Geo-Spatial Metrics for Insurance Risk Concentration and Diversification

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Abstract

(Re)insurance practitioners view geo-spatial variability of insurable losses and claims as a significant risk factor in the definition of key business processes and their outcomes. This second order, volatility type of a risk factor impacts the construction of insurance rates and reinsurance treaty premiums, the computation of reserve capital, the management of concentrations of physical and financial risk. With increased emphasis in the industry on analytics, modeling and measuring of all types of physical and financial variability and volatility, both temporal and geo-spatial, new effort in enriching the scope of metrics, which capture the nature of second order risk is much needed. In its first generation, second order metrics are pair-wise by nature. Significant effort by academics and practitioners is under way to develop a new generation of such metrics, which capture and express the complexities of concentration and inter-connectedness of multiple risk factors, of both physical, geo-spatial and of financial nature. Clarity and intellectual discipline of the definition of second order geo-spatial risk metrics helps (re)insurance practitioners to adopt these statistical and computational methodologies effectively and promptly. Further in significance, clarity, consistence and coherence of second order geo-spatial risk metrics allows practitioners to relate them efficiently to the main business workflows of (re)insurance firms, apply them in effective measurement, mitigating and hedging situations, and finally to propagate them easily to executive level decision makers.

Key Words

Geo-spatial risk metrics, diversification and concentration indices, insurance risk management

Introduction

Achieving sustainable scale of business operations is a key objective for insurance and reinsurance firms today. This drive towards market scale takes many forms. It is expressed in the motivation to acquire larger market shares of gross underwritten premiums by various means, most obvious being through acquisitions of business lines or whole books of insurance business. Tangible economies of scale are also an internal objective, and a driving force in pursuing of optimal cost and distribution of reserve capital among the firm's business units. Whichever is the business realization of the drive towards market scale, the achievement of this goal has two very different but intersecting implications for the insurance firm. One of them is the emerging demand for measuring and accounting for the effects of diversification in a (re)insurance book of business due to geo-spatial, insurance coverage or financial risk factor inter-dependencies. The second is measuring and accounting for concentration and clustering of insurance risks, which are subject to mutually, highly contingent outcomes. For top-level practitioners at a (re)insurance firm, to achieve scale in insurance operations it becomes a critical task to master the modeling and measurement of interdependence among risk factors in all forms, including the fundamental ones of exposure and expected loss diversification, and their counter-positioned expression of to risk concentration.

The objective of our paper is to review existing geo-spatial risk metrics, contribute to the development of new ones, and attempt to enhance the market applicability of such techniques and methodologies, for measuring of interdependence for insurance risk factors. It is also our objective to outline their direct use and applications in practical portfolio risk management for an insurer portfolio, and more particularly in managing underwriting concentration of insurance exposures, and henceforth defining and following with capital reserving tasks. For this very purpose we derive a set of metrics and indices from modeled and simulated insurance losses, for a detailed physical and geographical distribution of insured exposures, which comprise a book of business of a notional and regional firm.

In *section 1* of this paper we review the economic theory and motivation for developing diversification and concentration risk metrics, from the perspective of the insurance firm. We also outline the structure of the insured exposure of our case study, notional insurer, which we use to compute and construct our numerical risk metrics and indices. In *section 2*, we provide the details of the insurance loss modeling methodology and simulation, which produces the outputs for the development and construction of all numerical metrics and indices, which we present in *section 3*. In the following *section 3*, we develop and discuss three new geo-spatial risk metrics. Then in *section 4*, we show how the process of back-allocation of losses and risk metrics from a global portfolio level down to its single risk components impacts the inter-dependence structure of these same metrics within the overall risk profile of the insurance book of business. In *section 5*, we continue to focus on applied risk management by introducing three new diversification and concentration portfolio indices. With the same *section 5*, we conclude our article with analysis of the applied utility of these metrics for insurance risk management professionals, and outline some further research directions.

