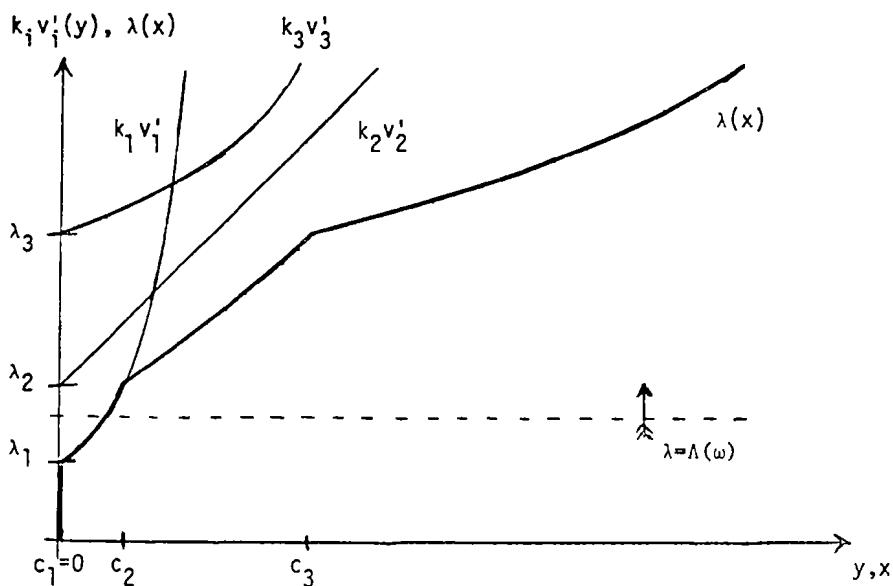


Fig. 1. Development of a constrained POREN for Given Pareto Multipliers, k_i

For an explicit proof, the reader is referred to Bühlmann (1978). Observe that for a non-degenerate POREN (i.e. when no company receives a share $Y_i = 0$), there is exactly one vector (k_1, \dots, k_n) (and its positive multiples) satisfying (4.7).

In the following, we shall always assume $B_i(\omega) \equiv \infty$ although the case where the upper bound is effective can be treated by similar methods. In particular, Theorem 3 can also be proved for an effective upper bound on the random vector $Y = (Y_1, Y_2, \dots, Y_n)$.

4.3. The Shape of a Constrained POREN

Condition (4.7) gives us an easy way to visualize and compute the shape of the optimal treaties $y_i(x)$. The Pareto multipliers k_i are fixed arbitrarily, and the curves $k_i v'_i(y)$ are plotted simultaneously for $y > 0$ with the level line $\lambda = \Lambda(\omega)$, as shown in Figure 1 for $n=3$. The indices are renumbered to give increasing intercepts $\lambda_i = k_i v'_i(0)$.

Now think of λ as parametrically increasing from zero. If $\lambda < \lambda_1 < \lambda_2 < \lambda_3$, it is clear from (4.7) that all y_i must equal the lower bound zero, and hence the total losses x are also zero. Increasing λ above the first intercept, so that $\lambda_1 < \lambda < \lambda_2 < \lambda_3$, permits $k_1 v'_1(y_1) = \lambda$ and $y_1 > 0$, but still $y_2 = y_3 = 0$; in other words, the first company takes all the losses $v_1 = x$. In the next interval, $\lambda_1 < \lambda_2 < \lambda < \lambda_3$, $k_1 v'_1(y_1) = k_2 v'_2(y_2) = \lambda$ and both y_1 and y_2 are positive, $y_1 + y_2 = x$, but still $y_3 = 0$, and so forth. In other words, by parametrically

increasing λ , we pass through different layers of risk-sharing in which first company # 1, then companies # 1 and # 2, etc., participate.

Furthermore, since at each level λ the values of the y_i are given directly from the curves, we can compute the total losses $x = \sum y_i$ by "horizontal addition". The heavy line in Fig. 1 shows the resulting parametric curve of λ versus x ; note the layering constants $c_1 = 0, c_2, c_3$, at which each new company begins to participate in x .

