

A Review of the CAS Report: “The Marginal Cost of Risk in a Multi-Period Risk Model” by Daniel Bauer & George Zanjani

-- Theory Development and Computer Program

Kailan Shang CFA, FSA, PRM, SCJP

Table of Contents

1. BACKGROUND	1
2. GENERAL COMMENTS	2
3. SPECIFIC COMMENTS.....	4
4. COMPUTER PROGRAM.....	10
APPENDIX A. R CODE: ALLOCATION BY PERCENTILE LAYER	12
APPENDIX B. R CODE: MULTI-PERIOD ALLOCATION	14
APPENDIX C. DATA INPUT FILE FORMAT	25

1. BACKGROUND

George Zanjani and Daniel Bauer of the Georgia State University Insurance Department faculty co-authored a report titled “The Marginal Cost of Risk in a Multi-Period Risk Model”. In the report, the authors extend the capital allocation model to a multi-period framework and use reinsurance experience to illustrate the allocation of capital to lines of insurance. The review focuses on the theory development and the computer program.

Capital allocation is an important management process of allocating both current and future potential financial resources to different sub-portfolios, projects, risks, and business units. It has been widely used in the insurance industry for risk management, pricing, performance measurement, and strategic planning.

The report explains the purpose of capital allocation and summarizes the available allocation approaches. It uses reinsurance experience data to test the stability of different approaches. It then develops the marginal cost of capital in a multi-period framework. The new approach takes into account not only the current financial resources but also the possibility of future changes such as raising new capital. It discussed the linkage of marginal cost of capital and capital allocation in the multi-period dynamic model. In the report, the new approach is applied to

capital allocation using the same set of reinsurance experience data and the result is compared to other capital allocation approaches.

2. GENERAL COMMENTS

In my view the authors have done a good job in summarizing various theoretical approaches of capital allocation. It has excellent educational value to practitioners to understand the available choices of capital allocation models. The extension to a multi-period dynamic model is enlightening and help advance the capital allocation models to a new level of sophistication that is necessary for reflecting certain management actions. The numerical comparison of capital allocation approaches using reinsurance experience data is also helpful for understanding the behavior of each approach.

As the authors mentioned in the paper, the new multi-period model “represents a step toward greater sophistication in firm valuation and risk pricing, but only a step.” While the model is theoretically sound, there are some factors that may need to be considered for a more meaningful representation of the real world. The following comments are about the general framework of capital allocation in the report. They should be viewed as discussions on the gap between the model and real practices. I think that these are areas for future research and do not expect the authors to address these issues in the report.

1. The report focuses on capital allocation among business lines. There are other dimensions of capital allocation such as risk types and business units. In reality, it is quite often to use risk type as the first layer of capital allocation. Based on the risk profile of each business line, the required capital of each risk type is further broken down to business lines and then aggregated at the business line level. While some capital allocation approaches can be applied to each dimension, the multi-period model developed in the report is not suitable for allocating capital according to the risk type. The reason is that it is difficult to divide the premium and claim among risk types.
2. The report considers capital allocation separately from the total capital level. Capital allocation methods are compared mainly based on the percentage allocated to each business line, rather than the amount of risk exposure. However, the total capital level of a company is determined according to its risk appetite. Capital allocated to a certain business line or risk type represents the measurement of the risk exposure. There will be inconsistency to allocate the capital by percentage using a method that generates a risk exposure assessment deviating much from the capital amount. It will cause a conflict with

the company's risk appetite.

3. The report assumes that the available capital is allocated to existing business lines. However, a company in normal cases usually retain extra capital for potential business development opportunities. The multi-period model does not account for it. This could change the amount of capital raised in distressed situations.
4. In addition to pricing and performance measurement as mentioned in the report, the purpose of capital allocation also includes business planning and risk management. Traditional capital allocation aims to maximize expected shareholder value. But the situation has changed. Risk has been considered explicitly when investors make decisions. Companies publish their risk appetite from time to time and investors can invest in companies that matching their risk tolerance. The target risk level is not included in the multi-period model explicitly.
5. A key assumption in the multi-period model is that a business line's losses indexed by time period are independent and identically distributed (i.i.d.). This simplified assumption makes the model less applicable. The loss experience may have cyclical patterns. Some risks exhibit temporal clustering which indicates a non-constant volatility term structure. The distribution may also change due to structural change of the business and its risk. In addition, for business lines with a long term benefit period, the period-losses are unlikely to have identical distribution. Reflecting these nuances could certainly make the model far more complicated than the current one. This assumption narrows the scope of model application.
6. There exist debates on the appropriate time horizon for capital management, including capital allocation. It is normal to see 1- year to 5-year capital plans but it is rare for longer periods. There are general two reasons against a long-term capital allocation. Firstly, the uncertainty of risk makes a long-term projection unreliable. Assumptions such as i.i.d. are normally unrealistic. Secondly, management actions can address new changes. Here, management actions include not only capital raising but also changes in the business strategy, etc. A multi-period model that considering these issues may be too complicated and its result is expected to be uncertain. It is important, yet difficult to determine an appropriate time horizon for the multi-period model.
7. The required calculation time may cause some reluctance of using the multi-period model. The numerical example requires several months for the calculation. There are only four aggregated business lines in the example. In reality, more business lines may be used and

it takes a longer time for calculation. Considering that there may be many rounds of assumption testing and adjustment, the calculation time is a big constraint. Parallel computing is necessary in real application. As mentioned in the report, in the optimization process, some manual adjustments were made. This may reduce the benefit of parallel computing. Some approximation techniques are required in real application to reduce the calculation time.

8. Finally, congratulations to the authors and the CAS on a very nice piece of work.

3. SPECIFIC COMMENTS

The following comments are on details such as model assumptions, model calibration, proof of propositions, typos, etc.

