RISK BEARING AND THE INSURANCE MARKET

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1. Introduction

Stimulated by Karl Borch's paper [3] we have tried to analyze the paper written by K. Arrow [1] in 1953. Contrary to Borch's opinion we have some doubt whether this work contains a theory of insurance as a special case. Nevertheless, it has inspired us to this note, which tries to develop a somewhat more realistic model. As a matter of fact, our development is more in the spirit of another paper by Arrow [2]. We, however, have chosen a more general setup, and we believe that our treatment is also different.

2. Arrow's Model (Interpreted Freely)

Arrow considers an economy of exchange with C *commodities* (labelled $c = 1, \ldots, C$) and a "world" that will be in one of S different *states* ($s = 1, \ldots, S$). The problem is to distribute the total supply of each commodity c in state s among I individuals in a Pareto-optimal fashion. According to a standard result in economic equilibrium theory every Pareto-optimal allocation can be realized by a system of perfectly competitive markets. The latter means that there are prices p_{sc} (the price for a unit of commodity c if state s occurs) and that each individual has a certain amount of money, which he then will spend to maximize his own utility. The beauty of this approach lies in its simplicity: Each individual has his own maximization problem (irrespective of the others). Thus it is enough to focus our attention on a *particular individual*. Let y denote his spendable money, let $x_{sc} \geq 0$ denote the amount of commodity c contingent to the occurrence of state s purchased, and let $V(x_{11}, \ldots, x_{SC})$ denote the "value" (or utility) of this decision.

Then the problem is to

$$\text{maximize } V(x_{11}, \ldots, x_{SC})$$
$$\text{subject to } \sum_{s=1}^{S} \sum_{c=1}^{C} x_{sc}\, p_{sc} \leq y. \tag{1}$$

Arrow's idea is to replace this market by a two stage market. Let $q_1 > 0, \ldots, q_S > 0$ be arbitrary numbers with $q_1 + \ldots + q_S = 1$. Here q_s is the price of a security ("policy" in insurance terminology) of type s, which pays one monetary unit if state s occurs and nothing otherwise. Let p_{sc} be the price of commodity c when state s has occurred. For consistency set

$$p_{sc} = \bar{p}_{sc}/q_s. \tag{2}$$

The two decisions are now:

a) *choice of the securities*. Buy $y_s \geq 0$ securities of type $s(s = 1, \ldots, S)$ such that $\sum_{s=1}^{S} y_s q_s \leq y$.

b) *Purchase of commodities after the state s has occurred*. Let x_{sc} denote the amount of commodity c that is purchased after the state s has occurred. We must have $\sum_{c=1}^{C} x_{sc} p_{sc} \leq y_s + y - \sum_{i=1}^{S} y_i q_i$. Again, we make our decision in a) and b) to maximize the resulting utility. Obviously, this two stage problem is equivalent to the original problem (1), equivalence meaning that the same commodity bundles can be bought with the same original money amount.

From now on let us assume that the function V is of the form (according to the axioms of vonNeumann-Morgenstern)

$$V(x_{11}, \ldots, x_{SC}) = \sum_{s=1}^{S} \pi_s V_s(x_{s1}, \ldots, x_{sC}). \tag{3}$$

Here π_s is the individual's subjective probability for state s, and V_s is the utility function that applies when state s occurs. Let

$$\begin{aligned} U_s(w) &= \text{maximum } V_s(x_{s1}, \ldots, x_{sC}) \\ &\text{subject to } x_{sc} \geq 0, \ \sum_{c=1}^{C} x_{sc} p_{sc} \leq w. \end{aligned} \tag{4}$$

Thus $U_s(w)$ is the utility of w monetary units in state s. With these definitions and assumptions problem a) (optimal choice of the securities) can be isolated as follows:

$$\begin{aligned} &\text{maximize } \sum_{s=1}^{S} \pi_s U_s\left(y + y_s - \sum_{i=1}^{S} y_i q_i\right) \\ &\text{subject to } y_s \geq 0, \ \sum_{i=1}^{S} y_i q_i \leq y. \end{aligned} \tag{5}$$

3. The Problems of Optimal Coverage

We shall study in detail the solutions of problems of the type (5). Our assumptions are as follows. a) The S utility functions $U_s(y)$ are twice differentiable, such that $U_s'(y) > 0$ and $U_s''(y) < 0$. Thus we assume that the utility functions are risk adverse. b) $q_1 + \ldots$

$+ q_S \geq 1$. If p_s is the probability that the market assigns to state s, certainly $q_s \geq p_s$. Summation over s yields the inequality above.

