Guaranteed annuity conversion options and their valuation*

Laura Ballotta and Steven Haberman

Faculty of Actuarial Science and Statistics,

Cass Business School, City University London

106 Bunhill Row, EC1Y 8TZ

E-mail address: L.Ballotta@city.ac.uk, S.Haberman@city.ac.uk

Abstract

In this chapter, we consider a theoretical model for the pricing and valuation of guaranteed annuity conversion options associated with certain unit-linked pension contracts in the UK. The valuation approach is based on the similarity between the payoff structure of the contract and a call option written on a coupon-bearing bond. The model makes use of a one-factor Heath-Jarrow-Morton framework for the term structure of interest rates, in order to obtain a closed-form analytical solution to the fair valuation of the liabilities implied by these contracts. Mortality risk is incorporated via a stochastic model for the evolution over time of the underlying hazard rates. Numerical results are investigated and the sensitivity of the price of the option to changes in the key financial and mortality parameters is also analyzed.

Keywords: guaranteed annuity option; hazard rates; reduction factor; longevity risk; risk neutral valuation; Heath-Jarrow-Morton model.

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1 Introduction

A guaranteed annuity option is a contract feature attached to personal (individual) pension policies, which provides the policyholder with the right to receive at retirement either a cash payment or an annuity, payable throughout the policyholder's remaining lifetime and calculated at a guaranteed rate, depending on which has the greater value. This guarantee of the conversion rate between cash and pension income was a common feature of pension policies sold in the UK during the (late) 1970s and 1980s, with more than 40 companies involved in this market.

Until recently, the UK experience has been that the cash benefit is more valuable than the guaranteed annuity payment since a higher pension could be obtained by using the cash payment to buy the best annuity rates available in the market (the so-called "open market option"). After the reductions in market interest rates in recent years, particularly since 1998, the position has changed and the guaranteed annuity has tended to be worth more than the cash benefit; unanticipated falls in mortality rates since these policies were issued have also made the guaranteed annuity more valuable to policyholders. As a result of these two combined effects, many UK insurance companies have experienced solvency problems requiring the setting up of extra reserves (using ad hoc methods) and leading one large mutual life insurer (Equitable Life, the world's oldest life insurance company) to be closed to new business in 2000. Although pension policies with these guarantees are no longer being sold in the UK, these are a common feature of corresponding policies in other countries, for example the US.

In this chapter, our objective is to explore a number of questions that arise from the UK experience and which are of current relevance. How should we value (and hence price) a guaranteed annuity option (GAO) in a market consistent manner (bearing in mind the trend towards fair valuation methodologies for insurance liabilities)? Is it possible to describe how the value of a GAO evolves over time, as the policyholder ages and as new financial information becomes available? What are the key sensitivities affecting the value of a GAO? How much of the historical problem in the UK can be explained by adverse changes in interest rates and how much by adverse changes in mortality rates? How do we allow for the uncertainty of mortality rate trends and what effect do these have on the value of a GAO?

We focus on unit-linked deferred annuity contracts purchased originally by a single premium. For simplicity, we ignore insurance company expenses, taxes, profit and pre-retirement death benefits in order to concentrate on the GAO. The analytical approach follows the financial economics literature and exploits the well-known option valuation theory in order to obtain results for the pricing, reserving and hedging of the GAO. Fuller details are given in Ballotta and Haberman (2002, 2003). An alternative approach based on modelling the dynamics of the annuity price, rather than the underlying term structure of interest rates, has been used by Bezooyen et al. (1998), Pelsser (2002) and Yang et al. (2002). However, we argue that a methodology based on the term structure of interest rates is more sound in that it facilitates the analysis of the effect on the GAO of changes in the underlying interest rates and their term structure.

We propose an option pricing approach which is based on the similarity between the payoff structure of the contract under consideration and a call option written on a coupon-bearing bond. We use a single-factor Heath-Jarrow-Morton framework for the term structure of interest rates. This choice is justified by the need to avoid dependence of the model on the market price of interest rate risk, which usually implies an arbitrary specification of the model parameters leading to arbitrage opportunities (Heath, Jarrow and Morton, 1992). Also, single-factor models have the advantage of allowing a mathematically tractable solution to the coupon bond option pricing problem (Jamshidian, 1989). Under the additional assumption of an unsystematic mortality risk, independent of the financial risk, a general pricing framework is proposed and closed-form analytical formulae for the value of the GAO are obtained. Further, we introduce a dynamic model for the evolution of mortality rates over time which requires the use of a numerical procedure for the computation of the survival probabilities involved in the valuation of the GAO. The valuation formulae derived implicitly contain the dynamic investment strategy that replicates the contract. Numerical results are investigated and the sensitivities of the price of the option to changes in the key parameters are also analyzed.

The paper is organized as follows: sections 2.1 and 2.2 develop the framework for the valuation of guaranteed annuity conversion options, with some of the mathematical details relegated to the Appendix. In sections 2.3 and 2.4, we introduce different approaches to modelling mortality risk. Section 3 provides numerical results and a sensitivity analysis. Concluding remarks are offered in section 4.

2 Fair valuation of guaranteed annuity options

2.1 General framework

As noted in section 1, a guaranteed annuity option provides the holder of the contract with the right to receive at retirement the greater of two choices: either a cash benefit (equal to the current value of the reference portfolio), or an annuity which would be payable throughout his/her remaining lifetime and which is calculated at a guaranteed rate, g.

Hence, if the policyholder is aged x at inception and N is the normal retirement age, the guaranteed annuity option pays out at maturity $T = N - x$:

$$\begin{aligned} C_T &= \max\left(gS_T a_{x+T}, S_T\right) \\ &= S_T + \left(gS_T a_{x+T} - S_T\right)^+ \\ &= S_T + gS_T\left(a_{x+T} - K\right)^+, \end{aligned}$$

where S is the market value of the equity fund backing the contract, $K = 1/g$ and a_{x+T} represents the "annuity factor", i.e. the expected present value at time T of a life annuity which pays £1 throughout the remaining lifetime of the policyholder, and the notation $(v)^+$ is equivalent to $\max(v, 0)$.

Consider a frictionless market with continuous trading. Assume that there are no taxes, no transaction costs, no restrictions on borrowing or short sales, all securities are perfectly divisible, and that the price process of the equity fund S follows an adapted, càdlàg and strictly positive semimartingale. Let r_t be the stochastic short rate. Applying risk-neutral valuation, the fair value at time T of the annuity can be calculated as

$$a_{x+T} = \hat{\mathbb{E}}\left[\left.\sum_{t=0}^{w-(T+x)} e^{-\int_T^{T+t} r_u du} 1_{(\mathcal{T}_{x+T} > t)} \right| \mathcal{F}_T\right],$$

where $\hat{\mathbb{E}}$ denotes the expectation under the risk-neutral probability measure $\hat{\mathbb{P}}$, w is the largest survival age, $\mathcal{F}_T$ is the information flow at maturity and $\mathcal{T}_y$ is a random variable representing the remaining lifetime of a policyholder aged y. Assume that the mortality risk, captured in the model by the random variable $\mathcal{T}_y$, is unsystematic and independent of any source of risk existing in the financial market. Then

$$\begin{aligned} a_{x+T} &= \sum_{t=0}^{w-(T+x)} \hat{\mathbb{E}}\left[1_{(\mathcal{T}_{x+T}>t)} \middle| \mathcal{F}_T \right] \hat{\mathbb{E}}\left[e^{-\int_T^{T+t} r_u du} \middle| \mathcal{F}_T \right] \\ &= \sum_{t=0}^{w-(T+x)} \mathbb{P}\left[1_{(\mathcal{T}_{x+T}>t)} \middle| \mathcal{F}_T \right] P_T\left(T+t\right) \\ &= \sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T\left(T+t\right) \end{aligned}$$

where $P_T\left(T+t\right)$ is the price a time T of a zero coupon bond with unit face value and redemption date $(T+t)$, and ${}_t p_{T+x}$ is the t-year survival probability of a person aged $\left(x+T\right)$. It follows that the terminal payoff of the guaranteed annuity option is:

$$C_T = S_T + g S_T \left(\sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T\left(T+t\right) - K \right)^+ . \tag{1}$$

Expressed in this way, the guaranteed annuity option contract shows a payoff structure which is analogous to that of an option written on a coupon bond, where the role of the coupon is played by the post-retirement survival probabilities.

