

1 A Nonparametric Method of Estimating Loss
2 Reserves

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5 **Abstract**

6 This paper introduces a new method for estimating loss reserves. The method
7 is different from other loss reserving methods because it explicitly assumes the
8 evolution of the incremental incurred claims for an accident year is the result of
9 a random split of the ultimate loss for that accident year into separate pieces of
10 losses that are observed in each development year over the claims settlement period.
11 The nature of the random split and the pattern of the evolution of incremental
12 incurred claims must be specified by the reserving actuary, thus giving the method
13 tremendous flexibility. The basic model used in this paper assumes the pieces of
14 losses form a decreasing sequence of order statistics. As this method provides loss
15 development factors without any knowledge of the distribution of the ultimate loss
16 or of the actual cumulative incurred claims, this methods is suitable for calculating
17 reserves for new lines of business, where there is little or no claims settlement data.

18 Key words and phrases: *incurred claims, pieces of losses, order statistics, random split, loss*
19 *development factor*

20 **1 Introduction**

21 The loss reserving problem can be briefly described as follows: let S_i denote the
 22 unknown ultimate incurred losses¹ for year of origin² i (excluding expected incomes
 23 from salvage and subrogation) and C_{ij} denote the best estimate of cumulative incurred
 24 claim amounts for year of origin (or accident year) i and development year³ j . The data
 25 used to estimate loss reserves are usually presented in the form of a “loss development
 26 triangle” as shown in Table 1. A basic assumption in loss reserving is that the data in the
 27 rows of Table 1 are mutually independent, i.e., C_{ij} and C_{rm} are independent if $i \neq r$.
 28 In other words, claims from different years of origin evolve independently. Another
 29 assumption is that all claims are settled within a certain number of calendar years, N
 30 years, say, from their date of occurrence, regardless of the year of occurrence. This
 31 means that $C_{ij} = C_{iN}$ for $j \geq N$ and $i = 1, 2, \dots$. Sometimes, however, the data in
 32 the loss development triangle consist of incremental incurred claims, c_{ij} , rather than
 33 cumulative incurred, where

$$c_{ij} = \begin{cases} C_{ij} - C_{i,j-1} & j = 1, 2, \dots \\ C_{i0} & j = 0. \end{cases}$$

34 The decision to use either incremental or cumulative values depends on the loss re-
 35 serving method used.

36 Given C_{ij} , the ultimate incurred loss for accident year i , S_i , is estimated as:

$$S_i = C_{ij} \times \text{LDF}_j \tag{1}$$

37 where LDF_j is the loss development factor for development year j to ultimate. When
 38 the total paid claims for occurrence year i at the end of development year j (TPC_{ij}) is
 39 known, the loss reserve at that point in time (LR_{ij}) is then given by

$$\text{LR}_{ij} = C_{ij} \times \text{LDF}_j - \text{TPC}_{ij}. \tag{2}$$

¹The loss reserving problem can also be described in terms of cumulative paid claims or incurred but not reported (IBNR) claims.

²The year of origin is the calendar year in which the claim actually occurred.

³The development year refers to the number of calendar years as measured from the year of origin so that $j = 0$ refers to the year of origin.

Table 1
Loss Development Triangle

Year of Origin (i)	Development Year (j)					
	0	1	2	\dots	$k-1$	k
1	C_{10}	C_{11}	C_{12}	\dots	$C_{1,k-1}$	C_{1k}
2	C_{20}	C_{21}	C_{22}	\dots	$C_{2,k-1}$	
\vdots	\vdots	\vdots	\vdots			
$k-1$	$C_{k-1,0}$	$C_{k-1,1}$				
k	$C_{k,0}$					

40 There are numerous methods for estimating loss development factors. These in-
 41 clude the chain ladder method and its many modifications, separation methods, prob-
 42 abilistic methods (such as Bühlmann, Straub and Schnieper (1980)), Bayesian methods
 43 (see, for example, de Alba (2002) and references therein), and many ad hoc methods
 44 such as the Bornhuetter-Ferguson method (Bornhuetter and Ferguson, 1972). For a de-
 45 tailed discussion of the practical issues involved in developing loss reserves, see, for
 46 example Berquist and Sherman (1977), Salzmann (1984), Wisner (1996), or Booth, et al.,
 47 (1999, Chapter 16). For an overview of many older actuarial loss reserving methods,
 48 see Van Eeghen (1981). A more modern treatment of loss reserves is given in Taylor
 49 (2000) and England and Verrall (2002).

50 The important common characteristic of the established loss reserving methods is
 51 their implicit reliance on the existence of a sufficiently long claims run-off triangle. This
 52 makes many of them unsuitable for estimating ultimate losses for new lines of busi-
 53 nesses, especially in the early years where the loss development process is immature.⁴

54 For new lines of business, the practical approach to loss reserving is as follows:

- 55 1. The actuary tries to get an understanding of the business by talking with the
 56 underwriters and claims-handlers; then
- 57 2. The actuary makes his/her best a priori guess based on this knowledge.

⁴One method that is suited for the early years is the Bornhuetter-Ferguson method.