If $q_1 + \ldots + q_S = 1$, (as in Arrows model) we can assume that $\sum_{i=1}^{S} y_i q_i = y$ without loss of generality in (5). However, in the more interesting case where $q_1 + \ldots + q_S > 1$, this is not true anymore. This suggests that we distinguish the following two problems.

Problem A

For a fixed z, $0 \leq z \leq y$, maximize $\sum_{s=1}^{S} \pi_s U_s(y + y_s - z)$ subject to the constraints that $y_s \geq 0$ and $\sum_{s=1}^{S} y_s q_s = z$.

Problem B

Maximize $\sum_{s=1}^{S} \pi_s U_s(y + y_s - \sum_{i=1}^{S} y_i q_i)$ subject to $y_s \geq 0$, and $\sum_{s=1}^{S} y_s q_s \leq y$.

Thus in Problem A the total amount spent for premiums, z, is prescribed, while in Problem B it is variable, subject only to the upper bound y.

In either case the existence of an optimal solution is clear: The quantity to be maximized is a continuous function of the decision variables $y_1, \ldots, y_S$, which (in both cases) vary over a compact set.

4. Solution of Problem A.

Theorem 1

For any $z (0 \leq z \leq y)$ there is a unique vector $\tilde{y}_1, \ldots, \tilde{y}_S$ satisfying

(i) $\sum_{s=1}^{S} \tilde{y}_s q_s = z, \tilde{y}_s \geq 0$ for all s

(ii) $\dfrac{\pi_s}{q_s} U'_s(y + \tilde{y}_s - z) \leq K$ for all s, such that $\tilde{y}_s = 0$ whenever this inequality is strict.

This vector, and only this vector, solves problem A.

Proof

For $z = 0$, the theorem is trivially true. Hence assume $z > 0$. To show the necessity of condition (ii), consider a vector $y_1, \ldots, y_S$

for which it is violated. Then there are indices s, t such that $y_t > 0$, $y_s \geq 0$ and

$$\frac{\pi_t}{q_t} U_t'(y + y_t - z) < \frac{\pi_s}{q_s} U_s'(y + y_s - z). \tag{6}$$

Then, by increasing y_s and decreasing y_t (such that the total premium remains z) the expected utility could be increased. (Note that for this part of the proof we did not need the assumption that the utility functions are risk averse.)

The necessity (and the existence of an optimal solution) show that there is at least one vector $\tilde{y}_1, \ldots, \tilde{y}_S$ that satisfies conditions (i) and (ii) above. Let $y_1, \ldots, y_S$ be any other vector that satisfies (i). First using concavity from below of the function U_s, and then (ii), we obtain the following estimate:

$$\begin{aligned} U_s(y + y_s - z) &\leq U_s(y + \tilde{y}_s - z) + U_s'(y + \tilde{y}_s - z) \cdot (y_s - \tilde{y}_s) \\ &\leq U_s(y + \tilde{y}_s - z) + K \frac{q_s}{\pi_s} (y_s - \tilde{y}_s). \end{aligned} \tag{7}$$

Note that the first inequality is strict unless $y_s = y_s$. By summing (7) over s we see that

$$\sum_{s=1}^{S} \pi_s U_s(y + y_s - z) \leq \sum_{s=1}^{S} \pi_s U_s(y + \tilde{y}_s - z), \tag{8}$$

with a strict inequality holding unless $y_s = \tilde{y}_s$ for all s. This completes the proof of Theorem 1.