Similarly, risk-neutral valuation implies that the value at time τ, $0 \leq \tau \leq T$, of the guaranteed annuity option contract entered by a policyholder aged x at time 0 is

$$\begin{aligned} V_x\left(x+\tau,\tau,T-\tau=N-x-\tau\right) &= \hat{\mathbb{E}}\left[e^{-\int_\tau^T r_u du} C_T 1_{(\mathcal{T}_{x+\tau}>T)} \middle| \mathcal{F}_\tau \right] \\ &= {}_{T-\tau}p_{x+\tau}\, \hat{\mathbb{E}}\left[e^{-\int_\tau^T r_u du} C_T \middle| \mathcal{F}_\tau \right] \\ &= {}_{T-\tau}p_{x+\tau} \left\{ S_\tau \right. \\ &\quad \left. + g\hat{\mathbb{E}}\left[e^{-\int_\tau^T r_u du} S_T \left(a_{x+T}-K\right)^+ \middle| \mathcal{F}_\tau \right] \right\} \end{aligned} \tag{2}$$

If $\tilde{\mathbb{P}}$ is a martingale probability measure equivalent to $\hat{\mathbb{P}}$ and defined by the density process (Geman, El Karoui, Rochet, 1995)

$$\eta_T := \left. \frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}} \right|_{\mathcal{F}_T} = e^{-\int_0^T r_u du} \frac{S_T}{S_0}, \tag{3}$$

then the expectation in equation (2) can be reduced to

$$
\begin{aligned}
\hat{\mathbb{E}}\left[e^{-\int_\tau^T r_u du} S_T \left(a_{x+T}-K\right)^+ \middle| \mathcal{F}_\tau \right] &= e^{\int_0^\tau r_u du} \hat{\mathbb{E}}\left[\eta_T S_0 \left(a_{x+T}-K\right)^+ \middle| \mathcal{F}_\tau \right] \\
&= e^{\int_0^\tau r_u du} S_0 \hat{\mathbb{E}}\left[\eta_T \middle| \mathcal{F}_\tau \right] \tilde{\mathbb{E}}\left[\left(a_{x+T}-K\right)^+ \middle| \mathcal{F}_\tau \right] \\
&= S_\tau \tilde{\mathbb{E}}\left[\left(a_{x+T}-K\right)^+ \middle| \mathcal{F}_\tau \right] \quad ,
\end{aligned}
$$

where $\tilde{\mathbb{E}}$ denotes the expectation under the stock-risk-adjusted probability measure $\tilde{\mathbb{P}}$. Therefore, we may demonstrate that the following holds:

$$
\begin{aligned}
& V_x\left(x+\tau, \tau, T-\tau\right) \\
= \quad & {}_{T-\tau}p_{x+\tau} S_\tau + {}_{T-\tau}\, p_{x+\tau} g S_\tau \tilde{\mathbb{E}}\left[\left(\sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T\left(T+t\right) - K \right)^+ \middle| \mathcal{F}_\tau \right],
\end{aligned} \tag{4}
$$

(see Ballotta and Haberman, 2002, 2003, for further background).

Both equations (1) and (4) show that the fair premium that the insurer should charge for a pension plan with a guaranteed annuity option feature attached is given by the current market value of the reference equity portfolio, plus the single premium for a coupon-bond like option, weighted by the probability of the policyholder to survive till retirement. A fair premium means, in this context, that if the actual premium charged were greater than V, the customers would not receive value for money; if instead the premium were less than V, then the insurer would be offering the guarantees too cheaply. In other words, the market would not be in equilibrium and arbitrage opportunities would arise.

The two equations, (1) and (4), also show that the guaranteed annuity option contract is affected by two sources of risk: the financial risk, in the form of the uncertainty related to future movements in both the equity fund and the market interest rate, and the mortality risk, captured by the random remaining lifetime of the policyholder. As we show in the remaining sections of this chapter, a closed analytical formula for the fair premium of the guaranteed annuity option can be obtained modelling the financial risk within the classical Black-Scholes economy, and making use of a single-factor Heath, Jarrow and Morton (1992) framework for the term structure of interest rates. Mortality effects are taken into consideration through two approaches: (a) basing the survival probabilities on the standard mortality tables which have been adopted by UK Life Insurance companies (for pricing and reserving calculations); or, alternatively, (b) calculating the survival probabilities from a stochastic model for the mortality risk, which is based on the reduction factor approach for modelling the time variations in hazard rates.

2.2 Pricing the guaranteed annuity option allowing for financial risk

In the frictionless market introduced in the previous section, we assume that the insurer invests the single premium paid by each policyholder at the start of the contract into an equity fund, whose risk-neutral dynamic is described by the following stochastic differential equation.

$$dS_t = r_t S_t dt + \sigma_S S_t d\hat{Z}_t, \tag{5}$$

where $\sigma_S \in \mathbb{R}^+$ and $\left(\hat{Z}_t : t \geq 0\right)$ is a standard one-dimensional $\hat{\mathbb{P}}$-Brownian motion. Thus, S_0 is the single premium. As mentioned above, we assume that the evolution of the term structure of interest rates is given by a single-factor HJM framework and we consider the specific case in which the forward rate volatility has an exponentially decaying structure which resembles an extended form of the model of Vasicek (1977). In other words, the risk-neutral dynamic of the forward rate is given by

$$df\left(t,T\right) = \left(\sigma^2 e^{-\lambda(T-t)} \int_t^T e^{-\lambda(u-t)} du\right) dt + \sigma e^{-\lambda(T-t)} d\hat{W}_t, \tag{6}$$

where $\left(\hat{W}_t : t \geq 0\right)$ is a standard one-dimensional $\hat{\mathbb{P}}$-Brownian motion correlated with $\hat{Z}$, so that

$$d\hat{W}_t d\hat{Z}_t = \rho dt$$

for any $\rho \neq 0$. This implies

$$\hat{Z}_t = \rho \hat{W}_t + \sqrt{1-\rho^2}\hat{W}'_t,$$

where $\left(\hat{W}'_t : t \geq 0\right)$ is a $\hat{\mathbb{P}}$-Brownian motion independent of both $\hat{W}$ and $\hat{Z}$. Hence, ρ represents the correlation coefficient between equity fund values and interest rates. In this framework:

$$r_t = \lim_{T \to t} f\left(t,T\right)$$

and

$$P_t\left(T\right) = e^{-\int_t^T f(t,u)du}.$$

Since we are using a single-factor model for the term structure of interest rates, and given the similarity between the guaranteed annuity option contract and a coupon bond option which we have previously observed (see equations 1 and 4), we can apply the Jamshidian (1989) decomposition and rewrite the annuity option payoff as the payoff generated by a portfolio of zero-coupon bond options with appropriate strike prices, K_t, and weights equal to the survival probabilities, ${}_t p_{T+x}$. More precisely, for $t = 0, 1, ..., w - (T + x)$, we find the critical value of the interest rate such that

$$\sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T (T + t) = K;$$

then we define a new "artificial" strike price K_t as the bond price which is calculated to correspond to this critical interest rate level, that is

$$K_t = P_T^* (T + t) .$$

Since the bond price is a monotonic function of the interest rate, it follows that

$$\left(\sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T (T + t) - K \right)^+ = \sum_{t=0}^{w-(T+x)} {}_t p_{T+x} \left(P_T (T + t) - K_t \right)^+ ,$$

which implies that (4) can be re-written as:

$$\begin{aligned}
& V_x (x + \tau, \tau, T - \tau) \\
= \quad & {}_{T-\tau} p_{x+\tau} S_\tau + {}_{T-\tau}\, p_{x+\tau} g S_\tau \sum_{t=0}^{w-(T+x)} {}_t p_{T+x} \tilde{\mathbb{E}} \left[\left(P_T (T + t) - K_t \right)^+ \middle| \mathcal{F}_\tau \right] \\
= \quad & {}_{T-\tau} p_{x+\tau} S_\tau \\
& + \frac{{}_{T-\tau} p_{x+\tau} g S_\tau}{P_\tau (T)} \sum_{t=0}^{w-(T+x)} {}_t p_{T+x} \left[P_\tau (T + t) e^{-\gamma(T,T+t) m_r (T-\tau)} N (d_t') - P_\tau (T) K_t N (d_t) \right] .
\end{aligned} \tag{7}$$

with

$$\begin{aligned}
d_t &= \frac{\ln \frac{P_\tau (T+t)}{K_t P_\tau (T)} - \frac{1}{2} \sigma_r^2 (T - \tau) \gamma^2 (T, T + t) - \gamma (T, T + t) m_r (T - \tau)}{\gamma (T, T + t) \sigma_r (T - \tau)}, \\
d_t' &= d_t + \gamma (T, T + t) \sigma_r (T - \tau) .
\end{aligned}$$

Details of the argument that lead to equation (7) are provided in the Appendix.

2.3 Mortality risk: deterministic approach

Although the procedure leading to the valuation equation (7) allows for stochastic mortality trends, our initial approach to the calculations is to use a set of three standard UK mortality tables for the calculation of the survival probabilities in (7) . Each of these incorporates an element of future mortality reduction computed on a deterministic basis.

These mortality tables have been produced by the Continuous Mortality Investigation Bureau of the Institute and Faculty of Actuaries for insurance company data on male pensioner mortality. They are extensively used for the calculation of premiums and reserves. The PA90 table is based on data for the period 1967-70 projected to 1990, PMA80-C10 is based on data for the period 1979-82 projected to 2010 and PMA92-C20 is based on data for the period 1991-94 projected to 2020. Because of the declining trend in mortality rates over time, and hence the increasing trend in survival probabilities, ${}_tp_{T+x}$, the expected present value of the life annuity increases as we move the assumption from PA90 to PMA80-C10 to PMA92-C20.

2.4 Mortality risk: stochastic approach

As an alternative to the deterministic approach, we introduce a stochastic model for mortality trends. In section 2.1, we defined $\mathcal{T}_x$ to be a random variable which represents the remaining lifetime of a policyholder aged x. The survival function of this random variable, $\mathcal{T}_x$, is given by

$$ {}_sp_x = \mathbb{P}\left(\mathcal{T}_x > s\right), $$

where $\mathbb{P}$ is the objective probability measure. If we explicitly allow for the time dependence of the hazard rate and we define $\mu_{x+t:t}$ to be the hazard rate for an individual at time t then aged $x+t$, then it follows that

$$ {}_sp_x = \mathbb{E}\left[e^{-\int_0^s \mu_{x+t:t}dt}\right]. \tag{8} $$

A widely used actuarial model for projecting mortality rates is the reduction factor model; this has been used in the UK and US for pensioner and annuitant populations for many years, see, for example, standard tables produced by the Continuous Mortality Investigation Bureau in the UK since 1967-70 (e.g. CMI Bureau 1999) and the General Annuity Valuation Tables in the US (Group Annuity Valuation Table Task Force, 1995).