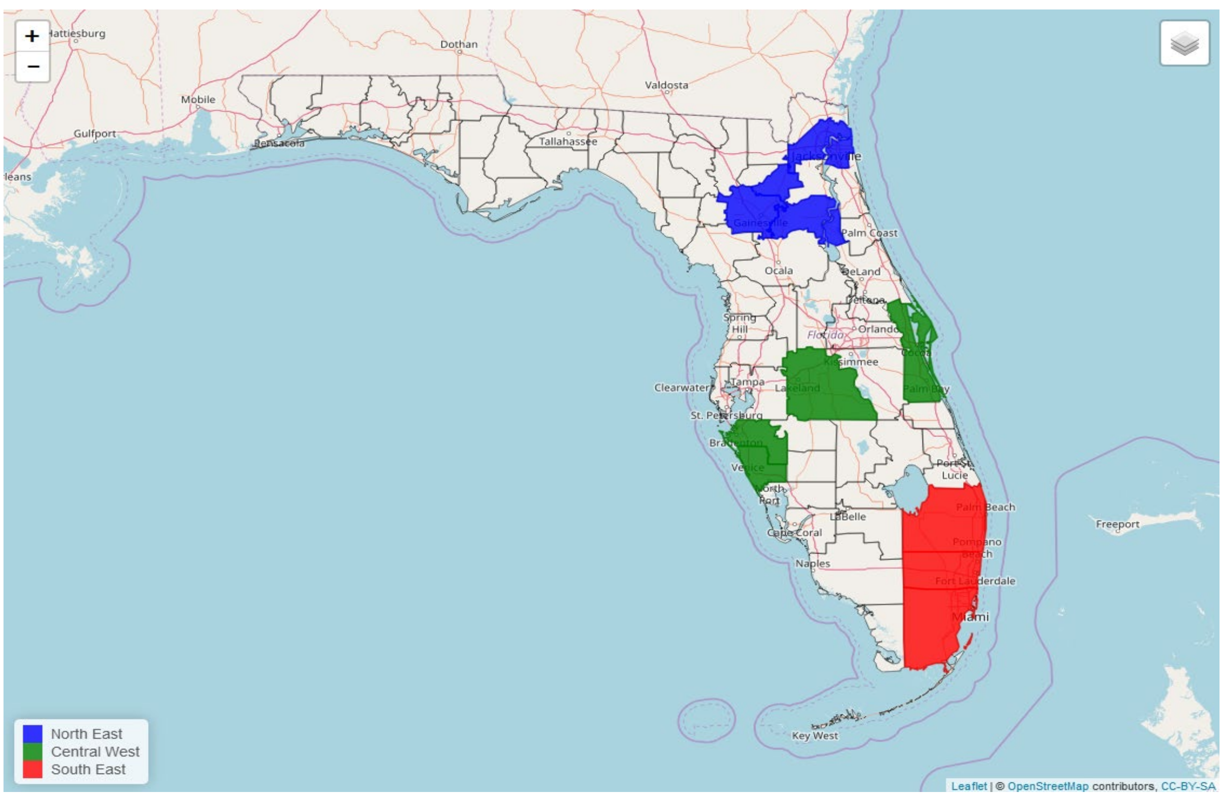
1.0 Economic Theory and Motivation

Connectedness, also described as interconnectedness, has become a central theme in modern risk management, still it has remained technically ambiguous, and less well defined than traditional statistical pairwise correlation metrics. Typically, modern financial and insurance portfolio correlation metrics are pair-wise, and are applied in linear Gaussian methodological and modeling frameworks. The impact of insurance risk interconnectedness on critical market and business tasks such as pricing, underwriting, concentration management, and reserving demands further improvements in our analytical and measurement practices. In this article we focus on measuring the inter-connectedness and its effects on a critical portfolio risk factor – the geo-spatial distribution of insurance claims and losses among the company business units of an insurance firm. Loss analytics, rate making, and premium pricing practices, among *company units* within an insurance firm, are inter-connected through the impact of claims from extreme intensity catastrophe events, with large geo-spatial footprints, which cause, what practitioners term, 'clustering or concentrations' in the accumulation of losses. Such undesirable loss accumulations may take place as well, during smaller catastrophe events, which occur in clustered patterns both spatially but also temporally in shorter spans of time – a phenomenon known as temporal clustering. Since premium prices, insurance rates and reserves to a great extend depend on a modeled expectation of insurance loss, practitioners are also concerned about the volatility, or various expressions of uncertainty in such loss expectations. In *section 3* of this paper we examine how identifiable and measurable connectedness impacts the variability in loss expectations, and henceforth the definition of key business metrics. Particularly we find that in the presence of strong and measurable connectedness among risk factors both volatility of loss, and variability of risk metrics are magnified by extreme catastrophe events, and by extreme disaster scenarios.

For the purpose of providing an analytical framework for the numerical tasks of deriving and testing risk metrics we use a notional case study of an insurance firm, based in the state of

Florida, and comprised of three business units, covering insured risks situated in twelve geo-administrative areas – in our case, defined as state counties. These three business units are named respectively as *South East (SE), Central Unit (CU) and North East (NE)*. The risks in two of the business units – *South East and North East,* are located in immediately neighboring administrative areas (counties), while in the case of the third business unit – Central *(CU)*, they are in proximity, but not strictly adjacent to each other, as illustrated in *figure 1*. In addition *SE* and *NE* business units are comprised of risks from four geographically bordering counties, while *CU* is comprised of risks from only three counties. Lastly, in turn these three business units constitute the whole book of business of this notional and regional insurer.

Figure 1: Three business units comprised of counties constitute the insurer book of business



The general purpose of our analysis becomes the modeling, measuring, and understanding of dependencies among business units in this notional insurance firm, and then presenting our results through a numerical and notional case study, which we develop in *sections 3 and 4*. To this effect we begin by discussing three new metrics, which we propose will facilitate practitioners in their tasks of the measurement of second order risk and inter-dependence among business units, but more generally among risk factors, in an insurance firm. The contribution to risk management of these new metrics is many twofold – but at least in our study we focus on their potential to: (1) enhance the accuracy of capital reserving; and (2) to facilitate optimal selection of second order, risk mitigating insurance and financial contracts and instruments.