5. Solution of Problem B.

If $\sum_{s=1}^{S} q_s = 1$, solve Problem A with $z = y$. Otherwise, the following result holds.

Theorem 2

Suppose that $\sum_{s=1}^{S} q_s > 1$. Then Problem B has a unique solution, which we denote by $\tilde{y}_1, \ldots, \tilde{y}_S$. a) If $\sum_{s=1}^{S} \tilde{y}_s q_s = y$, it can be characterized by conditions (i) and (ii) in Theorem 1 with $z = y$. b) If $\sum_{s=1}^{S} \tilde{y}_s q_s < y$, it is the only vector $\tilde{y}_1, \ldots, \tilde{y}_S$ that satisfies

i) $\tilde{y}_s \geq 0$ for all s and

ii) $\displaystyle \frac{\pi_s}{q_s} U_s'(y + \tilde{y}_s - \sum_{t=1}^{S} \tilde{y}_t q_t) \leq \sum_{j=1}^{S} \pi_j U_j'(y + \tilde{y}_j - \sum_{t=1}^{S} \tilde{y}_t q_t)$

for all s, such that $\tilde{y}_s = 0$ whenever the inequality is strict.

Proof

a) If there is an optimal $\tilde{y}_1, \ldots, \tilde{y}_s$ with $\sum_{s=1}^{S} \tilde{y}_s q_s < y$, it has to satisfy condition (ii) above. For, if it did not, there would either be an index s such that

$$\frac{\pi_s}{q_s} U'_s(y + \tilde{y}_s - \sum_{i=1}^{S} y_i q_i) > \sum_{j=1}^{S} \pi_j U'_j(y + \tilde{y}_j - \sum_{i=1}^{S} \tilde{y}_i q_i), \tag{9}$$

in which case the expected utility could be increased by increasing $\tilde{y}_s$, or there would be an index s such that $\tilde{y}_s > 0$ and the inequality in (ii) is strict, in which case the expected utility could be increased by a reduction of $\tilde{y}_s$. (For the necessity of (ii) we again did not need the assumption that the utility functions are risk averse).

b) Suppose now that $\tilde{y}_1, \ldots, \tilde{y}_S$ is a vector that satisfies conditions (i) and (ii) of part b) in Theorem 2. Any other decision, say $y_1, \ldots, y_S$ (where $\sum_s q_s y_s = y$ is also permissible), can be compared with it as follows: For any s,

$$U_s(y + y_s - z) \leq U_s(y + \tilde{y}_s - \tilde{z}) + U'_s(y + \tilde{y}_s - \tilde{z}) \cdot (y_s - \tilde{y}_s + \tilde{z} - z)$$

$$\leq U_s(y + \tilde{y}_s - \tilde{z}) + \frac{q_s}{\pi_s} (y_s - \tilde{y}_s) \sum \pi_j U'_j(y + \tilde{y}_j - \tilde{z}) +$$

$$+ U'_s(y + \tilde{y}_s - \tilde{z}) \cdot (\tilde{z} - z), \tag{10}$$

with the convenient notation $\tilde{z} = \sum \tilde{y}_i q_i$, $z = \sum y_i q_i$. Multiplying both sides by π_s, and summing over s, we get

$$\sum_{s=1}^{S} \pi_s U_s(y + y_s - z) \leq \sum_{s=1}^{S} \pi_s U_s(y + \tilde{y}_s - z). \tag{11}$$

Furthermore, this inequality is strict unless $y_s = \tilde{y}_s$ for all s, which shows the uniqueness of any optimal solution satisfying (ii).

6. How to Find the Solutions.

To find the solution of Problem A, first relabel the states such that

$$\frac{\pi_1}{q_1} U'_1(y - z) \geq \frac{\pi_2}{q_2} U'_2(y - z) \geq \ldots \geq \frac{\pi_S}{q_S} U'_S(y - z). \tag{12}$$

Now we choose y_1 such that

$$\frac{\pi_1}{q_1} U'_1(y + y_1 - z) = \frac{\pi_2}{q_2} U'_2(y - z). \tag{13}$$

Then we increase y_1 and choose y_2 such that

$$\frac{\pi_1}{q_1} U_1'(y + y_1 - z) = \frac{\pi_2}{q_2} U_2'(y + y_2 - z) = \frac{\pi_3}{q_3} U_3'(y - z) \quad (14)$$

etc. Thus, gradually we increase the coverage, from left to right, until the total premium reaches the level z. Clearly, the resulting coverage will satisfy properties (i) and (ii) of Theorem 1.