Given the form of (8), we propose a parsimonious model for the trajectory of the hazard rate over time. Following Sithole et al. (2000), we consider a reduction factor approach based on hazard rates, so that

$$ \mu_{y:t} = \mu_{y:0} RF\left(y, t\right) $$

where $\mu_{y:0}$ is the hazard rate for a person aged y in the base year (i.e. year 0) or period, and $\mu_{y:t}$ is the hazard rate for a person attaining age y in future year t (i.e. as measured from the base year or period), and the reduction factor $RF(y,t)$ is the ratio of the hazard rates. Using an approach based on generalized linear models, as proposed by Renshaw et al. (1996), Sithole et al. (2000) derive a series of models appropriate to UK annuitant and pensioner populations which simplify to the following structure:

$$RF(y,t) = e^{(\alpha+\beta y)t}.$$

The parameter α represents the rate of change in the hazard rate over time on the logarithmic scale and we would expect estimates to be negative. The parameter β represents an offset term that reflects a rate of change that could differ with age y.

Hence, with $y = x + t$, we propose the following model for the time evolution of the hazard rate:

$$\mu_{x+t:t} = \mu_{x+t:0} e^{(\alpha+\beta x+\beta t)t+\sigma_h Y_t}, \tag{9}$$

where $(Y_t : t \geq 0)$ is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$, which is introduced to model random variations in the forecast trend, and

$$\mu_{x+t:0} = a_1 + a_2 R + e^{b_1+b_2R+b_3\left(2R^2-1\right)}, \tag{10}$$

where

$$R = \frac{(x+t)-70}{50}, \qquad x \geq 50. \tag{11}$$

This model for $\mu_{x+t:0}$, the hazard rate for the base year, corresponds to the structure for the UK standard tables for annuitant and pensioner populations for the period 1991-94, as proposed by the CMI Bureau (1999). For numerical illustrations, we use the parametrization for the standard table for male pensioners (analysis based on lives, rather than pension amounts), and the corresponding values for α and β as estimated by Sithole et al (2000): see Table 2 later.

Further, we assume that Y follows an Ornstein-Uhlenbeck process; in other words, the process Y satisfies the following

$$\begin{cases} dY_t = -aY_t dt + dX_t \\ Y_0 = 0, \end{cases} \tag{12}$$

where $(X_t : t \geq 0)$ is a standard one-dimensional $\mathbb{P}$-Brownian motion, independent of the "financial" Wiener processes, $\hat{W}$ and $\hat{Z}$ (see equations 5 and 6). This is similar to the model of Milevsky and Promislow (2001), and has the desirable property of mean reversion, with the parameter a measuring the speed of mean reversion. The solution to (12),

$$Y_t = \int_0^t e^{-a(t-s)} dX_s,$$

implies that $Y_t \sim N\left(0, \xi^2\left(t\right)\right)$, with

$$\xi^2\left(t\right) = \frac{1 - e^{-2at}}{2a}.$$

Equation (8) does not lead to a closed form expression principally because the sum of lognormal random variables is not itself lognormal. Hence, for computational purposes, we need a numerical approximation and adopt a simulation procedure based on the Monte Carlo technique. Thus, we subdivide the time period $[0, s]$ into n equal subintervals of length $\Delta t = \frac{s}{n}$, and we define $t_i = \frac{i}{n}s = i\Delta t$, $i = 1, 2, ..., n$. At each step, we generate the path of the hazard rate as

$$\begin{aligned}
\mu_{x+t_i:t_i} &= \mu_{x+t_i:0} e^{(\alpha+\beta x+\beta t_i)t_i+\sigma_h \xi(t_i) z_i}, \\
\mu_{x+t_i:0} &= a_1 + a_2 R_i + e^{b_1+b_2 R+b_3\left(2R_i^2-1\right)} \\
R_i &= \frac{(x+t_i)-70}{50},
\end{aligned}$$

where z_i is a random sample from a standardized normal distribution. The integral in equation (8) is then approximated using the trapezoidal rule.

The model represented by equation (9) allows for a future mortality trend that is random. However, it is recognised in the literature that systematic deviations from the forecasted mortality rates may take place so that parameter risk is present. When this is applied to the trend at the older ages, the risk is usually referred to as “longevity risk”: see Marocco and Pitacco (1998), Olivieri (2001), Olivieri and Pitacco (2002), for example, for further discussion.

In order to incorporate longevity risk, we develop the model further by following the approach of Olivieri and Pitacco (2002), inter alia. For representing the range of possible evolutions of mortality, we consider a family of projected hazard rates, for a given age at entry x. Thus, we consider

$$\left[\mu_{x+t:t:H(x)}; H\left(x\right) \in \mathcal{H}\left(x\right)\right],$$

where $H\left(x\right)$ is a particular hypothesis concerning the trend of mortality for individuals entering an insured group at age x and $\mathcal{H}\left(x\right)$ represents a given

	$\tilde{\alpha} =$ with prob.	-0.08	-0.05	-0.02
Model	1.1	0.3	0.4	0.3
	1.2	0.2	0.6	0.2
	1.3	0.1	0.8	0.1
	1.4	1/3	1/3	1/3
(symmetric distribution)				
Model	2.1	0.4	0.3	0.3
	2.2	0.6	0.2	0.2
	2.3	0.8	0.1	0.1
(asymmetric distribution)				
Model	3	$\tilde{\alpha} \sim U(-0.08, -0.02)$		

Table 1: Parameter risk: range of values for $\tilde{\alpha}$ with corresponding probability functions.

set of such hypotheses. In particular, if we focus on equation (9) where the mortality trend is expressed by a set of parameters, then we could consider

$$\left[\mu_{x+t:t:\theta(x)}; \theta(x) \in \Theta(x)\right], \tag{13}$$

where $\theta(x)$ denotes a vector parameter and $\Theta(x)$ denotes the corresponding multi-dimensional parameter space. As a preliminary illustration of this methodology, we take $\theta(x) \equiv \alpha$ in equation (9) and consider the possible set of values for $\alpha : \Theta(x) \equiv \{-0.08, -0.05, -0.02\}$, which is shown in Table 1. We deal with the range of values for α by assuming that the parameter is a discrete random variable $\tilde{\alpha}$ and has the alternative probability functions shown in Table 1. Note that the estimate of α, $\hat{\alpha}$, from Sithole et al. (2000) is -0.02.

3 Numerical results and sensitivity analysis

In this section, we use the results developed in section 2.2 (namely, equation 7 for the GAO price), and the numerical procedures introduced in sections 2.3 and 2.4 to carry out a full sensitivity analysis for the value of the guaranteed annuity option contract.

Precisely, we concentrate on the extra premium that the insurer should charge at inception for the coupon bond-like option embedded in the contract,

Parameter set for numerical analysis
Design parameters:
$g = 11.1\%;$[1] $x = 50;$ $T + x = N = 65.$
Financial model
$S_0 = 100;$ $\sigma_S = 20\%$ p.a.; $\rho = 1;$ $f_0 = 4\%$ p.a.; $\sigma = 0.01;$ $\lambda = 0.15$
Mortality model (stochastic)
$\alpha = -0.02;$ $\beta = 0.0001;$ $\sigma_h = 10\%$ p.a.; $a = 0.5.$
$a_1 = -\frac{0.0081}{100};$ $a_2 = -\frac{0.07}{100};$ $b_1 = -4.67509;$ $b_2 = 5.629188;$ $b_3 = -1.2.$
[1] According to Bolton et al. (1997), $g = 11.1\%$ was the common guaranteed rate used in the UK by life insurance companies.

Table 2: Set of parameters used as benchmark for the comparative statics analysis. Parameters are subdivided into 3 blocks. The first group contains the parameters that characterize the individual policy; the second group contains the parameters representing the financial market components; the last group contains the parameters related to the mortality model.

namely, referring to equation (7) for $\tau = 0$:

$$\frac{{}_{T-\tau}p_{x+\tau}gS_\tau}{P_\tau(T)} \sum_{t=0}^{w-(T+x)} {}_tp_{T+x}\left[P_\tau(T+t)\,e^{-\gamma(T,T+t)m_r(T-\tau)}N(d_t') - P_\tau(T)\,K_tN(d_t)\right].$$

For this practical example, we incorporate a common design feature by assuming that the annuity has a 5-year guarantee period, so that the first five annual payments of the annuity scheme would be definitely payable, providing that the policyholder survives to retirement age.

We subdivide the analysis into two sections. The first one relates to the study of the behavior of the GAO when the parameters "imported" in the pricing formulae from the financial market are changed one at a time, ceteris paribus. The second set of results, instead, describes the behavior of the GAO when the mortality model parameters are changed, again individually ceteris paribus.

Unless otherwise stated, we use the benchmark set of parameters in Table 2 (and based on the stochastic mortality model of section 2.4). For the stochastic mortality model, we compute the survival probabilities by simulating 10,000 paths, with each path comprising 1 observation per month over each year.

3.1 GAO and the "financial" parameters

In this section, we illustrate the main comparative statics results for the sensitivity of the GAO to the financial parameters. In particular, we look at the sensitivity with respect to the market interest rate, the entry age of the policyholder, the volatility of the equity portfolio backing the policy and its correlation to interest rates, and the volatility coefficients of the forward rate. Fuller sets of results are presented by Ballotta and Haberman (2002), for the deterministic mortality case.

The sensitivity of the GAO to the initial redemption yield, f_0, used to calculate the initial bond prices $P_0(T)$ and $P_0(T+t), t = 0, 1, ..., w - (T + x)$, is shown in Figure 1. In particular, we observe a decreasing pattern due to the fact that higher current interest rates make the guaranteed annuity payments less attractive than the current rates available in the market.

Figure 2 shows the GAO profile versus the age of the policyholder at inception of the contract. The observed increasing pattern is explained by the different weights attaching to the expected future annuity payments in the case of a policyholder aged 20 at inception compared to the case of a policyholder entering the contract at later ages.