We examine traditional 'pairwise connectedness' measured between any two of the firm's business units, as well as between any two geo-administrative counties within the same company unit, and in parallel within the full portfolio. We also compute and analyze 'marginal connectedness', which is defined in our study, as the connectedness of a single company unit, or a single geographic area to the remainder of the insured portfolio. We will show how both metrics of pairwise connectedness, and marginal connectedness become valuable tools in execution of critical business tasks of managing concentration and risk ranking analysis, pricing and underwriting, and reserving and risk management.

2.0 Physical Peril Simulation for Insurance Loss

For the purpose of the numerical study of this article, we use simulated and modeled insurance losses produced by AIR Worldwide tropical cyclone model for North America, which includes coverage for the state of Florida, where the insured exposures of our notional firm are located. The model can be viewed as a sequence of conditional statistical algorithms, which we will describe at a very high level in this section. The primary objective of this work is not to review and provide detailed analysis of this type of natural catastrophe models for the insured peril of tropical cyclone, but to use their loss outputs to illustrate numerically, as well as theoretically, methodologies for second order, geospatial risk management, which in turn could be utilized by insurance industry practitioners.

The first set of modeling algorithms derives statistical probability distributions from historical annual frequency data of recorded storms. The critical variables in defining the physical model of a tropical cyclone are *central barometric pressure, radius of maximum winds, and forward speed*. Whereby central pressure is the primary physical determinant of the storm, and therefore the primary modeling function to define the key variables for the computation of its intensity. Theoretical statistical distributions for all of these three variables are used as baseline generators, and tested for goodness of fit to historical meteorological data. In a third stage of the model sequence of algorithms, a large sample of simulated and fully probabilistic storm tracks are generated. These storm tracks are generated from conditional probabilistic distributions derived from a large historical data set of recorded storm events. The sampled and simulated physical parameters are propagated through the storm track, and an analytical equation estimates the storm intensity on a predefined geo-spatial grid of a very high three dimensional metric and temporal granularity. In the last step of this sequence, modeled storm intensity is linked through a non-linear damageability response function to produce the final insurable loss values for a pre-defined insured exposure, with a known monetary value, located by latitude and longitude coordinates on the 'global' geo-spatial model grid. Then insured loss is computed by traditional actuarial formulas, which apply insurance policy limits and deductibles on the actual insurable loss coming from the natural peril model simulation.

$$insured\ loss = \min[limit, \max(insurable\ loss - deductible, 0)]$$

3.0 Towards Second Order Insurance Risk Management

Ultimately at the technical and quantitative level, we define the purpose of our work as directed towards the objective of developing principles, techniques and tools for comprehensive second order insurance risk management. In the context of a single insurance firm we define this problem as the measurement and management of single insurance risk, and business unit dependencies and connectedness. For a typical insurance firm such inter-dependencies in its book of business are manifold, including such of geo-spatial, physical, and financial nature. In parallel and at the macro-economic level, the risk measurement problem becomes one of defining and quantifying the systemic nature of the inter-connectedness of insurance and financial firms. In this article we focus on geo-spatial and physical inter-connectedness of insurance risks, at the micro - structure of a single insurance firm. Our data, computations, examples, and case studies are of this nature.

Theoretically we define a geo-spatial, second order risk metric to be a function ρ, which assigns a real number from the spatially aggregated distribution of insured losses $[X_{i,n}]$ by any predefined, geographic and administrative unit, which in our notional insurer corresponds to a business unit of the insurance firm. Further in our study we utilize non-coherent risk measures such as value-at-risk (VaR), also known as probable maximum loss (PML).

$$\rho[X_n] = VaR_\alpha(X_n) = \inf\{x \mid P(X_n > x)1 - \alpha\}$$

In the process of deriving geo-spatial risk metrics, we also make extensive use of coherent measures, and mostly of tail-value at risk (TVaR) defined as:

$$\rho'[X_n] = TVaR_\alpha(X_n) = \frac{1}{1-\alpha}\int_\alpha^1 VaR(X_n)dt$$

We expect that non-coherent risk measures, such as value-at-risk (VaR) and probable maximum loss (PML), will display both super-additive and sub-additive properties in accumulation, dependent on the numerical and empirical case, and the physical and statistical premises of the simulation. In short, they will not behave in a theoretically persistent manner.

$$super - additvity\colon \sum_n \rho[X_i] < \rho\left[\sum X_{i,n}\right]$$

$$sub - additvity\colon \rho\left[\sum X_{i,n}\right] < \sum_n \rho[X_i]$$

However, we expect that a coherent risk measure, such as tail value-at-risk (TVaR) to behave strictly in sub-additive manner in accumulations from portfolio units to the global risk profile of the book of business. We also make use of the positive homogeneity property of risk measures.

$$\rho[kX_i] = k\rho[X_i], k \geqq 0$$

The homogeneity property is expected to hold for both coherent and non-coherent classifications of second, order geo-spatial risk metrics. In our notional insurance book, comprised of three business units, we notate the loss distributions by unit respectively as:

$$for\ i = 1\ to\ n,\ \textit{South East}\ \text{as}\ [X_{SE,1}, \dots, X_{SE,n}];$$

$$for\ j = 1\ to\ n,\ \textit{Central Unit}\ \text{as}\ [X_{CW,1}, \dots, X_{CW,n}];$$

$$for\ k = 1\ to\ n,\ \textit{North East}\ \text{as}\ [X_{NE,1}, \dots, X_{NE,n}];$$

Where *n* is equal to 10,000 stochastic, simulated loss scenarios, for each of these spatial loss distributions.