58 The actuary's guess may be based on a simple loss ratio reserving method together
 59 with a rough conservative guess as to the development pattern (possibly based on the
 60 experience from some other "similar" business).

61 The objective of this paper is to provide reserving actuaries with a method or process
 62 to assist them with their "best guess" in the early years of development. The method
 63 introduced is different from other loss reserving methods because it explicitly assumes
 64 the evolution of the incremental incurred claims for an accident year is the result of a
 65 random split of the ultimate loss for that accident year into separate pieces of losses
 66 that are observed in each development year over the claims settlement period. The
 67 nature of the random split and the pattern of the evolution of incremental incurred
 68 claims must be specified by the reserving actuary, thus giving the method tremendous
 69 flexibility. The basic model used in this paper assumes the pieces of losses form a
 70 decreasing sequence of order statistics. As this method provides loss development
 71 factors without any knowledge of the distribution of the ultimate loss or of the actual
 72 cumulative incurred claims, it is suitable for calculating loss reserves for new lines of
 73 business, where there is little or no loss development data. This method is suitable for
 74 paid and incurred claims, and can also be used in conjunction with the Bornhuetter-
 75 Ferguson method by providing the necessary loss development (from j to ultimate)
 76 factors.

77 2 The Model

78 2.1 Assumptions

79 As is common in many models of the property/casualty loss reserving process, we
 80 assume:⁵

- 81 1) The maximum number of years it takes for incurred claims to be completely paid
 82 and settled is fixed and known to be N , i.e., a claim occurring in calendar year of
 83 origin i is settled by the end of calendar year $i + N$;
- 84 2) The incremental claim development process from different years of origin are
 85 mutually independent, i.e., c_{ij} and c_{kl} are independent if $i \neq k$; and

⁵This model can also be described in terms of paid claims.

86 3) The incremental incurred claims in each year of origin form a non-negative de-
 87 creasing sequence, i.e., $c_{ij} > c_{i,j+1}$ for $j = 0, 1, \dots, N - 1$. (The case where the
 88 incremental incurred claims for an arbitrary sequence is considered later in Sec-
 89 tion 5.2.)

90 Clearly, from the definition of N , S_i and the c_{ij} s,

$$S_i = c_{i0} + c_{i1} + \dots + c_{iN}. \quad (3)$$

91 where N is known. Equation (3) shows that S_i can be viewed as being split at random
 92 into $N + 1$ random pieces of losses $c_{i0}, c_{i1}, \dots, c_{iN}$ with the j^{th} piece of losses being
 93 revealed (i.e., made known) at the end of the j^{th} development year. On the other hand,
 94 assumption 3 implies that the sequence $c_{i0}, c_{i1}, \dots, c_{iN}$ is an ordered sequence. How-
 95 ever, it is unlikely that a purely random split will lead to an ordered sequence. Thus
 96 the precise nature of the split must be specified.

97 Suppose the total unknown incurred S_i is split at random under a uniform distribu-
 98 tion into $N + 1$ piece of losses parts $X_{i1}, X_{i2}, \dots, X_{i,N+1}$ such that

$$S_i = X_{i1} + X_{i2} + \dots + X_{i,N+1}. \quad (4)$$

99 We further assume that the pieces of losses are ordered and relabeled so that

$$X_{i(1)} \leq X_{i(2)} \leq \dots \leq X_{i(N+1)}.$$

100 By assumption 3 the incremental incurred claims are a realization of the ordered pieces
 101 of losses, i.e.,

$$c_{ij} = X_{i(N+1-j)} \quad \text{and} \quad C_{ij} = \sum_{k=0}^j X_{i(N+1-k)} \quad (5)$$

102 for $j = 0, 1, \dots, N$.

103 2.2 The Uniform Random Split

104 At this point it is important to clarify what is meant by the statement “ S_i is split at
 105 random under a uniform distribution.” Suppose we have N independent and identically

106 distributed random variables, U_1, U_2, \dots, U_N , which are uniformly distributed on $(0, 1)$.
 107 The $U_{(j)}$ s are ordered and relabeled as

$$0 < U_{(1)} \leq U_{(2)} \leq \dots \leq U_{(N)} < 1 \quad (6)$$

108 and the end points are the defined as $U_{(0)} = 0$ and $U_{(N+1)} = 1$. Next we define the
 109 spacings⁶ between the consecutive ordered U_j s as

$$Y_j = U_{(j)} - U_{(j-1)} \quad (7)$$

110 for $j = 1, 2, \dots, N + 1$. Then a random split of S_i into $N + 1$ pieces of losses $X_{i1}, X_{i2}, \dots,$
 111 $X_{i,N+1}$ means

$$X_{ij} = S_i \times Y_j \quad \text{for } j = 1, 2, \dots, N + 1. \quad (8)$$

112 Ordering the Y_j s as $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(N+1)}$ and an application of assumption 3
 113 immediately yields

$$c_{ij} = S_i Y_{(N+1-j)}, \quad \text{and} \quad (9)$$

$$C_{ij} = S_i \sum_{k=0}^j Y_{(N+1-k)} \quad (10)$$

114 for $j = 0, 1, \dots, N$. However, as the the cumulative incurred tends to be more stable
 115 than the incremental incurred, equation (9) is not used to estimate S_i . Instead, we use

$$S_i = \frac{C_{ij}}{\sum_{k=0}^j Y_{(N+1-k)}}. \quad (11)$$

116 3 Estimating Ultimate Loss

117 Recall that the $Y_{(j)}$ s are never known until S_i is known, hence they must be estimated.
 118 The obvious estimator of $Y_{(j)}$ is its mean, which leads to the first estimate of the total
 119 incurred for year of origin i given the incurred losses up to development year j :