For the further discussion, let $\tilde{y}_1, \ldots, \tilde{y}_S$ denote the optimal coverage if the premium equals z, hence

$$U(z) = \sum_{s=1}^{S} \pi_s U_s(y + \tilde{y}_s - z) \quad (15)$$

is the maximal utility at premium level z, and let $K = K(z)$ denote the upper bound in (ii) of Theorem 1. Finally, set

$$K_v(z) = \sum_{s=1}^{S} \pi_s U_s'(y + \tilde{y}_s - z). \quad (16)$$

Theorem 3

$U'(z)$ equals $K(z) - K_v(z)$ and is a non-increasing function.

Proof

Let z_1, z_2 be any two numbers, and let $\tilde{y}_s^{(i)}$ denote the optimal coverage for state s if the total premium should be z_i $(i = 1, 2)$. Using the concavity from below of U_s and property (ii) in Theorem 1, we find that

$$\begin{aligned} &U_s(y + \tilde{y}_s^{(2)} - z_2) - U_s(y + \tilde{y}_s^{(1)} - z_1) \\ &\leq U_s'(y + \tilde{y}_s^{(1)} - z_1) \cdot (\tilde{y}_s^{(2)} - \tilde{y}_s^{(1)} + z_1 - z_2) \quad (17) \\ &\leq \frac{q_s}{\pi_s} K(z_1) \cdot (\tilde{y}_s^{(2)} - \tilde{y}_s^{(1)}) - U_s'(y + \tilde{y}_s^{(1)} - z_1) \cdot (z_2 - z_1). \end{aligned}$$

Multiply both sides by π_s, and summing over s, we obtain the inequality

$$U(z_2) - U(z_1) \leq (K(z_1) - K_v(z_1)) \cdot (z_2 - z_1). \quad (18)$$

By interchanging the roles of z_1 and z_2, and inverting the sign, a lower bound is obtained for $U(z_2) - U(z_1)$. Finally, assume $z_1 < z_2$.

Then these two inequalities can be written as follows.

$$K(z_2) - K_v(z_2) \leq \frac{U(z_2) - U(z_1)}{z_2 - z_1} \leq K(z_1) - K_v(z_1). \quad (19)$$

Monotonicity of $K(z) - K_v(z)$ is seen immediately from (19), and the rest of theorem 3 follows by taking the limit for $z_2 \to z_1$. Now observe the following: Let $0 \leq \tilde{z} \leq y$ be the premium spent in the optimal solution $\tilde{y}_1, \tilde{y}_2, \ldots \tilde{y}_S$ of problem B (i.e. $\tilde{z} = \sum_{s=1}^{S} q_s \tilde{y}_s$). For this $\tilde{z}$ problem A must have the same solution as problem B and we conclude, that the two bounds appearing in the characterization of the solutions must be the same, hence

$$K(\tilde{z}) = K_v(\tilde{z}).$$

On the other hand theorem 3 leads to the following

Corollary

If	$K(0) \leq K_v(0)$	then	$\tilde{z} = 0$
	$K(y) \geq K_v(y)$	then	$\tilde{z} = y$

otherwise let z satisfy

$$K(z) = K_v(z) \qquad \text{then} \qquad \tilde{z} = z$$

Based on this corollary and the monotonicity of $K(z) - K_v(z)$, $0 \leq z \leq y$ one may find $\tilde{z} \neq 0$ by gradually increasing the level z of premium spent until $K(z) - K_v(z) = 0$, or if this does not happen for $z \leq y$, by putting $\tilde{z} = y$.