In Figure 3, we represent the behavior of the guaranteed annuity option as a function of the equity portfolio volatility, σ_S, for different values of the correlation coefficient ρ. As shown by the chart, the value of the GAO presents a different pattern depending on whether ρ is positive or negative. When ρ is negative, the policyholder might expect the equity market to move in the opposite direction from the interest rate. In this case, the annuity guaranteed by the pension plan becomes more and more attractive as the volatility of the reference portfolio increases. In fact, if market rates of interest drop, the policy locks in a competitive amount at a competitive rate. In the case of a rise in the level of the market rates, instead, the GAO might simply expire out-of-the-money. On the other hand, when ρ is positive and σ_S increases, the value of the annuity offered in the open market is more attractive, which reduces the value of the GAO. The same argument justifies the decreasing profile of the GAO as function of the correlation coefficient only (for fixed σ_S), and this may be deduced from vertical sections of the plots in Figure 3.

The changes in value of the GAO arising from changes in the parameters governing the volatility structure of the forward rate, and hence of the bond prices, are summarized in Figure 4, where the sensitivity to both the diffusion coefficient, σ, and the speed or adjustment, λ, are considered (for fixed ρ). The observed pattern finds an explanation in equations ($A3$) and ($A4$), which show that the the short rate volatility, σ_r, is a function which is increasing with σ and decreasing with λ. This means that if the interest rates are very

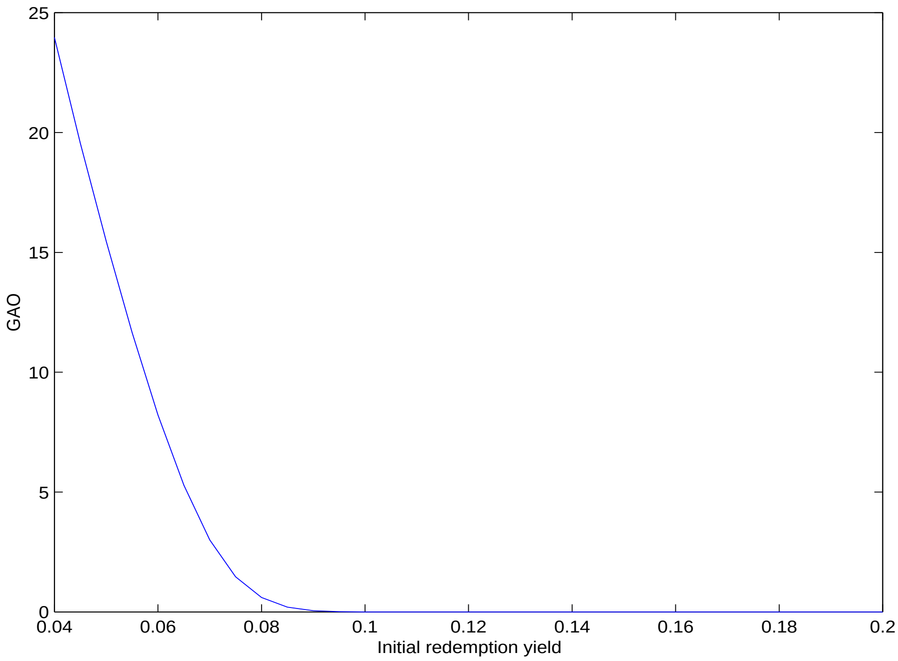

Figure 1: Sensitivity of the GAO to changes in the initial redemption yield used to calculate initial bonds prices.

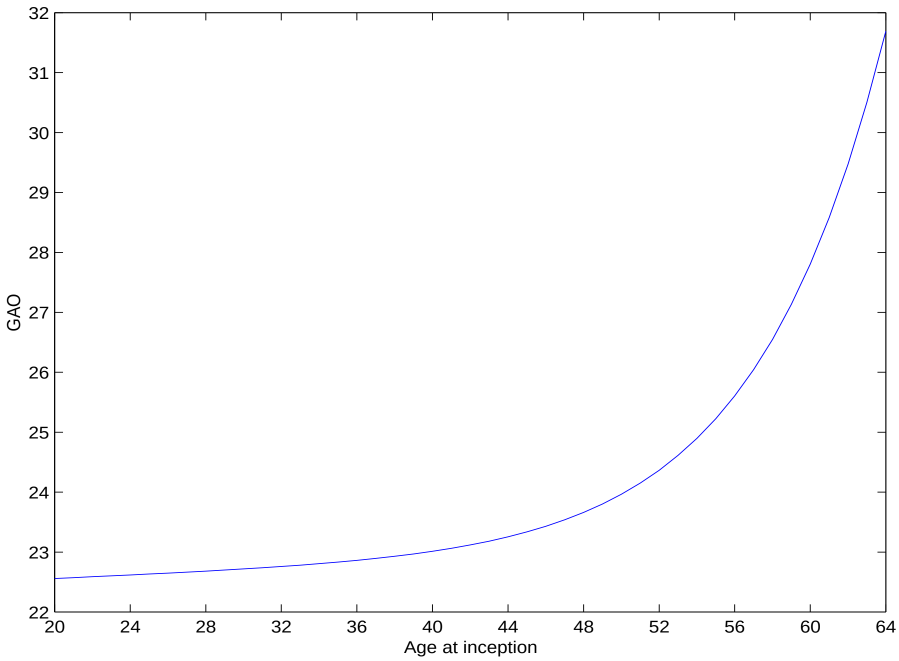

Figure 2: Values of the guaranteed annuity option for different entry ages and time evolution of the contract.

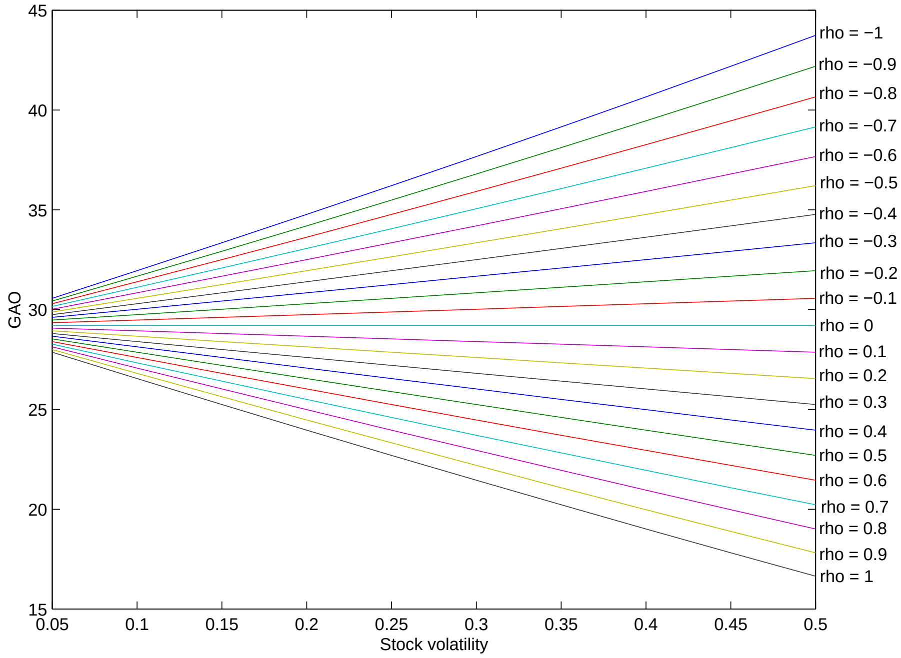

Figure 3: GAO vs volatility of the equity fund backing the policy, for different values of the correlation coefficient between equity and interest rates.

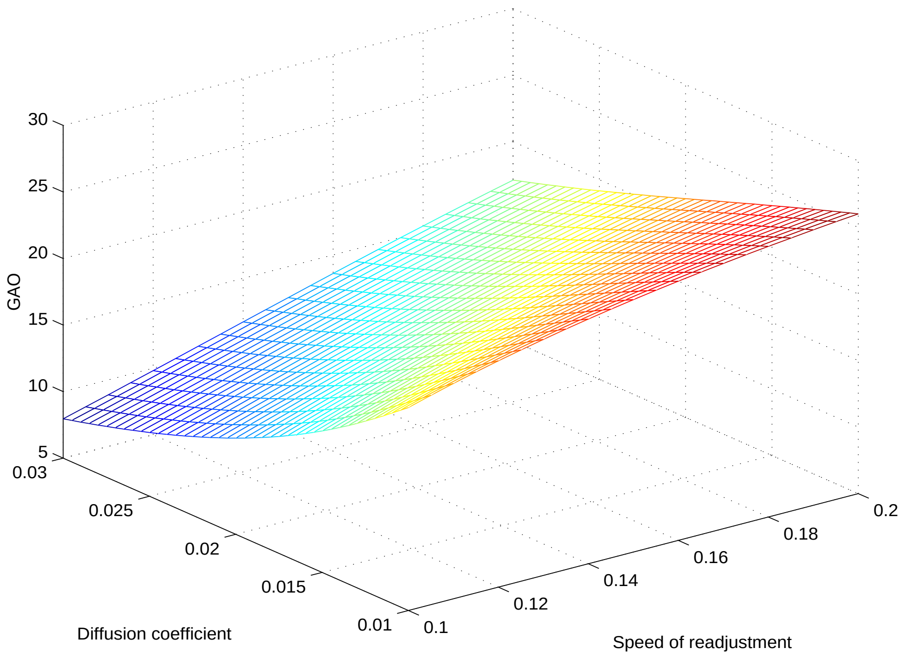

Figure 4: Sensitivity of the GAO value to the parameters governing the forward rate volatility.