⁶For comprehensive treatment of the distribution of the spacings between successive ordered random variables, see Pyke (1965).

$$\hat{S}_i^{(1)} = \frac{C_{ij}}{\sum_{r=0}^j \mathbf{E}[Y_{(N+1-r)}]} \quad (12)$$

i.e., the loss development factor from j to ultimate is

$$\text{LDF}_{j,N}^{(1)} = \frac{1}{\sum_{r=0}^j \mathbf{E}[Y_{(N+1-r)}]} \quad (13)$$

120 for $= 0, 1, \dots, N + 1$. Alternatively, we may use

$$\hat{S}_i^{(2)} = C_{ij} \mathbf{E} \left[\frac{1}{\sum_{r=0}^j Y_{(N+1-r)}} \right] \quad (14)$$

which yields the alternative loss development factor from j to ultimate

$$\text{LDF}_{j,N}^{(2)} = \mathbf{E} \left[\frac{1}{\sum_{r=0}^j Y_{(N+1-r)}} \right]. \quad (15)$$

121 From Jensen's inequality, $\text{LDF}_{j,N}^{(2)} \geq \text{LDF}_{j,N}^{(1)}$ for every j . Before calculating the values of
 122 $\text{LDF}_{j,N}^{(1)}$ and $\text{LDF}_{j,N}^{(2)}$ for various values of j and N , the distribution of the $Y_{(j)}$ s will be
 123 provided.

124 From the theory of the random division of an interval of unit length (for example,
 125 David (1981), chapter 5.4) or Feller (1970, chapter 1)), the variables Y_1, Y_2, \dots, Y_{N+1} form
 126 an exchangeable sequence of dependent random variables with joint pdf:

$$f(y_1, y_2, \dots, y_{N+1}) = \begin{cases} (N+1)! & \text{if } y_j \geq 0 \text{ and } \sum_{k=1}^{N+1} y_k = 1. \\ 0 & \text{otherwise.} \end{cases}$$

127 The marginal distribution of $Y_{(j)}$ is given by Maurer and Margolin (1976, equation (4.4))
 128 as

$$\Pr \left[Y_{(j)} > y \right] = \sum_{m=N+2-j}^{N+1} (-1)^{m+j-N-2} \binom{m-1}{N+1-j} \binom{N+1}{m} (1-my)^N I \left\{ y < \frac{1}{m} \right\}$$

129 where $I\{A\}$ is an indicator of the occurrence of the event A . In addition, the moments
 130 of $Y_{(j)}$ satisfy the recursion

$$(n - j)\mu_{j:n}^{(k)} + j\mu_{j+1:n}^{(k)} = n\mu_{j:n-1}^{(k)} \quad (16)$$

131 where $\mu_{j:n}^{(k)} = E[Y_{(j)}^k]$ for sample size n .

132 Due to the difficulties in deriving the inverse moments needed in $\text{LDF}_{j,N}^{(2)}$, however,
 133 Monte Carlo simulations⁷ are used to determine both sets of loss development factors.
 134 Table A1 in the Appendix shows the loss development factors from j to ultimate $\text{LDF}_{j,N}^{(1)}$
 135 and $\text{LDF}_{j,N}^{(2)}$, respectively.

136 4 Numerical Examples

137 Example 1: A Simple Data Set

138 Suppose a new product was introduced on January 1, 2000 and the loss development
 139 data as of December 31, 2002 are given in Table 2.

Table 2
Hypothetical Cumulative Incurred and Paid Claims (in 000's)

Year of Origin	Earned Premiums (in 000's)	Development Year (j)					
		Incurred Claims (C_{ij})			Paid Claims		
		0	1	2	0	1	2
2000	4,500	1,447	1,976	2,454	401	1,166	1,761
2001	8,500	3,578	3,911		906	2,573	
2002	16,000	4,754			1,719		

140 The total paid claims to date is thus $1,719 + 2,573 + 1,761 = 6,053$. To estimate the
 141 ultimate losses, we must first specify N . If we assume $N = 3$, then the total estimated
 142 ultimate claims for 2000–2002, as of December 31, 2002, is:

$$\hat{S}^{(1)} = 4,754 \times 1.9195 + 3,911 \times 1.2627 + 2454 \times 1.0662 = 16,680.18$$

$$\hat{S}^{(2)} = 4,754 \times 2.0379 + 3,911 \times 1.2826 + 2454 \times 1.0691 = 17,328.00$$

⁷The uniform (0, 1) random number generator **ran** in Press, et al., (1996, chapter B7, page 1142) is used to perform all simulations.

143 using the $N = 3$ rows of Table A1. The corresponding reserve estimates are $10,627.18 =$
 144 $16,680.18 - 6,053$ and $11,275.00 = 17,328.00 - 6,053$, respectively.