Note

It is sometimes more convenient, to follow the above procedure until the quotient $\dfrac{K(z)}{K_v(z)}$ reaches 1. To justify this alternative, we also prove that $\dfrac{K(z)}{K_v(z)}$ is nonincreasing $\left(\dfrac{K_v(z)}{K(z)} \text{ nondecreasing}\right)$ for $0 \leq z \leq y$.

Proof

Let $N = N(z)$ denote the set of indices for which $\tilde{y}_s = 0$. Then

$$K_v(z) = K(z) \left(\sum_{s \notin N} q_s \right) + \sum_{s \in N} \pi_s \, U_s'(y - z), \tag{20}$$

and therefore

$$\frac{K_v(z)}{K(z)} = \sum_{s \notin N} q_s + \frac{\sum_{s \in N} \pi_s \, U_s'(y - z)}{K(z)}. \tag{21}$$

Since $\dfrac{\pi_s \, U_s'(y - z)}{K(z)} \leq q_s$ for $s \in N$, this shows that $K_v(z)/K(z)$ is a nondecreasing function (the numerator in the last expression is a nondecreasing function, while $K(z)$ is nonincreasing).

In the following the procedure for finding the optimum in problem B is explicitly carried out for

exponential utility (Section 7)

quadratic utility (Section 8).

7. Exponential Utility

Let $U_s(x) = 1 - e^{-\alpha(x-y_s^*)}$, $U_s'(x) = \alpha e^{\alpha y_s^*} e^{-\alpha x}$. You may interpret y_s^* as the "need for money" in state s. Suppose then y sufficiently large, such that the following property holds for the optimum $\tilde{y}_1, \tilde{y}_2, \ldots \tilde{y}_S$ of problem B (according to theorem 2).

$$\frac{\pi_s}{q_s} e^{\alpha \tilde{y}_s^*} e^{-\alpha \tilde{y}_s} \leq \sum_j \pi_j e^{\alpha y_j^*} e^{-\alpha \tilde{y}_j} \qquad \text{for all } s \text{, with strict inequality only allowed if } \tilde{y}_s = 0. \tag{22}$$

With the notation

$$\pi_s^* = \pi_s e^{\alpha y_s^*} \tag{23}$$

and

$$C_s(y_1, y_2, \ldots y_S) = \frac{\frac{\pi_s^*}{q_s} e^{-\alpha y_s}}{\sum_{j=1}^{S} \pi_j^* e^{-\alpha y_j}} \tag{24}$$

(22) becomes

$$C_s(\tilde{y}_1, \tilde{y}_2, \ldots \tilde{y}_S) \leq 1 \qquad \text{for all } s \text{, with strict inequality only allowed if } \tilde{y}_s = 0. \tag{25}$$

Abbreviate $\hat{z}$ for $\sum_{j=1}^{S} q_j \tilde{y}_j$. (25) may hold for $z = 0$ and then $\hat{z} = 0$. Otherwise, increasing gradually the premium level z and adjusting $y_1, y_2, \ldots y_s$ at each level z according to the solution of problem A, $\max_{s \in S} C_s$ will monotonically decrease until it reaches 1 at $z = \hat{z}$. (See *note* after theorem 3.) Observe that in the exponential case the ordering

$$C_1(y_1, y_2, \ldots y_S) \geq C_2(y_1, y_2, \ldots y_S) \geq \ldots \geq C_S(y_1, y_2, \ldots y_s)$$

never changes during this process.

Let then m be the number of states, which are insured in the optimal solution of B (number of variables $\tilde{y}_s$ different from 0 in (25)).

