Mixture Distribution and Its Applications on P&C Insurance Data

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Agenda

- Introduction
- Mixture Distribution
- Finite Mixture Model
- Case Study
- Conclusions
- Q&A
Introduction

Skewed Insurance Data

- Skewed and asymmetric
- Heavy tails
- Mixed: typical and extreme
  - Investment return: normal and crisis
  - Claim amount: typical and large losses
Introduction

HO by-peril example: heavier tail than lognormal
Introduction

HO by-peril example: multiple peaks
Introduction

HO by-peril example: multiple peaks
Introduction

Investment example in DFA

- Assuming normal distribution, the likelihood of monthly loss over 14.1% (largest monthly drop in Deep Recession) is 0.02%; actual observation is 0.55%. 

Dow Jones Monthly Returns 1951-2011
Mixture Distribution

- Single distribution does not fit insurance data well
- A combination of multiple distributions can represent data better
- Mixture distributions:

\[ f(x, \pi_1, \pi_2, ... \pi_n, \beta_1, \beta_2, ... \beta_n) = \sum_{i}^{n} \pi_i \cdot f_i(x, \beta_i) \]

where \[ \sum_{i}^{n} \pi_i = 1 \]
Mixture Distribution

Typical mixture distributions in insurance

- Claims count: Zero + Poisson
- Claim amount: gamma + lognormal or gamma + Pareto

<table>
<thead>
<tr>
<th>Peril</th>
<th>$\pi$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fire</td>
<td>0.785</td>
<td>0.51</td>
<td>10500</td>
<td>11.5</td>
<td>0.83</td>
</tr>
<tr>
<td>Hail</td>
<td>0.148</td>
<td>1.19</td>
<td>520</td>
<td>8.8</td>
<td>0.61</td>
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Mixture Distribution

- Regime-Switching Models of Equity Returns;
- Two lognormal distributions with low and high volatilities;
- Two regimes may switch by a matrix of transition probabilities;

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<thead>
<tr>
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<th>Low Volatility</th>
<th>High Volatility</th>
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<tbody>
<tr>
<td>Mean</td>
<td>0.96%</td>
<td>-2.20%</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>3.59%</td>
<td>7.17%</td>
</tr>
<tr>
<td>Probability of Switching</td>
<td>3.37%</td>
<td>30.87%</td>
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The likelihood of penetrating -14.1% by regime-switching model is 0.41%.
Finite Mixture Model

\[ f(y \mid X; \pi_1, \pi_2, \ldots, \pi_n, \theta_1, \theta_2, \ldots, \theta_n) = \sum_{i} \pi_i \cdot f_i(X, \theta_i) \]

where \( \sum_{i} \pi_i = 1 \)

- \( y \): response variable; \( X \): explanatory variables
- A finite mixture model can be thought as a mixture of multiple GLMs
  - \( f_1(y \mid X; \theta_1) \) is a GLM for smaller fire loss assuming gamma
  - \( f_2(y \mid X; \theta_2) \) is a GLM for large fire loss assuming lognormal
- Often named as latent class model in economics
Finite Mixture Model

- Improvements on GLM
  - Expand distribution assumptions: Single exponential-family distribution vs. mixture
  - Expand model structure: Single regression formula vs. multiple models
  - Better fits on insurance data with heavy-tails, multimodal, excessive zeros, and other complex error distributions

<table>
<thead>
<tr>
<th>AOI Group</th>
<th>5% Deductible Factors for Hail</th>
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<tr>
<td></td>
<td>GLM gamma</td>
</tr>
<tr>
<td>2</td>
<td>0.359</td>
</tr>
<tr>
<td>18</td>
<td>0.187</td>
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</table>
Finite Mixture Model

**Numerical Solution**

- Solving maximum likelihood function
  
  $\text{Max}_{\pi, \theta} \sum_{j=1}^{N} \log(\sum_{i=1}^{n} \pi_i f_i(y_j | X_j ; \theta_i))$

  with constraint $\pi_i > 0$ and $\sum_{i}^{n} \pi_i = 1$

- EM (Expectation-Maximization) Algorithm
- Quasi-Newton Method
- Bayesian MCMC
Case Study: Data Description