The first geo-spatial risk metric is most appropriately defined and noted as a *covariance ratio*. This metric is computed as the ratio of the covariance of insured losses of any two of the business unit pairs to the sum of the variances of these same loss distributions. The covariance ratio metric for two business units, for example for *South East* and *Central Unit* is notated as $CR_{SE,CU}$; and is expressed in its traditional statistical form.

$$CR_{SE,CU} = \frac{COV[X_i : X_j]}{VAR[X_i] + VAR[X_j]}$$

The covariance ratio metric is computed for all three pairs of business units in *table 1*.

Table 1: Covariance Ratio, presented as a second order risk and geo-spatial metric, by each pair of business units

North East			
Central Unit			0.19
South East		0.24	0.05
	South East	**Central Unit**	**North East**

With this first risk metric, numerical results converge to first principles, from expectations of the importance of cumulative geo-spatial distances, being a main determinant in its explanatory strength. The distances $d_{i,k}$ between any two pairs of individual insured risks such as $r_{SE,i}$ *and* $r_{CU,k}$ are accumulated to measure the total conceptual and cumulative geo-spatial distance *D* between any two business units, in this case *South East* and *Central Unit.*

$$D_{SE,CU} = \sum_{i \neq k}^{n} d_{i,k}$$

The two business units of South East and North East, which have the largest cumulative geospatial distance between their respective comprising risks, as expected, have the lowest covariance ratio.

The second risk metric, is in part a transformation of the covariance ratio, and is notified as *covariance percent share, or covariance share* CS in brief. We define this metric as the percent share of the covariance of any two business units' losses to the full sum of the covariance matrix of all pairs of business units' losses. Again, in the example case of *South East* and *Central Unit* the new metric, notified as $CS_{SE,CU}$ takes the form:

$$CS_{SE,CU} = \frac{COV[X_i : X_j]}{\sum \sum COV[X_{i,j,k} : X_{k,j,i}]}$$

The covariance percent share geo-spatial risk metric is computed for all three notional business unit pairs in *table 2*.

Table 2: Covariance Percent Share, computed as a second order risk metric, for each pair of business units

North East			
Central Unit			0.05
South East		0.81	0.15
	South East	**Central Unit**	**North East**

Our third metric quantifies the marginal impact of second order insurance risk by measuring the geo-spatial inter-dependence of each business unit to the accumulated insurance loss of all other business units – in our case the remainder of the book of business is constructed of two units, as the whole book is constituted of three units. We notify this metrics as the *marginal covariance ratio*. To compute this metric, we create three marginal portfolio loss distributions, which are defined as the aggregated sum of each combination of any two out of all three-business' unit simulated losses. For example, the aggregated loss distribution without the *Northeast* business unit losses $[X_{NE,k}]$ is defined as the accumulated loss of the other two business units - *Central Unit and South East.* This combined, partial loss $Q_{1,n}$ is produced by a joint generating function q.

$$Q_1 = q[X_{1,CU}, X_{SE,1}], \ldots, Q_n = q[X_i, X_j]$$

The other two bi-regional loss distributions are expressed in similar joint, partial form with aggregate loss generating functions respectively $q' and\ q''$.

$$Central\ Unit\ and\ North\ East - [q'(X_{1,CU}, X_{1,NE}), \ldots, q'(X_j, X_k)]$$

$$South\ East\ and\ North\ East - \left[q''(X_{1,SE}, X_{1,NE}), \dots, q''(X_i, X_k)\right]$$

$$for\ i, j, k\ from\ 1\ to\ n = 10{,}000$$

With the two components of single unit losses and partial portfolio losses, the marginal covariance MC itself is computed between any single business unit and the aggregated loss of the remainder of the portfolio. In our example the marginal covariance MC_{NE} of losses for *North East,* $X_{NE,k}$ will be expressed to the aggregate loss $q(X_{SE,i}, X_{CU,j})$ from units *South East* and *Central.* Then the marginal covariance takes traditional statistical form:

$$MC_{NE} = COV[X_{NE,k}: q(X_{SE,i}, X_{CU,j})]$$

In turn, the marginal covariance ratio for any business unit, including in our example *North East* - MCR_{NE}, is formally defined as the ratio of the actual marginal covariance to the sum of the single unit variance and the partial variance of the remainder of the book of business.

$$MCR_{NE} = \frac{COV[X_{NE,k}: q(X_{SE,i}, X_{CU,j})]}{VAR\left[X_{NE,k}\right] + VAR\left[q(X_{SE,i}, X_{CU,j})\right]}$$

This new *marginal covariance ratio* metric is computed for the portfolio's three business units in *table 3*.