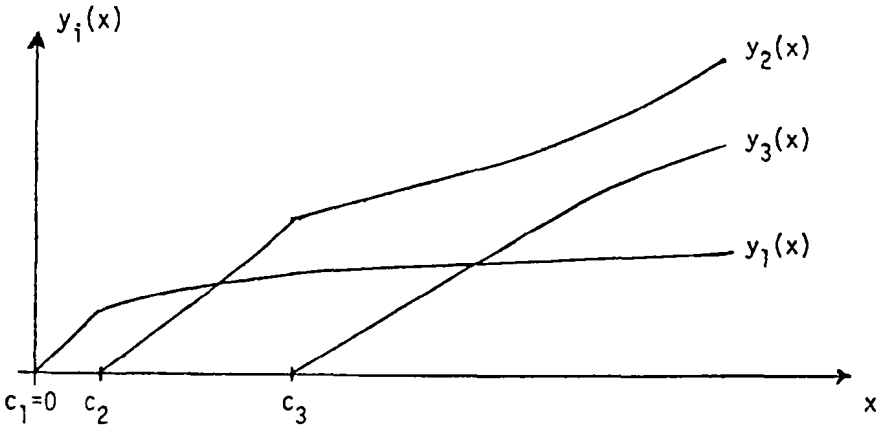


Fig 2 Layered POREX Treaties corresponding to fig. 1.

By reading fig. 1 sideways, we can easily visualize the fractional participation $y_i(x)/x$ at each level $\lambda(x)$; fig. 2 then shows the actual treaties, $y_i(x)$ versus x , corresponding to fig. 1.

To summarize, the constrained Pareto-optimal treaties consist of layered, non-linear functions determined parametrically as follows:

1. Given the Pareto multipliers $\{k_i\}$ and the disutility functions $\{v_i\}$, we renumber the companies so that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, where $\lambda_i = k_i v'_i(0)$. (Take $\lambda_{n+1} = +\infty$)
2. If $\lambda_j < \lambda \leq \lambda_{j+1}$, then only companies with indices $i \in J = \{1, 2, \dots, j\}$ participate in the losses, so $y_i(\lambda) > 0 (i \in J)$, and $y_i(\lambda) \equiv 0 (i \notin J)$, ($j = 1, 2, \dots, n$).
3. The participations y_i and total losses x in layers $(\lambda_j, \lambda_{j+1}]$ are found from the inverse w_i of v'_i :

$$(4.8a) \quad y_i(\lambda) = w_i \left(\frac{\lambda}{k_i} \right), \quad (i \in J),$$

and

$$(4.8b) \quad x(\lambda) = \sum_{i \in J} y_i(\lambda).$$

4. Since $v''_i > 0$, this inversion is unique, and the constrained POREXs are equal-

ly characterized by the ordered layering constants $c_1 = 0 \leq c_2 \leq c_3 \leq \dots \leq c_n$ with

$$(4.9) \quad c_i = \sum_{j=1}^{i-1} w_j \left(\frac{\lambda_i}{k_j} \right), \quad (i = 2, \dots, n).$$

Thus, in principle, the participants in a constrained POREX have to determine, through bargaining, the $n-1$ constants which determine the n layers of total pooled loss at which each company will begin to participate; once these values (and their order) are determined, the form of participation is uniquely determined by the individual utility functions. Generally speaking, a first company takes all the loss in the lowest layer, and successive companies start in succeeding layers, with relative participation in higher layers usually (but not necessarily!) diminishing.

4.4 The Exponential POREX

Because of simplicity and practical importance, we shall concentrate on exponential (dis)utilities in our later examples, for which all companies have u_i proportional to $-e^{-x/\alpha_i}$, and, say,

$$(4.10) \quad v_i(x) = \alpha_i e^{+x/\alpha_i}; \quad v'_i(x) = e^{x/\alpha_i}.$$

(This normalization makes $\lambda_i = k_i$.)