1. Page 7: The capital k is specified as “*the difference between the fair value of assets and the present value of claims.*” Present value of claims may cause some confusion. Is it the best estimate of liability, statutory reserve, or market consistent value of liability? Should the claims be based on best estimate assumption or padded assumption? What discount rate is used? It will have a big impact on the value of capital k to be allocated and the rationale of capital allocation.
2. Page 10: Formula 2.4 presents the optimization problem from Bauer and Zanjani (2013a) as the formula given below. While it is easy to guess what V and C represents, it may be helpful to explain their meanings in the report. In addition, I was not able to find the working paper referred here. Maybe a link can be provided as the report makes reference to this paper a couple of times.

$$\max_{k, q^{(1)}, q^{(2)}, \dots, q^{(N)}} \underbrace{\left\{ \sum_{i=1}^N p^{(i)}(q^{(i)}) - V(\min\{I, a\}) - C \right\}}_{=\Pi}$$

3. Page 15: At the bottom of the report, it says “*Instead, in our numerical approximations, we rely on a Bell-curve around the required quantile.*” This approach shows a more stable performance than simple VaR approach on Page 26. Some explanation of this “bell” approach will be helpful though it is not critical. By looking at the Excel file “Agg3Allocs.xlsx”, the formula below may be used for explanation.

$$VaR_p(Bell) = \frac{\sum_{x=-N \cdot p+1}^{N-N \cdot p} \phi\left(\frac{x}{\sigma}\right) V(N \cdot p + x)}{\sum_{x=-N \cdot p+1}^{N-N \cdot p} \phi\left(\frac{x}{\sigma}\right)}$$

where

N : no. of data points

p : confidence level

σ : standard deviation of the bell – curve

$\phi(x)$: probability density function of standard Normal distribution

4. Page 16: It is helpful to add the acronym of “Expected Shortfall” as “ES” in the first paragraph. “ES” was directly used later without definition.
5. Page 16: In the first paragraph of Section 2.4.4, it says “A less conventional risk measure that nonetheless satisfies the homogeneity property and **this** allows for an application of the Euler principle is the so-called exponential risk measure:” Should “this” be deleted?
6. Page 17: Should $L^{(i)}$ be $L^{(i)}$ in formula 2.11.

$$k^{(i)} = \frac{\partial \rho(I)}{\partial q^{(i)}} = \mathbb{E} \left[L^{(i)} \exp \left\{ \frac{cI}{\mathbb{E}[I]} \right\} \right] + c \frac{\mathbb{E}[L^{(i)}]}{\mathbb{E}[I]} \times \mathbb{E} \left[I \exp \left\{ \frac{cI}{\mathbb{E}[I]} \right\} \times \left(\frac{L^{(i)}}{\mathbb{E}[L^{(i)}]} - \frac{I}{\mathbb{E}[I]} \right) \right]. \quad (2.11)$$

7. Page 17: A duplication of the equal sign exists in the following formula.

$$k^{(i)} = = \mathbb{E} [L^{(i)} g'(\bar{F}_I(I))]$$

8. Page 18: The denominator of the term in the second row should be $\varphi(\Phi^{-1}(1-p))$?

$$\begin{aligned} g'(p) &= \varphi(\Phi^{-1}(1-p) - \lambda) \times \frac{1}{\varphi(\Phi^{-1}(1-p))} \\ &= \frac{\varphi(\Phi^{-1}(1-p) - \lambda)}{\varphi(\Phi^{-1}(1-p) - \lambda)} \\ &= \exp \left\{ \Phi^{-1}(1-p) \lambda - \frac{1}{2} \lambda^2 \right\}. \end{aligned}$$

9. Page 22: In the first paragraph it says “We believe this data offers a degree of realism missing from previous contributions where proposed allocation methods are only studied in the context of stylized examples or based on Normal distributions (which is particularly limiting as discussed in Section 2.4.10).” Section 2.4.10 talks about the connections of

allocation approaches. Should this reference be removed?

10. Page 26: “RTVaR allocations with $\alpha = 75\%$; 90% ; and 95% and $\beta = 2$ as for the covariance allocation.” A reference to Section 2.4.3 will be helpful, like what has been done for other approaches.
11. Page 27: In the examples of D’Arcy allocation method, CoC_{market} is equal to CoC_{normal} . The essence of this method is the volatility of CoC_{market} . Without varying the value of this variable, economic risk is not considered in the examples.
12. Page 28: In the first paragraph, it says “*Thus, in the second line for each method, Table 3.2 lists the allocations as a percentage of the aggregate risk measure. These are the percentages on which we will base our comparisons. This is not only because it facilitates comparisons, but also because this is in line with practice where the actual capital of a company may not be given in terms of a risk measure at all, or even if it is this may not be the measure used for allocation.*” It makes a good point here that these percentages reflect the portion of the total risk measure of each business line. On the other hand, actual capital, or available capital is unlikely to be allocated based on these percentages if the capital is not consistent with the risk measures. If the company operates according to a risk measure, its current capital should be at a level close to the risk measure. For example, if the company’s available capital is \$1 Billion and the risk measure generates a required capital of \$2 Billion, the company has a higher risk tolerance than implied from the risk measure. Allocation based on the mix result of the \$2 Billion risk measure will cause inconsistency with the company’s risk appetite.
13. Page 28: In the second paragraph, it says “*Indeed there are various realizations in the aggregate tail where the line realizations for **lines 1, 4, and 5** are under the expected loss,*” Should it be **lines 1, 3, and 4**?
14. Page 29: The second sensitivity test of the stability of capital allocation methods reduces the severity of extreme events to a lower level. It is mentioned that “*modification in the extreme tail should not have a tremendous impact on the allocation.*” However, as later discussed, tail based methods are heavily impacted by the adjustment of extreme events. They sound a little bit contradictory. In reality, the severity of extreme events matters a lot. For example, after several severe earthquakes (2010 in Chile, Haiti and New Zealand, and 2011 in Japan), RMS Inc., a market leader in catastrophe (CAT) modelling, released a new version of models with updated natural catastrophe model parameters. It is also helpful to comment on whether the stability of methods is a desired property or not. For a

conservative company, a high stability of risk measure in the second sensitivity test may not be wanted.

15. Page 34: The first paragraph talks about putting inflation into the model using a deterministic approach. It is likely to underestimate the volatility of the loss realization due to the uncertainty of inflation.
16. Page 34: Formula 4.2 for capital at time t . The capital accumulates at the risk free rate and the expected investment return of new premium income is risk free rate as well. An alternative choice is using the expected investment return based on the asset portfolio supporting the capital. Using the risk free rate to accumulate the capital and penalize the capital holding with a positive carrying cost seems too conservative and not very realistic. Capital normally has a risky investment strategy in the hope of higher investment income to support the target ROE. If a company invests only in risk free asset, the target ROE is likely to be lower and the carrying cost (cost of capital rate) should be close to zero. An investment in a safe company normally means a lower expected ROE. In addition, the carrying cost will change with the default probability. A lower default probability means a higher credit rating which reduces the cost of borrowing money and therefore the carrying cost of capital.
17. Page 36: In the report, the price level depends on the default probability and the business volume. High default probability or high business volume indicates a low price level. Based on the i.i.d. assumption, a high business volume means a high level of risk diversification and a low level of aggregated risk. Therefore, the company can withstand a low price level. However, if the i.i.d. assumption is relaxed, a high business volume does not necessarily leads to a low profit margin. Many individual risk exposures are highly correlated which makes the diversification benefit immaterial. Due to the entry barrier, if certain market is dominated by the insurer, it may use its monopoly power to charge a high price.
18. Page 37: The model has an important assumption about the cost of raising capital at different time points. The cost at the beginning of the period is lower than the cost at the end of the period. The idea is that with a distressed period, the company will raise new capital at a high cost to return to a solvent position. After that, the company can raise more capital, if necessary, at a low cost. The theoretical optimal strategy of capital raising is that *“at the end of the period, it only makes sense to either raise exactly enough capital to save the company or not raise any capital at all: Raising more will not be optimal since it*