From (21) we have

$$K_v(\tilde{z}) = \sum_{j=1}^{s} \pi_j^* e^{-\alpha\tilde{y}_j} = K(\tilde{z}) \sum_{j=1}^{m} q_j + \sum_{j=m+1}^{s} \pi_j^*$$

and hence from the corollary of theorem 3

$$1 = \sum_{j=1}^{m} q_j + \frac{1}{K(\tilde{z})} \sum_{j=m+1}^{s} \pi_j^*$$

$$\frac{1}{K(\tilde{z})} = \frac{1 - \sum_{j=1}^{m} q_j}{\sum_{j=m+1}^{s} \pi_j^*}$$

therefore (recall $K(\tilde{z}) = \frac{\pi_s^*}{q_s} e^{-\alpha\tilde{y}_s}$ for $s = 1, 2, \ldots m$)

$$\alpha\tilde{y}_s = \log \frac{\pi_s^*}{q_s} + \log \frac{1}{K(\tilde{z})} = \tag{26}$$

$$\log \frac{\pi_s^*}{q_s} + \underbrace{\log (1 - \sum_{j=1}^{m} q_j) - \log \sum_{j=m+1}^{s} \pi_j^*}_{\Delta m}$$

for $s \leq m$.

The optimal m is found as the first index for which

$$\frac{\pi_{m+1}^*}{q_{m+1}} \frac{1 - \sum_{j=1}^{m} q_j}{\sum_{j=m+1}^{s} \pi_j^*} \leq 1 \text{ or equivalently } \log \frac{\pi_{m+1}^*}{q_{m+1}} + \Delta_m \leq 0 \tag{27}$$

It is easily checked, that this condition also applies if $m = 0$.

Numerical Examples (In all examples the exponent $\alpha = 10^{-2}$)

First example

s	1	2	3	4	5
y_s^*	1000	100	50	10	5
π_s	0.1	0.2	0.3	0.2	0.2
q_s	0.3	0.3	0.3	0.3	0.3
π_s^*	2202.65	.544	0.495	0.221	0.210
$\frac{\pi_s^*}{q_s}$	7342.16	1.813	1.65	0.737	0.7

$1 - \sum_{j=1}^{s-1} q_j$	1	0.7
$\sum_{j=s}^{S} \pi_j^*$	2204.12	1.470
check (27)	3.33	*0.863*

Hence only state 1 is insured and from (26) $\tilde{y}_1 = 815.95$

$$q_1\tilde{y}_1 = 244.78.$$

Second example

Insurance becomes "horribly expensive" for $s = 1$, otherwise same as in first example.

s	*1*	*2*	*3*	*4*	*5*
y_s^*	1000	100	50	10	5
π_s	0.1	0.2	0.3	0.2	0.2
q_s	*1*	0.3	0.3	0.3	0.3
π_s^*	2202.65	0.544	0.495	0.221	0.210
$\frac{\pi_s^*}{q_s}$	2202.65	1.813	1.65	0.737	0.7
$\sum_{j=s}^{S} \pi_j^*$	2204.12				
check (27)	$<$ *1*				

Hence now *no* insurance is bought at all!

Third example

The "insurance need" is eliminated in state $s = 1$, otherwise still the same as before.

s	*2*	*3*	*4*	*5*	*1*
y_s^*	100	50	10	5	0
π_s	0.2	0.3	0.2	0.2	0.1
q_s	0.3	0.3	0.3	0.3	0.3
π_s^*	0.544	0.495	0.221	0.210	0.1
$\frac{\pi_s^*}{q_s}$	1.813	1.65	0.737	0.7	0.333
check (27)	1.155	1.126	*0.555*		

Hence insurance on $s = 2$ and 3 $\quad \tilde{y}_2 = 31.17 \; q_2\tilde{y}_2 = 9.35$

$$\tilde{y}_3 = 21.75 \; q_3\tilde{y}_3 = 6.52$$

8. Quadratic Utility

In this section

$$U_s(x) = \alpha(x - y_s^{**}) - \frac{(x - y_s^{**})^2}{2}; \; x - y_s^{**} \leq \alpha$$

$$U_s'(x) = \alpha + y_s^{**} - x$$

The condition corresponding to (22) in Section 7 is then

$$\frac{\pi_s}{q_s}(\alpha + y_s^{**} - y - \tilde{y}_s + \sum_j q_j\tilde{y}_j) \leq \alpha + \bar{y}_s^{**} - y - \bar{\tilde{y}} + \sum_j q_j\tilde{y}_j \quad (28)$$