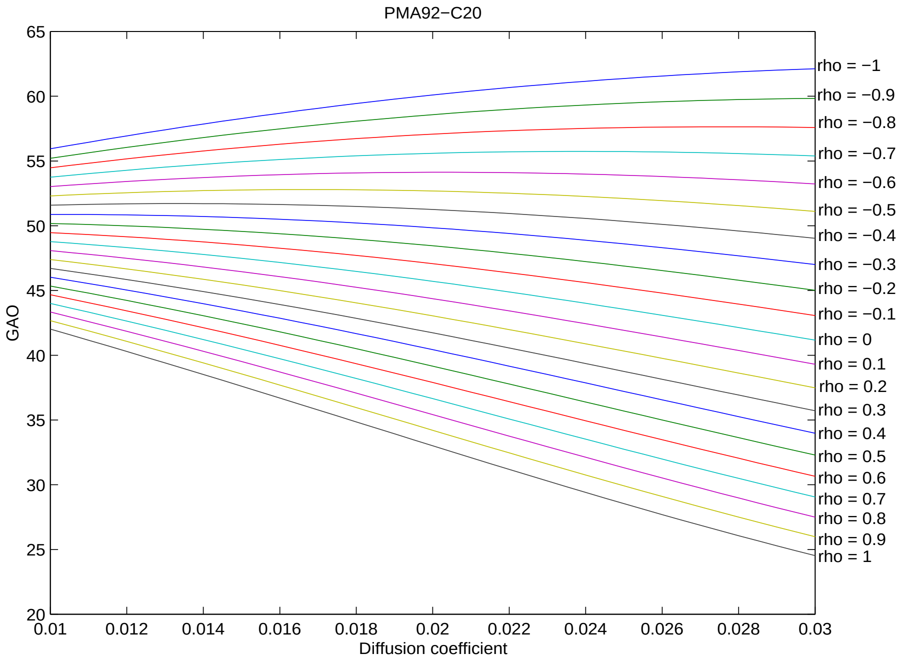

Figure 5: Exponentially decaying volatility model: guaranteed annuity option sensitivity to the diffusion term.

volatile, the expected present value of the annuity payments falls. So the policyholder seems to prefer not to exercise the option, but to take the cash payment instead and reinvest it at more favorable market conditions.

The sensitivity of the value of the GAO to changes in the diffusion coefficient, σ, and the correlation coefficient, ρ, is shown in Figure 5. As noted above, as σ increases, the present value of the annuity payments falls. When ρ is close to -1, this effect is offset by the attractiveness of the guarantee. Thus, as the volatility σ increases, the chance that interest rates will perform very well or very badly increases. The policyholder may benefit from falls in interest rates since the guaranteed payment appears more competitive; but the policyholder faces a limited downside risk in the event of a rise in interest rates because of the option scheme embedded in the contract. However, as the correlation between the equity market and interest rates becomes less and less negative, the decreasing effect induced by the drop in the expected present value of the annuity payments prevails.

Price of the GAO			
Life Table	$\sigma = 0.01, \lambda = 0.15$ (the benchmark case)	$\sigma = 0.001, \lambda = 0.001$	$\sigma_f = 0.001, \lambda \to 0$
PMA92-C20	42.0175	45.8667	45.8083
PMA80-C10	26.5497	29.7971	29.7541
PA90	13.7925	16.3651	16.3342

Table 3: Price of the guaranteed annuity option. Parameter set: $S_0 = 100; \sigma_S = 0.2; \rho = 1; g = 11.1\%; f_0 = 0.04; x = 50; N = 65$.

3.2 GAO and the "mortality" parameters

The value of the GAO when we use a deterministic mortality model is shown in Table 3, for the three standard UK pensioner-based life tables that have been investigated. These figures highlight the sensitivity of the results to the choice of the mortality assumptions and show, for the most recent mortality table, that the initial cost of the GAO is about $40-45\%$ of the original initial premium, S_0, paid by the policyholder at outset (for the parameters values shown). The limiting case of $\lambda \to 0$ corresponds to a constant volatility model, which is discussed more fully by Ballotta and Haberman (2002).

We now consider the impact on GAO values of changes in mortality trends, when survival probabilities are computed using the model described in section 2.4 and summarized by equations $(9)-(12)$. Figure 6 shows the impact on GAO values of the parameters related to the "deterministic" part of the model, i.e. the logarithmic rate of change in the hazard rate over time, α, and its offset term β. As α becomes more negative, so the downward trend in mortality rates becomes stronger. The final effect is then an improvement in survival probabilities and, as consequence, the GAO value contract rises in value. On the other hand, the value of the GAO decreases as β increases. This is consistent with the nature of this parameter, which is to mitigate the rate of decline in the mortality trend.

Figure 7 shows the behaviour of the GAO value as a function of the parameters related to the stochastic component of the model for the hazard rate. We observe that the value of the contract is an increasing function of the speed of convergence to the long-run mean, a. In particular, for small values of the diffusion parameter σ_h, the GAO is almost insensitive to changes in the parameter a. This means that the convergence of the random noise to its long-run mean is faster, the less "amplified" is the noise in the mortality process. We also observe that the GAO values decrease as σ_h increases; in other words, as uncertainty increases in the mortality trends, survival probabilities deteriorate and the contract value reduces consequently. However, as

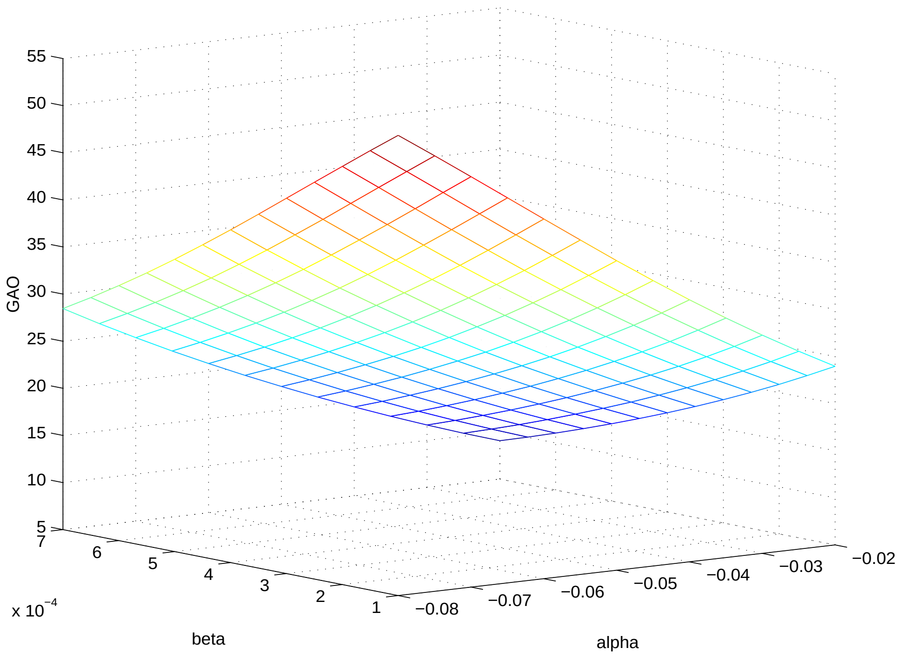

Figure 6: Effects on the guaranteed annuity option produced by the rate of change in mortality trends

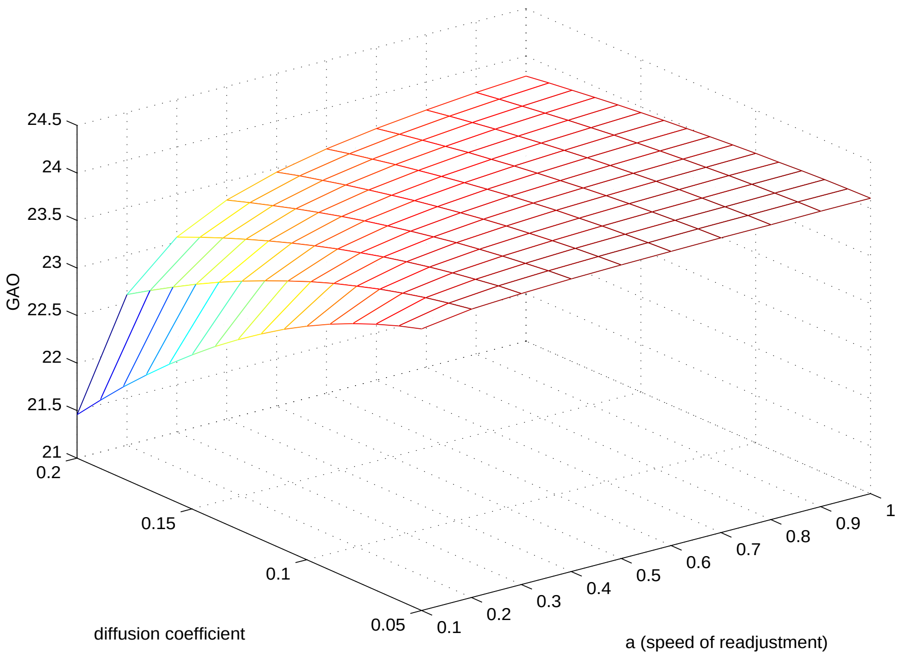

Figure 7: Sensitivity of the GAO to the parameters governing the stochastic component of the mortality model.

Model	GAO	$\mathbb{E}(\tilde{\alpha})$	$Var(\tilde{\alpha})$
Symmetric distribution case			
1.1	35.55	-0.05	0.00054
1.2	36.1314	-0.05	0.00036
1.3	36.9628	-0.05	0.00018
1.4	35.1092	-0.05	0.0006
Asymmetric distribution case			
2.1	36.4207	-0.053	0.000621
2.2	40.9213	-0.062	0.000576
2.3	46.5424	-0.071	0.000369
Uniform distribution case			
$\tilde{\alpha} \sim U(-0.08, -0.02)$	36.4237	-0.05	0.0003
3			
The benchmark			
$\tilde{\alpha} = -0.05$	37.8578		
$\tilde{\alpha} = -0.06$	43.1731		
$\tilde{\alpha} = -0.07$	48.5634		

Table 4: Sensitivity of the values of the GAO to the parameter error in the logarithmic rate of decline of hazard rates over time (α)

the mean-reversion effect becomes stronger, the effect of σ_h becomes almost negligible.