145 If, on the other hand, we assume $N = 5$, then the estimated ultimate claims for
 146 2000–2002, as of December 31, 2002, is:

$$\hat{S}^{(1)} = 4,754 \times 2.4564 + 3,911 \times 1.5412 + 2454 \times 1.2384 = 20,744.39$$

$$\hat{S}^{(2)} = 4,754 \times 2.6221 + 3,911 \times 1.5800 + 2454 \times 1.2505 = 21,713.57$$

147 using the $N = 5$ rows of Table A1. These lead to reserve estimates of $14,691.39 =$
 148 $20,744.39 - 6,053$ and $15,660.57 = 21,713.57 - 6,053$, respectively.

149 **Example 2: A Modified Bornhuetter-Ferguson Method**

150 This method can be used to provide the loss development to ultimate factors needed
 151 in applications of the Bornhuetter-Ferguson (B-F) method. For example, using the data
 152 in Table 3, the IBNR reserves are estimated using the traditional B-F method and a
 153 modified B-F method based on the $LDF_{j,N}^{(1)}$ given in Table A1. In deriving their estimates,
 154 Bornhuetter and Ferguson (1972) assume claims in the three most recent calendar years
 155 are settled in 3 years. Table 4 shows the annual loss development factors. Table 5
 156 provides a summary of the results.

Table 3

Hypothetical Premium and Loss Development Data (in 000's)

Year of Origin	Earned Premium	Development Year (j)					
		0	1	2	3	4	5
1997	5,000	2,500	3,650	4,200	4,325	4,335	4,330
1998	5,500	2,150	3,225	3,775	3,965	3,960	
1999	6,000	3,250	4,500	5,050	5,150		
2000	7,000	3,700	5,200	5,775			
2001	7,500	3,300	4,800				
2002	8,000	4,250					

157 *Source:* Based on the data in Bornhuetter and Ferguson, 1972, page 193, Exhibit A., with “Year of Origin” changed from 1966–1971 to 1997–2002.

Table 4
Annual Loss Development Factors for Table 3

Year of Origin	Earned Premium	Development Year (<i>j</i>)				
		0	1	2	3	4
1997	5,000	1.460	1.151	1.030	1.002	0.999
1998	5,500	1.500	1.171	1.050	0.999	
1999	6,000	1.385	1.122	1.102		
2000	7,000	1.405	1.111			
2001	7,500	1.455				
2002	8,000					

Table 5
Hypothetical IBNR Reserve Computation as of December 31, 2002

Year of Origin	(1)	LDFs		IBNR Factor		Indicated IBNR	
	Expected	(2)	(3)	(4)	(5)	(6)	(7)
	Losses	$LDF_j^{(BF)}$	$LDF_{j,3}^{(1)}$	$1 - 1/LDF_j^{(BF)}$	$1 - 1/LDF_{j,3}^{(1)}$	B-F	Mod. B-F
1999	5,700	1.000	1.0000	0.000	0.0000	-0-	-0-
2000	6,650	1.032	1.0662	0.031	0.0621	206	413
2001	7,125	1.166	1.2627	0.142	0.2080	1,012	1,482
2002	7,600	1.650	1.9195	0.394	0.4790	2,994	3,640
						4,212	5,535

Notes: Expected Losses is 95% of the earned premium. The information in Columns (2), (4) and (6) are provided by Bornhuetter and Ferguson, 1972, page 194, Exhibit B. The information in Columns (3) and (5) are derived from Table A1 with $N = 3$. Column (7) = Column (1) \times Column (5).

160 5 Generalizations and Practical Considerations

161 The loss reserving method introduced above is flexible and can be generalized in
 162 at least two directions: (i) by considering a non-uniform random split, and/or (ii) by
 163 considering an arbitrary ordered sequence (i.e., any alternative to the strictly decreasing
 164 ordered sequence) of random spacings to reflect the evolution of the incremental
 165 incurred loss.

166 5.1 A Non-Uniform Random Split

167 One can observe in Tables A1 and A2 that under the uniform random split $C_{i,0}$ is a
 168 relatively small percentage of S_i , then there is a fairly rapid development of incurred
 169 claims. For example, in Table A1, $1/\text{LDF}_{j,N}^{(1)}$ and $1/\text{LDF}_{j,N}^{(2)}$ are small for $j = 0, 1$ or 2 ,
 170 while Table A2 shows that the loss development factors for years 1 and 2 are high
 171 suggesting the rapid development of incurred claims.

172 If the actuary has loss development factors that are not similar to the quantities in
 173 Tables A1 and A2, another distribution defined on $(0, 1)$ must be used to form the basis
 174 of the split. Unfortunately, there is no obvious alternative distribution, especially one
 175 that is intuitively appealing. It is up to the reserving actuary to specify a continuous
 176 distribution with support on $(0, 1)$. Some alternative distributions with support on
 177 $(0, 1)$ include the beta, the truncated gamma, and the truncated Pareto distributions.
 178 One strategy that can be used is to have on hand tables similar to Tables A1 and A2
 179 for each potential alternative random split distribution. The actuary can then use the
 180 table (i.e., distribution) that best matches the observed loss development factors.