- Simulated Hurricane Model Output
- 8,500 of 10,000 years with hurricane losses.
- Mean Aggregate Severity = $57,000,000
- Standard Deviation = $136,000,000
- Skewness = 6.5

- Positive skewness suggests an asymmetric distribution
  - Lognormal
  - Gamma
Case Study: Simple Distributions Fit Poorly

- Lognormal: Determine Parameters
  - Maximum Likelihood Estimation (MLE)
  - Method of Moments (MOM)
- Intuitive Test: MLE and MOM parameter estimates differ implying Lognormal is not a good fit.
- Chi-Square Test:
  - Critical Value at 95% = 11.1
  - Test Statistic Value = 419.0
  - Since 419.0 > 11.1 we reject the null hypothesis that the data were drawn from a Lognormal distribution with the fitted parameters.
Case Study: Simple Distributions Fit Poorly

Lognormal MLE

- Mean of log(loss) is 16.03 and Standard deviation is 2.50
  - Implied Mean = $ 207,000,000
  - Implied Stdev = $4,681,000,000
  - Max observed value = $3,053,000,000

- Excess small losses (81 losses <= $3000) make the error from model misspecification extreme.
  - Lognormal assumes log(loss) are symmetric
  - Log($3000)=8.01. The symmetric point on the other side of mean is 24.05, or $27,800,000,000
  - The losses are positively skewed with a heavy right tail; log(loss) is negatively skewed with heavy left tail. Lognormal cannot address this specific shape of distribution.
Case Study: Simple Distributions Fit Poorly

- Gamma: Determine Parameters
  - MLE fit
  - MOM fit
- Intuitive Test: MLE and MOM parameter estimates differ implying Gamma is not a good fit.
- Chi-Square Test:
  - Critical Value at 95% = 11.1
  - Test Statistic Value = 683.3
  - Since 683.3 > 11.1 we reject the null hypothesis that the data were drawn from a Gamma distribution with the fitted parameters.
Case Study: Mixed Distributions Fit Better

- Mixed Gamma-Lognormal: Determine Parameters
  - Density:
    
    \[ f(x, \alpha_1, \beta_1, \pi_1, \mu_2, \sigma_2) = \pi_1 \cdot f_1(x, \alpha_1, \beta_1) + (1 - \pi_1) \cdot f_2(x, \mu_2, \sigma_2) \]

  - Likelihood:
    
    \[ L(\alpha_1, \beta_1, \pi_1, \mu_2, \sigma_2) = \prod_{i=1}^{8500} f(x_i, \alpha_1, \beta_1, \pi_1, \mu_2, \sigma_2) \]

  - Log-Likelihood:
    
    \[ l(\alpha_1, \beta_1, \pi_1, \mu_2, \sigma_2) = \sum_{i=1}^{8500} \ln(f(x_i, \alpha_1, \beta_1, \pi_1, \mu_2, \sigma_2)) \]
Case Study: Mixed Distributions Fit Better

- Mixed Gamma-Lognormal: MLE Parameters
  \[ \alpha_1 = .446, \beta_1 = 57.9M \]
  \[ \pi_1 = 0.884 \]
  \[ \mu_2 = 19.221, \sigma_2 = 0.789 \]

- Intuition: Aggregate Severity is drawn from:
  - 88.4% of time Gamma (Mean=26M, Stdev=39M)
  - 11.6% of time Lognormal (Mean=304M, Stdev=282M)

- Match to 1st two moments:
  - Mean of mixture matches data within 0.2%.
  - Standard deviation of mixture matches data within -0.7%. 
Case Study: Mixed Distributions Fit Better

- Mixed Gamma-Lognormal: Significance?

- Likelihood Ratio Test 95% Critical Value = 7.8
  - Mixed vs. Gamma Test Statistic = 668
  - Mixed vs. Lognormal Test Statistic = 1331

- Since test statistics > critical value the mixed distribution provides a significantly better fit to the data than either of the simple distributions.
Case Study: Fitting Mixtures

- Tools Available to Fit Mixed Distributions
  - Microsoft Excel SOLVER
  - R
  - SAS
  - Other

- Steps to Fit Mixed Distributions
  - Write the Mixed Density Function
  - Specify Initial Parameter Values
  - Write the Log-Likelihood Function
  - Maximize the Log-Likelihood by Changing Parameters
Case Study: Fitting Mixtures