3.3 Longevity risk

In this section, we consider the enhanced model given by equation (13), where we relax the assumption that the logarithmic rate of decline in mortality rates, α, is constant. α is the key mortality parameter as far as trends are concerned and has been difficult to forecast in practice, as many commentators have noted (CMI Bureau, 1998, Sithole et al., 2000, Olivieri, 2001). We reflect on this feature by assuming that this parameter is random. We use the specific distributions of Table 1 in order to illustrate the effect on the value of the GAO of allowing for this feature. In this discussion, we make the underlying assumption that $\tilde{\alpha}$ is independent of the other sources of randomness present in the model.

Table 4 thus shows the effect on the value of the GAO of allowing for parameter error in α. We firstly consider four symmetric distributions for $\tilde{\alpha}$ and the uniform distribution case, each with mean -0.05. Relative to the case with constant α case, the results indicate that the presence of parameter error leads to a reduction in the GAO value and that increased uncertainty (measured by $Var(\tilde{\alpha})$) leads to larger reductions. Given the one-sided option-like nature of C_T, the presence of symmetric uncertainty in $\tilde{\alpha}$ would be expected to lead to this result, which is comparable to the sensitivities with respect to σ_h (shown in Figure 7). The three asymmetric cases indicate the effect of underestimating the size of α on the value of the GAO - the comparison with the deterministic benchmark values at the foot of Table 4 shows again that the presence of uncertainty leads to a marginal reduction in the value of the GAO (see section 3.2).

3.4 UK historical analysis

In this section, we use the valuation formulae obtained above to understand how much of the GAO solvency problem experienced in the UK has been attributable to reductions in interest rates and how much to the reductions in mortality rates. The analysis carried out is an extension of a similar study performed by Ballotta and Haberman (2002) and is based on Ballotta and Haberman (2003). We start with a hypothetical contract issued in 1970 to a policyholder then aged 20, and follow the evolution of the value of this contract over time up to the present day. Thus, we use the annual average of retail banks' base rates (Bank of England, February 2002) for the initial term structure of interest rates. For convenience, the pre-retirement survival probability is computed using a fixed (life insurance based) mortality table, namely the AM92 mortality table. Post-retirement survival probabilities are computed using the stochastic mortality model (thereby allowing for post-mortality improvements) implemented in section 2.4. Results are presented in Figure 8. In Figure 9, we present the evolution over time of the implied guaranteed rate, i.e. the rate of interest such that the expected present value of the guaranteed annuity equals the principal amount, or

$$gS_T a_{x+T} = S_T. \tag{14}$$

Figure 9 shows that, between 1970 and 1972, the implied guaranteed rate of the hypothetical contract, for the chosen set of parameters, was 7.93% while the market interest rates were oscillating between 5% and 7%. Hence, the guaranteed annuity option was "in the money". After 1973, the market interest rate increased up to a maximum level of 15.50% in 1980; while the implied guaranteed rate of 7.93% was not competitive at all, and thus the guaranteed annuity option contract was far "out of the money". However, in 1993, market rates of interest dramatically decreased from 12% to 6%, which

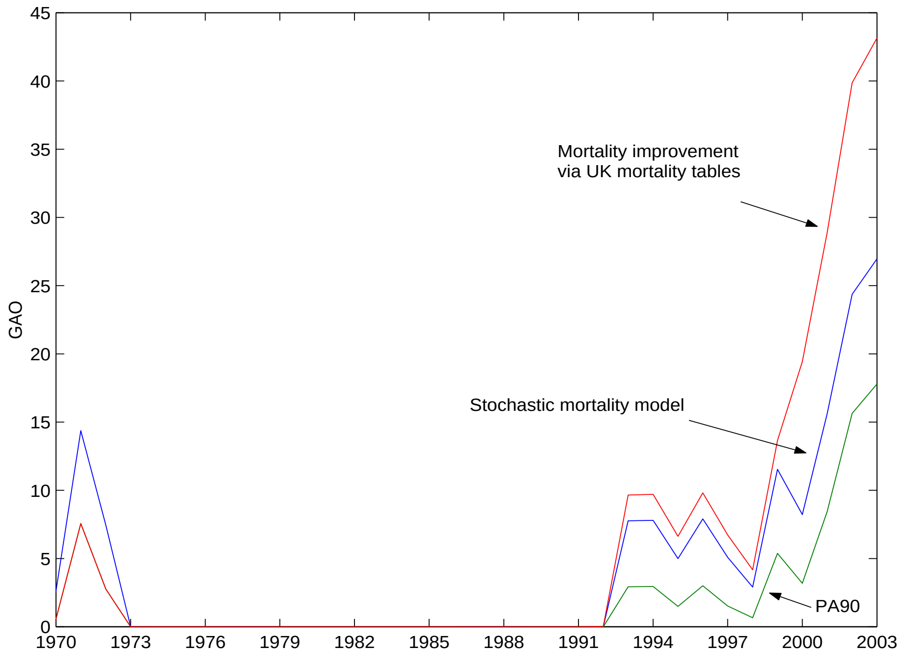

Figure 8: "Historical evolution" of a guaranteed annuity option contract issued in 1970 to a policyholder aged 20 for different specifications of the mortality model.

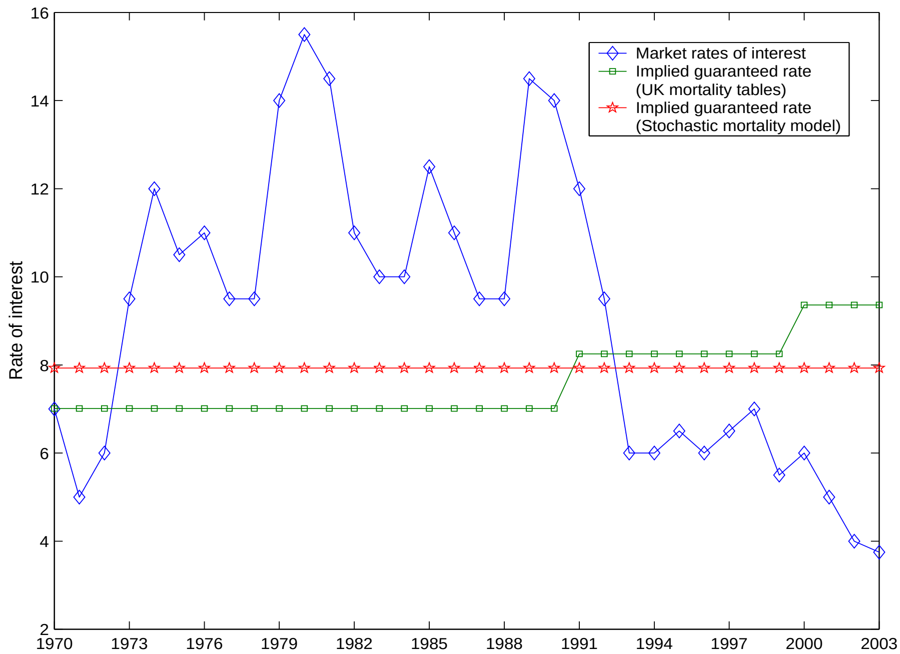

Figure 9: "Historical evolution" of the implied guaranteed rate for different specifications of the mortality model (equation 14).

brought the GAO contract back “in the money”. In the following years, market interest rates have fallen further to the current minimum value level of 3.75%, and the consequence has been a continuous rise in the value of the GAO. These results are confirmed by the trends in Figure 8.

Figures 8 and 9 show the historical evolution of the GAO value for the case in which mortality is modeled as described in section 2.4. Also shown, for comparison purposes, are the trends generated by the model in which the survival probabilities are computed using UK standard mortality tables (as described in section 2.3). Two such alternative models are analyzed. Firstly, we consider the case in which post-retirement survival probabilities are calculated using the PA90 mortality table only, as was the practice during the 1970s. The results are compared to those obtained for the case in which mortality improvements are incorporated by “switching” to more up-to-date mortality tables as these become available. Hence, when the valuation is performed during the period 1991-1999, the PMA80-C10 life table is used, while from the year 2000 onward, the PMA92-C20 life table is used[2].

The model based on the PA90 life table only provides a useful benchmark for distinguishing the effect of changing interest rates on the GAO values from the effect generated by improvements in mortality trends. This is because no mortality improvements (beyond 1990) are captured since a single fixed mortality table is used for valuation throughout the entire lifetime of the contract. Thus, neglecting improvements in mortality rates leads to an underpricing of the guaranteed annuity option of about 60% in year 2002 with respect to the values obtained using the model based on the three mortality tables. The corresponding level of underpricing is about 40% with respect to the stochastic mortality model used.

4 Concluding comments

In this paper we have introduced a theoretical model, based on the one-factor Heath-Jarrow-Morton term structure framework, for the valuation of guaranteed annuity conversion options attached to single premium unit-linked deferred annuity contracts. The approach depends on the correspondence between the contingent claim under consideration and an option contract written on a coupon paying bond. The behaviour of the GAO value with respect to changes in market conditions and mortality risk has been analyzed

[2]We note that each of these standard life tables is a “single entry” table with extrapolation up to a single future time point. Hence, they differ in character funadamentally from the stochastic model represented by equation (9) which allows for explicit projection of the hazard rate over each future potential survival period.

with numerical examples and the sensitivity analysis presented.