This utility function has the great advantage that the form of the POREX is independent of the initial wealth W_i (being absorbed in the k_i or the c_i);

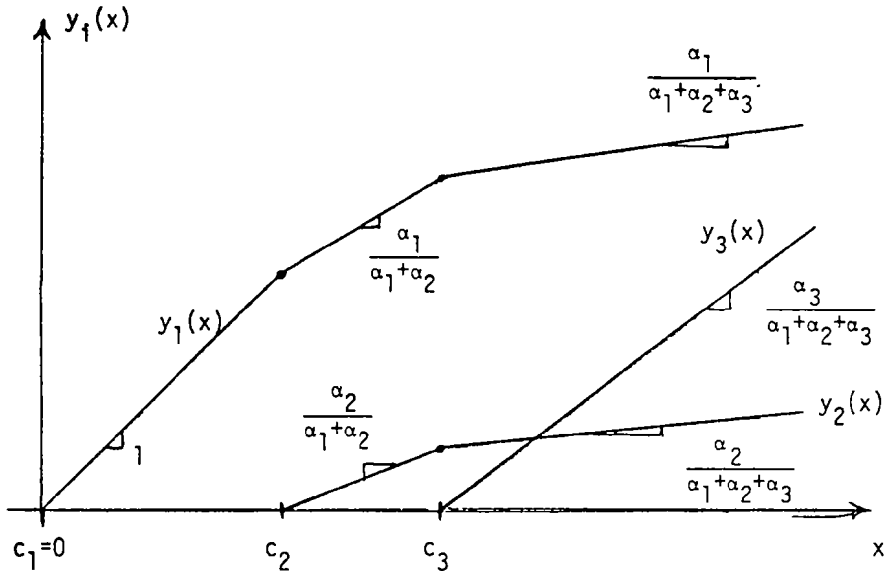


Fig. 3. POREX Treaties with Exponential Utilities

this is because the exponential is the only function with constant risk tolerance Wilson (1968) α_i is the risk tolerance unit--the larger is α_i , the more risk-tolerant is that company.

By simple calculations from the previous subsection (using $\ln v'_i$ and $\ln \lambda$), we find that, given an ordering $(1, 2, \dots, n)$ of the companies and the corresponding layering constants $c_1 = 0 \leq c_2 \leq \dots \leq c_n$,

$$(4.10) \quad y_i(x) = \sum_{j=1}^n \beta_{ij}(x - c_j)^+, \quad (i = 1, 2, \dots, n),$$

with

$$(4.11) \quad \beta_{ij} = \begin{cases} 0 & (j < i) \\ \alpha_i / \sum_{k=1}^i \alpha_k & (j = i) \\ \alpha_i [(\sum_{k=1}^j \alpha_k)^{-1} - (\sum_{k=1}^{j-1} \alpha_k)^{-1}] & (j > i) \end{cases}$$

independent of the c_j .

These piecewise-linear treaties are shown in fig. 3.

Of particular interest are the quota-share fractions ("stock functions" Wilson (1968)), $\beta_i(k)$, showing the incremental participation of company i in layer k :

$$(4.12) \quad \beta_i(k) = \frac{dy_i(x)}{dx} \Big|_{x \in [c_k, c_{k+1})} = \begin{cases} 0 & (k < i) \\ \alpha_i / \sum_{j=1}^k \alpha_j & (k > i). \end{cases}$$

In other words: in the exponential POREX, companies quota-share in layers, with their quota-fraction equal to their unit risk tolerance, divided by unit tolerances of all other companies participating in that layer.

5. THE FAIR, PARETO-OPTIMAL RISK EXCHANGE (FAIRPOREX)

We turn now, to the heart of our contribution. The basic difficulty of the POREX models is that they are indeterminate and do not completely describe the "best" treaties. Rather, they prescribe a subset of possible arrangements (which contains any "reasonable" REX), and leave the selection of the layering order and constants as the basic issue in the competitive bargaining. Borch has suggested the use of the Nash equilibrium point Borch (1960a), a market equilibrium mechanism using Arrow Certificates Borch (1960b, 1962) and game theory Borch (1962) to resolve this problem, but none of these is completely satisfactory.

As discussed in Section 3, we believe that mutual exchanges among insurance companies are governed both by a desire to modify short-term risk, and also by an understanding that, over the long run, no single company

shall profit from another in the sense of modifying the premium of contributed and assumed losses. Therefore, it is of interest to see if the FAIRREX requirement of Section 3 can be superposed on the POREXs of Section 4 to give one or several risk exchanges that are both fair and Pareto-optimal.

First we observe that there are effectively $n - 1$ free layering constants $\{c_i\}$ (if we drop the ordering convention), since one must be set to zero.