Table 3: Marginal Covariance Ratio, computed as a second order risk metric, for each business unit

North East			0.05
Central Unit		0.28	
South East	0.23		
	Central Unit &	**North East &**	**Central Unit &**

	North East	South East	South East

Each of these three metrics presents a different, but mutually complementing picture of active practices for measuring geo-spatial variability and inter-dependence in second order of insurance risk. We started by developing a purely pair-wise metric of *covariance ratio*; and then transformed it to still a pair-wise but now advantageously unitary, *covariance percent share* metric. Lastly, in our third step, we arrived at a marginal impact type of metric – *marginal covariance ratio*, which is no longer purely pairwise and is not unitary, but describes the relationship of every single unit to the accumulated remainder of the insurance book of business. Business units, which are comprised of counties, that are geographically neighboring, as expected produce metrics, which indicate higher degrees of dependence in second order of geospatial insurance risk. Then in a reverse relationship, we observe that business units with the largest cumulative spatial distances produce geo-spatial metrics of the lowest inter-dependence. These new metrics, in addition, allow for the explicit quantification and ranking of business units by magnitude of 'second order riskiness'. This ranking practice by a second order of geo-spatial risk metric can also be used very effectively within a business unit to understand relations and contributions of individual risks to the overall profile of the unit, and hence the metric becomes a valuable ranking tool for concentration and underwriting management. Thus once the exposure and expected loss outcomes of the portfolio geo-spatial risk distributions are well understood, by using such metrics practitioners would be in a very much improved environment for designing and structuring of mitigating insurance and financial products, as well as for underwriting and risk management tasks and strategies.

4.0 Covariance back-allocation, and Geospatial Inter-dependence of Metrics

In this section we proceed to develop explicitly the linkages between statistical measures of geo-spatial inter-dependence and the practical, business-minded understanding and measuring of portfolio diversification. We derive further spatial risk measures between insurance risk factors by using portfolio and business unit tail value-at-risk (TVaR), which is a coherent market risk metric, and comes with a broad practitioner acceptance and understanding. The individual business unit risk metrics of TVaR are derived by a procedure of back-allocation, executed downwards from the global portfolio level of the total, corporate 'global' metric. Back-allocating risk metrics, which practitioners often call a 'top-down approach', from an insurance or a financial portfolio total metric, reflects the notion that inter-dependence among risk factors, at the portfolio level itself, is present and captured in the modeled and global loss distribution of the portfolio risk factors. These impacts of risk factor interdependence themselves are contained in the modeled portfolio metric due to the very nature of the joint multi-variable insurance loss accumulation. In *section 2.0*, we described the physical peril model intensity simulation procedure, which models inter-dependent insurance losses in the geo-spatial domain on a grid based, granular system of 30 meters resolution. Two types of physical peril intensity inter-dependences are explicitly modeled: firstly of geo-spatial nature, which captures dependence among losses within a single simulated scenario; and secondly of temporal and inter-event nature, which captures dependencies across simulated, temporally clustered catastrophe scenarios. Analysis of the modeling principles and techniques of these effects is a large and separate research subject, and at present is beyond the scope of this article.

Expected and significant benefits of working with accurate and comprehensive second order metrics for risk management and underwriting guidelines' are that both pure technical premiums and risk measurements, such as value-at-risk and tail value-at-risk, are optimally back-allocated to individual risks and policies, and thus provide a sustainable and competitive edge in pricing and reserving. Capturing of interdependencies with back-allocation practices reflects the effects of diversification, or adversely of concentration, in an insurance book of business, which are both critical in making business decisions. For the rest of this study we will use the metric of tail value-at-risk, as the theoretically coherent choice for a portfolio and business unit risk metric. With respect to coherence, the most significant and fully required premise, for the theoretical purposes of our study, is the support of the sub-additive principle of accumulations of a risk metric ρ. In our notional case study, this principle is expected to hold in accumulations of the risk metric ρ itself from the business unit to the portfolio global level.

$$\rho[portfolio] \leq \sum_{i=1}^{k} \rho[business\ unit]$$

In the processes of corporate risk management, one practical translation of measuring inter-dependence is the identification and *measurement of portfolio diversification.* With good understanding and application of such principles, practitioners could then take advantage of measuring and understanding diversification while defining and optimizing cost savings from optimal capital reserves allocation, (see Zvezdov & Rath, 2017).

This algorithmic flow for *measuring portfolio diversification* takes a few steps of allocation and transformation. It begins by measuring a corporate, global $TVaR$ from the aggregated and full loss distribution Y_p of the entire insurance book of business, comprised of its three business units, with a joint aggregate loss generating function g:

$$Y_1 = g[X_{CW,1}, X_{SE,1}, X_{NE,1}], \dots, Y_n = g[X_{CW,n}, X_{SE,n}, X_{NE,n}]$$

$$for\ i,j,k\ from\ 1\ to\ n = 10{,}000$$

$$VaR_\alpha(Y_n) = \inf\{y \mid P(Y_n > y)1 - \alpha\}$$

$$TVaR_\alpha(Y_n) = \frac{1}{1-\alpha}\int_\alpha^1 VaR(Y_n)dt$$

After computing the numerical $TVaR(s)$ from the individual loss distributions of the business units, and then from the combined and joint loss distribution Y_p of the entire portfolio, we observe a sub-additive relationship, as it is expected by first theoretical principles of a coherent risk measure. We can now restate this sub-additive relationship in a form of $TVaR$ metric accumulation.

$$TVaR[portfolio] \leq \sum_{i=1}^{k} TVaR[business\ unit]$$

As a next step in the procedure to measure diversification, this total and global portfolio $TVaR_\alpha(Y_n)$ metric is allocated down to each single contributing risk – in our case to each business unit.