Although pension contracts with guaranteed annuity conversion options may no longer be being issued (eg. in the UK), there remains a significant practical problem of estimating appropriate reserves for those contracts sold in the past and where the option has not yet been exercised (Bolton et al., 1997). Thus, we believe that the above results (see equation 7) will be of considerable assistance to insurance companies for estimating such reserves, and for reporting and regulatory purposes.

Equation (7) provides some guidance as to the theoretical hedging strategy which should be employed. In fact, according to the valuation formula, the guaranteed annuity option can be seen as a portfolio consisting of a long position in the $(T+t)$-zero coupon bond which has to be funded by a short position in the T-zero coupon bond. However, we recognize that there are practical considerations to take into account, for example, the question of the availability of $(T+t)$-zero coupon bonds for such long maturities as the ones implied by the contract.

We have seen that the inclusion of stochastic mortality, through fluctuations around a trend, or longevity risk, leads to a reduction in the expected value of the GAO. However, we would advise some caution in the application of this result. Our valuation formula for V_x, equation (7), relates to an expected present value obtained using risk neutral valuation methods. However, an insurer might also be interested, for example for reserving purposes, in the full distribution of the random present value and, in particular, in upper tail values. These percentiles are likely to depend more directly on the stochastic mortality parameters introduced in the above models.

A Valuation formula for a guaranteed annuity option

The purpose of this Appendix is to show the derivation of the valuation formula (7) presented in section 2.1. Given the complexity of the terminal payoff of the guaranteed annuity option, we subdivide the calculations in three parts. In the first part, we give a quick overview of the market setup in the HJM framework; on the basis of this setup we price zero coupon bonds, and finally we use these preliminary set of results to complete the valuation process for the guaranteed annuity option.

A.1 The financial model

The market framework is described in section 2.1 and 2.2.

Consider the stock risk-adjusted probability measure $\tilde{\mathbb{P}}$ defined in (3) by

$$\begin{aligned}\eta_T &= e^{-\int_0^T r_u du}\frac{S_T}{S_0} = e^{-\frac{\sigma_S^2}{2}T+\sigma_S\hat{Z}_T} \\ &= e^{-\rho^2\frac{\sigma_S^2}{2}T-\left(1-\rho^2\right)\frac{\sigma_S^2}{2}T+\sigma_S\rho\hat{W}_T+\sigma_S\sqrt{1-\rho^2}\hat{W}_t'}.\end{aligned}$$

Since the Girsanov theorem implies that

$$\begin{aligned}\tilde{W}_t &: = \hat{W}_t - \rho\sigma_S t, \\ \tilde{W}_t' &: = \hat{W}_t' - \sigma_S\sqrt{1-\rho^2}t\end{aligned}$$

are $\tilde{\mathbb{P}}$-standard Brownian motions, the $\tilde{\mathbb{P}}$-dynamic of the forward rate is then

$$df\left(t,T\right) = \left(\sigma e^{-\lambda(T-t)}\left(\sigma\int_t^T e^{-\lambda(u-t)}du + \rho\sigma_S\right)\right)dt + \sigma e^{-\lambda(T-t)}d\tilde{W}_t.$$

Under these assumptions, it follows that the short rate process is

$$r_t = f\left(0,t\right) + \int_0^t \tilde{\mu}_f\left(v,t\right)dv + \sigma\int_0^t e^{-\lambda(t-v)}d\tilde{W}_v,$$

where

$$\begin{aligned}\tilde{\mu}_f\left(v,t\right) &= \sigma e^{-\lambda(t-v)}\left(\sigma\int_v^t e^{-\lambda(x-v)}dx + \rho\sigma_S\right) \\ &= \sigma e^{-\lambda(t-v)}\left[\frac{\sigma}{\lambda}\left(1-e^{-\lambda(t-v)}\right)+\rho\sigma_S\right].\end{aligned}$$

Therefore

$$r_t = f\left(0,t\right)+\left(1-e^{-\lambda t}\right)\left[\frac{\sigma^2}{2\lambda^2}\left(1-e^{-\lambda t}\right)+\frac{\rho\sigma\sigma_S}{\lambda}\right]+\sigma\int_0^t e^{-\lambda(t-v)}d\tilde{W}_v. \quad \text{(A1)}$$

As the last equation shows, the exponentially decaying structure of the forward rate volatility leads to a mean-reverting form of the short rate that closely resembles an extended version of the Vasicek (1977) model. Although equation ($A1$) is similar to the expressions derived by Vasicek (1977), it differs in the fact that it is obtained taking the initial term structure as exogenous, while for the Vasicek model the initial term structure is endogenous. According to ($A1$), under $\tilde{\mathbb{P}}$

$$\left(r_t - f\left(0,t\right)\right) \sim N\left(m_r\left(t\right), \sigma_r^2\left(t\right)\right) \quad \text{(A2)}$$

where

$$m_r\left(t\right) = \left(1-e^{-\lambda t}\right)\left[\frac{\sigma^2}{2\lambda^2}\left(1-e^{-\lambda t}\right)+\frac{\rho\sigma\sigma_S}{\lambda}\right]$$

and

$$\sigma_r^2(t) = \sigma^2 \left(\frac{1 - e^{-2\lambda t}}{2\lambda} \right). \tag{A3}$$

Therefore, we can show that

$$P_t(T) = \frac{P_0(T)}{P_0(t)} e^{-\frac{1}{2}\gamma^2(t,T)\sigma_r^2(t) - \gamma(t,T)(r_t - f(0,t))}, \tag{A4}$$

where

$$\gamma(t,T) = \left(\frac{1 - e^{-\lambda(T-t)}}{\lambda} \right).$$

A.2 Bond pricing

Let

$$F_\tau(T+t,T) := \frac{P_\tau(T+t)}{P_\tau(T)}$$

be the forward price at time $\tau \leq T$ for delivery at time T of a zero coupon bond expiring at time $(T+t)$, written as F_τ for short. From the HJM model, it follows that under $\hat{\mathbb{P}}$ the forward price satisfies the following stochastic differential equation:

$$dF_\tau = -\sigma_P(\tau,T)\, C_\tau(T,T+t)\, F_\tau d\tau + C_\tau(T,T+t)\, F_\tau d\hat{W}_\tau,$$

where

$$\sigma_P(\tau,T) = -\int_\tau^T \sigma e^{-\lambda(s-\tau)} ds = \frac{\sigma}{\lambda}\left(e^{-\lambda(T-\tau)} - 1\right)$$

is the volatility of a T zero coupon bond price process, and

$$\begin{aligned} C_\tau(T,T+t) &= \sigma_P(\tau,T+t) - \sigma_P(\tau,T) \\ &= \frac{\sigma}{\lambda} e^{-\lambda(T-\tau)} \left(e^{-\lambda t} - 1\right); \end{aligned}$$

so that

$$C_\tau(T,T+t) = -\sigma e^{-\lambda(T-\tau)} \gamma(T,T+t). \tag{A5}$$

Hence, under $\tilde{\mathbb{P}}$

$$\begin{aligned} dF_\tau &= -\sigma_P(\tau,T)\, C_\tau(T,T+t)\, F_\tau d\tau + C_\tau(T,T+t)\, F_\tau \left(d\tilde{W}_\tau + \rho\sigma_S d\tau\right) \\ &= C_\tau(T,T+t)\left(-\sigma_P(\tau,T) + \rho\sigma_S\right) F_\tau d\tau + C_\tau(T,T+t)\, F_\tau d\tilde{W}_\tau \\ &= A_\tau(T,T+t)\, F_\tau d\tau + C_\tau(T,T+t)\, F_\tau d\tilde{W}_\tau, \end{aligned}$$

with

$$A_\tau(T,T+t) = C_\tau(T,T+t)\left(-\sigma_P(\tau,T) + \rho\sigma_S\right),$$

or

$$A_\tau\left(T, T+t\right) = \gamma\left(T, T+t\right) e^{-\lambda(T-\tau)} \left[\frac{\sigma^2}{\lambda}\left(e^{-\lambda(T-\tau)} - 1\right) - \rho\sigma\sigma_S\right]. \qquad \text{(A6)}$$

Note that the forward price $F_\tau\left(T+t, T\right)$ admits the following representation under $\tilde{\mathbb{P}}$:

$$F_\tau\left(T+t, T\right) = F_0 e^{\int_0^\tau \left(A_v(T,T+t) - \frac{C_v(T,T+t)^2}{2}\right)dv + \int_0^\tau C_v(T,T+t)d\tilde{W}_v}.$$

In particular, consider $F_T\left(T+t, T\right) = P_T\left(T+t\right)$, then

$$\begin{aligned} F_T\left(T+t, T\right) &= F_0 e^{\int_0^T \left(A_v(T,T+t) - \frac{C_v(T,T+t)^2}{2}\right)dv + \int_0^T C_v(T,T+t)d\tilde{W}_v} \\ &= F_\tau e^{\int_\tau^T \left(A_v(T,T+t) - \frac{C_v(T,T+t)^2}{2}\right)dv + \int_\tau^T C_v(T,T+t)d\tilde{W}_v} \\ &= F_\tau e^{X_\tau(T)}, \end{aligned}$$

with

$$X_\tau\left(T\right) = \int_\tau^T \left(A_v\left(T, T+t\right) - \frac{C_v\left(T, T+t\right)^2}{2}\right) dv + \int_\tau^T C_v\left(T, T+t\right) d\tilde{W}_v. \qquad \text{(A7)}$$