From the n FAIRREX requirements (3.2) we subtract one because of the already satisfied conservation condition $\Sigma Y_i = \Sigma X_i$, leaving $n - 1$ effective constraints (3.2). Surprisingly, and very satisfyingly, it turns out that this is enough to completely specify a unique FAIRPOREX! Mathematically, this is seen as follows: For any REX (X, Y) defined in (2.11), let

$$(5.1) \quad \Phi(X, Y) = \sum_{i=1}^n E \left\{ G \cdot \int_1^{y_i} \ln v_i'(y) dy \right\}.$$

Define a FAIRREX as

$$(5.2) \quad E \{ G \cdot y_i(X) \} = E \{ G \cdot X_i \} = q_i, \quad (i = 1, 2, \dots, n),$$

where

$$E \{ GX \} = \Sigma q_i = q;$$

and define a FAIRPOREX as a FAIRREX that additionally satisfies (4.7).

Theorem 3

Suppose $q_i > 0$ and $v_i'(0) > 0$ for all $i = 1, 2, \dots, n$.

Then: For any bounded X ,

1. Among all FAIRREXs (X, Y) there is a FAIRREX (X, \tilde{Y}) that minimizes Φ ;
- (5.3) 2. The FAIRREX (X, \tilde{Y}) is a FAIRPOREX;
3. If at least one $v_i'' > 0$ over its range, then the FAIRPOREX is unique.

The proof is based upon a similar result of Gale (1977) who is concerned with the fair distribution of desirable economic goods. The proof, which relies on the convexity of Φ on the set of REXs, is rather delicate, and is developed in a separate paper Bühlmann (1978). There one finds also an extension in case X is not bounded.

The practical importance of this result is that a unique POREX can be found which also is fair in the long run, i.e., one can be both a "rational economic man" and a "rational actuary" in setting up a risk exchange.

6. FAIRPOREX ALGORITHMS

6.1. General n Company Algorithm

In 4.3 we have seen that a POREX can be characterized by its multiplier constants $k = (k_1, k_2, \dots, k_n)$ or equivalently by its layering constants

$c = (c_1, c_2, \dots, c_n)$. It is evident that finding a FAIRPOREX is a root-finding procedure where one tries to find either k or c such that for all i

$$(6.1) \quad \mu_i(c) = \text{Premium } [y_i(X/c)] \text{ becomes equal to the input premiums} \\ q_i = \text{Premium } [X_i]$$

or, equivalently, such that

$$(6.2) \quad v_i(k) = \text{Premium } [y_i(X/k)] \text{ becomes equal } q_i.$$

In the general n company case one works more easily with the multiplier constants. The search is then helped by the following observations:

1. Because of the conservation condition $\sum v_i(k) = \sum q_i$ for any k . So there is no problem of "escaping values" and if one v_i is too large at least one other must be too small.
2. If one increases k_j , then $v_i(k)$ is nondecreasing (strictly increasing) for $i \neq j$; is nonincreasing (strictly decreasing) for $i = j$.

The statements in parenthesis hold as soon as $Y_i^{(k)}$ is *not* identically zero.

Observe that these considerations do not depend on any ordering of the constants k_j , $j = 1, 2, \dots, n$.

6.2. General Two-Company Algorithm

The search is particularly simple when $n = 2$. Then, for $c_2 > 0$, $k_2 \uparrow$ if and only if $c_2 \uparrow$. If we pick $c^0 = (0, 0)$ initially, then it follows from the above that one $\mu_i(c^0)$ is too high, and the other is too low, and that μ_i decreases (increases) if its own (the other) layering constant increases. So the algorithm is simply:

1. Set $c^0 = (0, 0)$ and compute the μ_i . Renumber, if necessary, the company with $\mu_i > q_i$ as company # 2, the other as company # 1.
2. Keeping $c_1 = 0$, increase c_2 until μ_2 decreases to q_2 (and μ_1 increases to q_1).

6.3 Exponential Utility Algorithm

The computation of the fair layering constants is greatly simplified in the exponential case because the slope of $y_i(x)$ in layer k , $x \in [c_k, c_{k+1})$, remains constant at $\beta_i(k) = \alpha_i / (\alpha_1 + \alpha_2 + \dots + \alpha_k)$, and, in the top layer $[c_n, \infty)$, the n th company participates only in this layer: thus, $y_n(x) = \beta_n(n)(x - c_n)^+$.