is possible to raise in the beginning of the next period at better conditions; raising less will not be optimal since the company will go bankrupt and the policyholders are the residual claimants.” However, it is not realistic. Raising capital to merely meet the requirement of claim is optimal in the economic view but there are other constraints such as regulatory solvency requirements. In addition, the cost of capital raising at the beginning of the next period is definitely not “normal” if the company has zero capital. Corollary 4.1.1 also assumes a linear cost function to derive the optimal amount of new capital raised at the end of a time period. Cautions should be used as the marginal cost of raising capital is likely to rise with the amount of capital.

19. Page 40: The formula at the bottom contains some typos. A few right parentheses exist without their counterparties.
20. Page 44: In the calibration of the multi-period model, Moody’s and S&P experience are used for estimating the default rates of some A.M. Best ratings. It is unclear to me that in the regression analysis, which set(s) of default rates (Moody’s, S&P, A.M. Best) are used. If only the estimated default rates for A.M. Best ratings are used, it would be interesting to know the reason and what it looks like if Moody’s or S&P default rates are used?
21. Page 44: The price p_{it} is “calculated as the ratio of net premiums earned to the sum of loss and loss adjustment expenses incurred and company i in year t ;” The expected loss E_{it} is “calculated by applying the average net loss and loss adjustment expense ratio over the sample period for each firm to that year’s net premium earned.” The price uses actual company experience. The expected loss uses the industry average experience. In formula (4.5), the expected value of loss exposure is used in both places. Note that here p_{it} is the $p_t^{(i)}/E_{t-1}[I_t^{(i)}]$ in formula 4.5. This causes some inconsistency and some explanations will be helpful.

$$\begin{aligned}
 p_t^{(i)} &= e^{-r} \mathbb{E}_{t-1} \left[I_t^{(i)} \right] \\
 &\times \exp \left\{ \alpha - \beta \mathbb{P}_{t-1} \left(I_t > \left[a_{t-1}(1-\tau) + R_t^b - c_1(R_t^b) + \sum_{j=1}^N p_t^{(j)} \right] e^r + R_t^e - c_2(R_t^e) \right) - \gamma \mathbb{E}_{t-1}[I] \right\},
 \end{aligned} \tag{4.5}$$

22. Page 44: In Table 5.2, although the t-value indicates that α , β , and γ are not equal to zero, the adjusted R^2 of the regression is only 26%. This indicates the level of uncertainty of the price level. The resulting uncertainty the marginal cost of capital and capital allocation may be omitted in the multi-period model. In addition, the NAIC data for the period 2002 to 2010 are used while in the estimation of the default rates of A.M. Best ratings, data for the period of 2008-2012 are used. Due to the extreme credit crisis happened since 2008,

the discrepancy of data periods may cause some discrepancy.

23. Page 45: Table 5.3: the value of β in the base case is 392.96, 100 times the value of β in Table 5.2. It is likely caused by difference in scales but a small note could be helpful.
24. Page 45: In the first paragraph of Section 5.2. it says “*the cost of raising capital is represented by a quadratic cost function with the linear coefficient $c^{(1)}_1$ fixed at 7.5%; the cost of raising capital in distressed circumstances, ζ , varies from 20% to 75%;*” It is suggested to add “in normal circumstances” after the first “cost of raising capital” in the sentence.
25. Page 46: Calculation of the premium levels. The implicit default rate in the determination of premium level does not necessarily match the default rate implied from the capital model.
26. Page 57: It is mentioned that the scale of company, one of the terms in the marginal cost of risk, is not typically considered in insurance pricing practice. The scale of the company is considered in the price determination in Formula 4.11. Also in practice, the scale of a company plays an important role in insurance pricing. The diversification benefit is usually high for large companies which leads to a lower level of capital allocated to a business line. And it is normally reflected for pricing purpose.
27. Page 63: It says that “*the capital cost component share of non-payment costs ranges from more than 90% (in the base case” at an initial capitalization of $a = 0$).*” In Table 5.9 on page 61, the capital cost when $a = 0$ is 50,194,848. The non-payment costs is 105,910,928. The ratio is about **47%**.
28. Page 77: In the proof of the marginal cost for individual risk as in Proposition 4.2.1., the last term of the first order condition for $[q_i]$ is not quite straightforward to me.

$$\zeta \frac{\partial \rho}{\partial q^{(i)}} e^{-r}$$

It should equal to the Lagrange multiplier $\zeta \times \frac{\partial \text{constraint (4.12)}}{\partial q_i}$. But they do not match quite well. The terms in the first order conditions of p_i and R_b regarding constraint (4.12) make sense. Some explanation will be helpful.

$$s \left(\sum_j I^{(j)} \right) \leq \underbrace{a(1 - \tau) + R^b - c_1(R^b) + \sum_{j=1}^N p^{(j)} + (1 - \xi) V(0)}_{\text{Available Capital}}, \quad (4.12)$$