Practitioners use various numerical techniques for portfolio $TVaR$ back-allocation. In this study we examine the covariance, and the marginal covariance - variance back-allocation principles, which we develop, express, and modify specifically for the needs of our case study. These two capital and risk metric back-allocation techniques, to begin with, are constructed by expressing the decomposition of the full portfolio covariance matrix $\sum_{i=1}^{n}\sum_{k=1}^{n} COVAR(X_i, X_k)$ into its business unit covariance components.

$$\sum_{i=1}^{n}\sum_{k=1}^{n} Cov(X_i, X_k) = Cov\left(\sum_{i=1}^{n} X_i, \sum_{k=1}^{n} X_k\right) =$$

$$= \sum_{k=1}^{n} Cov(X_1, X_k) + \ldots + \sum_{k=1}^{n} Cov(X_n, X_k)$$

The marginal covariance of each individual business unit with the total portfolio is then expressed more robustly.

$$\sum_{k=1}^{n} Cov(X_i, X_{j+k}) = Cov(X_i, \sum_{k=1}^{n} X_{j+k})$$

An inter-dependent $TVaR$ for each business unit is constructed, with the contribution of a marginal covariance principle back-allocation weight w_i, computed in a ratio form.

$$w_i = \frac{Cov(X_i, \sum_{k=1}^{n} X_{j+k})}{Cov\left(X_i, \sum_{k=1}^{n} X_{j+k}\right) + Cov\left(X_j, \sum_{k=1}^{n} X_{i+k}\right) + Cov\left(X_k, \sum_{k=1}^{n} X_{j+i}\right)}$$

The business unit $TVaR$ back-allocation procedure itself is complete, with the expression below, showing the case for the South East SE unit risk metric, being dependent on the total portfolio $TVaR_\alpha(Y_n)$. This inter-dependent business unit metric is defined as a covariance back-allocated measure $TVaR_{CoVAR}[BU]$

$$TVaR_{CoVAR}[BU] = w_{i,BU} TVaR_\alpha(Y_n)$$

This covariance back-allocation relationship inevitably enforces a co-monotonic relationship between the sum of the allocated business unit metrics $TVaR_{CoVAR}[BU]$ and the total-global portfolio $TVaR_\alpha(Y_n)$.

$$TVaR_{\alpha}(Y_n) = \sum\nolimits_{i=1}^{k} TVaR[BU]_{CoVAR} \leq \sum\nolimits_{i=1}^{k} TVaR[BU]_{Independent}$$

This relation between the global metric, the back-allocated sum, and the sum of independent, stand-alone business unit metrics, described above, is a foundation for many daily tasks in the insurance portfolio risk management process. It also serves as a motivation and justification to search for optimal and cost-efficient distribution of capital reserves within the business.

Some practitioners use a back-allocation procedure, for both risk metrics and capital reserves, based on the marginal business unit covariance and the full variance of the total portfolio loss distribution Y_n.

$$TVaR[X_i]_{MCoVar/Var} = TVaR_{\alpha}(Y_n)\frac{Cov(X_i, \sum_{k=1}^{n} X_{j+k})}{VAR[Y_n]}$$

This second back-allocation technique does not support a strong theoretical and unitary relationship, as the sum of the marginal covariance of each unit: $Cov\left(X_i, \sum_{k=1}^{n} X_{j+k}\right) + Cov\left(X_j, \sum_{k=1}^{n} X_{i+k}\right) + Cov\left(X_k, \sum_{k=1}^{n} X_{j+i}\right)$, will not equal the variance of the global portfolio loss distribution – $VAR[Y_n]$. This is evident by pure statistical mechanics, which show that the sum of the marginal covariance(s) does not theoretically and necessarily equal the theoretical variance $VAR[Y_n]$ of the combined loss distribution.

$$VAR[Y_n] = VAR[X_i] + VAR\left[X_j\right] + VAR[X_k] + 2\sum_{i=1}^{n}\sum_{k=1}^{n} Cov(X_i, X_k)$$

$$VAR[Y_n] \neq Cov\left(X_i, \sum_{k=1}^{n} X_{j+k}\right) + Cov\left(X_j, \sum_{k=1}^{n} X_{i+k}\right) + Cov\left(X_k, \sum_{k=1}^{n} X_{j+i}\right)$$

With three types of metrics computed – one stand-alone and independent; a second back-allocated by the pure covariance principle; and a third back-allocated by the marginal covariance - variance principle, we explore their ranking and relative standing.

$$\sum TVaR[BU]_{MCoVar/Var} < \sum TVaR[BU]_{CoVar} < \sum TVaR[BU]_{Independent}$$