The key idea of the algorithm below is that the company which actually takes on this upper layer will be the one with largest resulting c_n . Because Premium $[(X - c)^+]$ is decreasing in c , this company can be found by finding the company i that minimizes q_i / α_i . Once this company is found (and renumbered # n), the losses handled in the topmost layer are removed from the pool, and the process repeated in the layer $[c_{n-1}, c_n)$, etc. Furthermore, the rankings $q_i^{(t)} / \alpha_i$, once made, are stable in every iteration t , so the ordering of the companies can be fixed once and for all! The algorithm also contains some short-cut stopping rules.

Exponential Utility Algorithm

1. Set $t \leftarrow n$, and index the companies so that:

$$\frac{q_1}{\alpha_1} \geq \frac{q_2}{\alpha_2} \geq \dots \geq \frac{q_n}{\alpha_n}.$$

Define $q_n^{(t)} = q_t = \text{Premium} [X_t]$.

2. At iteration t , c_t is the unique solution of

$$q_t^{(t)} = \beta_t(t) \cdot \{ \text{Premium} [(X - c_t)^+] - \text{Premium} [(X - c_{t+1})^+] \}$$

(Note: $c_{n+1} = \infty$).

3. If $c_t = 0$ (roundoff?), STOP! $c_1 = c_2 = \dots = c_{t-1} = 0$.

4. Define

$$q_i^{(t-1)} = q_i^{(t)} - \beta_t(t) \cdot \{ \text{Premium} [(X - c_t)^+] - \text{Premium} [(X - c_{t+1})^+] \},$$

$(i = 1, 2, \dots, t-1).$
(Note: $q_i^{(t-1)} = 0$ for $i \geq t$)

5. If $q_i^{(t-1)} = 0$ (roundoff?) for all $i < t$, STOP! There is no probability mass below $X = c_t$, and $c_1 = c_2 = \dots = c_{t-1} = c_t$.
6. Set $t \leftarrow t - 1$. If $t = 1$, set $c_1 = 0$ and STOP!
Otherwise, GOTO Step 2.

An equivalent, and somewhat simpler algorithm is as follows:

Alternate Form

1. Set $t \leftarrow n$, and index the companies so that:

$$\frac{q_1}{\alpha_1} \geq \frac{q_2}{\alpha_2} \geq \frac{q_3}{\alpha_3} \geq \dots \geq \frac{q_n}{\alpha_n}.$$

2. At iteration t , c_t is the unique solution of

$$q_t = \beta_t(t) \{ \text{Premium} [(X - c_t)^+] - \sum_{i=1}^n q_i \}.$$

3. If $c_t = 0$ (roundoff?), STOP, $c_1 = c_2 = \dots = c_{t-1} = 0$.

4. If the term in braces is zero for c_t , STOP!

There is no probability mass below $X = c_t$, and $c_1 = c_2 = \dots = c_{t-1} = c_t$.

5. Set $t \leftarrow t - 1$. If $t = 1$, set $c_1 = 0$ and STOP!

Otherwise, GOTO Step 2.

This algorithm has been implemented in APL for arbitrary $p(x)$ over $X \in [0, 1, 2, \dots]$; a copy of the program may be obtained by writing the

authors. A numeral example, computed according to the first algorithm and using a Pareto distribution is given in Appendix A.

7. INTERPRETATIONS OF SPECIAL CASES

With the FAIRPOREX model developed, it is of interest to show how certain practical risk pools arise as special cases. Some of these interpretations have been given previously by Borch (1960a) and Gerber (1978). We shall emphasize the exponential utility case.

The first observation is that with exponential utilities we will have $q_1/\alpha_1 = q_2/\alpha_2 = \dots = q_n/\alpha_n$ when the "volume" of each company is the same fraction of its unit risk tolerance. In this case, it is easy to see that $c_1 = c_2 = \dots = c_n = 0$, and we have a quota claims pool (2.7), with $\theta_i = \alpha_i / (\alpha_1 + \alpha_2 + \dots + \alpha_n) = q_i / (q_1 + q_2 + \dots + q_n)$. All companies participate in all losses, but the most risk-tolerant (highest premium) takes the greatest share; this justifies (2.8). (Of course this result is trivially true if all (general) utility functions are identical and $q_1 = q_2 = \dots = q_n$, so $\theta_i = n^{-1}$.)