Equations $(A5)$ and $(A6)$ imply that:

$$\begin{aligned} \int_\tau^T C_v\left(T, T+t\right)^2 dv &= \sigma^2\gamma^2\left(T, T+t\right)\int_\tau^T e^{-2\lambda(T-v)}dv \\ &= \gamma^2\left(T, T+t\right)\sigma^2\left(\frac{1-e^{-2\lambda(T-\tau)}}{2\lambda}\right) \\ &= \gamma^2\left(T, T+t\right)\sigma_r^2\left(T-\tau\right); \end{aligned}$$

and

$$\begin{aligned} \int_\tau^T A_v\left(T, T+t\right) dv &= \gamma\left(T, T+t\right)\frac{\sigma^2}{\lambda}\int_\tau^T e^{-\lambda(T-v)}\left(e^{-\lambda(T-v)} - 1\right) dv \\ &\quad -\gamma\left(T, T+t\right)\rho\sigma\sigma_S\int_\tau^T e^{-\lambda(T-v)}dv \\ &= \gamma\left(T, T+t\right)\frac{\sigma^2}{\lambda}I_1 - \gamma\left(T, T+t\right)\rho\sigma\sigma_S I_2. \end{aligned}$$

The integrals I_1 and I_2 can be evaluated thus:

$$\begin{aligned}
I_1 &= \int_\tau^T e^{-\lambda(T-v)}\left(e^{-\lambda(T-v)}-1\right)dv \\
&= -\frac{1}{2\lambda}\left(1-e^{-\lambda(T-\tau)}\right)^2; \\
I_2 &= \int_\tau^T e^{-\lambda(T-v)}dv \\
&= \frac{1}{\lambda}\left(1-e^{-\lambda(T-\tau)}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_\tau^T A_v(T,T+t)\,dv &= -\gamma(T,T+t)\frac{\sigma^2}{2\lambda^2}\left(1-e^{-\lambda(T-\tau)}\right)^2-\gamma(T,T+t)\frac{\rho\sigma\sigma_S}{\lambda}\left(1-e^{-\lambda(T-\tau)}\right) \\
&= -\gamma(T,T+t)\left(1-e^{-\lambda(T-\tau)}\right)\left[\frac{\sigma^2}{2\lambda^2}\left(1-e^{-\lambda(T-\tau)}\right)+\frac{\rho\sigma\sigma_S}{\lambda}\right] \\
&= -\gamma(T,T+t)\,m_r(T-\tau).
\end{aligned}$$

Hence, from $(A7)$ it follows that

$$X_\tau(T) = -\gamma(T,T+t)\,m_r(T-\tau)-\frac{1}{2}\gamma^2(T,T+t)\,\sigma_r^2(T-\tau)-\sigma\gamma(T,T+t)\int_\tau^T e^{-\lambda(T-v)}d\tilde{W}_v.$$

This implies that:

$$X_\tau(T)\sim N\left(-\gamma(T,T+t)\,m_r(T-\tau)-\frac{1}{2}\gamma^2(T,T+t)\,\sigma_r^2(T-\tau),\gamma^2(T,T+t)\,\sigma_r^2(T-\tau)\right).$$

A.3 Pricing the guaranteed annuity option

In section 2.1, we showed that the valuation formula for the guaranteed annuity option is

$$\begin{aligned}
& V_x(x+\tau,\tau,T-\tau) \\
= \quad & {}_{T-\tau}p_{x+\tau}S_\tau +{}_{T-\tau}\,p_{x+\tau}gS_\tau \sum_{t=0}^{w-(T+x)} {}_tp_{T+x}\tilde{\mathbb{E}}\left[\left.(P_T(T+t)-K_t)^+\right|\mathcal{F}_\tau\right].
\end{aligned}$$

Therefore, in order to determine a closed analytical formula for the option contract, we need to solve the following expectation:

$$\tilde{\mathbb{E}}\left[\left.(P_T(T+t)-K_t)^+\right|\mathcal{F}_\tau\right],$$

where $\tilde{\mathbb{E}}$ denotes the expectation under the stock risk-adjusted probability measure $\tilde{\mathbb{P}}$ and

$$K_t = P_T^* (T + t)$$

is the artificial strike price calculated as the bond price such that

$$\sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T (T + t) = K.$$

The expectation under consideration can be solved using the results obtained in the previous sections. In fact, it can be rewritten as:

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\left(P_T (T + t) - K_t \right)^+ \middle| \mathcal{F}_\tau \right] \\
= & \tilde{\mathbb{E}} \left[\left(F_T (T + t, T) - K_t \right)^+ \middle| \mathcal{F}_\tau \right] \\
= & \int_{-d_t}^{\infty} \left(F_\tau e^{-\frac{1}{2}\gamma^2(T,T+t)\sigma_r^2(T-\tau)-\gamma(T,T+t)(m_r(T-\tau)-\sigma_r(T-\tau)y)} - K_t \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\end{aligned}$$

where $y \sim N(0, 1)$ and

$$d_t = \frac{\ln \frac{P_\tau(T+t)}{K_t P_\tau(T)} - \frac{1}{2}\sigma_r^2 (T - \tau) \gamma^2 (T, T + t) - \gamma (T, T + t) m_r (T - \tau)}{\gamma (T, T + t) \sigma_r (T - \tau)}.$$

Therefore

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\left(P_T (T + t) - K_t \right)^+ \right] \\
= & F_\tau e^{-\frac{1}{2}\gamma^2(T,T+t)\sigma_r^2(T-\tau)-\gamma(T,T+t)m_r(T-\tau)} \int_{-\infty}^{d_t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y^2-2\gamma(T,T+t)\sigma_r(T-\tau)y\right)} dy \\
& -K_t \int_{-\infty}^{d_t} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
= & F_\tau e^{-\gamma(T,T+t)m_r(T-\tau)} \int_{-\infty}^{d_t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\gamma(T,T+t)\sigma_r(T-\tau))^2} dy \\
& -K_t N (d_t) \\
= & F_\tau e^{-\gamma(T,T+t)m_r(T-\tau)} N (d_t') - K_t N (d_t) \\
= & \frac{P_\tau (T + t)}{P_\tau (T)} e^{-\gamma(T,T+t)m_r(T-\tau)} N (d_t') - K_t N (d_t)
\end{aligned}$$

where

$$\begin{aligned}
d_t' &= d_t + \gamma (T, T + t) \sigma_r (T - \tau) \\
&= \frac{\ln \frac{P_\tau(T+t)}{K_t P_\tau(T)} + \frac{1}{2}\sigma_r^2 (T - \tau) \gamma^2 (T, T + t) - \gamma (T, T + t) m_r (T - \tau)}{\gamma (T, T + t) \sigma_r (T)},
\end{aligned}$$

which leads to equation (7).

References

[1] Ballotta, L. and Haberman S. (2002). Valuation of guaranteed annuity conversion options, under review.

[2] Ballotta, L. and Haberman S. (2003). The fair valuation problem of guaranteed annuity options: the stochastic mortality environment case, under review.

[3] Bank of England Annual Statistical Abstract, Part 1, September 2001, updated to February 2002.

[4] Bezooyen, J. T. S., Exley C. J. E. and Mehta S. J. B. (1998). Valuing and hedging guaranteed annuity options. Presented to Institute and Faculty of Actuaries Investment Conference, September 1998, University of Cambridge.

[5] Bolton, M. J., Carr, D.H., Collis, P.A. et al (1997). Reserving for Annuity Guarantees – The Report of the Annuity Guarantees Working Party, Institute of Actuaries, London.

[6] Continuous Mortality Investigation Bureau, (1998).The mortality of pensioners in insured group pension schemes 1991-94. Continuous Mortality Investigation Report N° 16, *The Institute and Faculty of Actuaries, UK.*

[7] Continuous Mortality Investigation Bureau, (1999). Standard tables of mortality based on the 1991-94 experiences. Continuous Mortality Investigation Report N° 17, *The Institute and Faculty of Actuaries, UK.*

[8] Geman, H., El Karoui N. and Rochet J. C. (1995). Changes of numeraire, changes of probability measure and option pricing, *Journal of Applied Probability*, 32, 443-458.

[9] Group Annuity Valuation Task Force (1995). 1994 Group annuity mortality and annuity reserving tables. *Transactions of the Society of Actuaries*, 47, 865-913.

[10] Heath, D., Jarrow R. A. and Morton A. (1992). Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation, *Econometrica*, 60, 77-105.

[11] Jamshidian, F. (1989). An exact bond option formula, *The Journal of Finance*, 44, 1, 205-209.

[12] Marocco, P. and Pitacco E. (1998). Longevity risk and life annuity reinsurance. Transactions of the 26^{th} International Congress of Actuaries, 6, 453-479.

[13] Milevsky, M. A. and Promislow S. D. (2001) . Mortality derivatives and the option to annuitise, *Insurance: Mathematics and Economics*, 29, 299-318.

[14] Olivieri, A. (2001). Uncertainty in mortality projections: an actuarial perspective. *Insurance: Mathematics and Economics*, 29, 231-245.

[15] Olivieri, A. and Pitacco E. (2002). Inference about mortality improvements in life annuity portfolios. Presented to 27^{th} International Congress of Actuaries, Cancun, Mexico.

[16] Pelsser, A. (2002). Pricing and hedging guaranteed annuity options via static option replication. Conference Proceedings of the 6^{th} International Congress on Insurance: Mathematics and Economics.

[17] Renshaw, A., Haberman S. and Hatzoupoulos P. (1996). The modelling of recent mortality trends in United Kingdom male assure lives, *British Actuarial Journal*, 2, 449-477.

[18] Sithole, T. Z.,Haberman S. and Verrall R. J. (2000) . An investigation into parametric models for mortality projections, with applications to immediate annuitants' and life office pensioners' data, *Insurance: Mathematics and Economics*, 27, 285-312.

[19] Vasicek, O. (1977). An equilibrium characterization of the term structure, Journal of Financial Economics, 5, 177-188.

[20] Yang, S., Waters H. R. and Wilkie A. D. (2002). Hedging for guaranteed annuity options, Conference Proceedings of the 6^{th} International Congress on Insurance: Mathematics and Economics.