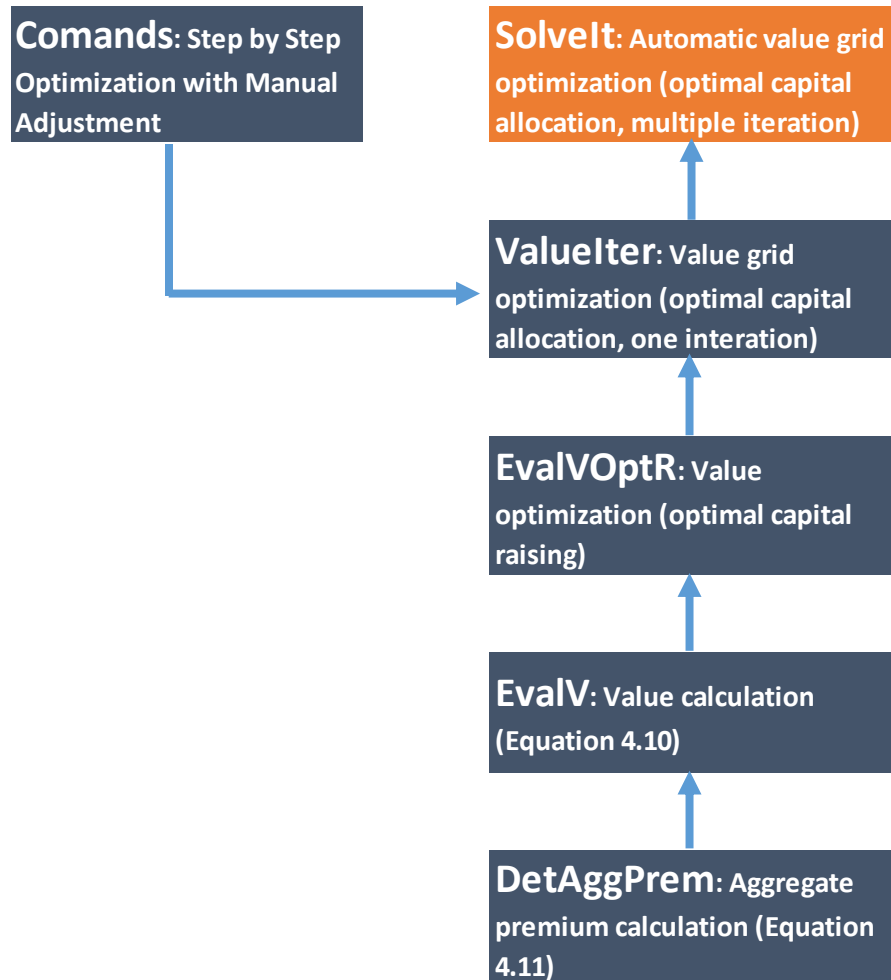
29. Page 79: “Proof of Corollary 4.2.1. In the absence of a regulatory constraint, $\zeta =$ so that Equation (B.1) yields:” Should it be “Proof of Corollary 4.2.1. In the absence of a regulatory constraint, $\zeta = \mathbf{0}$ so that Equation (B.1) yields:”?

4. COMPUTER PROGRAM

Accompanying the report are tools/programs that apply many capital allocation methods to the reinsurance experience, as illustrated in the report. Both Excel and Matlab were used in the calculations. These tools, or their adapted versions, can serve educational purpose or be used in practice. Since Matlab is not an open-sourced software, it may be valuable to implement the Matlab programs using an open-sourced software to make the tools more accessible. Octave is an alternative to Matlab that has similar functionalities and coding syntax. But R was chosen due to its popularity and a better user interface than Octave. However, the users can always make use of the matlab codes using Octave with minor changes. Some efforts were made to have a richer documentation as well.

1. File “Agg3Allocs.xlsx” includes the asset allocation examples in Section 4. All the methods except the allocation by percentile layer are realized in the Excel tool. To better illustrate the capital allocation by percentile layer, a new Excel file named “Agg3Allocs with Percentile Layer.xlsx” was created to illustrate the calculation. However, due to the constraints of Excel, it may not be able to handle a larger dataset efficiently. R code with documentation is also provided. The code is provided in Appendix A. A text file (PercLayerAlloc.txt) containing the code is also provided.
2. The optimization of multi-period dynamic model was implemented by Matlab in the report. With the optimized capital amount, capital allocation, and firm’s value, other results such as RAROC were calculated using Excel. The R code that implements the Matlab program is provided in Appendix B. Figure 1 shows the structure of the R code. It is the same as the structure of the Matlab code. In the report, the automatic optimization with multiple iterations is not used. Rather the optimization is done step by step. Function ValueIter is called for 320 times in an optimization process. The output of each step is used for the input of the next step. Some manual adjustments are made to early iterations. Therefore, using function “SolveIt” may not give us a global minimum and may take longer to run given the lack of necessary manual adjustments.

Figure 1. R Code Structure



However, unlike Matlab, standard R setting does not include parallel computing, either implicit or explicit. Therefore, for a multi-core computer or a computer cluster, the R code needs to be adapted for enabling high performance computing. The value grids depending on different capital levels make it easy to split the calculation among CPUs. This is also useful and necessary given the long run time of the multi-period model. R-project.org maintains a list of options for high-performance and parallel computing.¹ They can be used to speed up the optimization process with more computing resources.

The code is provided in Appendix B. A text file (MultiPeriodModel.txt) containing the code is also provided.

¹ High-Performance and Parallel Computing with R.

<http://cran.r-project.org/web/views/HighPerformanceComputing.html>

3. A small bug has been found in the Matlab program while it does not have any impact on the report. In file "SolveIt.m", the value grid is not updated after each iteration. In the code snippet below, after running the function ValueIter, Vals need to be updated as begV to feed into the next iteration. The blue line is what is missing in the original code.

```
while diff>tolerance

    [begV,qs,R_b] = ValueIter(Is,Els,Vals,qs,tau,c_1,c_1sq,xi,intr,alpha,beta,gamma,
    AGRID,ADEL);

    diff = norm(begV - Vals);

    Vals = begV;

    k=k+1;

end;
```

APPENDIX A. R CODE: ALLOCATION BY PERCENTILE LAYER

```
# Capital allocation by Percentile Layer (ZB Paper Page 14)
```

```
# Read in claim data, which includes byline and aggregate claims.
```

```
# The directory need to be changed for the data file to be used.
```

```
data <- read.csv("C:/ClaimData.csv")
```

```
# Input the confidence levels to be analysed.
```

```
alpha <- c(0.9, 0.95, 0.99)
```

```
# Function alloc calculate the allocation by percentile layer with two inputs: confidence levels and the claim data
```

```
# The function outputs the allocation byline. Each confidence level generates two lines: the value of risk measure for each line and the percentages
```

```
alloc <- function(alpha, claimdata){
```

```
    # Descending sort the data by total claim
```

```
    idx = order(claimdata$Total, decreasing=T)
```

```
    data_sorted = claimdata[idx,]
```

```
    # Calculate the percentage of total claim for each line
```

```
    prop <- data_sorted/data_sorted$Total
```

```
    propr <- subset(prop, select = -Total)
```

```
    propr <- data.matrix(propr)
```

```
    data_sorted = as.matrix(data_sorted)
```

```
    # number of records and lines
```

```
    nline = ncol(propr)
```

```
    nrecord = nrow(propr)
```