If companies have the same premiums q_i , but differing α_i , then they will take higher layers, the larger their risk tolerance is. If both q_i and α_i vary, then it is the ranking of q_i/α_i that selects the layering order. It is difficult to give closed formulae for the layers, but one can show that,

$$(7.1) \quad \text{Premium} [(X - c_t)^+] - \text{Premium} [(X - c_{t+1})^+] = \sum_{i=1}^t \alpha_i \left[\frac{q_i}{\alpha_i} - \frac{q_{i+1}}{\alpha_{i+1}} \right]$$

so the larger α_{t+1} , the wider the t th layer, $[c_t, c_{t+1})$.

As the α_n of one company becomes very large compared to the risk tolerance of other companies, it is clear that it will take up the uppermost layer, and take a larger and larger share of it, reaching finally $Y_n = 1$. $(X - c_n)^+$, with $\text{Premium} [(X - c_n)^+] = q_n$. This is, of course, a (full) stop loss cover, with this company acting like a reinsurer. However, we shall see in the next section that such a company might not be interested in participating this pool on fair terms.

The opposite case is a highly risk-adverse company, with α_1 very small. In the limit $\alpha_1 = 0$ it takes all the losses in $[0, c_1)$, leaving $(X - c_1)^+$ to the other companies.

If there are many companies participating, then, of course, there are many layers. However, in practice similar companies (with comparable q_i/α_i) could probably share the same layers, and lower-level companies could probably be involved in only a few layers (rather than continuing to the end with very small $\beta_i(j)$), without seriously violating both Pareto-optimality and fairness. This would greatly simplify the administration of the pool. On the other hand our model seems to explain the many practical interlocking hierarchies of pools of different sized carriers.

8. THE DESIRABILITY OF FAIRPOREX

By merging its original loss X_i with the pool, company i accepts a new loss $Y_i = y_i(X)$, where utility is the "best possible" in the Pareto-optimal sense, i.e. for the *grand coalition* of all companies. Also for all values of X , the treaty terms have been adjusted so that the premiums of all companies are unchanged in the long run. But, is it, in the *individual* sense, always advantageous to join a FAIRPOREX? Unfortunately, the answer is, *no*.

If we re-examine our development, we see that nowhere was the range of our Pareto-optimal fraction restricted to guarantee that $V_i = E\{v_i(Y_i)\} \leq V_i^0 = E\{v_i(X_i)\}$ for each participant; in fact, there is no reference to the original marginal distribution (except for the means q_i), and the distribution of the pooled losses X is used only to set the layering constants. The strict requirement of fairness may select a Pareto-optimal solution which has $V_i > V_i^0$ for some company, who would then prefer to "go it alone". Typically this happens to the larger, risk-tolerant company, who is asked to team up with a smaller, risk-averse company. This forces the larger company to take the tail of the total losses, which may appear worse than not joining the pool.

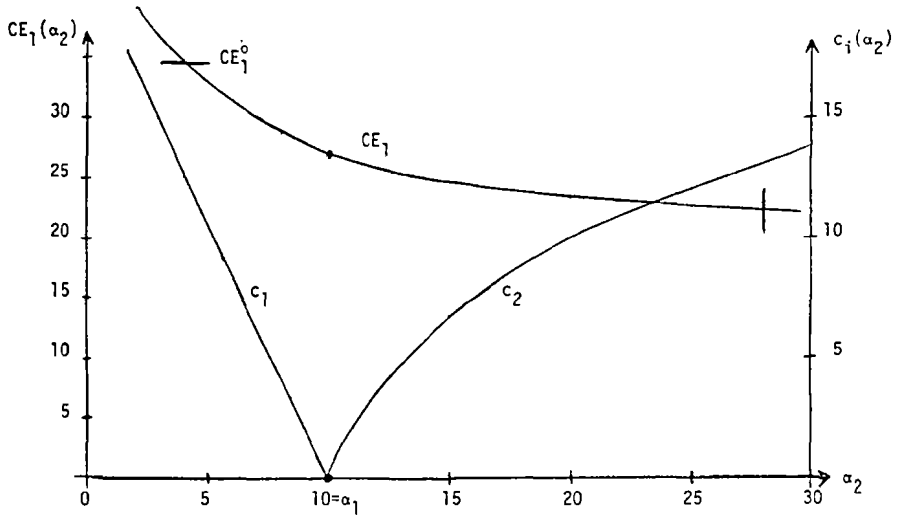


Fig. 4. Numerical Example with X_1, X_2 Geometrically Distributed, $q_1 = q_2 = 20$, and Exponential Utilities, Showing Layer Values and Certainty Equivalent for Company #1.