for all s, with strict inequality only allowed if $\tilde{y}_s = 0$

Abbreviations

$$\bar{y}^{**} = \sum_j \pi_j y_j^{**}$$

$$\bar{\tilde{y}} = \sum_j \pi_j \tilde{y}_j$$

Redefine

$$\alpha + y_s^{**} - y = y_s^* \qquad \text{and you obtain}$$

$$\frac{\frac{\pi_s}{q_s}(y_s^* - \tilde{y}_s + \sum_j q_j\tilde{y}_j)}{\bar{y}^* - \bar{\tilde{y}} + \sum_j q_j\tilde{y}_j} \leq 1 \quad (29)$$

for all s, with strict inequality only allowed if $\tilde{y}_s = 0$

Observe that as long as the numerator of the left side in (29) is positive, we are in the region where U_s' is positive. The numbering of the sides is defined in decreasing order of

$$C_s = \frac{\pi_s}{q_s}\frac{y_s^*}{\bar{y}_s}, \text{ hence } C_1 \geqslant C_2 \geqslant \ldots \geqslant C_S \quad (30)$$

These quantities are the inital values at $y_1 = y_2 = \ldots = y_S = 0$ of the functions

$$C_s(y_1, y_2, \ldots y_S) = \frac{\frac{\pi_s}{q_s}(y_s^* - y_s + \sum q_j y_j)}{y^* - \bar{y} + \sum q_j y_j} \quad (31)$$

We again gradually increase $z = \sum_j q_j y_j$ and for each z adapt $y_1, y_2, \ldots y_S$ according to the solution of problem A; $\max_{s \in S} C_s$ will then again monotonically decrease to 1, but unfortunately the ordering of the $C_s(y_1, y_2, \ldots y_S)$ (for those s which are not yet insured) may change! So while it is clear that insurance, if any, must always be bought on $s = 1$, we must if necessary try several combinations of other states to find out the optimum.

Numerical Examples

First example

s	1	2	3	4	5	
y_s^*	1000	100	50	10	5	
π_s	0.1	0.2	0.3	0.2	0.2	$\bar{y}^* = 138$
q_s	0.3	0.3	0.3	0.3	0.3	
C_s	2.415	0.483	0.362	0.048	0.024	

We try to insure state number 1 only. If this does achieve an optimum we must have

$$C_1(y_1, 0, 0, \ldots 0) = \frac{1}{3} \frac{1000 - y_1 + 0.3\, y_1}{138 - 0.1\, y_1 + 0.3\, y_1} = 1$$

from which we find

$$y_1 = 450.77$$

$$q_1 y_1 = 135.23$$

It remains to be checked whether $C_s(y_1, 0, 0, \ldots 0) \leq 1$ for $s \geq 2$

$$C_2(y_1, 0, 0, \ldots 0) = \frac{2}{3} \frac{100 + 135.23}{228.15} = 0.69$$

$$C_3(y_1, 0, 0, \ldots 0) = \frac{3}{3} \frac{50 + 135.23}{228.15} = 0.81 \text{ (has surpassed } C_2\text{!)}$$

As states 4 and 5 have the same probabilities and premiums as state 2 their C-values must be lower than that of state 2 also. This shows that just insuring state 1 with the above amounts is optimal.

Second example

If we change in the first example only q_1 from 0.3 to 1 (insurance on the state insured in the first example becomes "horribly expensive"), then all initial C-values drop below 1 which means that no insurance should be bought.

Third example

"Insurance need" in state 1 is eliminated (i.e. $y_1^* = 0$). Otherwise same as first example.

s	2	3	4	5	1	
y_s^*	100	50	10	5	0	
π_s	0.2	0.3	0.2	0.2	0.1	$\bar{y}^* = 38$
q_s	0.3	0.3	0.3	0.3	0.3	
C_s	1.75	1.32	0.18	0.09	0	

It is obvious that some insurance must be bought, certainly on $s = 2$ and probably also on some other states, $s = 3$ being a very likely candidate.