Incorporating Cyclical Effect with Hidden Markov Models

```
# Calculate the average proportion given the total claim is greater than a threshold (li/l given l>= lj)
avg <- matrix(, nrow = nrecord, ncol = nline)
for (i in 1:nline) {
  for (j in 1:nrecord){
    avg[j,i] = sum(propr[1:j, i])/j
  }
}

# Calculate the average proportion * layer
avglayer <- matrix(, nrow = nrecord, ncol = nline)
for (i in 1:nline) {
  for (j in 1:nrecord){
    if (j == nrecord) {
      avglayer[j,i] = avg[j, i] * data_sorted[j, nline+1]
    } else {
      avglayer[j,i] = avg[j, i] * (data_sorted[j, nline+1] - data_sorted[j+1, nline+1])
    }
  }
}

# Calculate the allocation by percentile layer
ncl = length(alpha)
allocvec = matrix(,nrow = ncl*2,ncol = nline)
for (k in 1: ncl) {
  for (i in 1:nline) {
    allocvec[(k-1)*2+1,i] = sum(avglayer[as.integer(nrecord*(1-alpha[k])):nrecord,i])
  }
  for (i in 1:nline) {
    allocsum = sum(allocvec[(k-1)*2+1,])
    allocvec[(k)*2,i] = allocvec[(k-1)*2+1,i]/allocsum
  }
}

# Add headers for the output matrix
colnames(allocvec) <- c(paste("Line", 1: nline))
rownames <- vector(, ncl*2)
for (i in 1:ncl){
  rownames[(i-1)*2+1] = paste("Amount with Conf. Level =", alpha[i])
  rownames[(i)*2] = paste("Percentage with Conf. Level =", alpha[i])
}
rownames(allocvec) <- rownames
return(allocvec)
}

# Run the function
alloc(alpha,data)
```

APPENDIX B. R CODE: MULTI-PERIOD ALLOCATION

```
# Multi-period capital allocation (Section 5.2, ZB Paper)
# This is to replicate the Matlab code provided by Zanjani and Bauer. The terms used here are kept the
# same as in the Matlab code as much as possible.
# Print functions are commented out but can be used for testing the functions.
```

```
# The optimization requires R package "neldermead"
library(neldermead)
# Fuction ginv in MASS is used for a better inverse matrix calculation than using solve(M)
library(MASS)
```

```
# Read in claim data, which includes byline and aggregate claims.
# The directory need to be changed for the data file to be used.
data <- read.csv("C:/ClaimData.csv")
# A file selection window will pop up if the next line is executed
# data <- read.csv(file.choose())
```

```
# Sorting the data by total claim
idx <- order(data$Total)
data <- data[idx,]
```

```
#####
#####
```

```
# Model parameters, function variables, etc.
```

```
#-----
# AgglIs: Aggregate losses
# EAggl: Expected value of aggregate loss
# amntIN: Existing capital in the company at the beginning of period:  $a(1-\tau) + R_b - C_1(R_b)$ 
# raiseADD: Additional capital to be raised:  $R_e - C_2(R_e)$ 
# intr: Interest rate
# alpha, beta, gamma: price model parameters in equation 4.11
# Vs: Value function in the next period
# a: capital
# R_b: capital raised at the beginning of period
# tau: carrying cost of capita
# c_1, c_1sq: parameters of the cost function of raising capital at the beginning of the period
# xi: parameter of the linear cost function of raising capital at the end of the period
# breg is the interpolation coefficient from linear regression
# AGRID and ADEL are the grid size and the delta in the (equally spaced) grid for a
# ls: ls: Losses for the four lines, dimension # of records x # of lines
# ElS: Expected value of byline losses
# oldqs: Starting values for q, dimension # of grids x # of lines
# tolerance: tolerance of the error
```

```
# Set initial values. Some are used for illustration but not necessary for the optimization.
```

```
#-----
```

Incorporating Cyclical Effect with Hidden Markov Models

```
ls <- subset(data, select = -Total)
Aggl_ <- as.vector(data$Total)
EAggl_ <- mean(Aggl_)
ls_ <- data.matrix(subset(data, select = -Total))
Els_ <- as.vector(colMeans(ls_))
amntIN_ <- 100000000
raiseADD_ <- 10000
intr_ <- 0.03
alpha_ <- 0.315624284
beta_ <- 392.958
gamma_ <- 1.48e-10
a_ <- 1e10
R_b_ <- 8.012236598865196e+09
tau_ <- 0.03
c_1_ <- 0.075
c_1sq_ <- 1e-10
xi_ <- 0.5
breg_ <- 1.5
AGRID_ <- 25
ADEL_ <- 2.5e8
qvec_ <- c(0.1, 0.3, 0.25, 0.35)
tolerance_ <- 1000

# Read values of V
# Read from a specified file directory
# Vs_ <- read.csv("C:/March 2014/Research/Allocation/ForCAS/ConventionalAllocations/R
Program/vs.csv")
# A file selection window will pop up if the next line is executed
# Vs_ <- read.csv(file.choose())
# Change Vs_ from a data frame to a vector
# Vs_ <- as.vector(Vs_$vs)
# The initial values of V can also be set to zero
Vs_ <- rep(0, AGRID_+1)
startVs_ <- Vs_

# Read values of q
# Read from a specified file directory
# startqs <- read.csv("C:/March 2014/Research/Allocation/ForCAS/ConventionalAllocations/R
Program/startqs.csv")
# A file selection window will pop up if the next line is executed
startqs <- read.csv(file.choose())
# Change oldqs_ from a data frame to a matrix
oldqs_ <- data.matrix(startqs)
startqs_ <- oldqs_

#####
#####
```

Incorporating Cyclical Effect with Hidden Markov Models

```
# Function DetAggPrem (Equation 4.11 price model)
# Model Inputs
# -----
# Aggls: Aggregate losses
# EAggl: Expected value of aggregate loss
# amntIN: Existing capital in the company at the beginning of period:  $a(1-\tau) + R_b - C_1(R_b)$ 
# raiseADD: Additional capital to be raised:  $R_e - C_2(R_e)$ 
# intr: Interest rate
# alpha, beta, gamma: price model parameters in equation 4.11
# -----

DetAggPrem <- function(Aggls,EAggl,amntIN, raiseADD,intr,alpha,beta, gamma){

  k = length(Aggls)
  breakcon = TRUE

  # The minimum aggregate price when the default rate is zero ( $Aggls = (pstar + amntIN) * \exp(intr) +$ 
  # raiseADD )
  pstar =  $\exp(-intr) * (Aggls[k] - raiseADD) - amntIN$ 

  while (breakcon) {
    # check if the policyholders are willing to pay the minimum aggregate price given the default rate
    if ( $\exp(-intr) * EAggl * \exp(\alpha - \beta * (50000 - k)/50000 - \gamma * EAggl) < pstar$ ) {
      if (k==1) {
        pstar = 0
        breakcon = FALSE
      } else {
        k = k-1
      }
    }
    # update the minimum aggregate price given the new default rate that is one level higher
    # Here one level refers to 1/# of aggregate loss data
    pstar =  $\exp(-intr) * (Aggls[k] - raiseADD) - amntIN$ 
  }
  # If the policyholders are willing to pay the minimum aggregate price given a default rate of zero,
  # then aggregate price is calculated using formula 4.11 assuming the default rate is zero.
  } else if (k==50000) {
    pstar =  $\exp(-intr) * EAggl * \exp(\alpha - \gamma * EAggl)$ 
    breakcon = FALSE
  }
  # If the policyholders are willing to pay the minimum aggregate price given a positive default rate,
  # then aggregate price based on formula 4.11 is calculated. It is higher than the minimum aggregate
  # price.
  # Therefore, the default rate is adjusted up.
  # price based on formula 4.11 - minimum price at the given default rate
  # d = -----
  # (Aggregate loss at the default rate one level higher - aggregate loss at the given default rate)
  # It is expected to be no greater than 1. Otherwise, the adjusted default rate two levels higher the given
  # default rate.
  } else {
```