The right side of this inequality expresses support for sub-additive principles in accumulation, while the left demonstrates lack of a unitary relationship described earlier above. On one hand, theoretical principles of sub-additive accumulations guarantee that the independent sum of risk metrics exceeds the dependent and back-allocated sums, through covariance participations in the granularity of business unit risk metrics. On the other hand, the relationship between the pure covariance business unit metrics and those computed through back-allocation by the marginal covariance - variance principle cannot be strongly guaranteed by theoretical and statistical principles. The numerical ratios of the back-allocated business unit metrics to the independent stand-alone business unit metric are summarized in *table 4*. The theoretical expressions of the ratios in *table 4* take the statistical forms, described hereby.

$$ratio\ of\ pure\ Covar\ to\ independnet\text{:}\ \frac{TVaR[BU]_{CoVar}}{TVaR[BU]_{Independent}}$$

$$ratio\ of\ marginal\ Covar\ to\ independnet\text{:} \frac{TVaR[BU]_{MCoVar/Var}}{TVaR[BU]_{Independent}}$$

Table 4: Numerical ratios of covariance and marginal covariance / variance back-allocated business unit $TVaR[BU]$ to the independent and stand-alone business unit metric

Ratios of Business Unit TVaR(s)		
Independent	**to pure CoVar**	**to MCoVar / Var**

North East	2.73	1.27
Central Unit	2.11	0.98
South East	0.63	0.29

The numerical behavior of sub-additive accumulations of insurance portfolio $TVaR(s)$ by business unit supports an economic motivation and proposition that there are economies of scale and cost-savings' benefits in capital reserving, due to risk dispersion and geo-spatial diversification. The pure covariance back-allocation principle is most widely accepted among practitioners, and it is also the more coherent in mathematical risk theory.

5.0 Insurance Portfolio Diversification and Concentration Indices

Many of the requirements for cost effective and optimal capital reserving are dependent on measuring and quantifying portfolio diversification, which is theoretically justified under the mathematical principle of sub-additive accumulations. To provide numerical support and proof for this analysis, we continue to use our three business units - *South East (SE), North East (NE), and Central Unit (CU)*, which compose our single notional insurance firm. The first two business units are comprised of risks from bordering and clustered geo-administrative geographies – US counties. The risks in the last one – *CU*, are less spatially concentrated, as evident in *figure 1*.

In context of our case study of a notional insurance firm, we examine some indexed diversification measures DI derived from business unit and portfolio covariance and $TVaR$ metrics. For all three business units, and for the entire portfolio we measure $TVaR$ at $\alpha = 0.05$.

The index construction relies on the mathematical properties of sub-additive accumulation of $TVaR$, which we reviewed in the previous section. Computed from the loss distributions of these three business units, respectively $[X_i, X_j \; and \; X_k]$, and the distribution of the entire insurance book of business $[Y_n]$, these risk metrics numerically do provide support for first theoretical principles, expressed in a generalized form.

$$TVaR[Y_n] = TVaR\left[\sum_{i=1}^{n} X_{i,j,k}\right] \leqq \sum_{i,j,k=1}^{n} TVaR\left[X_{i,j,k}\right]$$

The first index, notified as $DI_{VaR/TVaR}$, is comprised of two types of components – (1) business unit and portfolio value-at-risk VaR and (2) tail value-at-risk $TVaR$. This first index includes VaR metrics, both at business unit and at portfolio level, which theoretically do not satisfy the requirements for mathematical coherence, and in particular of risk metric sub-additivity. Value at risk metrics, also known as probable maximum loss *PML*, in context of natural catastrophe risk management, are not coherent and sub-additive, as are their $TVaR$ counterparts. For the same reason of lack of coherence, $VaR \; and \; PML$ types of metrics are not subject to portfolio back-allocation, and are generally computed independently from their underlying, single factor loss distribution for the exposure or unit at risk. For this same reason, for the construction of the first diversification index we use risk metrics computed only from the stand-alone and independent loss distributions of the business units $[X_i, X_j \; and \; X_k]$, with $i, j \; and \; k \; from \; 1 \; to \; 10{,}000 \; simulation \; scenarios$, and not derived through the 'top-down' back-allocation procedure, developed in *section 4*. To detail the expression, with an example case for the North-East business unit, the index is constructed as the ratio of the difference of its independent VaR_{Ind} *and* $TVaR_{Ind}$ at $\alpha = 0.05$, to the sum of the same differences of independent metrics in all three units.

$$DI_{VaR/TVaR} = \frac{VaR_{Ind}[X_i] - TVaR_{Ind}[X_i]}{\sum_{i,j,k=1}^{n}\{VaR_{Ind}[X_i] - TVaR_{Ind}[X_i]\}}$$

The second diversification index is constructed from inter-dependent business unit $TVaR(s)$. These metrics are back allocated, by the top-down procedure, from the portfolio level global metric by the covariance principle, outlined in *section 4*. Logically the index is directly dependent on sub-additive relations, and hence we notify it appropriately as $DI_{CoVaR/TVaR}$. We show in the previous section, that due to mathematical coherence of back-allocation, the business unit metrics become co-monotonic additive to the portfolio level metric. Again for the Northeast business, the index is formally expressed hereby.

$$SE\ DI_{CoVaR/TVaR} = \frac{TVaR_{CoVAR}[X_i]}{TVaR\left[\sum_{i=1}^{n} X_{i,j,k}\right]}$$