181 Suppose the actuary chooses a specific cumulative distribution function $F_U(u)$ with
 182 continuous support on $(0, 1)$. As the length of the claims settlement period is $N +$
 183 1 years, we sample N independent and identically distributed random variables, $U_1,$
 184 \dots, U_j, \dots, U_N from $F_U(u)$.⁸ The sampled U_j s are then ordered and relabeled as before.
 185 The resulting spacings, $Y_j = U_{(j)} - U_{(j-1)}$, with $U_{(0)} = 0$ and $U_{(N+1)} = 1$, are then
 186 used to define the random split. As before, simulations are then used to determine
 187 the expectations needed to determine the loss development factors. As an example,

⁸For more on techniques for generating random variables from continuous distributions see, for example, Bartley, Fox, and Schrage (1983, Chapters 5 and 6); Kalos and Whitlock (1986, Chapter 3); and Fishman (1996, Chapter 3) and Ross (2002, Chapters 3 to 5).

188 Tables A3 and A4 provide the loss development factors to ultimate and the annual loss
 189 development factors in the case of the truncated exponential pdf of U_j defined by

$$f_U(u) = \frac{\lambda e^{-\lambda u}}{1 - e^{-\lambda}}, \quad (17)$$

190 for $0 \leq u \leq 1$ and $\lambda > 0$.

191 5.2 An Arbitrary Ordered Sequence

192 Recall equation (9) in which we defined the sequence of incremental incurred claims
 193 as $c_{i0} \geq c_{i1} \geq \dots \geq c_{iN}$. If the actuary believes, however, that the pattern of incremen-
 194 tal incurred claims is different, then tables of loss development factors to ultimate and
 195 annual loss development factors that are consistent with the specified pattern must be
 196 derived.

197 To be precise, for $j = 0, 1, \dots, N$ let θ_j denote the order of the set of order statistic
 198 $Y_{(1)}, Y_{(2)}, \dots, Y_{(N+1)}$ that is used to define c_{ij} . Note that $\theta_0, \theta_1, \dots, \theta_N$ is a permutation of
 199 the elements of the set $\{1, 2, \dots, N+1\}$. (For example, equation (9) implies $\theta_j = N+1-j$.
 200 As another example, the actuary may specifically believe that $\theta_0 = N-1, \theta_1 = N, \theta_2 =$
 201 $N+1, \theta_j = N+1-j$ for $j = 3, \dots, N$, which implies $c_{i0} \leq c_{i1} \leq c_{i2} \geq c_{i3} \geq \dots \geq c_{iN}$.)
 202 It follows that c_{ij} and C_{ij} are defined as

$$c_{ij} = S_i Y_{(\theta_j)}, \quad \text{and} \quad (18)$$

$$S_i = \frac{C_{ij}}{\sum_{k=0}^j Y_{(\theta_k)}}. \quad (19)$$

203 for $j = 0, 1, \dots, N$.

204 In general, the loss development factors can be obtained via a simulation of sample
 205 size M as follows:

206 **Step 1:** For given settlement period of $N+1$ years, set $\text{TEMP}_{j,N}^{(1)} = 0$ and $\text{TEMP}_{j,N}^{(2)} = 0$
 207 for $j = 0, 1, 2, \dots, N$.

208 **Step 2:** Create an $(N+1)$ dimensional permutation vector $\theta = (\theta_0, \theta_1, \dots, \theta_N)$ con-
 209 taining the actuary's specified pattern of incremental incurred losses.

210 **Step 3:** Generate N random variables $U_1, \dots, U_j, \dots, U_N$ from the actuary's specified
 211 random splitting distribution, $F_U(u)$.

212 **Step 4:** Order the sampled U_j s as $U_{(1)} \leq U_{(2)} \dots \leq U_{(N)}$.

213 **Step 5:** For $j = 1, 2, \dots, N + 1$, define $Y_j = U_{(j)} - U_{(j-1)}$, with $U_{(0)} = 0$ and $U_{(N+1)} = 1$.

214 **Step 6:** Order the $N + 1$ Y_j s as $Y_{(1)} \leq Y_{(2)} \dots \leq Y_{(N+1)}$.

Step 7: For $j = 0, 1, 2, \dots, N$, set

$$\begin{aligned} \text{TEMP}_{j,N}^{(1)} &= \text{TEMP}_{j,N}^{(1)} + \sum_{r=0}^j \mathbf{E}[Y_{(\theta_r)}] \\ \text{TEMP}_{j,N}^{(2)} &= \text{TEMP}_{j,N}^{(2)} + \frac{1}{\sum_{r=0}^j Y_{(\theta_r)}}. \end{aligned}$$

215 **Step 8:** Repeat Steps 3 to 7 a total of M times.