Incorporating Cyclical Effect with Hidden Markov Models

```
d = (exp(-intr) * EAggl * exp(alpha - beta * (50000 - k)/50000 - gamma * EAggl) -
max(pstar,0))/(exp(-intr) * Aggls[k+1] - exp(-intr) * Aggls[k])
if (d > 1) {
  #sprintf("Oooops:")
  #sprintf("k = %d, d = %f", k, d)
}
pstar = exp(-intr) * EAggl * exp(alpha - beta * (50000 - k + d)/50000 - gamma *
EAggl)
breakcon = FALSE
}
}
#sprintf("k: %d pstar: %f", k, pstar)
return(pstar)
}
```

#Example of calling the function

```
#s <- DetAggPrem(Aggls_,EAggl_,amntIN_, raiseADD_,intr_,alpha_,beta_, gamma_)
#s
```

```
#####
#####
```

Function EvalV (Equation 4.10: Bellman equation)

Model Inputs

Aggls: Aggregate losses

EAggl: Expected value of aggregate loss

Vs: Value function in the next period

a: capital

R_b: capital raised at the beginning of period

tau: carrying cost of capita

c_1, c_1sq: parameters of the cost function of raising capital at the beginning of the period

xi: parameter of the linear cost function of raising capital at the end of the period

intr: interest rate

alpha, beta, gamma: price model parameters in equation 4.11

breg is the interpolation coefficient from linear regression

AGRID and ADEL are the grid size and the delta in the (equally spaced) grid for a

```
EvalV <- function(Aggls, EAggl, Vs, a, R_b,tau,c_1, c_1sq, xi,intr,alpha, beta, gamma, breg, AGRID,
ADEL){
```

The cost will occur only additional capital is raised. There is no cost for reducing capital such as dividend payments.

```
if (R_b > 0) {
  cost_1 = c_1
  cost_2 = c_1sq
} else {
```

Incorporating Cyclical Effect with Hidden Markov Models

```
cost_1 = 0
cost_2 = 0
}

pstar = DetAggPrem(Aggls,EAggl,a * (1-tau) + R_b - R_b * cost_1 - R_b * R_b * cost_2, (1-xi) *
Vs[1], intr, alpha, beta, gamma)

# Equation 4.14
D = (a * (1-tau) + R_b - cost_1 * R_b - cost_2 * R_b * R_b + pstar) * exp(intr) + (1-xi) * Vs[1]
# Equation 4.13
S = (a * (1-tau) + R_b - cost_1 * R_b - cost_2 * R_b * R_b + pstar) * exp(intr)

Val = 0;
datasize = length(Aggls)

for (k in 1:datasize) {
  if (S >= Aggls[k]) {
    Val = Val + (pstar - exp(-intr) * Aggls[k] - tau * a - cost_1 * R_b - cost_2 * R_b *
R_b)
    index = floor((S - Aggls[k])/ADEL)
    if (index < AGRID) {
      # Linear interpolation
      Val = Val + (exp(-intr) * (Vs[index + 1] + ((S - Aggls[k])/ADEL - index) *
(Vs[index + 2] - Vs[index + 1])))
    } else {
      # Linear extrapolation
      Val = Val + exp(-intr) * (Vs[AGRID+1] + breg * (S - Aggls[k] - AGRID *
ADEL))
    }
  } else if (D >= Aggls[k]) {
    Val = Val + (((1/(1-xi)) * (S - Aggls[k]) + Vs[1]) * exp(-intr) - (a + R_b))
  } else {
    Val = Val - (a + R_b)
  }
}
Val = Val / datasize
return(Val)
}

#####
#####
#Example of calling the function
#t = EvalV(Aggls_, EAggl_, Vs_, a_, R_b_, tau_, c_1_, c_1sq_, xi_, intr_, alpha_, beta_, gamma_, breg_,
AGRID_, ADEL_)
#t

# Function EvalVOptR (For a given portfolio q_1,q_2, ... , this function calculates the optimal
```

Incorporating Cyclical Effect with Hidden Markov Models

```
# value function and capital raising decision)
# Model Inputs
# -----
# Is: Losses for the four lines, dimension # of records x # of lines
# Els: Expected value of byline losses
# Vs: Value function in the next period
# a: capital
# R_b: capital raised at the beginning of period
# tau: carrying cost of capita
# c_1, c_1sq: parameters of the cost function of raising capital at the beginning of the period
# xi: parameter of the linear cost function of raising capital at the end of the period
# intr: interest rate
# alpha, beta, gamma: price model parameters in equation 4.11
# breg is the interpolation coefficient from linear regression
# AGRID and ADEL are the grid size and the delta in the (equally spaced) grid for a
# -----

EvalVoptR <- function(Is, Els, qvec, Vs, a, tau, c_1, c_1sq, xi, intr, alpha, beta, gamma, breg, AGRID,
ADEL){

  # Given the portfolio mix, calculating the aggregate claims.
  nline = length(Els)
  nrecord = nrow(Is)
  Aggls <- vector(, nrecord)
  EAggl = sum(qvec * Els)
  for (j in 1:nrecord){
    for (i in 1:nline){
      Aggls[j] = Aggls[j] + qvec[i] * Is[j, i]
    }
  }

  Aggls = sort(Aggls)

  # Set the step size for Rb
  intercept = max(a * (1-tau) / 20,5000000)
  # Value with Rb = -a*(1-tau)
  Vopt = EvalV(Aggls, EAggl, Vs, a, -a * (1-tau), tau, c_1, c_1sq, xi, intr, alpha, beta, gamma, breg,
AGRID, ADEL)
  kopt = 0

  # Calculate the value using 60 different values of Rb. This is to avoid getting stuck with a local maximum.
  # By finding the maximum of the 60 values of V, Rb used in the maximum V is used as the starting point
  # for value maximization.
  for (k in 1:60) {
    Vloc = EvalV(Aggls, EAggl, Vs, a, -a * (1-tau) + intercept * k, tau, c_1, c_1sq, xi, intr, alpha, beta,
gamma, breg, AGRID, ADEL)
    if (Vloc > Vopt) {
```

