MBA Actuaries
CONSULTING SERVICES

ST-2: Extreme Events: Statistical Extreme Value Theory and Its Applications

2010 Casualty Loss Reserve Seminar

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Developing Limit Distributions

- Can convergence of the distribution of the normalized M_n be achieved?
- What are optimal choices of normalizing sequences $\{a_n\}$ and $\{b_n\}$
- Which member of the GEV family of distributions obtained?
- In application, generally the underlying CDF, F, of the iid sequence $X_1,\ldots,X_n$ is unknown

Developing Limit Distributions

1. Let $F(x) = 1 - e^{-x}$ for $x > 0$ (Exponential)

- In determining the limit GEV distribution, let $a_n = 1$ and $b_n = n$
- The limit distribution as $n \to \infty$ of $\Pr\{(M_n - b_n)/a_n \leq z\}$ is $G(z) = \exp(-e^{-z})$, the Gumbel distribution corresponding to $\xi = 0$ in the GEV family
- The limit distribution of threshold exceedance is $H(y) = e^{-y}$, the exponential distribution which corresponds to a GPD with shape parameter $\xi = 0$ and adjusted scale parameter $\tilde{\sigma} = 1$

Developing Limit Distributions

2. Let $F(x) = \exp(-1/x)$ for $x > 0$ (Fréchet)

- In determining the limit GEV distribution, let $a_n = n$ and $b_n = 0$
- The limit distribution as $n \rightarrow \infty$ of $\Pr\{(M_n - b_n)/a_n \leq z\}$ is

  $G(z) = \exp(-1/z)$, the Fréchet distribution corresponding to

  $\xi = 1$ in the GEV family
- The limit distribution of threshold exceedance is $H(y) = (1 + y/u)^{-1}$,

  which corresponds to a GPD with shape parameter $\xi = 1$ and adjusted

  scale parameter $\tilde{\sigma} = u$

Developing Limit Distributions

3. Let $F(x) = x$ for $0 \le x \le 1$ (Uniform [0, 1])

- In determining the limit GEV distribution, let $a_n = 1/n$ and $b_n = 1$
- The limit distribution as $n \to \infty$ of Pr $\{(M_n - b_n)/a_n \le z\}$ is $G(z) = e^z$, a Weibull type distribution corresponding to $\xi = -1$ in the GEV family
- The limit distribution of threshold exceedance is $H(y) = 1 - y * (1 - u)^{-1}$, which corresponds to a GPD with shape parameter $\xi = -1$ and adjusted scale parameter $\tilde{\sigma} = 1 - u$

Relaxing Independence Condition

- A random process $X_1, \ldots, X_n$ is said to be **stationary** if, given any set of integers $\{i_1, \ldots, i_n\}$ and any integer m, the joint distributions of $\{\boldsymbol{X}_{i1}, \ldots, \boldsymbol{X}_{ik}\}$ and $\{\boldsymbol{X}_{i1+m}, \ldots, \boldsymbol{X}_{ik+m}\}$ are identical.
- A stationary series $X_1, \ldots, X_n$ is said to satisfy the D(u_n) condition if, for all $i_1 < \ldots < i_p < j_1 < \ldots < j_q$ with $j_1 - i_p > l$, |

  $\Pr\{X_{i1} < u_n, \ldots, X_{ip} < u_n, X_{j1} < u_n, \ldots, X_{jq} < u_n\}$ - $\Pr\{X_{i1} < u_n, \ldots, X_{ip} < u_n\}$ * $\Pr\{X_{j1} < u_n, \ldots, X_{jq} < u_n\} \leq \alpha(n, l)$, where $\alpha(n, l_n) \rightarrow 0$ for some sequence l_n such that $l_n / n \rightarrow 0$ as $n \rightarrow \infty$

Relaxing Independence Condition

Let $X_1, \ldots, X_n$ be a stationary process, and define

$M_n = \max\{X_1, \ldots, X_n\}$ Then if $\{a_n > 0\}$ and $\{b_n\}$ are sequences of constants such that

$$\Pr\{(M_n - b_n)/a_n \le z\} \to G(z)$$

as

$n \to \infty$

where G is a non-degenerate distribution function, and the $D(u_n)$ condition is satisfied with $u_n = a_n z + b_n$ for all real z, G is a member of the GEV family

Relaxing Independence Condition

- Do the results for the threshold excesses hold under the relaxed independence conditions?
- Main concern is clustering of maxima (dependence over a short time interval) which could tend to invalidate the regularity conditions for asymptotic normality of the MLE
- In other words, the likelihood estimates of the parameters can be poor or inappropriate for the model

Relaxing Independence Condition

One approach to obtain a set of threshold excesses that are approximately independent is **declustering** or filtering of dependent observations

1) Use an empirical rule to define clusters of exceedances
2) Identify the maximum excess within each of the clusters
3) Assume the cluster maxima to be independent, with conditional excess distribution given by the GPD
4) Fit the GDP to the cluster maxima

Relaxing Independence Condition

Limitations to declustering method:

1. Sensitive to arbitrary choices in cluster determination (cluster maxima)
2. Wastage of data by disregarding portions of data when selecting a single maxima for a cluster
3. Trade off in determining the distance between clusters
   a. Too little introduces to much bias in parameter estimation
   b. Too much reduces the number of independent maximas, thus increasing the variance

Parameter Estimation

There are various techniques which can be used for parameter estimation for the GEV and threshold exceedance models

- Graphical - Probability plots, quantile plots, return level plots, density plots
- Moment based techniques – Function of model moments compared to empirical equivalents
- Parameter estimated as a specified function of order statistics
- Likelihood techniques

Parameter Estimation

- The choice of technique or techniques applied should consider the trade off between bias and variance and the accuracy of the results of each technique given the observed properties of the underlying process
- All-around utility and adaptability to complex model building make it an attractive choice
- Drawback to MLE is the regularity conditions required for the usual asymptotic properties for validation

Parameter Estimation

- Regularity conditions are not satisfied by the GEV model because the endpoints are functions of the parameter values:
  a. $\mu - \sigma/\xi$ is an upper endpoint when $\xi < 0$ (Weibull type)
  b. $\mu - \sigma/\xi$ is an lower endpoint when $\xi > 0$ (Fréchet)
- Smith addresses some of these concerns in "Maximum Likelihood Estimation in a Class of Non-regular cases" (1985)

Parameter Estimation

- Smiths findings:
  1. If $\xi > 0.5$, the MLE are regular in the sense of having usual asymptotic properties
  2. If $-1 < \xi < 0.5$, the MLE are obtainable but do not have standard asymptotic properties
  3. If $\xi < -1$, the MLE is unlikely to be obtainable

  BUT

  For $\xi \leq -0.5$ the distribution has a very short, bounded upper tail and is not usually encountered in EVM

QUESTIONS?