216 **Step 9:** For $j = 0, 1, 2, \dots, N$, the loss development factors are estimated as:

$$\widehat{\text{LDF}}_{j,N}^{(1)} = \frac{M}{\text{TEMP}_{j,N}^{(1)}} \quad (20)$$

$$\widehat{\text{LDF}}_{j,N}^{(2)} = \frac{\text{TEMP}_{j,N}^{(2)}}{M}. \quad (21)$$

217 5.3 Other Practical Considerations

218 In practice, other potential problems may occur such as different years of origin
 219 having different claim settlement periods, i.e., N depends on i , and the existence of
 220 negative incremental incurred claims. For example, Tables 6 and 7 display two hypo-
 221 theoretical data sets with several problems evident. In Table 6, one can assume that claims
 222 are settled in three years, i.e., $N = 3$. However, the claims do not exhibit the pattern
 223 assumed by Tables A1 and A2. In fact, even though the claims are settled in 3 years, the
 224 total incurred claim changes only slightly after development year 1, and for year 2001
 225 the cumulative incurred is decreasing. In this case our method is not ideally suited to
 226 the data in Table 6.

227 Table 7 presents similar challenges as some incremental incurred claims are zero
 228 and some are negative. The sequence of incurred losses generated in 1996 and 1997
 229 appear to have a pattern distinct from those in subsequent years. In addition, it may
 230 be incorrect to assume the years of origin all have the same settlement period, i.e., N

Table 6
Second Hypothetical Cumulative Incurred Loss Data Set (in 000's)

Year of Origin	Development Year (j)						
	0	1	2	3	4	5	6
1997	2,237	2,369	2,376	2,376	2,376	2,376	
1998	2,899	2,942	2,936	2,934	2,934		
1999	2,225	2,330	2,322	2,325			
2000	2,145	2,205	2,207				
2001	1,513	1,499					
2002	1,168						

231 depends on i . In such cases where there are non-positive incremental incurred claims
 232 and/or different claim settlement periods, the ultimate incurred losses for accident
 233 year i , S_i , can still be estimated as

$$S_i = \frac{C_{ij}}{\sum_{k=0}^j Y_{(N_i+1-k)}}. \quad (22)$$

234 Note, the length of the claim settlement period can be approximated by observing
 235 cumulative loss development factors. If the assumptions of this model (as stated in
 236 Section 2) hold then the j^{th} annual cumulative loss development factor for year of
 237 origin i , $C_{i,j+1}/C_{i,j}$, should satisfy

$$\frac{C_{i,j+1}}{C_{i,j}} \approx \frac{\text{LDF}_{j,N}^{(k)}}{\text{LDF}_{j+1,N}^{(k)}}. \quad (23)$$

238 for $k = 1, 2$ and $j = 0, 1, \dots$. Table A2 shows the values $\text{LDF}_{j,N}^{(k)}/\text{LDF}_{j+1,N}^{(k)}$ for $k = 1, 2$,
 239 $j = 0, 1, \dots, 6$ and $N = 1, 2, \dots, 9$. The annual cumulative loss development factors
 240 should then be compared with those in Table A2. Table 8 shows the actual cumulative
 241 loss development factors generated by Table 7. Comparing the first two columns of
 242 Table 8 with those expected in Table A2 show that patterns of actual cumulative loss
 243 development factors for years 1997 to 2001 are too low, making the data in Table 7
 244 not consistent with the assumption of a uniform random split. Notice that the results

Table 7
Third Hypothetical Cumulative Incurred Loss Data (in \$000's)

Year of Origin	Development Year (j)						
	0	1	2	3	4	5	6
1996	1,076	927	927	951	960	1,087	1,087
1997	957	1,193	1,312	1,295	1,220	1,392	
1998	1,421	1,788	2,086	2,236	2,252		
1999	1,473	1,910	2,235	2,192			
2000	1,447	1,976	2,454				
2001	3,578	3,911					
2002	4,754						

245 of Tables A3 and A4 for $\lambda = 5$ seem to better fit the data from the later years in Tables
 246 7 and 8 than the uniform random split.

247 **6 Summary and Closing Comments**

248 For new lines of business, the practical approach to loss reserving requires the ac-
 249 tuary to make his/her best guess of the reserve level based on prior knowledge. The
 250 actuary's guess may be based on a simple loss ratio reserving method together with
 251 a rough conservative guess as to the development pattern (possibly based on the ex-
 252 perience from some other "similar" business). This paper provides reserving actuaries
 253 with a tool to assist them with their "best guess" of the reserves, especially in the early
 254 years of development. The method essentially uses an a priori pattern in the table of
 255 expected loss development factors to determine the claims reserves. The pattern of
 256 expected loss development factors is independent of the distribution of the cumula-
 257 tive incurred claims in the accident year and can be varied depending on the actuary's
 258 estimate of the length of the claim settlement period, and the random split used.

259 When there is a sufficiently large amount of data in the loss development triangle,
 260 the actuary can use the method of this paper to generate tables of expected loss devel-

Table 8
Loss Development Factors ($C_{i,j+1}/C_{i,j}$) from Table 7

Year of Origin	Development Year ($j/j + 1$)					
	0/1	1/2	2/3	3/4	4/5	5/6
1996	0.8615	1.0000	1.0259	1.0095	1.1323	1.0000
1997	1.2466	1.0997	0.9870	0.9421	1.1410	
1998	1.2583	1.1667	1.0719	1.0072		
1999	1.2967	1.1702	0.9808			
2000	1.3656	1.2419				
2001	1.0931					
2002						

261 opment factors to see which ones match the observed loss development factors. The
 262 best matched tables can be used to estimate the loss reserves.