Fig. 4 shows the result of several computer analyses for two companies with exponential disutilities; α_1 is fixed at 10 units, and α_2 varies parametrically from 2 to 30 units. The distributions of X_1 and X_2 are geometric over the integers, with means $q_1 = q_2 = 20$ units; X_1 and X_2 are assumed independent, so the distribution of X is negative binomial. Thus for $\alpha_2 < 10$, company # 1

takes the upper layer, with the values of c_1 shown; for $\alpha_2 > 10$, company # 2 takes the upper layer, with the values of c_2 shown. (c_i on righthand scale).

Expected disutilities are expressed in terms of certainty equivalents:

$$(8.1) \quad v_i(CE_i) = E\{v_i(y_i(X))\}; \quad v_i(CE_i^0) = E\{v_i(X_i)\};$$

or, in the case of exponential disutilities:

$$(8.2) \quad CE_i = \alpha_i \ln E\{\exp(y_i(X)/\alpha_i)\}; \quad CE_i^0 = \alpha_i \ln E\{\exp(X_i/\alpha_i)\}.$$

Without a pool, $CE_1^0 = CE_2^0 = 34.14$, i.e. then original random losses are equivalent to a deterministic loss of 34.14 units.

In a pool with $\alpha_2 = \alpha_1 = 10$, the certainty equivalents drop to $CE_1 = CE_2 = 24.57$, a large improvement for both companies. Now, let α_2 decrease, i.e. let company # 2 become more risk-averse. Fig. 4 shows that CE_1 begins to increase as it is forced to take a larger fraction of a higher layer; finally, at about $\alpha_2 = 4$, $CE_1 = CE_1^0$, and company # 1 decides the FAIRPOREX is not worth it.

Conversely, if company # 2 becomes more risk tolerant, then CE_1 decreases somewhat from 24.57 as α_2 increases, until finally at about $\alpha_2 = 28$, company # 2 decides to leave the pool. (Calculations not shown.)

This change in the desirability of a FAIRPOREX can lead to interesting questions of coalition stability with three or more companies. For example, let companies A, B, C , each have geometric losses with $q_A = q_B = q_C = 10$, and exponential utilities $\alpha_A = 3$, $\alpha_B = 10$, and $\alpha_C = 20$. Initially, $CE_A^0 = 20.53$, $CE_B^0 = 13.71$, and $CE_C^0 = 11.77$.

Now, through the use of the algorithm, one can show that company A is *not* acceptable to B , and certainly *not* acceptable to C in a two-party agreement. B and C are compatible, however, and it turns out that: $c_B = 0$, $c_C = 5.19$, $CE_B = 10.84$, $CE_C = 11.48$, when they share the sum of two random geometric losses.

And surprisingly, the three-party exchange is *also* satisfactory to all parties concerned since: $c_A = 0$, $c_B = 7.07$, $c_C = 14.10$, $CE_A = 10.38$, $CE_B = 11.08$, $CE_C = 11.72$, when sharing the sum of three random geometric losses.

Note that the "outcast" A gained the most over his initial independent position in the triad, and that B and C were forced to give up some gains from their diad, although they are still better off than going it alone. This suggests that a weak partner should always try to get in at the beginning of the negotiations, and prevent his stronger partners from computing what they can do without him!

We have been unable to form a stable triad from among companies who did not wish to form any diads among themselves.

9. THE POSSIBILITY OF SIDE PAYMENTS

If a FAIRPOREX will not form because of inequities in the sizes of the participants, or because some company feels that it profits insufficiently, a possible generalization of our model is to introduce fixed side payments π_i , such that company i actually pays $\pi_i + y_i(X)$, with $\sum \pi_i = 0$ and with $y_i(X)$ as before. For an exponential utility, this has the quick comparison advantage of increasing the certainty equivalent by π_i . In many cases there is a range of side payments that will bring recalcitrant members back into the exchange, and still leave an improvement in utility for everyone.