We try to find an optimal solution, where y_2 and y_3 are different from zero

$$C_2(y_2, y_3, 0, \ldots 0) = \frac{2}{3}\,\frac{100 - y_2 + 0.3(y_2 + y_3)}{38 - 0.2y_2 - 0.3y_3 + 0.3\,(y_2 + y_3)} = 1$$

$$\text{or } 860 - 14\,y_2 + 3\,y_3 = 0$$

$$C_3(y_2, y_3, 0, \ldots 0) = \frac{50 - y_3 + 0.3(y_2 + y_3)}{38 - 0.2y_2 - 0.3y_3 + 0.3(y_2 + y_3)} = 1$$

$$\text{or } 120 - 7\,y_3 + 2y_2 = 0$$

$$y_2 = 69.35 \qquad y_3 = 36.96$$

$$q_2y_2 = 20.80 \qquad q_3y_3 = 11.09 \qquad \text{total premium} \quad 31.89$$

We must check that $C_4(y_2, y_3, 0, 0, 0) \leq 1$. This check suffices since $\frac{\pi_s}{q_s} \leq \frac{\pi_4}{q_4}$ for $s = 5$, 1 (C_5 and C_1 will then automatically be below 1).

$$\text{Check: } C_4(y_2, y_3, 0, 0, 0) = \frac{2}{3}\,\frac{10 + 31.89}{38 - 24.96 + 31.89} = 0.62,$$

which proves optimality.

9. Literature

[1] Arrow, K. J., (1964), The Role of Securities in the Optimal Allocation of Risk Bearing. Review of Economic Studies (translation of original paper in French, 1953).

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PARETO-OPTIMAL RISK EXCHANGES AND RELATED DECISION PROBLEMS

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1. SUMMARY

In various branches of applied mathematics the problem arises of making decisions to reconcile conflicting criteria. One example is the classical statistical problem, where a type 1 error cannot be arbitrarily reduced without increasing the probability for a type 2 error. Another example, quite familiar to actuaries, is graduation, where a compromise between smoothness and fit has to be reached This motivates the concept of Pareto-optimal decisions, which is discussed in section 2 There is a simple method, maximizing a weighted average of the scores, to obtain certain Pareto-optimal decisions. In section 3 a condition is given, which is satisfied in most applications, that guarantees that all the Pareto-optimal decisions can be found by this method This is applied in section 4, where the problem of risk exchange between n insurance companies is considered. The original model of Borch is generalized: it is assumed that some of the companies are not willing to contribute more than a certain fixed amount towards the aggregate loss of the other companies The theorem in section 4 gives a characterization of all the Pareto-optimal risk exchanges Because of the restrictions, these risk exchanges do not just depend on the combined surplus (which would amount to pooling) in general, and can be found by an algorithm. One benefit of this generalization of Borch's Theorem is that two seemingly unrelated results (optimality of a stop loss contract, and optimality of certain dividend formulas in group insurance) follow from it as special cases.

2. EVALUATION OF DECISIONS UNDER CONFLICTING VIEW POINTS

Often one is faced with the situation where a decision has to be made in the presence of several criteria. Mathematically, the problem can be formulated as follows.

Let D be the set of all possible decisions. We are given n real-valued functions $s_1(d), \ldots, s_n(d)$, $d \in D$. If $d_1, d_2 \in D$ and $s_i(d_1) \geq s_i(d_2)$, this means that decision d_1 is *better* than (or at least as good as) decision d_2 *with respect to criterion i*. Let

$$s(d) = (s_1(d), \ldots, s_n(d)), \; d \in D \tag{1}$$

and

$$S = \{x/x = s(d) \text{ for some } d \in D\} \tag{2}$$

denote the range of the "score function" $s(\cdot): D \rightarrow R^n$. A decision d_1 is said to be *strictly better* than a decision d_2, if $s_i(d_1) \geq s_i(d_2)$ for $i = 1, \ldots, n$, and if at least one of these inequalities is strict. A decision d is called *Pareto-optimal*, if there is not a decision that