Incorporating Cyclical Effect with Hidden Markov Models

```
kopt = k
Vopt = Vloc
}
}

# Set the model options. For testing purpose, the maximum iteration number can be reduced.
options = optimset(MaxFunEvals = 1000, MaxIter = 1000)

# Define the value function with the parameter R_b
fct <- function(x){
  temp = -EvalV(Aggl, EAggl, Vs, a, exp(x)-a * (1-tau), tau, c_1, c_1sq, xi, intr, alpha, beta, gamma,
breg, AGRID, ADEL)
  return(temp)
}

# Optimization using Nelder-Mead Simplex Method
sol = fminsearch(fun = fct, x0 = log(intercept * kopt + 0.001), options)

# Note that in the function the variable is changed to its logarithm
R_b = sol$optbase$xopt

# The flag associated with exist status of the algorithm. The following values are available:
# -1 The maximum number of iterations has been reached.
# 0 The maximum number of function evaluations has been reached.
# 1 The tolerance on the simplex size and function value delta has been reached.
exitflag = sol$exitflag
Val = - sol$optbase$fopt

# optimal value of Rb
R_b = exp(R_b) - a*(1-tau)

#if (exitflag == 0) {
#   sprintf("Evaluations in EvalOptR exceeded for a %f qs %f %f %f %f temp val %f tem Rb %f", a,
qvec[1], qvec[2], qvec[3], qvec[4], Val, R_b)
#} else if (exitflag == -1) {
#   sprintf("Iterations in EvalOptR exceeded for a %f qs %f %f %f %f temp val %f tem Rb %f", a,
qvec[1], qvec[2], qvec[3], qvec[4], Val, R_b)
#} else {
#   sprintf("A maximum value has been found.")
#}

return(list(R_b, Val))
}

#Example of calling the function
#u = EvalVOptR(Is_, Els_, qvec_, Vs_, a_, tau_, c_1_, c_1sq_, xi_, intr_, alpha_, beta_, gamma_, breg_,
AGRID_, ADEL_)
```

Incorporating Cyclical Effect with Hidden Markov Models

#u

```
#####  
#####
```

Function ValueIter

This function calculates the value iteration for one period, that is it
determines optimal portfolios.

Model Inputs

Is: Is: Losses for the four lines, dimension # of records x # of lines

Els: Expected value of byline losses

Vs: Value function in the next period

oldqs: Starting values for q, dimension # of grids x # of lines

tau: carrying cost of capita

c_1, c_1sq: parameters of the cost function of raising capital at the beginning of the period

xi: parameter of the linear cost function of raising capital at the end of the period

intr: interest rate

alpha, beta, gamma: price model parameters in equation 4.11

AGRID and ADEL are the grid size and the delta in the (equally spaced) grid for a

```
ValueIter <- function(Is,Els,Vs,oldqs,tau,c_1, c_1sq, xi,intr,alpha,beta, gamma, AGRID,ADEL){
```

```
  nline = ncol(Is)
```

```
  #value grid
```

```
  Vals = matrix(0, nrow = AGRID+1, ncol = 1)
```

```
  # portfolio grid
```

```
  qs = matrix(0, nrow = AGRID+1, ncol = nline);
```

```
  # new capital grid
```

```
  R_b = matrix(0, nrow = AGRID+1, ncol = 1)
```

```
  # Linear regression for extrapolation
```

```
  myxs = matrix(, nrow = 6, ncol = 2)
```

```
  myVs = matrix(, nrow = 6, ncol = 1)
```

```
  for (k in 1:6) {
```

```
    myxs[k,1] = 1;
```

```
    myxs[k,2] = (AGRID - 5 + k) * ADEL
```

```
    myVs[k,1] = Vs[AGRID - 5 + k]
```

```
  }
```

```
  reg = ginv(t(myxs) %*% myxs) %*% t(myxs) %*% myVs
```

```
  #sprintf('reg: %d %d', reg[1], reg[2])
```

```
# Set options for optimization process. For testing purpose, the maximum iteration number can be
```

Incorporating Cyclical Effect with Hidden Markov Models

reduced.

```
options = optimset(MaxFunEvals = 1000, MaxIter = 1000)

# Find the optimal portfolio and value grids(One iteration)
for (k in 0:AGRID) {
  fct <- function(x) {
    temp = EvalVOptR(Is,Els,as.vector(exp(x)), Vs, k * ADEL, tau,c_1, c_1sq,
xi,intr,alpha,beta,gamma, reg[2],AGRID, ADEL)
    return(-as.numeric(temp[2]))
  }

  optpars = vector(,nline)
  for (i in 1:nline) {
    optpars[i] = log(oldqs[k+1,i])
  }

  # Optimization using Nelder-Mead Simplex Method
  sol = fminsearch(fun = fct, x0 = optpars, options)

  # Note that in the function the variable is changed to its logarithm
  optpars = as.vector(sol$optbase$xopt)
  for (j in 1:nline) {
    qs[k+1,j] = exp(optpars[j])
  }
  Vals[k+1,1] = - as.numeric(sol$optbase$fopt)

  # The flag associated with exist status of the algorithm. The following values are available:
  # -1 The maximum number of iterations has been reached.
  # 0 The maximum number of function evaluations has been reached.
  # 1 The tolerance on the simplex size and function value delta has been reached.
  exitflag = sol$exitflag

  #if (exitflag == 0) {
  #   sprintf("Iterations in Valuelter exceeded for a %f temp qs %f %f %f %f val %f tem
Rb %f", k * ADEL, qs[k+1,1], qs[k+1,2], qs[k+1,3], qs[k+1,4], Vals[k+1, 1], R_b[k+1, 1])
  #}

  solveR = EvalVOptR(Is,Els, as.vector(qs[k+1,1:nline]), Vs, k * ADEL, tau, c_1, c_1sq, xi,
intr, alpha,beta,gamma, reg[2], AGRID, ADEL)
  Vals[k+1,1] = as.numeric(solveR[2])
  R_b[k+1,1] = as.numeric(solveR[1])

  #sprintf("Step %d: Port %6.2f %6.2f %6.2f %6.2f Raise %6.2f Val %6.2f",
k+1,qs[k+1,1],qs[k+1,2],qs[k+1,3],qs[k+1,4],R_b[k+1,1],Vals[k+1,1])
  }

return(list(Vals, qs, R_b))
```

```
}  
  
#Example of calling the function  
#v = Valuelter(Is_,Els_,Vs_,oldqs_,tau_,c_1_, c_1sq_, xi_,intr_,alpha_,beta_, gamma_, AGRID_,ADEL_)  
#v  
  
#####  
#####  
# Commands  
# -----  
# First iteration  
  v = Valuelter(Is_,Els_,Vs_,oldqs_,tau_,c_1_, c_1sq_, xi_,intr_,alpha_,beta_, gamma_, AGRID_,ADEL_)  
  
# Vs_ and oldqs_ are then used for the next iteration. Manual adjustments may be made for quicker  
convergence.  
  Vs_ = v[1]  
  Vs_ = as.vector(Vs_[[1]])  
  oldqs_ = v[2]  
  oldqs_ = as.matrix(oldqs_[[1]])  
  
# Export Vs_ and oldqs_ to a file  
  write.csv(Vs_, file = "C:/v1.csv")  
  write.csv(oldqs_, file = "C:/q1.csv")  
  
# -----  
# Second iteration  
  v = Valuelter(Is_,Els_,Vs_,oldqs_,tau_,c_1_, c_1sq_, xi_,intr_,alpha_,beta_, gamma_, AGRID_,ADEL_)  
  
# Vs_ and oldqs_ are then used for the next iteration. Manual adjustments may be made for quicker  
convergence.  
  Vs_ = v[1]  
  Vs_ = as.vector(Vs_[[1]])  
  oldqs_ = v[2]  
  oldqs_ = as.matrix(oldqs_[[1]])  
  
# Export Vs_ and oldqs_ to a file  
  write.csv(Vs_, file = "C:/v2.csv")  
  write.csv(oldqs_, file = "C:/q2.csv")  
  
# More iterations ...  
# In the example in the report, 320 iterations are used in the optimization.  
  
#####  
#####  
# Function Solvelt  
# This function calculates the value iteration multiple periods.  
# Model Inputs
```