- Mixed Gamma-Gamma:
  - Density:
    \[
    f(x, \alpha_1, \beta_1, \pi_1, \alpha_2, \beta_2) = \pi_1 * f_1(x, \alpha_1, \beta_1) + (1 - \pi_1) * f_2(x, \alpha_2, \beta_2)
    \]
  - Specify Initial Parameter Values

- Likelihood:
  \[
  L(\alpha_1, \beta_1, \pi_1, \alpha_2, \beta_2) = \prod_{i=1}^{8500} f(x_i, \alpha_1, \beta_1, \pi_1, \alpha_2, \beta_2)
  \]

- Log-Likelihood:
  \[
  l(\alpha_1, \beta_1, \pi_1, \alpha_2, \beta_2) = \sum_{i=1}^{8500} \ln(f(x_i, \alpha_1, \beta_1, \pi_1, \alpha_2, \beta_2))
  \]
Case Study: Fitting Mixtures

- Maximize Log-Likelihood: Excel SOLVER

Excel spreadsheet showing:
- GAMMADIST function with parameters and calculations.
- Initial starting values for parameters: Alpha1 = 0.295, Beta1 = 193,000,000, p = 0.50, Alpha2 = 1,000, Beta2 = 57,000,000.
- Year 1 data: AggLoss = 46,452,953, Likelihood = $H5 \times \text{GAMMADIST}(E11,$H$3,$H$4,FALSE)+(1-$H5) \times \text{GAMMADIST}(E11,$H$6,$H$7,FALSE)$, Log-Likelihood = 150,165.8.
Case Study: Fitting Mixtures

Maximize Log-Likelihood: Excel SOLVER
Case Study: Fitting Mixtures

- Maximize Log-Likelihood: R
- [http://www.r-project.org/](http://www.r-project.org/)

```r
> hd<-read.csv("HurrData.csv")
> AggLoss<-hd[,3]
>
> gamgamST<-c(0.295,193000000,0.50,1.000,57000000)
>
> gamgamLL<-function(x) -sum(log(x[3])*dgamma(AggLoss,shape=x[1],scale=x[2]) +
+ (1-x[3])*dgamma(AggLoss,shape=x[4],scale=x[5])))
>
> optim(gagmaST,gamaLL,method="L-BFGS-B",
+ lower=c(0,0,0,0,0),upper=c(Inf,Inf,1,Inf,Inf))
```
Case Study: Fitting Mixtures

- **Parameter Risk: Sample Data**
  - The second distribution could have low credibility.
  - Sensitivity test with slight data changes.
  - Parameter uncertainties in cat modeling firms (AIR, RMS, EQECAT).

- **Parameter Risk: Initial Values**
  - Could lead to local maxima.
  - Try different starting values:
    - Start with 90%/10% weights.
    - Use same distribution to infer starting means such as a mixture of 2-Gamma distributions.
Case Study: Fitting Mixtures

- Parameter Risk: Robustness
  - Remove 81 losses less than $3000, and refit MLE lognormal and gamma-lognormal distributions.
  - For lognormal, the fitted mean decreased by 29%; the fitted standard deviation decreased by 54%.
  - For gamma-lognormal, the fitted mean increased 2%, the fitted standard deviation decreased by 0.1%.
  - Mixture distribution is more robust!

\[
\mu = \frac{1}{n} \sum_{i=1}^{n} \ln(x_i) \quad E[X] = e^{\mu + \sigma^2/2}
\]
Case Study: Implications

- **Expected Reinsurance Recovery**
  - Low credibility for high layers
    - Hurricane output only contained 56 losses over $800M.
    - Only 5 losses over $1.6B.
  - Fitted distribution can help evaluate cost for higher layers

- **Alternative Tail Estimates**
  - Percentiles/VaR
  - TVaR
Conclusions

- Insurance data are skewed and heavy tailed.
- Single distribution in general cannot fit data well.
- Mixture distribution can represent insurance data with excess zeroes, multiple modes, and heavy tails.
- Finite mixture model improves GLM by assuming mixture distribution.
- Many insurance applications: ERM (PML, TVaR), asset management, reinsurance (cat, per risk), high deductible (worker comp, property), predictive modeling (frequency, severity).