263 In closing, there are several important attributes of this method:

- 264 1. It can be used for new and old business;
- 265 2. It can be used in conjunction with other methods such as the Bornhuetter-Ferguson
 266 method;
- 267 3. It makes no assumptions about the underlying distribution of the ultimate losses;
- 268 4. The ultimate losses are estimated using only the most recent cumulative loss data;
- 269 5. The method can be used if the length of the settlement period varies by year of
 270 origin;
- 271 6. The factors $LDF_{j,N}^{(2)}$, $LDF_{j,N}^{(2)}$ and their ratios $LDF_{j,N}^{(1)}/LDF_{j+1,N}^{(1)}$ and $LDF_{j,N}^{(1)}/LDF_{j+1,N}^{(1)}$
 272 do not depend on the actual claims development pattern; and
- 273 7. Tables of factors and ratios can be created and saved for each combination of
 274 development year j and settlement period N , and for various types of random
 275 splits such as the uniform and beta distributions. The appropriate table can be
 276 chosen to:

- 277 (a) Match the observed pattern of loss development factors; or to
278 (b) Match the actuary's or underwriter's best guess of what the pattern of loss
279 development factors should be.

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Table A1
Loss Development Factor from j to Ultimate
For Various Development Years and Settlement Periods(N)

N	Development Year j									
	0	1	2	3	4	5	6	7	8	9
	Results for $LDF_{j,N}^{(1)}$:									
0	1.0000									
1	1.3345	1.0000								
2	1.6353	1.1253	1.0000							
3	1.9195	1.2627	1.0662	1.0000						
4	2.1903	1.4015	1.1495	1.0416	1.0000					
5	2.4564	1.5412	1.2384	1.0980	1.0288	1.0000				
6	2.6959	1.6715	1.3259	1.1590	1.0689	1.0208	1.0000			
7	2.9402	1.8007	1.4137	1.2232	1.1152	1.0516	1.0159	1.0000		
8	3.1853	1.9333	1.5038	1.2894	1.1650	1.0881	1.0403	1.0126	1.0000	
9	3.4169	2.0584	1.5905	1.3542	1.2153	1.1270	1.0691	1.0321	1.0101	1.0000
	Results for $LDF_{j,N}^{(2)}$:									
0	1.0000									
1	1.3871	1.0000								
2	1.7247	1.1347	1.0000							
3	2.0379	1.2826	1.0691	1.0000						
4	2.3333	1.4312	1.1569	1.0428	1.0000					
5	2.6221	1.5800	1.2505	1.1015	1.0294	1.0000				
6	2.8804	1.7182	1.3422	1.1649	1.0707	1.0211	1.0000			
7	3.1417	1.8547	1.4343	1.2316	1.1185	1.0526	1.0161	1.0000		
8	3.4033	1.9938	1.5281	1.3003	1.1698	1.0901	1.0409	1.0127	1.0000	
9	3.6511	2.1259	1.6188	1.3676	1.2219	1.1301	1.0704	1.0325	1.0102	1.0000

Notes: Development year 0 refers to the year in which the claim was incurred. N is the number of calendar years it takes to settle all claims occurring in the same calendar year. Thus $N = 0$ implies claims are settled in the calendar year of their occurrence.

Table A2
The Ratio Loss Development Factors For Various Development Years and Settlement Periods

N	Development Year $j/j + 1$									
	0/1	1/2	2/3	3/4	4/5	5/6	6/7	7/8	8/8	9/10
	Results for $LDF_{j,N}^{(1)}/LDF_{j+1,N}^{(1)}$:									
1	1.3345									
2	1.4531	1.1253								
3	1.5201	1.1843	1.0662							
4	1.5628	1.2192	1.1036	1.0416						
5	1.5938	1.2445	1.1279	1.0673	1.0288					
6	1.6129	1.2606	1.1440	1.0843	1.0472	1.0208				
7	1.6328	1.2737	1.1558	1.0968	1.0605	1.0351	1.0159			
8	1.6476	1.2856	1.1663	1.1068	1.0707	1.0460	1.0273	1.0126		
9	1.6600	1.2941	1.1745	1.1143	1.0784	1.0541	1.0359	1.0218	1.0101	
	Results for $LDF_{j,N}^{(2)}/LDF_{j+1,N}^{(2)}$:									
1	1.3871									
2	1.5200	1.1347								
3	1.5889	1.1997	1.0691							
4	1.6303	1.2371	1.1094	1.0428						
5	1.6595	1.2635	1.1353	1.0700	1.0294					
6	1.6764	1.2801	1.1522	1.0880	1.0486	1.0211				
7	1.6939	1.2931	1.1646	1.1011	1.0626	1.0359	1.0161			
8	1.7070	1.3047	1.1753	1.1115	1.0732	1.0472	1.0278	1.0127		
9	1.7175	1.3133	1.1836	1.1193	1.0813	1.0557	1.0367	1.0221	1.0102	

Notes: Development year 0 refers to the year in which the claim was incurred. N is the number of calendar years it takes to settle all claims occurring in the same calendar year. Thus $N = 0$ implies claims are settled in the calendar year of their occurrence.