To revisit the back-allocation process in context of index construction, we restate that the global portfolio metric $TVaR[Y_n]$ is measured from the stochastically simulated, multivariate and joint distribution of all insurance losses, in all businesses of the entire portfolio. Geo-spatial inter-dependencies among the individual business units, and individual risks are captured by the covariance back-allocation principle itself, applied to the global portfolio metric. As outlined in the previous *section 4.0*, this principle constructs a back-allocation ratio, also known as back-allocation weight, from each business unit's marginal covariance $Cov\left(X_i, \sum_{k=1}^{n} X_{j+k}\right)$ share of the sum total of the portfolio marginal units' covariance matrix $\sum_{i=1}^{n}\sum_{k=1}^{n} Cov(X_i, X_k)$. The $DI_{CoVaR/TVaR}$ index reflects the presence of these correlation effects, while the $DI_{VaR/TVaR}$ index does not. Both diversification indices $DI_{VaR/TVaR}$ and $DI_{CoVaR/TVaR}$ are computed in *table 5*.

Table 5: $VaR - TVaR$ *and* $Covariance - TVaR$ *based diversification indices computed for each business unit.*

Diversification Indices	VaR / TVaR	Covar / TVaR
North East	0.06	0.10
Central Unit	0.28	0.42
South East	0.67	0.48

Another two diversification indices, based on covariance-allocated, and on independently computed risk metric, are proposed by Tasche (2006). The indices measure the relationship between a marginal covariance allocated risk metric, which fits into the definition of our business unit $TVaR[BU]_{CoVar}$, which is back-allocated through the covariance principle, and a fully independent risk metric, such as those which we compute and define as $TVaR[BU]_{Independent}$. The latter, independent metric, fits into our definition of a stand-alone business unit $TVaR$, computed from the stand-alone and independent loss distributions of its accumulated and aggregated insurable risks. This diversification metric is defined formally, in the example of the Northeast business hereby bellow, using both Tasche's, in the first expression, and ours in the second expression, adopted notations.

$$DI(X_k|Y_n)_{TVaR\ Ratio} = \frac{TVaR[X_k|Y_n]}{TVaR_{Ind}[X_k]}, Tasche\ (2006)$$

$$DI(X_k|Y_n)_{TVaR\ Ratio} = \frac{TVaR_{CoVAR}[X_k]}{TVaR_{Ind}[X_k]}, Zvezdov\ (2018)$$

For the whole insurance book of business, with a joint modeled loss distribution $[Y_n]\ with\ n = 10{,}000$, simulation scenarios, the same metric respectively is expressed in similar mathematical logic.

$$DI(Y_n) = \frac{TVaR[Y_n]}{\sum_{i,j,k=1}^{n} TVaR_{Ind}\left[X_{i,j,k}\right]}$$

This set of five indices, for each business unit, and for the whole book of business, is computed in *table 6*.

Table 6: Business unit diversification ratio indices based on covariance and independent $TVaR(s)$

Diversification Indices	Covar-TVaR / Independent Ratio
North East	2.73
Central Unit	2.11
South East	0.63
Portfolio	0.92

To put our numerical analysis in summary - we have produced, generalized, and quantified three diversification and concentration indices: $DI_{VaR/TVaR}$, $DI_{CoVaR/TVaR}$, and $DI_{TVaR\ Ratio}$. All three indices point to the same relative ranking of business units, by these quantified patterns and metrics of risk clustering and risk concentration, summarized in *table 7.*

Table 7: Relative ranking of risk concentration and clustering by business unit, with highest concentration & clustering marked by rank 1, and lowest marked by rank 3.

	VaR - TVaR	Covar TVaR	Covar TVaR Ratio
North East	3	3	3

Central West	2	2	2
South East	1	1	1

6.0 Some Conclusions

The interpretation of diversification indices and geo-spatial metrics provides meaningful analytics and supporting tool for practitioners to systemically examine both actual and observed claims and simulated loss relationships, of sub-additive accumulations, geo-spatial inter-dependence, and diversification. Such indices and metrics inform practitioners of risk clustering in the spatial and temporal domains, which is an opposing effect to aims and promises of portfolio diversification, Risk clustering and concentration is a highly undesirable effect of insurance underwriting by all risk managers, and business unit managers. Furthermore, these effects of concentration and clustering of risk in physical, geo-spatial, and temporal pattern, are measurable, quantifiable, and furthermore manageable, with various exposure redistributions techniques, as well as, with risk dispersion and transferring reinsurance and capital market contracts.

It is quite clear that traditional pair-wise metrics of Gaussian model domains are insufficiently equipped to capture and describe the complexities of risk factor interconnectedness, which new generations of computationally powerful models render possible to derive in enterprise IT environments today. The analytical and numerical efforts in this paper attempt to contribute towards the development, and propagation to industry practitioners, of a set of geo-spatial and second order risk metrics, which capture in a more coherent manner the effects of diversification, concentration and connectedness among complex risk factors in an insurance book of business.

Coherence, consistence, and clarity are also an indispensable requirement for such metrics, to be able to be easily consumed, and to have an impact on decisions made by executive stakeholders.

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