For instance, in the example above, A could give 0.24 units to B and 0.24 units to C to induce them to form a triad, and still have a certainty equivalent left of $CE_A = 10.38 + 0.48 = 10.82$, well below being left out at $CE_A = 20.53$. In fact, he has $20.53 - 10.82 = 9.71$ more units of side payment left to "sweeten the pot" if B and C prove stubborn.

Of course, by introducing side payments we abandon the basic idea of fairness. If you want so, this is where the snake bites its own tail. We started by constructing fair risk exchanges and now we modify them to become *unfair* again!

This just proves that—in our model as in the real world—fair pools will not always form!

APPENDIX A

Numerical Example (Exponential disutilities)

1. We demonstrate the calculation of the unconstrained FAIRPOREX (business pool), the constrained FAIRPOREX (claims pool), for a total input variable $X = Z - 1$ where Z has density $f_Z(x) = 2x^{-3}$, ($x \geq 1$), ($Z \sim$ Pareto on $[1, \infty)$ with parameter 3 to guarantee a finite mean $E[Z] = 2$). We also define Premium $[Y] = E[Y]$ for any random variable Y .

2. For the unconstrained FAIRPOREX we have [see (4.6)]

$$y_i(x) = \frac{\alpha_i}{\sum_{j=1}^n \alpha_j} x + \gamma_i$$

where γ_i is computed from the fairness condition

$$E[Y_i] = q_i \Rightarrow \gamma_i = q_i - \beta_i.$$

3. For the constrained FAIRPOREX we compute according to the "Exponential Utility Algorithm" (first version) as explained in 6.3.

THE FAIRPOREN IS COMPUTED FOR FIVE COMPANIES WITH GIVEN α_i AND q_i :

i	α_i	$q_i^{(5)}$	$\beta_i^{(5)}$	$q_i^{(4)}$	$\beta_i^{(4)}$	$q_i^{(3)}$	$\beta_i^{(3)}$	$q_i^{(2)}$	$\beta_i^{(2)}$	$q_i^{(1)}$	$\beta_i^{(1)}$
5	100	3	.585	—	—	—	—	—	—	—	—
4	50	.2	.292	.050	.704	—	—	—	—	—	—
3	15	2	.088	.155	.211	.140	.714	—	—	—	—
2	5	2	.029	.185	.071	.180	.238	.133	.833	—	—
1	1	1	.006	.097	.014	.096	.048	.087	.167	.06	.1
			$C_5 = .95$	$C_4 = .71$		$C_3 = .28$		$C_2 = .063$		$C_1 = .0007$	

Note that in our example

$$\text{Premium } [(X - c_t)^+] = \frac{1}{c_t + 1}$$

$$q_i^{(t)} = \beta_i(t) \left(\frac{1}{c_t + 1} - \frac{1}{c_{t+1} + 1} \right) = > \frac{1}{c_t + 1} = \frac{q_i^{(t)}}{\beta_i^{(t)}} + \frac{1}{c_{t+1} + 1}$$

and for $i \leq t - 1$

$$q_i^{(t-1)} = q_i^{(t)} - \beta_i(t) \left(\frac{1}{c_t + 1} - \frac{1}{c_{t+1} + 1} \right)$$

The computation of c_1 serves for checking, of course we must have $c_1 = 0$.

4. The results can be rearranged in the following final form:

Business pool

company	claims quota	fixed payment (+ pament) (- receipt)
5	58.5%	-28.5%
4	29.2%	- 9.2%
3	8.8%	+11.2%
2	2.9%	+17.1%
1	0.6%	+ 9.4%

Claims pool

company	layer 5	layer 4	layer 3	layer 2	layer 1
5	58.8%	—	—	—	—
4	29.2%	70.4%	—	—	—
3	8.8%	21.1%	71.4%	—	—
2	2.9%	7.1%	23.8%	83.3%	—
1	0.6%	1.4%	4.8%	16.7%	100%
layer interval	[.95, ∞)	[.71, .95)	[.28, .71)	[.063, .28)	[0, .063)

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