Table A3
LDF_{j,N}⁽¹⁾ for the Truncated Exponential Distribution with Parameter λ

N	Development Year j									
	0	1	2	3	4	5	6	7	8	9
	$\lambda = 1:$									
0	1.0000									
1	1.3261	1.0000								
2	1.6130	1.1209	1.0000							
3	1.8811	1.2514	1.0634	1.0000						
4	2.1304	1.3823	1.1424	1.0396	1.0000					
5	2.3776	1.5139	1.2264	1.0931	1.0273	1.0000				
6	2.6049	1.6376	1.3101	1.1513	1.0655	1.0197	1.0000			
7	2.8284	1.7576	1.3920	1.2117	1.1093	1.0487	1.0150	1.0000		
8	3.0482	1.8786	1.4756	1.2737	1.1562	1.0834	1.0381	1.0119	1.0000	
9	3.2585	1.9959	1.5570	1.3349	1.2036	1.1200	1.0654	1.0303	1.0095	1.0000
	$\lambda = 5:$									
0	1.0000									
1	1.2041	1.0000								
2	1.3468	1.0672	1.0000							
3	1.4597	1.1323	1.0331	1.0000						
4	1.5554	1.1912	1.0716	1.0200	1.0000					
5	1.6469	1.2461	1.1103	1.0456	1.0133	1.0000				
6	1.7291	1.2954	1.1467	1.0725	1.0315	1.0094	1.0000			
7	1.7950	1.3388	1.1802	1.0989	1.0515	1.0231	1.0070	1.0000		
8	1.8582	1.3814	1.2124	1.1244	1.0717	1.0385	1.0176	1.0054	1.0000	
9	1.9291	1.4239	1.2447	1.1508	1.0932	1.0555	1.0304	1.0141	1.0044	1.0000
	$\lambda = 10:$									
0	1.0000									
1	1.1094	1.0000								
2	1.1747	1.0346	1.0000							
3	1.2211	1.0656	1.0169	1.0000						
4	1.2589	1.0923	1.0357	1.0100	1.0000					
5	1.2895	1.1155	1.0536	1.0226	1.0066	1.0000				
6	1.3190	1.1360	1.0702	1.0355	1.0157	1.0047	1.0000			
7	1.3454	1.1551	1.0861	1.0486	1.0258	1.0117	1.0036	1.0000		
8	1.3649	1.1711	1.0997	1.0603	1.0355	1.0193	1.0089	1.0028	1.0000	
9	1.3904	1.1879	1.1136	1.0722	1.0456	1.0275	1.0152	1.0071	1.0022	1.0000

Table A4
LDF_{j,N}⁽²⁾ for the Truncated Exponential Distribution with Parameter λ

N	Development Year j									
	0	1	2	3	4	5	6	7	8	9
	$\lambda = 1:$									
0	1.0000									
1	1.3791	1.0000								
2	1.7023	1.1300	1.0000							
3	2.0004	1.2710	1.0662	1.0000						
4	2.2764	1.4115	1.1494	1.0408	1.0000					
5	2.5492	1.5530	1.2382	1.0964	1.0279	1.0000				
6	2.7976	1.6850	1.3263	1.1571	1.0673	1.0200	1.0000			
7	3.0423	1.8129	1.4124	1.2199	1.1124	1.0498	1.0152	1.0000		
8	3.2778	1.9405	1.5000	1.2843	1.1609	1.0853	1.0387	1.0120	1.0000	
9	3.5078	2.0654	1.5854	1.3480	1.2099	1.1229	1.0666	1.0307	1.0096	1.0000
	$\lambda = 5:$									
0	1.0000									
1	1.2424	1.0000								
2	1.4106	1.0717	1.0000							
3	1.5424	1.1419	1.0343	1.0000						
4	1.6542	1.2056	1.0744	1.0205	1.0000					
5	1.7630	1.2651	1.1150	1.0468	1.0135	1.0000				
6	1.8570	1.3179	1.1531	1.0745	1.0320	1.0095	1.0000			
7	1.9343	1.3650	1.1882	1.1018	1.0525	1.0234	1.0071	1.0000		
8	2.0093	1.4113	1.2220	1.1282	1.0732	1.0390	1.0177	1.0054	1.0000	
9	2.0895	1.4569	1.2559	1.1555	1.0953	1.0564	1.0307	1.0142	1.0044	1.0000
	$\lambda = 10:$									
0	1.0000									
1	1.1246	1.0000								
2	1.1978	1.0360	1.0000							
3	1.2486	1.0682	1.0172	1.0000						
4	1.2896	1.0959	1.0364	1.0101	1.0000					
5	1.3235	1.1199	1.0547	1.0229	1.0067	1.0000				
6	1.3546	1.1409	1.0716	1.0360	1.0158	1.0048	1.0000			
7	1.3847	1.1608	1.0879	1.0493	1.0260	1.0118	1.0036	1.0000		
8	1.4054	1.1773	1.1018	1.0612	1.0359	1.0194	1.0090	1.0028	1.0000	
9	1.4342	1.1947	1.1160	1.0733	1.0460	1.0277	1.0153	1.0071	1.0022	1.0000