Incorporating Cyclical Effect with Hidden Markov Models

```
# -----  
# Is: Is: Losses for the four lines, dimension # of records x # of lines  
# Els: Expected value of byline losses  
# startVs: Starting values of the value function  
# startqs: Starting values for q, dimension # of grids x # of lines  
# tau: carrying cost of capita  
# c_1, c_1sq: parameters of the cost function of raising capital at the beginning of the period  
# xi: parameter of the linear cost function of raising capital at the end of the period  
# intr: interest rate  
# alpha, beta, gamma: price model parameters in equation 4.11  
# AGRID and ADEL are the grid size and the delta in the (equally spaced) grid for a  
# tolerance: tolerance of the error  
# -----  
  
Solvelt <- function(Is, Els, startVs, startqs, tau, c_1, c_1sq, xi, intr, alpha, beta, gamma, AGRID, ADEL,  
tolerance){  
  
  nline = ncol(Is)  
  Vals = rep(0, AGRID+1)  
  qs = matrix(0, nrow = AGRID+1, ncol = nline);  
  R_b = rep(0, AGRID+1)  
  
  Vals = startVs  
  qs = startqs  
  
  diff = tolerance + 1;  
  
  k = 1  
  
  # Iterations until the change in values is small enough.  
  while (diff>tolerance) {  
    solve = ValueIter(Is,Els,Vals,qs,tau,c_1,c_1sq,xi,intr,alpha,beta,gamma, AGRID,ADEL)  
    begV = solve[1]  
    begV = as.vector(begV[[1]])  
    qs = solve[2]  
    qs = as.matrix(qs[[1]])  
    R_b = solve[3]  
    R_b = as.vector(R_b[[1]])  
  
    x = begV - Vals  
    diff = sqrt(t(x)%*%x)  
    Vals = begV  
    #sprintf("Step %d (Diff %f):", k, diff);  
  
    #for (j in 0:AGRID) {  
      #   sprintf("Port %6.2f %6.2f %6.2f %6.2f Raise %6.2f Val %6.2f", qs[j+1,1],  
qs[j+1,2],qs[j+1,3],qs[j+1,4],R_b[j+1],Vals[j+1])
```


Incorporating Cyclical Effect with Hidden Markov Models

```
#}  
k=k+1  
}  
  
return(list(Vals, qs, R_b))  
}
```

#Example of calling the function

```
# w = Solvelt(Is_, Els_, startVs_, startqs_, tau_, c_1_, c_1sq_, xi_, intr_, alpha_, beta_, gamma_, AGRID_,  
ADEL_, tolerance_)  
# w
```

APPENDIX C. DATA INPUT FILE FORMAT

Several files are used as input to the R program. Their formats are provided below for reference.

ClaimData.csv (The claim data by line and in aggregate)

In the example in the report, 50,000 data records are used.

Line1	Line2	Line3	Line4	Total
8226240	1987777539	54262714	72291051	2122557544
6024930	1679567680	24316500	43904140	1753813249
6.92E+08	1021541344	21048569	3589903	1738649361
6.04E+08	951349103.8	1.52E+08	19141012	1727041941
7738891	1706704935	4240603	4060778	1722745207
7002996	1554459184	82585419	47840881	1691888481
3.03E+08	1295090953	30943746	7830667	1636541212
18508937	1472161500	1.13E+08	9725430	1613066438
66084911	1350629625	1.66E+08	5154582	1588233450
6677791	1416351704	1.38E+08	15456738	1576109126

vs.csv (The initial values of the firm's value)

In the example in the report, 26 rows corresponding to different levels of capital are used.

```
vs  
1171540963  
1200149723  
1223898613  
1243115689  
1257520137  
1266296795  
1269086658  
1267494314  
1262852082
```

Incorporating Cyclical Effect with Hidden Markov Models

1256341718

1248918163

Startqs.csv (The initial values of the q with 2 dimensions: # of product lines and # of grids)

In the example in the report, 26 rows corresponding to different levels of capital are used of 4 product lines are used.

q1	q2	q3	q4
0.284015	0.235828	0.112161	1.69904
0.568029	0.471656	0.224322	3.39808
0.957708	1.086047	0.847649	6.625276
1.283897	1.650622	1.470977	9.852472
1.737066	2.173433	2.094304	13.07967
2.533581	2.446732	3.66E-05	16.30686
2.985075	2.973314	3.489034	19.43765
2.998459	3.339475	6.978031	19.43765
3.729151	4.006588	7.967162	19.43765
3.678143	4.388207	8.956292	19.43765
4.545081	4.928503	9.526214	19.43765